Robustness Analysis of Polynomials with Linearly Correlated Uncertain Coefficients in l^p-normed Balls

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Abstract

The aim of this paper is to obtain the uncertain value set in the complex plane for systems with real and complex parameters that are known to lie inside a ball in a weighted l^p-norm. It generalizes previously available results and may be used to test the robust stability of polynomials whose coefficients lie in a weighted l^p ball.

1 Introduction

The aim of this paper is to obtain the uncertain value set in the complex plane for systems with real and complex parameters that are known to lie inside a ball in a weighted l^p-norm. This work is motivated by the work of Tsypkin and Polyak [1], [2] where the stability question for uncertain families of polynomials with real coefficients is resolved. This research generalized many results for the cases $p = 1, 2, \infty$ [3], [4], [5]. In fact, it was shown that the problem of determining the Hurwitz stability of a polynomial with coefficients in a ball is reducable to a one-parameter search. On the other hand, the case for which the true polynomial coefficients depend affinely on a set of structured parameters is very important from the control point of view [6]. In [2] and [7] it is shown that the Schur and affine Hurwitz cases need a one-parameter optimization when $p = 1, 2, \infty$ and a two-parameter optimization otherwise. In these papers, the authors apply the separating line principle by rotating the value set in all possible complex plane directions. Unfortunately, this does not give any information about the shape of the value set for the general case. Our approach is to fix a complex plane direction and then to obtain the maximum of the uncertain value set for that direction (after the set is translated so it is centered at the origin). The solution follows the work in [8] where the directional maximum is obtained by rotating the value set and maximizing it along the real axis. We show that in the most general case, a two-parameter optimization problem needs to be solved, but in many cases of interest, the problem can be reduced to a one-parameter search problem. Our results are extendable to a robust stability

test in a straightforward manner, giving at the same time the critical member of the family for which stability is first lost. We show that if the value set is maximized along the center polynomial direction, a simple check of the "maximum polynomial" is necessary in order to test the stability of the family. Moreover, this allows us to compute the stability margin. Since the problem is solved for any s in the complex plane, the two-parameter approach is valid for any Γ-stability region.
This paper is structured as follows; In section 2, we treat the real-coefficients case. In section 3 we treat the complex-coefficients case. Some applications are presented in section 4. Our conclusions are given in section 5. All proofs are provided in the appendices. In the remaining of the paper, $\mid x \mid$ denotes the absolute value of x for a real x.

2 Real-Coefficients Case

First, we consider the family of polynomials

$$P_\gamma(s) = \sum_{i=0}^{m} p_i(s) q_i \tag{1}$$

where $p_i(s),\ i = 0, \cdots, m$ are real-coefficients polynomials and the q_i are restricted to be in an l^p-normed ball, i.e,

$$\parallel \mathbf{q} \parallel_p = \sum_{i=0}^{m} \left| \frac{q_i - q_i^0}{\alpha_i} \right|^p \leq \gamma^p \tag{2}$$

where $1 \leq p < \infty$. For the case where $p = \infty$, the usual definition of

$$\parallel \mathbf{q} \parallel_\infty = \max_{1 \leq i \leq n} \left| \frac{q_i - q_i^0}{\alpha_i} \right| \tag{3}$$

is used. All the weights α_i are positive and $p_0(s) = \sum_{i=0}^{m} p_i(s) q_i^0$. Then we can write (1) and (2) using $r_i = (q_i - q_i^0)/\alpha_i$ as

$$P_\gamma(s) = p_0(s) + \sum_{i=0}^{m} p_i(s) \alpha_i r_i, \quad \sum_{i=0}^{m} |r_i|^p \leq \gamma^p \tag{4}$$

Given a complex plane direction $e^{j\phi} = \cos\phi + j \sin\phi$, we are interested in computing the maximum of $\sum_{i=0}^{m} p_i(s) \alpha_i r_i$ for a fixed s along the given direction $e^{j\phi}$. Define $A_i(s, \phi)$ and $B_i(s, \phi)$ as, respectively, the real and imaginary parts of $p_i(s) e^{-j\phi}$, then it becomes clear that the problem is equivalent to maximizing

$$F(s, \phi) = \sum_{i=0}^{m} \alpha_i r_i A_i(s, \phi) \tag{5}$$

subject to

$$\sum_{i=0}^{m} \alpha_i r_i B_i(s, \phi) = 0, \quad \text{and} \sum_{i=0}^{m} |r_i|^p \leq \gamma^p \tag{6}$$

To solve this problem, the Lagrange multipliers technique is used by forming

$$G(\mathbf{r}, \lambda, \eta) = \sum_{i=0}^{m} \alpha_i r_i A_i(s, \phi) + \eta \sum_{i=0}^{m} \alpha_i r_i B_i(s, \phi) - \lambda(\sum_{i=0}^{m} |r_i|^p - \gamma^p) \tag{7}$$

where $\mathbf{r} = (r_1, \cdots, r_m)^T$. We then have the following result.

