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Robust non-fragile LQ controllers: the static state feedback case

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This paper describes the synthesis of non-fragile or *resilient* regulators for linear systems. A general framework for fragility is described using state-space methodologies, and the LQ/$\mathcal{H}_2$ static state-feedback problem is examined in detail. We discuss the multiplicative structured uncertainties case, and propose remedies of the fragility problem using an optimization programming framework via matrix inequalities. A special case that leads to a convex optimization framework via linear matrix inequalities (LMIs) will be considered. The benchmark problem is taken as an example to show how special controller gain variations can affect the performance of the closed-loop system.

1. Introduction

In the literature there are different theoretical approaches and several computational techniques which treat the classical problem shown in figure 1:

Given a linear plant P with additive uncertainties ΔP, find a feedback controller K which internally stabilizes the family $P + \Delta P$ and satisfies a given performance measure.

In a recent paper, however, Keel and Bhattacharyya (1997) have shown that, in the case of unstructured uncertainties in the plant, the controllers resulting from either weighted $\mathcal{H}_\infty$,] or l_1 synthesis techniques exhibit a poor stability margin if not implemented exactly. This so-called ‘fragility’ is displayed even though these controllers are optimal when implemented using their nominal parameters. Another example of a compensator that cannot be exactly implemented is from Rosenthal and Wang (1996) where a dynamic controller is going to be designed in order to place the closed loop poles of a linear plant: it can be easily shown that, in one numerical example, the 15th digits numerical implementation of the controller matrices results in an unstable closed-loop system!

Keel and Bhattacharrya (1997) give the following suggestions to overcome the fragility problem:

(1) Develop synthesis algorithms which take into account the uncertainties in the controllers and search for the ‘best’ solution that guarantees a compromise between optimality and fragility;

(2) Parametrize the controller in an appropriate way (lower-order or fixed-structure controllers).

Haddad and Corrado (1997) address and solve a special case of the fragility problem by considering a *structured uncertain* dynamic compensator for a noise-driven linear plant: with the use of classical quadratic Lyapunov bounds (Bernstein and Haddad 1990), Haddad and Corrado (1997) obtain a controller which is proven to be ‘resilient’ in the sense that stability and some measure of performance are maintained even when the controller is not exactly implemented.

It is true that other authors have hinted at the problem of fragility (see, for example, Ackermann 1993, p. 75) and that many critics have dismissed the issue, since robust controllers are not designed to be resilient. On the other hand, the problem is reminiscent of the linear quadratic Gaussian (LQG) optimal controllers which were only useful when implemented on the exact plant, and had no guaranteed robustness margins if the plant was uncertain. This lack of robustness was corrected using linear quadratic Gaussian synthesis with loop transfer recovery (LQG/LTR) (Dorato *et al.* 1995). In addition, even robust controllers will eventually have to be implemented on an actual system using digital hardware, and should be resilient both to implementation errors and to tuning (Ackermann 1993).

The aim of this paper is to extend the ideas in Keel and Bhattacharyya (1997), Haddad and Corrado (1997) and, with reference to the scheme of figure 2, to analyse the robust fragility problem for a static full-state feedback controller synthesis problem by considering the combined effect of structured uncertainties in the plant and in the compensator. Note that it is reasonable to consider only structured uncertainties in the controller since the designer can exactly choose the structure even though he may not be able to implement that nominal configuration. The basic idea is that, instead of computing the controller as a single point in the parameters space, we look for a set of controllers allowing the par-

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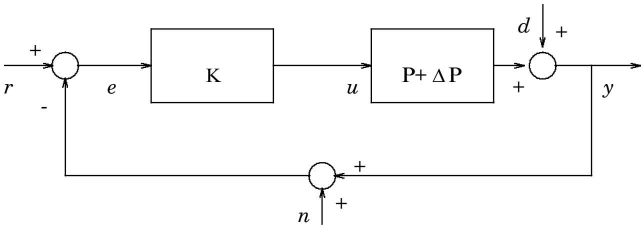

Figure 1. Robust control scheme.

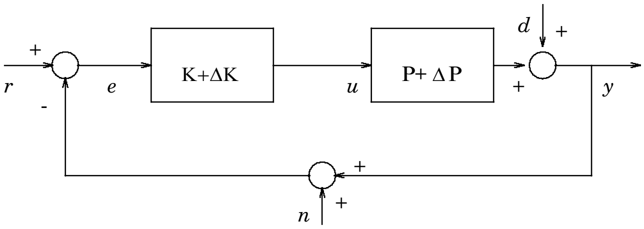

Figure 2. Robust fragility control scheme.

ameters to lie in a region of uncertainty. This is reminiscent of the design of Ackermann (1993) and Barmish *et al.* (1992).

