Computation of Bit-Error Probabilities for Optical Receivers using Thin Avalanche Photodiodes

Byonghyok Choi, *Student Member, IEEE* and Majeed M. Hayat, *Senior Member, IEEE*

***Abstract*— The large-deviation-based asymptotic-analysis and importance-sampling methods for computing bit-error probabilities for avalanche-photodiode (APD) based optical receivers, developed by Letaief and Sadowsky [IEEE Trans. Inform. Theory, vol. 38, pp. 1162–1169, 1992], are extended to include the effect of dead space, which is significant in high-speed APDs with thin multiplication regions. It is shown that the receiver's bit-error probability is reduced as the magnitude of dead space increases relative to the APD's multiplication-region width. The calculated error probabilities and receiver sensitivities are also compared with those obtained from the Chernoff bound.**

***Index Terms*—Avalanche photodiodes, asymptotic analysis, error probability, large deviation, importance sampling, dead space, optical receiver, receiver sensitivity.**

I. Introduction

AVALANCHE photodiodes (APDs) are often preferred over p-i-n photodiodes in high-speed receivers because of their internal optoelectronic gain, which results in converting each incoming photon, absorbed by the APD, into a cascade of electron-hole pairs [1]. This internal-gain mechanism results in an amplified photocurrent, which combats the Johnson noise present in the pre-amplifier stage of the receiver, thereby improving the receiver sensitivity drastically [1]. The optoelectronic gain results from the cascade (or avalanche) of carrier impact ionizations that take place in the high-field intrinsic multiplication layer of the APD. Due to its stochastic nature, however, this avalanche multiplication process is inherently noisy, resulting in random fluctuations in the gain and the time response of APDs [2], [3]. Moreover, following a photoexcitation, the avalanche-buildup time, which is the time required for the cascade of impact ionizations to complete, can be a limiting factor in modern transmission systems that operate near 10-Gbps transmission rates. Fortunately, APDs with thin multiplication layers (i.e., $<$200 nm) have been shown to offer reduced buildup times and reduced gain fluctuations [4], [5], making them suitable for high-speed optical receivers [1].

Indeed, it is known that the excess noise factor, which is a measure of gain uncertainty, is significantly lowered in thin APDs. This is known to be a result of the dead space, which is the minimum distance that a newly generated carrier must travel before becoming capable of impact ionizing [4]. Additionally, as the APD becomes thinner, dead space occupies a larger fraction of the multiplication region (e.g., 20–25%) and its effect on the gain statistics becomes progressively more substantial [4]. A key consequence of dead space is that the carrier multiplication process is no longer Markovian [6]. Therefore, the traditional theory for gain statistics in thick APDs, originally developed by McIntyre [7], is not applicable to thin APDs. Subsequently, a generalized theory, based on renewal relations, was developed to characterize the mean, excess noise factor, and the gain's moment generating function (mgf) in the presence of dead space [2], [8].

Exact calculation of the bit-error probability (P_b) has been a challenge due to the lack of explicit, closed-form expressions for the probability distribution functions of the APD's gain and the test-statistic used by the receiver. This led to the development of a number of approximate methods for calculating the error probability over the years. These include the Gaussian approximation [9], the saddle-point approximation [10], and Chernoff bounds [9]. Generally, the accuracy of the Gaussian approximation is questionable while the Chernoff bound is usually not sufficiently tight (although it is exponentially tight). The saddle-point method, on the other hand, is an analytical technique that provides an accurate approximation of the error probability. More recently, Letaief and Sadowsky [11] developed a probabilistic asymptotic-analysis method, based on large-deviation theory, that facilitates the approximation of the error probability. Their technique yields an elegant probabilistic equivalent to the saddle-point approximation [11] and also offers an efficient Monte-Carlo simulation method for estimating the error probability. However, the mgf for the APD's gain used in their work does not account for the dead space. In this letter, we generalize the asymptotic analysis and efficient Monte-Carlo technique by Letaief and Sadowsky [11] to accommodate APDs with dead space, thereby making their technique applicable to thin, high-speed APDs.

II. Preliminaries

Consider an on-off-keying (OOK) optical communication scheme. The decision of the received signal in each bit is made by means of a simple threshold test based on the statistic $D = \sum_{k=1}^{M} G_k + N$ [11], where $G_k \in \mathbb{N}$ denotes the avalanche gain associated with the kth primary electron, M is a Poisson random variable representing the total number of primary electrons produced in a bit, and N is the receiver's zero-mean Gaussian thermal (Johnson) noise with variance σ^2 [3]. The sequence $\{G_k\}$ is i.i.d., and its univariate gain distribution is characterized by the mgf reported in [2], [8].