Theorem 1 *Let $1 < p < \infty$. The maximum $G(s, \eta)$ is given by*

$$G^*(s, \phi) = \gamma \left(\sum_{i=0}^{m} |\alpha_i A_i(s, \phi) + \eta^* \alpha_i B_i(s, \phi)|^{p/(p-1)} \right)^{(p-1)/p} \tag{8}$$

where $\eta^* \in \mathbf{R}$ is the solution to the equation

$$\sum_{i=0}^{m} \alpha_i^{p/(p-1)} B_i(s, \phi) |A_i(s, \phi) + \eta B_i(s, \phi)|^{1/(p-1)} \mathrm{sgn}[A_i(s, \phi) + \eta B_i(s, \phi)] = 0 \tag{9}$$

Proof: See Appendix A. ■ The solution η^* always exists and is unique (except in the trivial case $B_i(s, \phi) = 0, i = 0, \cdots, m$). In order to prove this claim first note that that the term on the left of (9) is continuous and tends to ∞ as $\eta \to \infty$ and to $-\infty$ as $\eta \to -\infty$. The uniqueness of the solution comes from the fact that each of the terms of the sum in (9) is a monotonically increasing function of η and so is the sum. Furthermore, it is easy to see that

$$\min_{i=0,\cdots,m} \left\{ -\frac{A_i(s, \phi)}{B_i(s, \phi)} \right\} \leq \eta^* \leq \max_{i=0,\cdots,m} \left\{ -\frac{A_i(s, \phi)}{B_i(s, \phi)} \right\} \tag{10}$$

For some special cases, an analytic solution to the previous problem is available as described next.

Lemma 1 *[9] When $p = 2$, (9) is satisfied by*

$$\eta^* = \frac{-\sum_{i=0}^{m} \alpha_i^2 B_i(s, \phi) A_i(s, \phi)}{\sum_{i=0}^{m} \alpha_i^2 B_i^2(s, \phi)} \tag{11}$$

Proof: Obvious since $p - 1 = 1$. ■ In order to simplify the treatment of the case $p = \infty$ it is customary to assume without loss of generality that the polynomials $p_i(s)e^{j\phi}$ are ordered such that

$$\frac{-A_0(s, \phi)}{B_0(s, \phi)} \leq \frac{-A_1(s, \phi)}{B_1(s, \phi)} \leq \cdots \leq \frac{-A_m(s, \phi)}{B_m(s, \phi)} \tag{12}$$

Equation (9) now transforms into

$$\sum_{i=0}^{m} \alpha_i B_i sgn(A_i + \eta B_i) = 0 \tag{13}$$

The function above is piecewise constant, so we have to look for a sudden change of sign in (13). This may be carried out by simple inspection of the function $\sum_{i=0}^{m} \alpha_i |B_i| \epsilon_i$ where all the ϵ_i are initially set to -1 and orderly changed to 1. If we denote by $l \in \{0, \cdots, m\}$ the index for which the function changes its sign as ϵ_l changes from -1 to 1, the solution will clearly be $\eta^* = -A_l/B_l$, and since for this case

$$G^* = \gamma \sum_{i=0}^{m} \alpha_i |A_i + \eta^* B_i| \tag{14}$$

we obtain the following lemma.

Lemma 2 *If $p = \infty$, then*

$$G^* = \frac{\gamma}{B_l} \sum_{i=0}^{m} \alpha_i |A_i B_l - A_l B_i|$$

■ For the case $p = 1$ the function in (9) is piecewise linear although not necessarily continuous. Let $l(\eta) \in \{0, \cdots, m\}$ be the index of the maximum of $|\alpha_i A_i + \eta \alpha_i B_i|$, $i = 0, \cdots, m$ for a given real η. Then, equation (9) becomes

$$A_{l(\eta)} + \eta B_{l(\eta)} = 0 \tag{15}$$

It is obvious from the definition of $l(\eta)$ that this is a piecewise constant function. Since the maximum of $|\alpha_i A_i + \eta \alpha_i B_i|$ can not be zero, we must look to a sudden change of sign in $A_{l(\eta)} + \eta B_{l(\eta)}$ which can take place only at the discontinuities of $l(\eta)$. Let η^* be the value for which this change of sign occurs.

