Statistical Properties of the Impulse Response Function of Double-Carrier Multiplication Avalanche Photodiodes Including the Effect of Dead Space

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***Abstract*— The statistical properties of the impulse response function of double-carrier multiplication avalanche photodiodes (APD's) is determined including the effect of dead space. A theory is developed based on recurrence equations. The dead space is the minimum distance that a newly generated carrier must travel in order to acquire sufficient energy to become capable of causing an impact ionization. We derive recurrence equations for the first moment, second moment, and the probability distribution function of a set of random variables that are related, in a deterministic way, to the random impulse response function of the APD. These equations are solved numerically to produce the mean impulse response, the standard deviation, and the signal-to-noise ratio (SNR), all as functions of time. Dead space reduces the mean and standard deviation of the impulse response function for all time. For stable APD's, the mean impulse response function exhibits an exponentially decaying tail. Comparing two APD's with the same mean gain, one with dead space and the other without dead space, it is found that the former has an impulse-response-function SNR that is higher than the latter for all time. Dead space also reduces the exponential decay rate of the impulse-response-function SNR. This means that dead space reduces the noise in the tail of the impulse response.**

I. Introduction

AVALANCE photodiodes (APD's) with a variety of structures have been used as detectors in fiber-optic communication systems. APD's affect the performance of such systems in two ways. First, they introduce noise as a result of the random fluctuations in the gain, since the number of multiplications is random. Second, the randomness of the times at which multiplications occur produce another source of randomness in the impulse response of the APD as a function of time. It is the latter that is responsible for inter-symbol interference in high data rate systems.

A fundamental assumption implicit in most models of noise of conventional APD's [1]–[10] is that the probability of impact ionization by a carrier is independent of its ionization history, so that the ionization rate is the same at all times, including the instant following the carrier's own generation. From a physical point of view, however, a newly generated carrier must travel some distance in order to build up sufficient energy to enable it to initiate an ionization [11]–[13]. To take this effect into account, a model must be formulated in which the ionization probability of a carrier is set to zero for a certain distance, called the dead space, immediately following its generation. More realistically, the ionization probability would be expected to increase gradually, beginning from zero and reaching a steady-state value after some distance called the "sick space."

The statistical properties of the gain of double-carrier multiplication (DCM) APD's have been determined recently taking into account the effect of dead space [13]–[15]. These statistical properties include the mean gain, the excess noise factor, and the probability distribution function of the gain. In high data rate communication systems, however, the statistical properties of the impulse response function of the APD play a crucial role in estimating the performance of such systems. Saleh *et al.* [14] treated the dead-space model using the theory of age-dependent branching processes and computed the mean and standard deviation of the impulse response function of the APD as functions of time following a photoexcitation. This theory, however, is limited to APD's with only single-carrier multiplication. Later, Kahraman *et al.* [10] determined the autocorrelation function of the impulse response function of a DCM APD model for an APD with no dead space.

In this paper we develop a theory for the statistics of the impulse response function of a DCM APD assuming a sick-space model. The theory assumes that a parent carrier, at an arbitrary location in the multiplication region, initiates the multiplication process. A set of recurrence equations are then derived for the probability distribution function, the first and second moments, of the number of electrons and holes generated at a prescribed time as a result of the parent carrier. These random variables are related to the random impulse response function of the APD in deterministic way. As a special case, the dead-space model is applied to these recurrence equations and numerical solutions are obtained for the mean impulse response function and its standard deviation. We have demonstrated that the mean impulse response function exhibits an exponentially decaying tail for stable APD's, i.e., APD's with finite mean gain. This exponential decay rate is analytically determined and the condition for stability is derived. Dead space tends to reduce both the mean impulse response function and its standard deviation. This is result of the orderliness brought about by dead space. Dead space also increases the signal-to-noise ratio of the impulse response function when the mean gain is held fixed. This is accompanied, however, by a smaller decay rate

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for the mean impulse response. These results suggest that dead space should be included in the performance analysis of optical receivers.

In order to simplify computations, our analysis assumes a uniform electric field; nonetheless, generalization to non-uniform fields is straight forward. This can be accomplished by employing the method which was used to generalize the gain statistics to non-uniform fields [13]. In this case, carrier ionization coefficients, dead space, and carrier velocities can be position and field dependent. Since we are interested in determining the effect of dead space on the time dynamics of avalanche multiplication process, our analysis focuses on the multiplication region of the APD. The rest of the much wider depletion region is thus not included in the model.