Manuscript received July 7, 2005. The associate editor coordinating the review of this letter and approving it for publication was Prof. Maode Ma. This work was supported by the National Science Foundation through awards ECS-0334813 and ECS-0196569.

B. Choi is with the ECE Dept., University of Florida, Gainesville, FL (e-mail: aru0080@ecel.ufl.edu).

M. M. Hayat is with the ECE Dept., University of New Mexico, Albuquerque, NM (e-mail: hayat@ece.unm.edu).

Digital Object Identifier 10.1109/LCOMM.2006.01004.

Under hypotheses H_0 and H_1, the mean of M is assumed to be λ_0 and λ_1, respectively, where $\lambda_0 < \lambda_1$. These parameters can be calculated from the signal intensities and the dark current parameters. The type-I and type-II error probabilities are given by $P_{0,\gamma} \triangleq \mathsf{P}\{D \geq \gamma | H_0\}$ and $P_{1,\gamma} \triangleq \mathsf{P}\{D < \gamma | H_1\}$, respectively, where γ is the test threshold; consequently $P_b = 0.5(P_{0,\gamma} + P_{1,\gamma})$.

III. Asymptotic Analysis

Letaief and Sadowsky have provided a probabilistic method, based on Cramer's theorem [12], for the asymptotic analysis of the bit-error probability for the detection model described above. For its close relevance to our work, we will briefly recall germane aspects of it. Define $\bar{s} = \sup\{s \in \mathbb{R} : \Lambda_G(s) < \infty\}$, where $\Lambda_G \triangleq \log \phi_G$ is log-mgf of the APD's gain, G, and $\phi_G(s) \triangleq \mathsf{E}[e^{sG}]$ is its mgf. Note that $\bar{s} \geq 0$ since $\Lambda_G(0) = 0$. Also, since $G > 0$ a.s. (since $G \in \mathbb{N}$), we have $\Lambda_G(s) < \infty$ for all $s < \bar{s}$, and moreover, $\Lambda_G(s)$ is strictly increasing and analytic on $(-\infty, \bar{s})$ [11]. The function $\Lambda_G(\cdot)$ is said to be *steep* if $\Lambda_G'(s) \uparrow \infty$ as $s \to \bar{s}$ [12]. In what follows, it is assumed that $\gamma = c_i^{-1}\lambda_i$, $i = 0, 1$, where c_0 and c_1 are positive constants [11]. The term asymptotic refers to the parameter γ being large.

Theorem 1: (Letaief and Sadowsky [11]) *Suppose that (i) $c_0 < \mu_G^{-1} < c_1$, where $\mu_G \triangleq \mathsf{E}[G]$, and (ii) $\Lambda_G(s)$ is steep. Then, $P_{i,\gamma} \sim C_i^* \gamma^{-1/2} e^{-I_i^* \gamma}$, where $C_i^* = \Big(2\pi w_i^* s_i^{*2}\big[\Lambda_G'(s_i^*)^2 + \Lambda_G''(s_i^*) + (c/w_i^*)\big]\Big)^{-1/2}$, $I_i^* = s_i^* - c_i(e^{\Lambda_G(s_i^*)} - 1) - \frac{1}{2}cs_i^{*2}$, $w_i^* = c_i e^{\Lambda(s_i^*)}$, $c = \sigma^2/\gamma$, and s_i^* is the unique solution of the equation $c_i\Lambda_G'(s)\exp\big(\Lambda_G(s)\big) + cs = 1$.*

Now, according to [2], the dead-space-generalized mgf ϕ_G of the APD's gain is given by

$$\phi_G(s) = e^{s + r_{s/2}(d-1)}, \tag{1}$$

where for any $z \in \mathbb{C}$, r_z is the solution to the equation $r_z + \alpha w\big(e^{2z + r_z(3d-1)} - 1\big) = 0$. Here, w is the width of the multiplication region of the APD, α is the electron ionization coefficient, and the dimensionless quantity d is the dead space normalized by w. (In the above mgf, it is implicitly assumed that (i) the hole ionization coefficient is also equal to α, viz., the hole-to-electron ionization ratio $k = 1$, a condition that is well approximated in thin APDs, and (ii) electrons and holes have equal dead spaces.) Finally, we assume that the avalanche multiplication is initiated by an electron at the edge of the multiplication region. In order to apply Theorem 1 to this mgf, we must show that $\Lambda_G(s) = s + r_{s/2}(d-1)$ is steep.