Lemma 3 *If $p = 1$, then*

$$G^* = \gamma \alpha_{l(\eta^*)} |A_{l(\eta^*)} + \eta^* B_{l(\eta^*)}| \tag{16}$$

Proof: G^* can be obtained from (8) as

$$G^* = \gamma \lim_{\eta \to \eta^*} \alpha_{l(\eta)} |A_{l(\eta)} + \eta B_{l(\eta)}| \tag{17}$$

But since $|A_{l(\eta)} + \eta B_{l(\eta)}|$ is continuous everywhere it is possible to write (16). ■

Another well-known case [1] is obtained when $s = j\omega$ and $p_i(j\omega) = (j\omega)^i, \ i = 0, \cdots, m$. In this case, we let

$$\begin{array}{rclrcl} A_i(j\omega) & = & (j\omega)^i \cos\phi, & B_i(j\omega) & = & (j\omega)^i \sin\phi, \ i \text{ even} \\ A_i(j\omega) & = & -j^{i-1}\omega^i \sin\phi, & B_i(j\omega) & = & j^{i-1}\omega^i \cos\phi, \ i \text{ odd} \end{array} \tag{18}$$

and

$$X = \sum_{i \text{ even}} |\alpha_i \omega^i|^{p/(p-1)}, \quad Y = \sum_{i \text{ odd}} |\alpha_i \omega^i|^{p/(p-1)} \tag{19}$$

We then obtain the following result.

Lemma 4 *[1] For* $s = j\omega$, $p_i(j\omega) = (j\omega)^i$, $i = 0, \cdots, m$, *the directional maximum in (8) is found to be*

$$G^* = \gamma \left[\frac{(XY)^{p-1}}{Y^{p-1}|\cos\phi|^p + X^{p-1}|\sin\phi|^p} \right]^{1/p} \tag{20}$$

Proof: Note first that equation (9) can be rewritten such that

$$|\cos\phi + \eta^* \sin\phi| = |-\sin\phi + \eta^* \cos\phi| \frac{X^{p-1}|\sin\phi|^{p-1}}{Y^{p-1}|\cos\phi|^{p-1}} \tag{21}$$

$$sgn(-\sin\phi + \eta^* \cos\phi) \cdot sgn(\cos\phi) = -sgn(\cos\phi + \eta^* \sin\phi) \cdot sgn(\sin\phi) \tag{22}$$

After some straightforward calculations it is possible to write

$$|-\sin\phi + \eta^* \cos\phi| = \frac{X^{p-1}|\sin\phi|^{p-1}}{Y^{p-1}|\cos\phi|^p + X^{p-1}|\sin\phi|^p} \tag{23}$$

and (21) and (23) can be used in (8) to obtain (20), which gives the polar description of an l^p ellipse with the axes parallel to the coordinate axes. ■ Note that the cases where $\phi = 0$ and $\phi = \pi/2$ are particularly interesting since they lead to

$$\begin{aligned} G^* &= \gamma X^{(p-1)/p}; \quad \phi = 0 \\ G^* &= \gamma Y^{(p-1)/p}; \quad \phi = \pi/2 \end{aligned} \tag{24}$$

3 Complex-Coefficients Case

It has already been pointed out by some authors that different couplings between the real and imaginary part can be considered in the complex case. For instance, Katbab and Jury [10], consider the case in which the parameter $q_i = q_{iR} + jq_{iI}$ is structured in a way such that $|q_{iR}|$ and $|q_{iI}|$ are independent; Kogan [11], following Chapellat et al.[12], considers the l^2-norm. The work of Soh [13] deals with any l^p norm correlating the real and imaginary parts of the parameters considered to be independent. In this section, we will generalize and extend all the previous results. In order to provide enough flexibility in coupling the real and imaginary parts as well as the different coefficients, we consider the following l^p ball

$$\sum_{i=0}^{m} \left(\left| \frac{q_{iR} - q_{iR}^0}{\alpha_i} \right|^r + \left| \frac{q_{iI} - q_{iI}^0}{\beta_i} \right|^r \right)^{p/r} \leq \gamma^p \tag{25}$$