II. Model

We consider a double-carrier multiplication APD. An electron injected at $x = 0$ travels in the x direction with a fixed velocity v_e under the effect of an electric field. After a random distance X_e (called the electron life span) an impact ionization occurs. Upon ionization, an electron-hole pair is generated, so that the original electron is replaced by two electrons and a hole. The two electrons behave in a statistically identical and independent manner. The hole, on the other hand, travels in the $-x$ direction with a fixed velocity v_h and ionizes after traveling a random distance X_h (called the hole life span), resulting in two holes and an electron. The electrons and holes repeat the process as they travel through the multiplication region, and so on. The multiplication region extends from $x = 0$ to $x = w$. When an electron reaches the right edge of the multiplication region its role ends. Similarly, a hole ceases to ionize when it reaches the left edge. If the process terminates, it does so when all possible carriers have reached the edges of the multiplication region.

The random variables X_e and X_h are assumed to be statistically independent and have arbitrary probability density functions $h_e(.)$ and $h_h(.)$, respectively. This allows us to take into account the age of carriers. For instance in the dead-space model

$$h_e(x) = \begin{cases} 0 & x < d_e \\ \alpha e^{-\alpha(x-d_e)} & x \geq d_e \end{cases} \tag{1a}$$

and

$$h_h(x) = \begin{cases} 0 & x < d_h \\ \beta e^{-\beta(x-d_h)} & x \geq d_h \end{cases} \tag{1b}$$

where d_e and d_h are the electron and hole dead spaces, respectively, and α and β are the ionization rates for electrons and holes that have traveled beyond the dead space, respectively.

When an electron at location $x = 0$ and at time $t = 0$ initiates the multiplication process, an electric current is induced, at time t, by the moving electrons and holes within the multiplication region. This current, which is a function of time, is the random impulse response function $I(t)$ of the APD. Our objective is the determine the statistics of $I(t)$.

III. Statistics of the Impulse Response

Our approach is based on writing recurrence equations for the number of carriers of one kind (i.e., electrons or holes) at time t after the initiation of the multiplication process by a single carrier of a certain kind at an arbitrary location within the multiplication region. These type of recurrence equations arise naturally in branching processes [16], in which the occurrence of an event independently replicates a statistically identical process of event generation. In fact, a similar approach underlies the theory developed by Hayat *et al,* [15] in which the statistical properties of the random gain of avalanche photodiodes are determined for the dead space model.

A. The Mean

Consider an electron at location x and at time $t = 0$ initiating a multiplication process. Assume that the electron is responsible for the production of a random number $Z_e(t, x)$ of electrons, including the parent electron itself, at time t. Similarly, $Z_h(t, x)$ is the random number of holes produced by the electron and its offsprings at time t. On the other hand, for a hole initiating the multiplication process at location x and at time $t = 0$, $Y_e(t, x)$ is defined to be the number of electrons at time t and $Y_h(t, x)$ is the corresponding number of holes. The impulse response can be expressed in terms of $Z_e(t, x)$ and $Z_h(t, x)$ by adding up the contributions from electrons and holes in the multiplication region at time t:

$$I(t) = \frac{q}{w}\{v_e Z_e(t, 0) + v_h Z_h(t, 0)\} \tag{2}$$

where q is the charge of the electron. Once the statistical properties of $Z_e(t, x)$ and $Z_h(t, x)$ are determined, the statistics of $I(t)$ can be deduced from (2).

We now proceed to develop recurrence equations for the random variables $Z_e(t, x)$, $Z_h(t, x)$, $Y_e(t, x)$, and $Y_h(t, x)$. Suppose that an electron at location x initiates the multiplication process and let us examine the quantity $Z_e(t, x)$ associated with this single electron. To simplify the derivation, consider first the case $t \leq (w - x)/v_e$ which corresponds to the time before the electron reaches the boundary at $x = w$. If the first ionization for this electron occurs at a location $s \in [x, x + v_e t]$, then we will have two electrons and a hole at location s. Associated with these two electrons and a hole are the random variables

$$Z_{e1}\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big), \quad Z_{e2}\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big), \quad \text{and} \quad Y_e\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big)$$

respectively. These random variables are statistically independent since each of these carriers acts independently. Furthermore, $Z_{e1}\big(t - \big(\frac{s-x}{v_e}\big), s\big)$ and $Z_{e2}\big(t - \big(\frac{s-x}{v_e}\big), s\big)$ are identically distributed since the two electrons act identically. Conditioning on the first ionization occurring at a location $s \in [x, x + v_e t]$, the number of electrons at time t produced by the original electron that initiated the process is

$$Z_e(t, x \mid s) = Z_{e1}\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big)$$

$$+ Z_{e2}\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big) + Y_e\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big). \quad (3)$$