This paper is organized as follows. In § 2, we present the synthesis of static state feedback controllers for linear systems while allowing structured uncertainties in the feedback gain matrix. We then further restrict our study to multiplicative structured uncertainties in the plant. In § 3, a numerical example using linear matrix inequalities as a computational tool is given. Our conclusions and directions for future research are given in § 4.

2. Outline of the problem

Consider the following time-varying linear system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0 \\ z(t) = C_z x(t) + D_z u(t) \end{cases} \tag{1}$$

where

- $x(t) \in \mathbb{R}^n$, is the state vector, $u(t) \in \mathbb{R}^m$, is the control input, $z(t) \in \mathbb{R}^l$ is the objective measurements vector,
- $A(t)$, $t \geq 0$, contains affine uncertainties (see Gahinet *et al.* 1995) of the form

$$A(t) = A_0 + \sum_{r=1}^{q} \alpha_r(t) A_r = A_0 + \delta A$$

where the scalar coefficients $\alpha_r(t)$, $t \geq 0$, are Lebesgue measurable functions on $[0, \infty)$ representing unknown coefficients grouped in the vector

$$\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_q]$$

whose values belong to an uncertainty interval Ξ

$$\Xi = \{\alpha_r(t) | \underline{\alpha}_r \leq \alpha_r(t) \leq \overline{\alpha}_r, \quad 1 \leq r \leq q, \quad t \geq 0\} \tag{2}$$

In order to simplify the notation, we suppress the time dependence of the α_rs when no confusion arises. The system (1) can then be written in the form

$$\begin{cases} \dot{x}(t) = (A_0 + \delta A)x(t) + Bu(t) \\ z(t) = \quad C_z x(t) \quad + D_z u(t) \end{cases} \tag{3}$$

Now, we assume that the initial condition $x(0) = x_0$ is a random variable with zero mean and covariance matrix equal to I_n and proceed to find a state-feedback compensator $u(t) = Kx(t)$ which minimizes the linear quadratic (LQ) performance index, given by

$$\mathcal{J} = \mathcal{E}\left[\int_0^\infty z^{\mathrm{T}}(t) z(t)\, \mathrm{d}t\right] \tag{4}$$

where $\mathcal{E}$ denotes the expectation with respect to the initial state x_0. As standard assumptions, we suppose that the matrices C_z and D_z are such that $C_z^{\mathrm{T}} D_z = 0$ and $D_z^{\mathrm{T}} D_z > 0$.

2.1. *Non-fragile controller synthesis scheme*

Although one finds a controller $u = Kx$ to minimize (4), the controller actually implemented is

$$u(t) = (K + \delta K)x(t) \tag{5}$$

where K is the nominal controller gain, and the term δK, which belongs to a closed and bounded subset of $\mathbb{R}^{m \times n}$, Δ_K, and contains the element $\delta K = 0$, it represents controller gain variations. In this case, the performance index (4) is a function of K, the uncertain term δK, and the uncertainties $\boldsymbol{\alpha}$ in (3) and hence $\mathcal{J} = \mathcal{J}(K, \delta K, \boldsymbol{\alpha})$.

The following standard robustness analysis procedure can then be applied to test the fragility of the controller.

(1) Letting $\delta K = 0$, a 'nominal' controller $\bar{K}$ is designed so that the guaranteed-cost (Bernstein and Haddad 1990)

$$\bar{J}(K) = \sup_{\boldsymbol{\alpha} \in \Xi} \mathcal{J}(K, 0, \boldsymbol{\alpha})$$

is minimized.

(2) Suppose the controller to be 'nominal', $\bar{K}$, and compute the LQ/$\mathcal{H}_2$ guaranteed cost taking into account the uncertainty in the controller.

The design which explicitly takes into account the fragility issue is instead: compute a new controller $\tilde{K}$ by solving the new guaranteed cost problem

$$\min_{K \in \mathbb{R}^{m\times n}} \tilde{J}(K) \tag{6}$$

where

$$\tilde{J}(K) = \sup_{\delta K \in \Delta_K, \boldsymbol{\alpha} \in \Xi} \mathcal{J}(K, \delta K, \boldsymbol{\alpha})$$

With this scheme in mind, we now study the multiplicative uncertainty case of equation (5) in greater detail.