so the parameters $q_i = q_{iR} + jq_{iI}$ define the family

$$P_\gamma = \sum_{i=0}^{m} p_i(s)(q_{iR} + jq_{iI}) \tag{26}$$

where $p_i(s)$, $i = 0, \cdots, m$ are real-coefficients polynomials. Let $t_{iR} = (q_{iR} - q_{iR}^0)/\alpha_i$ and $t_{iI} = (q_{iI} - q_{iI}^0)/\beta_i$, and $p_0(s) = \sum_{i=0}^{m} p_i(s)(q_{ir}^0 + jq_{iI}^0)$, then given a complex plane direction

$e^{j\phi} = \cos\phi + j\sin\phi$, we are interested in computing the maximum of $\sum_{i=0}^m p_i(s)(\alpha_i t_{iR} + j\beta_i t_{iI})$ along that direction for a fixed $s \in \mathbf{C}$, where

$$\sum_{i=0}^m (|t_{iR}|^r + |t_{iI}|^r)^{p/r} \leq \gamma^p \tag{27}$$

Defining A_i and B_i as, respectively, the real and imaginary parts of $p_i(s)e^{-j\phi}$, the problem is again to maximize the real part with the imaginary part being zero and the coefficients confined in the l^p-normed complex ball. Then consider the problem of maximizing

$$\begin{aligned} G(\mathbf{t}, \lambda, \eta) &= \sum_{i=0}^m (\alpha_i t_{iR} A_i - \beta_i t_{iI} B_i) + \eta \sum_{i=0}^m (\alpha_i t_{iR} B_i + \beta_i t_{iI} A_i) \\ &\quad -\lambda[\sum_{i=0}^m (|t_{iR}|^r + |t_{iI}|^r)^{p/r} - \gamma^p] \end{aligned} \tag{28}$$

and define W_i as

$$W_i = \left(|A_i\beta_i + \eta^* B_i\alpha_i|^{r/(r-1)} + |-B_i\beta_i + \eta^* A_i\alpha_i|^{r/(r-1)}\right)^{(r-1)/r} \tag{29}$$

We then obtain the following result.

Theorem 2 *The maximum of the real part is given by*

$$G^* = \gamma \left(\sum_{i=0}^m (W_i)^{(r-1)p/r(p-1)} \right)^{(p-1)/p} \tag{30}$$

where η^* is the solution to the equation

$$\begin{aligned} &\sum_{i=0}^m B_i\alpha_i |A_i\alpha_i + \eta B_i\alpha_i|^{1/(r-1)} \mathrm{sgn}(A_i\alpha_i + \eta B_i\alpha_i) \\ + &\sum_{i=0}^m A_i\beta_i |\eta A_i\beta_i - B_i\beta_i|^{1/(r-1)} \mathrm{sgn}(\eta A_i\beta_i - B_i\beta_i) = 0 \end{aligned} \tag{31}$$

Proof: See Appendix B. ■ Following the same type of argument as in the previous section, equation (31) can be shown to always have a unique solution that can be bounded in the following way

$$\min_{i=0,\cdots,m} \left\{ \min\left\{ -\frac{B_i}{A_i}, -\frac{A_i}{B_i} \right\} \right\} \leq \eta^* \leq \max_{i=0,\cdots,m} \left\{ \max\left\{ -\frac{B_i}{A_i}, -\frac{A_i}{B_i} \right\} \right\} \tag{32}$$

This equation is analytically solved in some special cases such as $r = 2$ or $p_i(s) = (j\omega)^{i-1}$. In fact, with the same definitions as in (18) we have the following

Lemma 5 *For $p_i(s) = (j\omega)^{i-1}$,*

$$G^* = \gamma \left(\sum_{i \text{ even}} [|\alpha_i\omega^i|^{r/(r-1)} Y^r |\cos\phi|^r + |\beta_i\omega^i|^{r/(r-1)} X^r |\sin\phi|^r]^{(r-1)p/(p-1)r} \right.$$

$$+\sum_{i\ \mathrm{odd}}[|\alpha_i\omega^i|^{r/(r-1)}X^r|\sin\phi|^r+|\beta_i\omega^i|^{r/(r-1)}Y^r|\cos\phi|^r]^{(r-1)p/(p-1)r}\Bigg)^{(p-1)/p}$$
$$\cdot(Y^{r-1}|\cos\phi|^r+X^{r-1}|\sin\phi|^r)^{-1} \qquad (33)$$