The conditioning can be removed by averaging over all possible s in the interval $[x, x + v_e t]$. Thus, given that the original electron ionizes before time t has elapsed,

$$\begin{aligned}\langle Z_e(t,x)\rangle &= \int_x^{x+v_e t} \langle Z_e(t,x \mid s)\rangle h_e(s-x)\,ds \\ &= \int_x^{x+v_e t} \Big\langle Z_{e1}\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big) + Z_{e2}\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big) \\ &\qquad + Y_e\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big)\Big\rangle h_e(s-x)\,ds \\ &= \int_x^{x+v_e t} \Big[2\Big\langle Z_e\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big)\Big\rangle \\ &\qquad + \Big\langle Y_e\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big)\Big\rangle\Big] h_e(s-x)\,ds. \quad (4)\end{aligned}$$

It is also possible, however, that the original electron does not ionize at all by time t (i.e., $Z_e(t,x) = 1$). The probability of this event is $1 - H_e(v_e t)$, where

$$H_e(x) = \int_{-\infty}^{x} h_e(x')\,dx' \quad (5)$$

is the distribution function of the electron life span random variable X_e. Consider next the case $t > (w - x)/v_e$ which corresponds to the time following the instant when the electron reaches the boundary at $x = w$. In this case, the upper limit of (4) is changed to w and $Z_e = (t, x) = 0$ with probability $1 - H_e(w - x)$. Let us define the mean quantities

$$z_e(t,x) = \langle Z_e(t,x)\rangle$$

and

$$y_e(t,x) = \langle Y_e(t,x)\rangle.$$

In view of the discussion above, we can write a recurrent expression for $z_e(t,x)$ for all t:

$$\begin{aligned}z_e(t,x) &= u_1(t,x)\big[1 - H_e(v_e t)\big] + \int_x^{\min(w, x+v_e t)} \\ &\quad \cdot \Big[2z_e\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big) + y_e\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big)\Big] \\ &\quad \cdot h_e(s-x)\,ds \quad (6a)\end{aligned}$$

where

$$u_1(t,x) = \begin{cases} 1 & \text{if } t \le (w-x)/v_e \\ 0 & \text{otherwise} \end{cases}. \quad (6b)$$

Let us now examine the number of electrons at time t after a single hole at x initiates the multiplication process at time $t = 0$. Once again, for simplicity, consider first the case $t \le (x/v_h)$ which corresponds to the time before the hole reaches the boundary at $x = 0$. If we condition on the first ionization occurring at $s \in [x - v_h t, x]$, then we will have two newly generated holes and a newly generated electron at location s; and

$$\begin{aligned}Y_e(t,x \mid s) &= Y_{e1}\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big) \\ &\quad + Y_{e2}\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big) + Z_e\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big) \quad (7)\end{aligned}$$

where the first two terms on the right correspond to the two holes, and the third term corresponds to the electron. Therefore, if we assume that the original hole ionizes before reaching the edge at $x = 0$, then by averaging over all possible s in the interval $[x - v_h t, x]$ we obtain:

$$\begin{aligned}\langle Y_e(t,x)\rangle &= \int_{x-v_h t}^{x} \langle Y_e(t,x \mid s)\rangle h_h(x-s)\,ds \\ &= \int_{x-v_h t}^{x} \Big\langle Y_{e1}\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big) + Y_{e2}\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big) \\ &\qquad + Z_e\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big)\Big\rangle h_h(x-s)\,dx. \\ &= \int_{x-v_h t}^{x} \Big[2\Big\langle Y_e\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big)\Big\rangle \\ &\qquad + \Big\langle Z_e\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big)\Big\rangle\Big] h_h(x-s)\,ds. \quad (8)\end{aligned}$$

The probability that the hole does not ionize at all in the time interval t (i.e., $Y_e(t,x) = 0$) is $1 - H_h(v_h t)$, where

$$H_h(x) = \int_{-\infty}^{x} h_h(x')\,dx' \quad (9)$$

is the distribution function for the hole life span random variable X_h. Consider next the case $t > x/v_h$. The lower limit of (8) becomes 0 since, in this case, the parent hole has already reached the boundary at $x = 0$. Thus for all t, the recurrent expression for $y_e(t,x)$ becomes

$$\begin{aligned}y_e(t,x) &= \int_{\max(0, x-v_h t)}^{x} \Big[2y_e\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big) \\ &\quad + z_e\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big)\Big] h_h(x-s)\,ds. \quad (10)\end{aligned}$$