2.2. *Multiplicative structured uncertainties*

Let the nominal state-feedback matrix K be an $m \times n$ $(m < n)$ matrix. If we allow relative percentage drift from the nominal entries of the matrices K and represent each entry of the perturbed matrix as a multiplicative scalar uncertainty, we have

$$(K + \delta K) = \begin{bmatrix} k_{11}(1+\delta_{11}) & \cdots & k_{1n}(1+\delta_{1n}) \\ \vdots & \ddots & \vdots \\ k_{m1}(1+\delta_{m1}) & \cdots & k_{mn}(1+\delta_{mn}) \end{bmatrix} = \begin{bmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{m1} & \cdots & k_{mn} \end{bmatrix} + \begin{bmatrix} k_{11}\delta_{11} & \cdots & k_{1n}\delta_{1n} \\ \vdots & \ddots & \vdots \\ k_{m1}\delta_{m1} & \cdots & k_{mn}\delta_{mn} \end{bmatrix} \tag{7}$$

where $\delta_{ij} \in \Delta = \{\delta_{ij} | -1 < \underline{\delta}_{ij} \le \delta_{ij} \le \overline{\delta}_{ij} < 1, i = 1, \dots, m, j = 1, \dots, n\}$. Equation (7) can be rewritten as

$$K + \delta K = \sum_{i=1}^{m} e_i [1+\delta_{i1} \quad \cdots \quad 1+\delta_{in}] \begin{bmatrix} k_{i1} & & \\ & \ddots & \\ & & k_{in} \end{bmatrix} = \sum_{i=1}^{m} e_i v_i^{\mathrm{T}} \mathcal{K}_i \tag{8}$$

where e_i, $i = 1, \dots, m$, is the canonical basis column vector of $\mathbb{R}^m$ and v_i^{T}, $i = 1, \dots, m$ is an n-dimensional row vector which collects the terms $[1+\delta_{i1} \ \dots \ 1+\delta_{in}]$. In this case, the closed-loop system is given by

$$\dot{x}(t) = \left(A_0 + \sum_{i=1}^{q} \alpha_i A_i + B \sum_{i=1}^{m} e_i v_i^{\mathrm{T}} \mathcal{K}_i\right) x(t) = \left(A_0 + \sum_{r=1}^{q} \alpha_r A_r + \sum_{i=1}^{m} b_i v_i^{\mathrm{T}} \mathcal{K}_i\right) x(t) \tag{9}$$

where b_i is the ith n-dimensional column of the matrix B. The product $b_i v_i^{\mathrm{T}}$ can be arranged as

$$\mathcal{B}_{i0} + \delta_{i1}\mathcal{B}_{i1} + \cdots + \delta_{i2}\mathcal{B}_{in} = \mathcal{B}_{i0} + \sum_{j=1}^{n} \delta_{ij}\mathcal{B}_{ij}$$

where $\mathcal{B}_{i0} = [b_i \ \dots \ b_i]$ and $\mathcal{B}_{ij} = b_i e_j^{\mathrm{T}}$. Finally, the closed-loop system matrix has the following structured uncertainty form

$$A_0 + \sum_{r=1}^{q} \alpha_r A_r + \sum_{i=1}^{m} \left(\mathcal{B}_{i0} + \sum_{j=1}^{n} \delta_{ij}\mathcal{B}_{ij}\right)\mathcal{K}_i = \tilde{\mathcal{A}}(\boldsymbol{\alpha}) + \sum_{i=1}^{m} \tilde{\mathcal{B}}_i(\boldsymbol{\delta}_i)\mathcal{K}_i \tag{10}$$

where $\boldsymbol{\alpha} = [\alpha_1, \dots, \alpha_q]$ and $\boldsymbol{\delta}_i = [\delta_{i1}, \dots, \delta_{in}]$, $i = 1, \dots, m$. The same scheme is applied to the expression of the objective measurements $z(t)$

$$z(t) = C_z x(t) + \sum_{i=1}^{m} \left(\mathcal{D}_{z,i0} + \sum_{j=1}^{n} \delta_{ij}\mathcal{D}_{z,ij}\right)\mathcal{K}_i x(t) = C_z x(t) + \sum_{i=1}^{m} \tilde{\mathcal{D}}_{z,i}(\boldsymbol{\delta}_i)\mathcal{K}_i x(t)$$