Proof: Let

$$\begin{aligned} X &= \sum_{i\ \mathrm{even}}|\alpha_i\omega^i|^{r/(r-1)}+\sum_{i\ \mathrm{odd}}|\beta_i\omega^i|^{r/(r-1)}, \\ Y &= \sum_{i\ \mathrm{odd}}|\alpha_i\omega^i|^{r/(r-1)}+\sum_{i\ \mathrm{even}}|\beta_i\omega^i|^{r/(r-1)} \end{aligned} \qquad (34)$$

we may rewrite (31) as

$$\begin{aligned} 0 &= X\sin\phi|\cos\phi+\eta\sin\phi|^{1/(r-1)}sgn(\cos\phi+\eta\sin\phi) \\ &\quad +Y\cos\phi|\eta\cos\phi-\sin\phi|^{1/(r-1)}sgn(\eta\cos\phi-\sin\phi) \end{aligned} \qquad (35)$$

The equation above looks very much like (21) so it is possible to write

$$|-\sin\phi+\eta^*\cos\phi|=\frac{X^{r-1}|\sin\phi|^{r-1}}{Y^{r-1}|\cos\phi|^r+X^{r-1}|\sin\phi|^r} \qquad (36)$$

and

$$|\cos\phi+\eta^*\sin\phi|=\frac{Y^{r-1}|\cos\phi|^{r-1}}{Y^{r-1}|\cos\phi|^r+X^{r-1}|\sin\phi|^r} \qquad (37)$$

Recalling that now W_i can be written in the form

$$\begin{aligned} W_i &= |\alpha_i\omega^i|^{r/(r-1)}|\cos\phi+\eta\sin\phi|^{r/(r-1)}+|\beta_i\omega^i|^{r/(r-1)}|-\sin\phi+\eta\cos\phi|^{r/(r-1)},\ i\ \mathrm{even} \\ &= |\alpha_i\omega^i|^{r/(r-1)}|-\sin\phi+\eta\cos\phi|^{r/(r-1)}+|\beta_i\omega^i|^{r/(r-1)}|\cos\phi+\eta\sin\phi|^{r/(r-1)},\ i\ \mathrm{odd} \end{aligned} \qquad (38)$$

and substituting (36) and (37) into (38), G^* boils down to the expression in (33). ■

A simpler particular case, considered by Kogan in [14], appears when $p=r$ since G^* can be readily transformed into (20), where X and Y are defined in (34). Another interesting situation was explored in [13] where $p=\infty$ and $\alpha_i=\beta_i,\ i=0,\cdots,m$ as presented next.

Lemma 6 *For* $p=\infty$ *and* $\alpha_i=\beta_i,\ i=0,\cdots,m$

$$G^*=\frac{\left(\sum_{i\ \mathrm{even}}|\alpha_i\omega^i|^{r/(r-1)}\right)^{(r-1)/r}+\left(\sum_{i\ \mathrm{odd}}|\alpha_i\omega^i|^{r/(r-1)}\right)^{(r-1)/r}}{(|\cos\phi|^r+|\sin\phi|^r)^{1/r}} \qquad (39)$$

Proof: Here $X=Y$ and the expression of G^* is obvious. ■

4 Applications

The previous results can be used to check the robust stability of a family of polynomials as described in (4) and (26). For this, let $\phi_0(s)$ the phase of $p_0(s)$ for some fixed $s \in \partial\Gamma$, where $\partial\Gamma$ is the continuous boundary of the open stability set Γ. Then, the family of polynomials is Γ-stable if and only if the following two conditions hold.

1. The polynomial $p_0(s)$ is Γ-stable, and

2. $G^*(s, \phi_0(s)) < |p_0(s)|$, for all $s \in \partial\Gamma$.

This result follows by simple application of the zero exclusion principle [6]. On the other hand, the results that have been just presented allow us to use the concept of Quantitative Feedback Theory, intoduced by Horowitz [15], where the knowledge of the value set templates on the Nichol's chart at certain frequencies is fundamental. The main interest of our approach is that it is possible to obtain the value set boundary at equally spaced angles if G^* is computed for the phases $\phi_i = \{0, 2\pi/N, \cdots, 2\pi k/N, \cdots, 2\pi(N-1)/N\}$ which allows a very useful uniform gridding. However, if some derivatives information is present, the phases set can be modified to increase the resolution around the critical ones. Another benefit of the freedom in the phase selection is that more complex value sets can be dealt with. See [16], [17], [18]. For instance, in the computation of the value sets of parametric rational functions [19] it is important to set the phase of either the numerator or the denominator at specific values in order to significantly reduce the number of required calculations. Also, with the proposed strategy, it is possible to consider much more general uncertainties by means of the recently proposed Tree Strcutured Decomposition [20]. For further examples and references, see [21].