We follow the same steps to derive recurrence equations similar to (6) and (10) for the mean quantities $z_h(t,x) = \langle Z_h(t,x)\rangle$ and $y_h(t,x) = \langle Y_h(t,x)\rangle$; these equations are

$$\begin{aligned}z_h(t,x) &= \int_x^{\min(w, x+v_e t)} \Big[2z_h\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big) \\ &\quad + y_h\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big)\Big] h_e(s-x)\,ds, \quad (11)\end{aligned}$$

$$y_h(t,x) = u_2(t,x)\left[1 - H_h(v_h t)\right] + \int_{\max(0,x-v_h t)}^{x} \left[2y_h\left(t - \left(\frac{x-s}{v_h}\right), s\right) + z_h\left(t - \left(\frac{x-s}{v_h}\right), s\right)\right] h_h(x-s)\, ds \tag{12a}$$

where

$$u_2(t,x) = \begin{cases} 1 & \text{if } t \le x/v_h \\ 0 & \text{otherwise} \end{cases}. \tag{12b}$$

Denoting the mean impulse response $\langle I(t)\rangle$ by $i(t)$ and using (2) we obtain

$$i(t) = \frac{q}{w}\{v_e z_e(t,0) + v_h z_h(t,0)\}. \tag{13}$$

The dead-space model can be applied by inserting the expressions for the functions $h_e(x)$ and $h_h(x)$, as given by (1), into (6), (10), (11), and (12).

B. Standard Deviation

The standard deviation of the impulse response $\sigma(t) = \sqrt{i^2(t) - \langle I^2(t)\rangle}$ can be determined by developing recurrent expressions for the second order statistics of $Z_e(t,x)$, $Z_h(t,x)$, $Y_e(t,x)$, and $Y_h(t,x)$ using the same technique used for the means. The second moment of the impulse response $i_2(t) = \langle I^2(t)\rangle$ can be computed by taking an ensemble average of the square of both sides of (2). Thus,

$$i_2(t) = \frac{q^2}{w^2}\left\{\langle v_e^2 Z_e^2(t,0)\rangle + \langle v_h^2 Z_h^2(t,0)\rangle + 2v_e v_h \langle Z_e(t,0) Z_h(t,0)\rangle\right\} \tag{14}$$

and the standard deviation of $I(t)$ can then be readily computed using

$$\sigma(t) = \sqrt{i_2(t) - i^2(t)}. \tag{15}$$

We now proceed to determine the second-order statistics of $Z_e(t,x)$, $Z_h(t,x)$, $Y_e(t,x)$, and $Y_h(t,x)$. Assume first that $t \le (w-x)/v_e$. We start by squaring both sides of (3) to obtain

$$\{Z_e(t,x\,|\,s)\}^2 = \left\{Z_{e1}\left(t - \left(\frac{s-x}{v_e}\right), s\right) + Z_{e2}\left(t - \left(\frac{s-x}{v_e}\right), s\right) + Y_e\left(t - \left(\frac{s-x}{v_e}\right), s\right)\right\}^2. \tag{16}$$

Thus, given that the original electron ionizes before a time interval t has elapsed

$$\langle Z_e^2(t,x)\rangle = \int_x^{x+v_e t} \langle Z_e^2(t,x\,|\,s)\rangle h_e(s-x)\, ds$$
$$= \int_x^{x+v_e t} \left\langle \left\{Z_{e1}\left(t - \left(\frac{s-x}{v_e}\right), s\right) + Z_{e2}\left(t - \left(\frac{s-x}{v_e}\right), s\right) + Y_e\left(t - \left(\frac{s-x}{v_e}\right), s\right)\right\}^2\right\rangle h_e(s-x)\, ds. \tag{17}$$

Using the fact that Z_{e1}, Z_{e2}, and Y_e are statistically independent and that Z_{e1} Z_{e2} are identically distributed we can rewrite (17) as

$$\langle Z_e^2(t,x)\rangle = \int_x^{x+v_e t} \left\{2\left\langle Z_e^2\left(t - \left(\frac{s-x}{v_e}\right), s\right)\right\rangle + \left\langle Y_e^2\left(t - \left(\frac{s-x}{v_e}\right), s\right)\right\rangle + 2\left\langle Z_e\left(t - \left(\frac{s-x}{v_e}\right), s\right)\right\rangle^2 + \left\langle Y_e\left(t - \left(\frac{s-x}{v_e}\right), s\right)\right\rangle^2 + 4\left\langle Y_e\left(t - \left(\frac{s-x}{v_e}\right), s\right)\right\rangle \cdot \left\langle Z_e\left(t - \left(\frac{s-x}{v_e}\right), s\right)\right\rangle\right\} h_e(s-x)\, ds. \tag{18}$$