If the 'fictitious' set of inputs $\tilde{u}_i(t) = \mathcal{K}_i x(t) \in \mathbb{R}^n$, $i = 1, \dots, m$, is introduced, the LQ/$\mathcal{H}_2$ synthesis problem can be restated as: *given the linear uncertain system*

$$\begin{cases} \dot{x}(t) = \tilde{\mathcal{A}}(\boldsymbol{\alpha}) x(t) + \sum_{i=1}^{m} \tilde{\mathcal{B}}_i(\boldsymbol{\delta}_i)\, \tilde{u}_i(t) \\ z(t) = C_z x(t) + \sum_{i=1}^{m} \tilde{\mathcal{D}}_{z,i}(\boldsymbol{\delta}_i)\, \tilde{u}_i(t) \end{cases} \tag{11}$$

where $\tilde{\mathcal{A}}(\boldsymbol{\alpha})$ *and* $\tilde{\mathcal{B}}_i(\boldsymbol{\delta}_i)$ *are from* (10), *find diagonal matrices* $\mathcal{K}_1, \dots, \mathcal{K}_m$, *such that, if* $\tilde{u}_i(t) = \mathcal{K}_i x(t)$, *the guaranteed cost*

$$\tilde{J} = \sup_{\boldsymbol{\alpha} \in \Xi, \boldsymbol{\delta}_i \in \Delta} \mathcal{E}_{x_0}\left[\int_0^{\infty} z^{\mathrm{T}}(t) z(t)\, \mathrm{d}t\right] \tag{12}$$

attains its minimum value.

A matrix inequality formulation (Boyd *et al.* 1994) of this problem is as follows.

Problem MI: Find matrices $\Omega > 0$, $Q > 0$, and $Y_1, \dots, Y_m$ such that $\mathrm{tr}(\Omega)$ is minimized and (see (13), bottom of page 162), for all $\tilde{\alpha}$ and $\tilde{\delta}_i$ such that

$$\tilde{\alpha} \in \tilde{\Xi} = \{\{\underline{\alpha}_1, \overline{\alpha}_1\} \times \cdots \times \{\underline{\alpha}_q, \overline{\alpha}_q\}\}$$

$$\tilde{\delta}_i \in \Delta = \left\{\{\underline{\delta}_{i1}, \overline{\delta}_{i1}\} \times \cdots \times \{\underline{\delta}_{in}, \overline{\delta}_{in}\}\right\}$$

$$\begin{bmatrix} \Omega & I \\ I & Q \end{bmatrix} > 0 \tag{14}$$

$$\begin{aligned} (Y_1 Q^{-1})_{i,j} &= 0 \\ \vdots \quad & \vdots \quad \vdots \\ (Y_m Q^{-1})_{i,j} &= 0 \end{aligned} \tag{15}$$

$$i = 1, \ldots, n, \quad j = 1, \ldots, n, \quad i \neq j.$$

The entries k_{ij} of the controller K are obtained from

$$\begin{aligned} (Y_1 Q^{-1})_{i,i} &= k_{1i} \\ \vdots \quad & \vdots \quad \vdots \\ (Y_m Q^{-1})_{i,i} &= k_{mi} \end{aligned}$$

$$i = 1, \ldots, n$$

Note that the $2^{q+m\times n}$ convex constraints (13) take into account all the possible combinations between the upper and lower values of the uncertainty intervals. The above formulation does not precisely correspond to the dynamic optimization problem expressed by equations (11) and (12). This formulation is a computational paradigm that gives an upper bound to the performance index (12) because of the sufficiency criterion expressed by the evaluation of the inequalities (13) over the upper and lower values of the sets where the uncertainties α_r and δ_{ij} are defined. This means that if the constraints are found not to be fully satisfied during a particular search algorithm nothing can be said about a possible solution of the problem.

The most important fact on this optimization problem relies on its non-*convexity* characteristics. This observation can be argumented by observing the set of constraints (15), a matrix product between $Y_1, \ldots, Y_m$ and the *inverse* of Q which imposes that the off-diagonal terms of $\mathcal{K}_i$ must be zero. The matrix product (15) between independent variables of the optimization process (see Boyd *et al.* (1994) for a comprehensive classification of convex constraints and functions) is no longer a convex function in its arguments (see, as a simple example, the scalar function $f(x,y) = x/y$, where x and y are scalars that belong to an arbitrary $\mathbb{R}^2$ box) and this means that the feasible set of the optimization problem **MI** is non-*convex*. The obvious consequence of this fact is that algorithms based on the linear matrix inequalities computational paradigm (Boyd *et al.* 1994) cannot treat this problem.