Theorem 2: *For $d < 1/3, \Lambda_G(s)$ is steep and the conclusion of Theorem 1 thus holds for the mgf given by* (1).

Proof: We begin by calculating $\bar{s}$ associated with $\Lambda_G(s) = s + r_{s/2}(d-1)$. It can be shown by direct substitution that $r_{s/2} = \alpha w - \psi\big(\alpha w(3d-1)e^{\alpha w(3d-1)+s}\big)/(3d-1)$, where $y = \psi(x)$ is the zeroth-branch solution to the equation $ye^y = x$ (also called the Lambert W function). Since $\psi(x) \leq 0$ whenever $x < 0$, the hypothesis $d < 1/3$ implies that $r_{s/2} < 0$. Hence, $\Lambda_G(s)$ has maximum value when $r_{s/2}$ is minimum (since $d \leq 1$ by definition). Now, $\psi(x)$ has a minimum value of -1 when $x = -e^{-1}$. Thus, the minimum $r_{s/2}$ occurs precisely when $\alpha w(3d-1)e^{\alpha w(3d-1)+s} = -e^{-1}$. We can now solve for s at which $r_{s/2}$ is minimized. Thus, $\bar{s} = \alpha w(1-3d) - \log\big(e\alpha w(1-3d)\big)$. Next, it is easy to see that $\Lambda_G'(s) = 1 + r_{s/2}'(d-1)$ and $r_{s/2}' = -\alpha w(1 + r_{s/2}'(3d-1))e^{s + r_{s/2}(3d-1)}$. The latter yields

$$r_{s/2}' = \frac{-\alpha w e^{s + r_{s/2}(3d-1)}}{1 + \alpha w e^{s + r_{s/2}(3d-1)}(3d-1)}.$$

By setting $s = \bar{s}$, we obtain $r_{\bar{s}/2} = \alpha w + 1/(3d-1)$, and the numerator of $r_{\bar{s}/2}'$ becomes $1/(3d-1) < 0$. However, as $s \uparrow \bar{s}$, the denominator of $r_{\bar{s}/2}'$ converges to 0 monotonically from above. Hence, $\lim_{s\uparrow\bar{s}} r_{\bar{s}/2}' = -\infty$ and consequently $\lim_{s\uparrow\bar{s}} \Lambda_G'(s) = \lim_{s\uparrow\bar{s}} 1 - r_{s/2}'(1-d) = \infty$. □

Remark The mean gain μ_G and the parameter αw are related by $\alpha w = (1-\mu_G)/\big((3d-1)(\mu_G - 2d)\big)$ [13]. This relationship is used in Section V in determining the parameter αw that would yield a certain mean gain.

IV. Efficient Monte-Carlo Calculation of the Bit-error Probability

In [11], an efficient Monte-Carlo estimation method was adopted for calculating the bit-error probabilities based on importance sampling. A sequence $D^{(l)} = (M^{(l)}, G^{(l)}, N^{(l)})$, $l = 1, 2, \ldots, L$ of realizations of all the random quantities is first generated according to their twisted distributions. The error probability is then estimated using the unbiased estimator $\hat{P}_{i,\gamma}^* = L^{-1}\sum_{l=1}^{L} \mathbf{1}_i(D^{(l)})W(D^{(l)})$, where $\mathbf{1}_i(\cdot)$ is the indicator function for error events, viz., $\mathbf{1}_1(D) = 1$ if $D < \gamma$ and zero otherwise, and $\mathbf{1}_0(D) = 1$ if $D \geq \gamma$ and zero otherwise. $W(\cdot)$ is the importance-sampling weighting function, defined as the ratio of true distribution of D with respect to the twisted sampling distribution [11],

$$\begin{aligned} W(D) &= \exp\big(\lambda_i^* - \lambda_i + s_i^{*2}\frac{\sigma^2}{2}\big)\exp\Big(-s_i^*\big[\sum_{k=1}^{M} G_k + N\big] \\ &\quad + \big[\Lambda_G(s_i^*) + \log\big(\lambda_i/\lambda_i^*\big)\big]M\Big), \end{aligned}$$

where $\lambda_i^* = w_i^*\gamma$ is the biased dominant value of λ_i [11].