As before, if $t > (w-x)/v_e$, then the upper limit of (17) is changed to w and $Z_e(t,x) = 0$ with probability $1 - H_e(w-x)$. Hence, we define the second moments $\xi_e(t,x) = \langle Z_e^2(t,x)\rangle$ and $\varepsilon_e(t,x) = \langle Y_e^2(t,x)\rangle$, and write the following recurrent expression for all t:

$$\xi_e(t,x) = u_1(t,x)\left[1 - H_e(v_e t)\right] + \int_x^{\min(w,x+v_e t)} \left\{2\xi_e\left(t - \left(\frac{s-x}{v_e}\right), s\right) + \varepsilon_e\left(t - \left(\frac{s-x}{v_e}\right), s\right) + 2z_e^2\left(t - \left(\frac{s-x}{v_e}\right), s\right) + y_e^2\left(t - \left(\frac{s-x}{v_e}\right), s\right) + 4z_e\left(t - \left(\frac{s-x}{v_e}\right), s\right) \cdot y_e\left(t - \left(\frac{s-x}{v_e}\right), s\right)\right\} h_e(s-x)\, ds \tag{19}$$

and similarly we obtain

$$\varepsilon_e(t,x) = \int_{\max(0,x-v_h t)}^{x} \left\{2\varepsilon_e\left(t - \left(\frac{x-s}{v_h}\right), s\right) + \xi_e\left(t - \left(\frac{x-s}{v_h}\right), s\right) + 2y_e^2\left(t - \left(\frac{x-s}{v_h}\right), s\right) + z_e^2\left(t - \left(\frac{x-s}{v_h}\right), s\right) + 4z_e\left(t - \left(\frac{x-s}{v_h}\right), s\right) \cdot y_e\left(t - \left(\frac{x-s}{v_h}\right), s\right)\right\} h_e(x-s)\, ds. \tag{20}$$

The same reasoning yields recurrent expressions for the quantities

$$\xi_h(t,x) = \langle Z_h^2(t,x)\rangle \quad \text{and} \quad \varepsilon_h(t,x) = \langle Y_h(t,x)\rangle:$$

$$\xi_h(t,x) = \int_x^{\min(w,x+v_e t)} \left\{2\xi_h\left(t - \left(\frac{s-x}{v_e}\right), s\right) + \varepsilon_h\left(t - \left(\frac{s-x}{v_e}\right), s\right) + 2z_h^2\left(t - \left(\frac{s-x}{v_e}\right), s\right) + y_h^2\left(t - \left(\frac{s-x}{v_e}\right), s\right) + 4z_h\left(t - \left(\frac{s-x}{v_e}\right), s\right) \cdot y_h\left(t - \left(\frac{s-x}{v_e}\right), s\right)\right\} h_e(s-x)\, ds. \tag{21}$$

and

$$\varepsilon_h(t,x) = u_2(t,x)\big[1 - H_h(v_h t)\big] + \int_{\max(0,x-v_h t)}^{x} \Big\{2\varepsilon_h\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big) + \xi_h\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big) + 2y_h^2\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big) + z_h^2\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big) + 4z_h\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big) \cdot y_h\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big)\Big\} h_h(x-s)\,ds. \quad (22)$$

Finally, recurrent expressions for the quantities $R_Z(t,x) = \langle Z_e(t,x)Z_h(t,x)\rangle$ and $R_Y(t,x) = \langle Y_e(t,x)Y_h(t,x)\rangle$ can be derived using the same method and we only provide the results:

$$R_Z(t,x) = \int_{x}^{\min(w,x+v_e t)} \cdot \Big\{2R_Z\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big) + R_Y\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big) + 2z_e\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big) z_h\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big) + 2z_e\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big) y_h\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big) + 2y_e\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big) z_h\Big(t - \Big(\frac{s-x}{v_e}\Big), s\Big)\Big\} \cdot h_e(s-x)\,ds \quad (23)$$

$$R_Y(t,x) = \int_{\max(0,x-v_h t)}^{x} \cdot \Big\{2R_Y\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big) + R_Z\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big) + 2y_e\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big) y_h\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big) + 2z_e\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big) y_h\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big) + 2y_e\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big) z_h\Big(t - \Big(\frac{x-s}{v_h}\Big), s\Big)\Big\} h_h(x-s)\,ds. \quad (24)$$

Once (19)–(24) are solved to generate $\xi_e(t,x), \xi_h(t,x)$, and $R_Z(t,x)$, the standard deviation can be computed using (14) and (15).