5 Conclusions

In this paper, we have generalized previous results on the stability of polynomials whose coefficients lie in a normed l^p ball. The approach taken relies on the Lagrange multipliers approach, and illustrates the fact that in general, a two-parameter optimization problem has to be solved. In many relevant cases however, the optimization may be reduced to a one-parameter search and an explicit solution may even be obtained. The value sets obtained from the solution of the optimization problem can be used to check the stability of the given family of polynomials or in a design context by means of the Quantitative Feedback Theory.

APPENDIX A

Obviously, the maximum is attained when equality in (6) holds, therefore we can construct the function

$$G(\mathbf{r},\lambda,\eta) = \sum_{i=0}^{m} \alpha_i r_i A_i(s,\phi) + \eta \sum_{i=0}^{m} \alpha_i r_i B_i(s,\phi) - \lambda(\sum_{i=0}^{m} |r_i|^p - \gamma^p) \quad (40)$$

where $\mathbf{r} = (r_1, \cdots, r_m)^T$. Setting the partial derivatives to zero and dropping the explicit s and ϕ dependence, yields

$$\frac{\partial G}{\partial r_i} = \alpha_i A_i + \eta \alpha_i B_i - \lambda p |r_i|^{p-1} sgn(r_i) = 0 \quad (41)$$

$$\frac{\partial G}{\partial \eta} = \sum_{i=0}^{m} \alpha_i B_i r_i = 0 \quad (42)$$

$$\frac{\partial G}{\partial \lambda} = -(\sum_{i=0}^{m} |r_i|^p - \gamma^p) = 0 \quad (43)$$

Assuming that $\lambda > 0$, (41) can be written as

$$|r_i| = \left| \frac{\alpha_i A_i + \eta \alpha_i B_i}{\lambda p} \right|^{1/(p-1)} \quad (44)$$

$$sgn(r_i) = sgn(\alpha_i A_i + \eta \alpha_i B_i) \quad (45)$$

Substituting (44) into (43) and rearranging terms we obtain

$$\lambda p = \frac{\left[\sum_{i=0}^{m} |\alpha_i A_i + \eta \alpha_i B_i|^{p/(p-1)}\right]^{(p-1)/p}}{\gamma^{p-1}} \quad (46)$$

which when replaced in (44) results in

$$|r_i| = \gamma \frac{|\alpha_i A_i + \eta \alpha_i B_i|^{1/(p-1)}}{\left[\sum_{i=0}^{m} |\alpha_i A_i + \eta \alpha_i B_i|^{p/(p-1)}\right]^{1/p}} \quad (47)$$

The value of η that gives the maximum is obtained by substituting (47) and (45) into (42). Then, the resulting equation is (9). The maximum value of G now is

$$G^* = \gamma \frac{\sum_{i=0}^{m} \alpha_i A_i |\alpha_i A_i + \eta^* \alpha_i B_i|^{1/(p-1)} sgn(\alpha_i A_i + \eta^* \alpha_i B_i)}{\left(\sum_{i=0}^{m} |\alpha_i A_i + \eta^* \alpha_i B_i|^{p/(p-1)}\right)^{1/p}} \quad (48)$$

However, since $\eta^* \sum_{i=0}^{m} \alpha_i r_i B_i$ equals to zero, this quantity can be added to the expression in (48), so a more compact form results

$$\begin{aligned} G^* &= \gamma \frac{\sum_{i=0}^{m} (\alpha_i A_i + \eta^* \alpha_i B_i) |\alpha_i A_i + \eta^* \alpha_i B_i|^{1/(p-1)} sgn(\alpha_i A_i + \eta^* \alpha_i B_i)}{\left(\sum_{i=0}^{m} |\alpha_i A_i + \eta^* \alpha_i B_i|^{p/(p-1)}\right)^{1/p}} \\ &= \gamma \left(\sum_{i=0}^{m} |\alpha_i A_i + \eta^* \alpha_i B_i|^{p/(p-1)} \right)^{(p-1)/p} \\ &= \gamma^p \lambda p \end{aligned} \quad (49)$$