The above simulation procedure can be applied to the mgf given by (1) by virtue of Theorem 2. To generate simulations of the APD's gain from the twisted distribution, we first generate random samples $\tilde{G}^{(l)}$ from the probability mass function of the gain, $P_G(k) \triangleq \mathsf{P}\{G = k\}$, $k = 1, 2, \ldots$. These are calculated from the gain's characteristic function $\phi_G(ju)$ using the efficient inversion method described in [14]. We then employ the acceptance/rejection procedure with acceptance probability $\exp\big(s_i^*\tilde{G}^{(l)} - \Lambda_G(s_i^*)\big)$ using the numerically calculated values of s_i^* according to Theorem 1. With this procedure the accepted samples are distributed according to the twisted probability mass function $P_G^{(s_i^*)}(k) = e^{s_i^* k - \Lambda_G(s_i^*)}P_G(k)$. The random variable M is also simulated using a twisted Poisson distribution with parameter λ_i^*, and N is a twisted Gaussian random variable with mean $s_i^*\sigma^2$ and variance σ^2. We omit the details.

V. RESULTS

We calculated the error probability P_b using (i) efficient Monte-Carlo simulation (with $L = 20,000$), (ii) asymptotic analysis (using $P_{i,\gamma} \sim C_i^* \gamma^{-1/2} e^{-I_i^* \gamma}$), and (iii) the Chernoff bound, given by $P_{i,\gamma} \leq e^{-I_i^* \gamma}$ [11]. We considered $\lambda_0 = \eta \lambda_1$, where the transmitter extinction ratio is $\eta \approx 0.02$ [1], and used the test threshold $\gamma = (\mathsf{E}_0[D] + \mathsf{E}_1[D])/2$.

Fig. 1 depicts the dependence of P_b on the normalized dead space, d, for fixed average mean numbers of primary electrons, $\lambda = (\lambda_0 + \lambda_1)/2$ and $\mu_G = 10$. The calculations show that the receiver performance is improved as d increases. Clearly, the Chernoff bound yields an upper bound for P_b. On the other hand, the efficient Monte-Carlo results differ only slightly from the asymptotic analysis.

In optical communications, it is customary to measure the performance of the receiver by its *sensitivity*, S_o, which is defined as the minimum mean number of photons per bit necessary to produce $P_b = 10^{-9}$. The lower the sensitivity, the better the receiver is. Fig. 2 depicts the dependence of the receiver sensitivity on the mean APD gain μ_G, parameterized by d. It is seen that for any μ_G, the presence of dead space lowers the sensitivity. Moreover, for a fixed d, there exists an optimal value for μ_G that minimizes the value of the sensitivity. For the case considered, the optimal mean gain is approximately 80. At this mean gain, S_o improves from 1300 to 890 photons per bit (a 31% improvement) as the relative dead space d increases from 0 to 0.2. Beyond the optimal gain, gain-fluctuation noise begins to outweigh the benefit of the gain and the sensitivity begins to deteriorate. This behavior is consistent with the dependence of the signal-to-noise ratio of the photocurrent on the APD's mean gain [1], [8].

VI. CONCLUSIONS

We extended the efficient Monte-Carlo and asymptotic-analysis techniques for conventional APD-based receivers to the class of thin APDs, which is of significant practical importance in high-speed optical receivers. For these APDs, the dead space plays an important role in the statistics of the gain. Our calculations showed that the receiver performance improves as the dead space increases relative to the width of the APD's multiplication region.

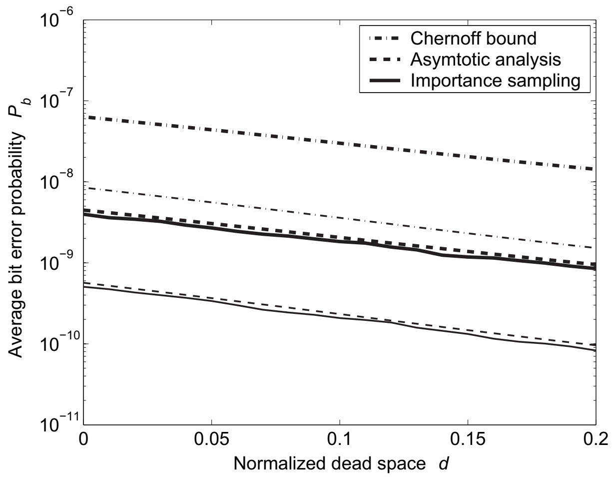

Fig. 1: Average bit-error probability as a function of the normalized dead space. For each method, thick and thin curves correspond to mean numbers of primary electrons λ of 4000 and 4500, respectively. The variance of the Gaussian thermal noise is assumed as $\sigma^2 = 3.6 \times 10^7$ [1].