When a single input system is considered $(m = 1)$, we have only one diagonal matrix $\mathcal{K}$ whose diagonal elements represent the controller coefficients k_i, $i = 1, \ldots, n$ but the structure of the problem **MI** still holds.

2.2.1. *A special case.* The only case when the problem **MI** can be reduced to a set of linear matrix inequalities is when the relative drifts from the nominal entries of the controller matrix K are the same

$$\delta_{ij} = \delta, \qquad i = 1, \ldots, n, \qquad j = 1, \ldots, m$$

as was assumed in Haddad and Corrado (1997). The optimization problem can be stated as follows.

Problem LMI1: Find matrices $\Omega > 0$, $Q > 0$ and Y such that $\mathrm{tr}(\Omega)$ is minimized and (see (16))

$$\begin{bmatrix} \Omega & I \\ I & Q \end{bmatrix} > 0 \tag{17}$$

Finally, the controller K is equal to $K = YQ^{-1}$. Note that $\tilde{\alpha}_r \in \tilde{\Xi}$, $\tilde{\delta} \in \Delta = \{\underline{\delta}, \overline{\delta}\}$. In this case, the number of convex constraints (16) is equal to 2^{q+1} because an affine uncertainties structure with $q+1$ parameters in the closed loop system is obtained.

It must be noted that in Haddad and Corrado (1997) the non-fragile design of reduced order dynamic compensators was discussed and, as a final computational result, a guaranteed cost problem which turned out to be a non-*convex* optimization problem has been obtained.

$$\begin{bmatrix} Q\tilde{\mathcal{A}}^{\mathrm{T}}(\tilde{\alpha}) + \tilde{\mathcal{A}}(\tilde{\alpha})Q + \sum_{i=1}^{m}\left(Y_i^{\mathrm{T}}\tilde{\mathcal{B}}_i^{\mathrm{T}}(\tilde{\delta}_i) + \tilde{\mathcal{B}}_i(\tilde{\delta}_i)Y_i\right) & \left(C_z Q + \sum_{i=1}^{m}\tilde{\mathcal{D}}_{z,i}(\tilde{\delta}_i)Y_i\right)^{\mathrm{T}} \\ \left(C_z Q + \sum_{i=1}^{m}\tilde{\mathcal{D}}_{z,i}(\tilde{\delta}_i)Y_i\right) & -I \end{bmatrix} \leq 0 \tag{13}$$

$$\begin{bmatrix} Q\left(A_0 + \sum_{r=1}^{q}\tilde{\alpha}_r A_r\right)^{\mathrm{T}} + \left(A_0 + \sum_{r=1}^{q}\tilde{\alpha}_r A_r\right)Q + (1+\tilde{\delta})(Y^{\mathrm{T}}B^{\mathrm{T}} + BY) & (C_z Q + (1+\tilde{\delta})D_z Y)^{\mathrm{T}} \\ (C_z Q + (1+\tilde{\delta})D_z Y) & -I \end{bmatrix} \leq 0 \tag{16}$$

In our approach instead, we considered only full state static feedback compensator but it has been shown that the non-fragile synthesis problem reduces to a simpler and easier computational paradigm based on linear matrix inequalities.

Also, if the structure of the controller in this particular case is going to be examined

$$(1+\delta)K$$

where K is going to be found using **LMI1**, the non-fragile synthesis problem can be viewed as a guaranteed cost synthesis with respect to the uncertainties in the plant but with an imposed *gain margin* on the controller entries. Moreover, this synthesis problem is strictly similar to the mathematical problem shown in Chang and Peng (1972) where a guaranteed cost approach for a quadratic performance index was studied for a linear system of the type

$$\dot{x} = A(q)x + B(q)u$$

where $A(q) = A_0 + \sum_{i=1}^{n'} q_i(t)A_i, \quad -1 \le q_i(t) \le 1, \quad i = 1,2,\ldots,n'$ and $B(q) = q_b(t)B,\ 1 \le q_b(t) \le b$. It is easy to see that the scalar uncertainty in the controller can be regarded as a scalar uncertainty of the input matrix of the linear system (1) when the closed loop equations are considered.