C. Probability Distribution

Assume for the sake of simplicity that $v_e = N v_h$, where N is fixed integer, then (2) asserts that $I(t) = (v_e q/w)\big[Z_e(t,0) + N Z_h(t,0)\big]$. Define $K(t,x) = N Z_h(t,x)$; therefore, the normalized impulse response $J(t) = I(t)w/v_e q$ is an integer valued random variable given by the sum of $Z_e(t,0)$ and $K(t,0)$. Let us define the probability distribution function $P_J(n,t) = Pr\{J(t) = n\} = Pr\{I(t)w/v_e q = n\}$ for which we desire to develop a recurrent expression. Let $P_{Ze}(n,t,x) = Pr\{Z_e(t,x) = n\}, P_{Ye}(n,t,x) = Pr\{Y_e(t,x) = n\}$, and $P_K(n,t,x) = Pr\{K(t,x) = n\}$ be the probability distribution functions for $Z_e(t,x)$, $Y_e(t,x)$ and $K(t,x)$, respectively. Since $Z_e(t,x)$ and $Z_h(t,x)$ are statistically independent, so are $Z_e(t,x)$ and $K(t,x)$. Equation (2) tells us, therefore, that once $P_{Ze}(n,t,x)$ and $P_K(n,t,x)$ are known then the probability distribution of $J(t)$ can be easily determined by the discrete convolution formula for the sums of independent random variables:

$$P_J(n,t) = \sum_{k=0}^{n} P_{Ze}(k,t,0) P_K(n-k,t,0) \equiv P_{Ze}(n,t,0) \otimes P_K(n,t,0). \quad (25)$$

Note that

$$Pr\{K(t,x) = n\} = Pr\{N Z_h(t,x) = n\} = \begin{cases} Pr\{Z_h(t,x) = m\} & \text{if } n = mN \\ 0 & \text{otherwise} \end{cases}$$

where $m = 0, 1, 2, \cdots$, so that $P_K(n,t,x)$ is determined once $P_{Zh}(n,t,x)$ is known.

We now proceed to determine $P_{Ze}(n,t,x)$ and $P_{Zh}(n,t,x)$. Let $P_{Yh}(n,t,x) = Pr\{Y_h(t,x) = n\}$ be the probability distributions functions of $Y_h(t,x)$. Note that once a carrier ionizes, then its immediate offsprings act independently and those of the same type act identically. With this in mind, we use the same recurrent technique used in deriving the recurrence equations for the mean and the second moments to derive four recurrence equations for the probability distribution functions:

$$P_{Ze}(n,t,x) = \delta_{0,n}[1 - H_e(w-x)] \cdot [1 - u_1(t,x)] + \delta_{1,n}[1 - H_e(v_e t)]u_1(t,x) + \int_{x}^{\min(w,x+v_e t)} \Big[P_{Ze}\Big(n, t - \Big(\frac{s-x}{v_e}\Big), s\Big) \otimes P_{Ze}\Big(n, t - \Big(\frac{s-x}{v_e}\Big), s\Big) \otimes P_{Ye}\Big(n, t - \Big(\frac{s-x}{v_e}\Big), s\Big)\Big] h_e(s-x)\,ds, \quad (26a)$$

$$P_{Ye}(n,t,x) = \delta_{0,n}[1 - H_h(\min(v_h t, x))] + \int_{\max(0,x-v_h t)}^{x} \Big[P_{Ye}\Big(n, t - \Big(\frac{x-s}{v_h}\Big), s\Big) \otimes P_{Ye}\Big(n, t - \Big(\frac{x-s}{v_h}\Big), s\Big) \otimes P_{Ze}\Big(n, t - \Big(\frac{x-s}{v_h}\Big), s\Big)\Big] h_h(x-s)\,ds \quad (26b)$$

$$P_{Zh}(n,t,x) = \delta_{0,n}[1 - H_e(\min(v_e t, w-x))] + \int_{x}^{\min(w,x+v_e t)} \Big[P_{Zh}\Big(n, t - \Big(\frac{s-x}{v_e}\Big), s\Big) \otimes P_{Zh}\Big(n, t - \Big(\frac{s-x}{v_e}\Big), s\Big) \otimes P_{Yh}\Big(n, t - \Big(\frac{s-x}{v_e}\Big), s\Big)\Big] h_e(s-x)\,ds \quad (27a)$$