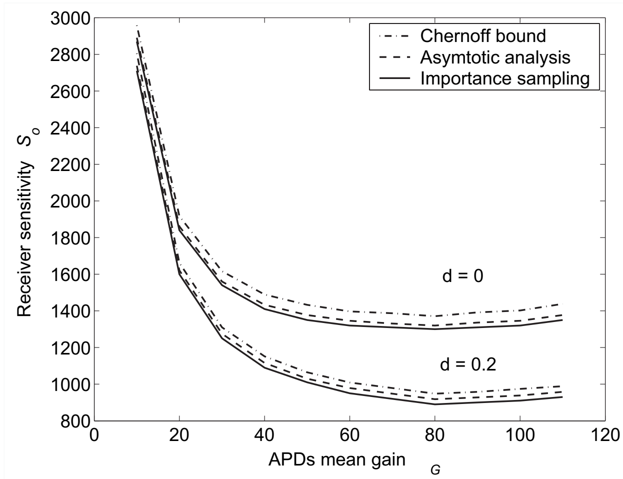

Fig. 2: Receiver sensitivity as a function the APD's mean gain μ_G for $d = 0$ and $d = 0.2$. It is assumed that $\sigma^2 = 8 \times 10^6$ and that each incident photon results in a primary electron.

REFERENCES

[1] G. P. Agrawal, *Fiber-optic Communication Systems.* New York: John Wiley & Sons, 2002.

[2] M. M. Hayat *et al.*, "Bit-error rates for optical receivers using avalanche photodiodes with dead space," *IEEE Trans. Commun.*, vol. 43, pp. 99–106, Jan. 1995.

[3] S. D. Personick, "Statistics of a general class of avalanche detectors with applications to optical communications," *Bell Syst. Tech. J.*, vol. 50, pp. 3075–3095, 1971.

[4] K. F. Li *et al.*, "Avalanche multiplication noise characteristics in thin GaAs $p^+ - i - n^+$ diodes," *IEEE Trans. Elect. Dev.*, vol. 45, pp. 2102–2107, Oct. 1998.

[5] G. S. Kinsey *et al.*, "Waveguide avalanche photodiode operating at 1.55μm with a gain-bandwidth product of 320 GHz," *IEEE Photon. Technol. Lett.*, vol. 13, pp. 842–844, Aug. 2001.

[6] B. E. A. Saleh *et al.*, "Effect of dead space on the excess noise factor and time response of avalanche photodiodes," *IEEE Trans. Elec. Dev.*, vol. 37, pp. 1976–1984, Sept. 1990.

[7] R. J. McIntyre, "Multiplication noise in uniform avalanche photodiodes," *IEEE Trans. Elect. Dev.*, vol. ED-13, pp. 164–168, Jan. 1966.

[8] M. M. Hayat *et al.*, "Effect of dead space on gain and noise of double-carrier multiplication avalanche photodiodes," *IEEE Trans. Elect. Dev.*, vol. 39, pp. 546–552, Mar. 1992.

[9] S. D. Personick *et al.*, "A detailed comparision of four approaches to the calculation of the sensitivity of optical fiber system receivers," *IEEE Trans. Commun.*, vol. COM-25, pp. 541–548, May 1977.

[10] C. W. Helstrom, "Computing the performance of optical receivers with avalanche diode detectors," *IEEE Trans. Commun.*, vol. 36, pp. 61–66, Jan. 1988.

[11] K. B. Letaief and J. S. Sadowsky, "Computing bit-error probabilities for avalanche photodiode receivers by large deviation theory,"*IEEE Trans. Inform. Theory*, vol. 38, pp. 1162–1169, May 1992.

[12] J. A. Bucklew, *Large Deviations Techniques in Decision, Simulation, and Estimation.* New York: Wiley, 1990.

[13] M. M. Hayat *et al.*, "An analytical approximation for the excess noise factor of avalanche photodiodes with dead space," *IEEE Elect. Dev. Lett.*, vol. 20, pp. 344–347, July 1999.

[14] J. A. Gubner and M. M. Hayat, "A method to recover counting distributions from their characteristic functions," *IEEE Signal Proc. Lett.*, vol. 3, pp. 184–186, June 1996.