The following numerical experiments compare the controller designed using the computational paradigm **LMI1** with respect to that obtained by using the robust LQ/$\mathcal{H}_2$ synthesis.

3. Numerical experiments

Consider the mechanical system shown in figure 3, known as the 'benchmark problem' (Gahinet *et al.* 1995), where

(1) $u(t)$ is the control input and $x_1(t), x_2(t)$ are the positions, with respect to a reference system, of the masses m_1, m_2, respectively;

(2) the masses† m_1, m_2 are equal to 1 and the stiffness† $p(t),\ t \ge 0$, is an uncertain parameter whose values belong to the interval $[0.5, 2]$.

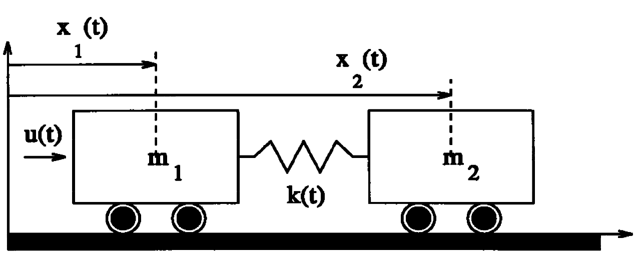

Figure 3. Benchmark problem.

† Note that all quantities are measured in appropriate units.

The linear time-varying model which describes the behaviour of the system is given by

$$\left\{\begin{aligned} \begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\\ \dot{x}_4(t)\end{bmatrix} &= \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -p(t) & p(t) & 0 & 0\\ p(t) & -p(t) & 0 & 0\end{bmatrix} \times \begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix} + \begin{bmatrix}0\\0\\1\\0\end{bmatrix} u(t) \\ z(t) &= \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} u(t)\end{aligned}\right. \tag{18}$$

It is easy to see that we can represent (18) as an affine uncertain model where the matrix $A(t)$, $t \ge 0$, is given by

$$\begin{aligned} A(t) &= \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} + p(t) \begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\end{bmatrix} \\ &= A_0 + p(t)A_1 \end{aligned}$$

and the matrices B, C_z and D_z are constant.

Using the MATLAB™ LMI toolbox, the `lmiedit` editor and the `mincx` function, we designed a nominal LQ/$\mathcal{H}_2$ static state-feedback controller by solving the problem **LMI1** with $\delta = 0$, $q = 1$, $\bar{\alpha} = 2$ and $\underline{\alpha} = 0.5$. We found that the guaranteed LQ/$\mathcal{H}_2$ performance was equal to 7.1913 and the controller gain vector

$$\begin{aligned}\bar{K} &= [k_1 \quad k_2 \quad k_3 \quad k_4] \\ &= [-4.1066 \quad 3.1081 \quad -2.8628 \quad -1.4067]\end{aligned} \tag{19}$$

3.1. *First experiment*

As a first experiment, we tested the fragility of the nominal controller: an affine family of uncertain controllers given by

$$(1+\delta)\bar{K} \tag{20}$$

was generated, where $\delta \in [-\bar{\delta}, \bar{\delta}]$ is a parameter which corresponds to a drift in the nominal values k_i, $i = 1, \dots, 4$. In this case each component of K was considered to have same relative uncertainty range (Haddad and Corrado 1997). A *gain margin analysis* (Anderson and Moore 1990, Dorato *et al.* 1995), by varying $\bar{\delta}$ has been performed using MATLAB™ LMI Toolbox standard routines `quadstab` and `pdlstab`: the values of $\bar{\delta}$ for which the closed-loop system is no longer quadratically stable (Boyd *et al.* 1994, Gahniet *et al.* 1994, 1995) or, less conservatively, does not admit a parameter-dependent Lyapunov function (Boyd *et al.* 1994, Gahinet *et al.* 1994, 1995) were checked. With reference to the closed-loop system

$$\dot{x}(t) = (A_0 + pA_1 + (1+\delta)B\bar{K})x(t)$$

$$p \in [0.5,\ 2], \qquad \delta \in [-\bar{\delta}, \bar{\delta}]$$

we observed that quadratic stability is lost if $\bar{\delta} > 0.49$, and the system does not admit a parameter-dependent Lyapunov function if $\bar{\delta} > 0.88$.