and

$$P_{Yh}(n,t,x) = \delta_{0,n}[1 - H_h(x)] \cdot [1 - u_2(t,x)] + \delta_{1,n}[1 - H_h(v_h t)]u_2(t,x) + \int_{\max(0,x-v_h t)}^{x} \Big[P_{Yh}\Big(n, t - \Big(\frac{x-s}{v_h}\Big), s\Big) \otimes P_{Yh}\Big(n, t - \Big(\frac{x-s}{v_h}\Big), s\Big) \otimes P_{Zh}\Big(n, t - \Big(\frac{x-s}{v_h}\Big), s\Big)\Big] h_h(x-s)\, ds \quad (27b)$$

where $\otimes$ denotes discrete convolution defined in (25) and $\delta_{k,n}$ is the Kronecker delta function ($\delta_{k,n} = 1$ if $k = n$ and 0 otherwise). The two sets of coupled integral equations (26), (27) can be solved, in principle, to determine $P_{Ze}(n,t,x)$, $P_{Zh}(n,t,x)$, $P_{Ye}(n,t,x)$, and $P_{Yh}(n,t,x)$.

The complexity of (26) and (27) arises in part from the appearance of discrete convolution under the integrals; nonetheless, these convolution can be transformed into multiplications with the aid of generating functions. Let $F_{Ze}(s,t,x)$, $F_{Zh}(s,t,x)$, $F_{Ye}(s,t,x)$, and $F_{Yh}(s,t,x)$ be the generating functions of $Z_e(t,x)$, $Z_h(t,x)$, $Y_e(t,x)$, and $Y_h(t,x)$, respectively, for example

$$F_{Ze}(s,t,x) = \sum_{k=0}^{\infty} P_{Ze}(n,t,x)s^k, \quad |s| \le 1. \quad (28)$$

Thus, (26) and (27) can be written as

$$F_{Ze}(s,t,x) = [1 - H_e(w-x)] \cdot [1 - u_1(t,x)] + [1 - H_e(v_e t)]u_1(t,x)s + \int_{x}^{\min(w,x+v_e t)} \Big[F_{Ze}\Big(s, t - \Big(\frac{s'-x}{v_e}\Big), s'\Big)^2 \cdot F_{Ye}\Big(s, t - \Big(\frac{s'-x}{v_e}\Big), s'\Big)\Big] h_e(s'-x)\, ds', \quad (29a)$$

$$F_{Ye}(s,t,x) = 1 - H_h(\min(v_h t, x)) + \int_{\max(0,x-v_h t)}^{x} \Big[F_{Ye}\Big(s, t - \Big(\frac{x-s'}{v_h}\Big), s'\Big)^2 \cdot F_{Ze}\Big(s, t - \Big(\frac{x-s'}{v_h}\Big), s'\Big)\Big] h_h(x-s')\, ds' \quad (29b)$$

$$F_{Zh}(s,t,x) = 1 - H_e(\min(v_e t, w-x)) + \int_{x}^{\min(w,x+v_e t)} \Big[F_{Zh}\Big(s, t - \Big(\frac{s'-x}{v_e}\Big), s'\Big)^2 \cdot F_{Yh}\Big(s, t - \Big(\frac{s'-x}{v_e}\Big), s'\Big)\Big] h_e(s'-x)\, ds' \quad (30a)$$

and

$$F_{Yh}(s,t,x) = (1 - H_h(x)) \cdot [1 - u_2(t,x)] + [1 - H_h(v_h t)]u_2(t,x)s + \int_{\max(0,x-v_h t)}^{x} \Big[F_{Yh}\Big(s, t - \Big(\frac{x-s'}{v_h}\Big), s'\Big)^2 \cdot F_{Zh}\Big(s, t - \Big(\frac{x-s'}{v_h}\Big), s'\Big)\Big] h_h(x-s')\, ds'. \quad (30b)$$

The foregoing coupled nonlinear integral equations may be solved numerically. Once a generating function is evaluated on the unit circle, then its Fourier series coefficients give the respective probability distribution function:

$$P(n,t,x) = \frac{1}{2\pi} \int_0^{2\pi} F(e^{j\omega}, t, x) e^{-j\omega n}\, d\omega$$

with the proper subscripts on P and F. Furthermore, these integral equations can be used to derive the recurrence equations for the first moments $z_e(t,x)$, $z_h(t,x)$, $y_e(t,x)$, and $y_h(t,x)$ ((6), (10)–(13)); and for the second moments $\xi_e(t,x), \varepsilon_e(t,x)\xi_h(t,x)$ and $\varepsilon_h(t,x)$ ((18)–(21). This may be accomplished by use of the standard relations $z(t,x) = \big(\partial/\partial s F_Z(s,t,x)\big)_{s=1}$ and $\xi_e(t,x) = z(t,x) + \big(\partial^2/\partial s^2 F_Z(s,t,x)\big)_{s=1}$