APPENDIX B

The Lagrange function becomes for this case

$$\begin{aligned} G(\mathbf{t},\lambda,\eta) &= \sum_{i=0}^{m}(\alpha_i t_{iR} A_i - \beta_i t_{iI} B_i) + \eta \sum_{i=0}^{m}(\alpha_i t_{iR} B_i + \beta_i t_{iI} A_i) \\ &\quad -\lambda[\sum_{i=0}^{m}(|t_{iR}|^r + |t_{iI}|^r)^{p/r} - \gamma^p] \end{aligned} \tag{50}$$

where $\mathbf{t} = (t_{0R} + jt_{0I}, \cdots, t_{mR} + jt_{mI})^T$. Setting the partial derivatives to zero yields

$$\frac{\partial G}{\partial t_{iR}} = \alpha_i A_i + \eta\alpha_i B_i - \lambda p |t_{iR}|^{r-1}(|t_{iR}|^r + |t_{iI}|^r)^{(p-r)/r} sgn(t_{iR}) = 0 \tag{51}$$

$$\frac{\partial G}{\partial t_{iI}} = \alpha_i A_i + \eta\alpha_i B_i - \lambda p |t_{iI}|^{r-1}(|t_{iR}|^r + |t_{iI}|^r)^{(p-r)/r} sgn(t_{iI}) = 0 \tag{52}$$

$$\frac{\partial G}{\partial \eta} = \sum_{i=0}^{m} \alpha_i B_i t_{iR} + \beta_i A_i t_{iI} = 0 \tag{53}$$

$$\frac{\partial G}{\partial \lambda} = -(\sum_{i=0}^{m}(|t_{iR}|^r + |t_{iI}|^r)^{p/r} - \gamma^p) = 0 \tag{54}$$

Let $C_i = (|t_{iR}|^r + |t_{iI}|^r)$. Then, assuming that $\lambda > 0$, it is possible to write equations (51) and (52) as

$$|t_{iR}| = \left|\frac{\alpha_i A_i + \eta\alpha_i B_i}{\lambda p C_i^{(p-r)/r}}\right|^{1/(r-1)} \tag{55}$$

$$|t_{iI}| = \left|\frac{-\beta_i B_i + \eta\beta_i A_i}{\lambda p C_i^{(p-r)/r}}\right|^{1/(r-1)} \tag{56}$$

$$sgn(t_{iR}) = sgn(\alpha_i A_i + \eta\alpha_i B_i) \tag{57}$$

$$sgn(t_{iI}) = sgn(\eta\beta_i A_i - \beta_i B_i) \tag{58}$$

Substituting the expressions above into (53) it is readily seen that (31) results, from which η^* is obtained. On the other hand, equations (55) and (56) can be raised to r and added, resulting in

$$\begin{aligned} C_i &= \frac{|\alpha_i A_i + \eta^*\alpha_i B_i|^{r/(r-1)} + |-\beta_i B_i + \eta^*\beta_i A_i|^{r/(r-1)}}{(\lambda p)^{r/(r-1)} C_i^{(p-r)/(r-1)}} \\ &= \frac{W_i}{(\lambda p)^{r/(r-1)} C_i^{(p-r)/(r-1)}} \end{aligned} \tag{59}$$

or, alternatively,

$$C_i^{p/r} = \frac{W_i^{(pr-p)/(pr-r)}}{(\lambda p)^{p/(p-1)}} \tag{60}$$

which when substituted in (54) gives the following expression for λp

$$\lambda p = \frac{\left[\sum_{i=0}^{m} W_i^{(pr-p)/(pr-r)}\right]^{(p-1)/p}}{\gamma^{p-1}} \tag{61}$$

In order to obtain the maximum of (28) we replace the expressions for t_{iR} and t_{iI} in (55-58) into (28) to obtain

$$G^* = \frac{1}{(\lambda p)^{1/(r-1)}} \cdot \sum_{i=0}^{m} \frac{W_i}{C_i^{(p-r)/r(r-1)}} \tag{62}$$

which, after substituting the expressions for λp and C_i and performing some tedious algebraic manipulations becomes

$$G^* = \gamma \left(\sum_{i=0}^{m} W_i^{(pr-p)/(pr-r)} \right)^{(p-1)/p} \tag{63}$$

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