3.2. *Second experiment*

The aim of the second experiments was to compare the guaranteed performance of the nominal controller with the performance of a controller designed by taking explicitly into account the fragility issue. We computed the guaranteed LQ/$\mathcal{H}_2$ cost for 100 uncertainty intervals $[-\bar{\delta}, \bar{\delta}]$ where $\bar{\delta} \in \{0, 0.01, 0.02, \dots, 0.98, 0.99\}$. The following computational procedure for each value of $\bar{\delta}$ was realized:

(1) the controller K was fixed to its nominal value (19), $\bar{K}$, and the guaranteed $\mathcal{H}_2$ performance was computed by solving the following convex optimization problem;

Problem LMI2: Find matrices $\Omega > 0$, $Q > 0$ such that $\mathrm{tr}(\Omega)$ is minimized and (see (21))

$$\begin{bmatrix} \Omega & I \\ I & Q \end{bmatrix} > 0 \tag{22}$$

where $\tilde{\delta} \in \Delta$ and $\tilde{p} \in \{0.5,\ 2\}$. This problem has been solved using the `mincx` function.

(2) A 'non-fragile' controller and the corresponding guaranteed $\mathcal{H}_2$ performance was computed using the convex optimization paradigm **LMI1** and the function `mincx`. Figure 4 shows a plot of the guaranteed cost versus $\bar{\delta}$ for the nominal and the 'non-fragile' controller.

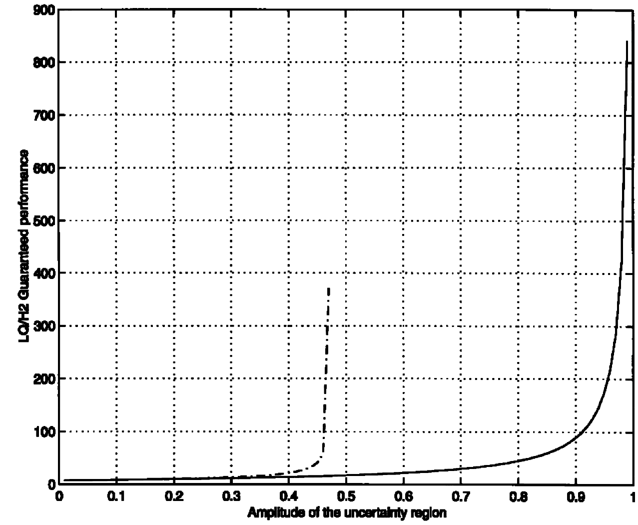

Figure 4. LQ/$\mathcal{H}_2$ guaranteed cost vs. $\bar{\delta}$: *nominal controller* → dashed line; *non-fragile controller* → solid line.

Obviously the guaranteed cost for case (1) can be computed until the convex constraints (21), (22) are feasible; this means the the closed loop system is quadratically stable. On the other hand, for $\bar{\delta} = 0.49$ the controller computed using the convex optimization formulation **LMI1** given by equations (16) and (17) was

$$K = [-6.0620 \quad 5.3883 \quad -4.3872 \quad -0.1621]$$

and the guaranteed LQ/$\mathcal{H}_2$ performance in this case was equal to 16.8341.

It is easy to observe from figure 4 that a design which takes into account the uncertainties in the controller guarantees quadratic stability and 'acceptable' guaranteed performance in a wider range of regulator parameter variations.

4. Conclusions

In this paper the effect of LQ/$\mathcal{H}_2$ robust synthesis of uncertain, static state feedback controllers, for linear systems with structured uncertainties in the dynamic matrix was considered. A simple but significant result regarding the computational equivalence with a non-convex problem has been obtained when multiplicative structured uncertainties are allowed in the controller. A guaranteed-cost approach which turned out to be a worst-case gain margin optimal synthesis problem was formulated in a particular case using a linear matrix

$$\begin{bmatrix} Q(A_0 + \tilde{p}A_1 + (1+\tilde{\delta})B\bar{K})^{\mathrm{T}} + (A_0 + \tilde{p}A_1 + (1+\tilde{\delta})B\bar{K})Q & (C_z + (1+\tilde{\delta})D_z\bar{K})^{\mathrm{T}}Q \\ Q(C_z + (1+\tilde{\delta})D_z\bar{K}) & -I \end{bmatrix} \le 0 \tag{21}$$

inequalities computational paradigm, and used in the numerical experiments involving the benchmark control problem. The results show that the non-fragile controller exhibits a larger stability margin and there exists a trade-off between controller resilience and system performance.

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