D. Asymptotic Behavior of the Mean Impulse Response Function

The knowledge of the behavior of $i(t)$ for large values of t is important for two reasons. First, it tells us about the stability of the APD, namely, if the area under the tail of $i(t)$ is finite, then the device is stable in the sense that it has finite mean gain. Second, if the decay rate of $i(t)$ is known for large values of t then this will help making conclusions on the maximum allowable data rate to keep the effect of intersymbol interference minimal. Equation (2) tells us that once the asymptotic behavior of $z_e(t,0)$ and $z_h(t,0)$ are known, then the asymptotic behavior of $i(t)$ can be determined. It can be shown in the literature [17] on recurrence equations that all of the quantities $z_e(t,x)$, $z_h(t,x)$, $y_e(t,x)$, and $y_h(t,x)$ have exponential tails with a common rate independent of x. Hence, we can write

$$\lim_{t\to\infty} z_e(t,x)e^{\gamma t} = a_e(x), \quad (31a)$$

$$\lim_{t\to\infty} y_e(t,x)e^{\gamma t} = b_e(x), \quad (31b)$$

$$\lim_{t\to\infty} z_h(t,x)e^{\gamma t} = a_h(x), \quad (31c)$$

and

$$\lim_{t\to\infty} y_h(t,x)e^{\gamma t} = b_h(x), \quad (31d)$$

for some unknown value γ which we take to be the exponential rate of decay (or growth). Multiplying both sides of (6) and (10) by $e^{\gamma t}$ and taking limits at $t \to \infty$ we obtain

$$a_e(x) = \int_x^w [2a_e(s) + b_e(s)] e^{\gamma([s-x]/v_e)} h_e(s-x)\, ds, \quad (32a)$$

and

$$b_e(x) = \int_0^w \left[2b_e(s) + a_e(s)\right] e^{\gamma([x-s]/v_h)} h_h(x-s)\, ds. \quad (32b)$$

Clearly,

$$a_e(w) = 0, \quad (33a)$$

and

$$b_e(0) = 0. \quad (33b)$$

We are interested in the special value for γ that makes the system of equations in (32) have a nontrivial solution. In order to apply the dead space model, the expressions for h_e and h_h given by (1) are used in (32) to give:

$$a_e(x) = \int_{x+de}^{w} \left[2a_e(s) + b_e(s)\right] \alpha e^{\gamma([s-x]/ve) - \alpha(s-x-de)} ds, \quad 0 \le x \le w - d_e, \quad (34a)$$

and

$$b_e(x) = \int_0^{x-dh} \left[2b_e(s) + a_e(s)\right] \beta e^{\gamma([x-s]/vh) - \beta(x-s-dh)} ds, \quad d_h \le x \le w, \quad (34b)$$

with the boundary conditions

$$a_e(w - d_e) = 0, \quad (35a)$$

and

$$b_e(d_h) = 0. \quad (35b)$$

Differentiating (34) with respect to x, we obtain

$$a_e'(x) = -\alpha\left[2a_e(x+d_e) + b_e(x+d_e)\right] e^{\gamma(d_e/v_e)} + (\alpha - \gamma/v_e) a_e(x), \quad (36a)$$

$$b_e'(x) = \beta\left[2b_e(x-d_h) + a_e(x-d_h)\right] e^{\gamma(d_h/v_h)} + (\gamma/v_h - \beta) b_e(x), \quad (36b)$$

Let us assume a solution of the form $a_e(x) = c_1 e^{rx}$ and $b_e(x) = c_3 e^{rx}$. Substituting in (36), we obtain (see (37) at bottom of the page). For a nontrivial solution the matrix in (37) must be singular (i.e., its determinant must equal zero). Therefore, r must satisfy:

$$\left(r + 2\alpha e^{de(r+\gamma/v_e)} - \alpha + \gamma/v_e\right)\left(r - 2\beta e^{dh(-r+\gamma/v_h)} - \beta - \gamma/v_h\right) + \alpha\beta e^{de(r+\gamma/ve)+dh(-r+\gamma/v_h)} = 0. \quad (38)$$

For each γ, let $r_1(\gamma)$ and $r_2(\gamma)$ be the two zeros of (38), then simple algebra gives

$$c_3 = -\alpha^{-1} e^{d_e(r_1(\gamma)+\gamma/v_e)} \cdot \left[r_1(\gamma) + 2\alpha e^{d_e(r_1(\gamma)+\gamma/v_e)} - \alpha + \gamma/v_e\right] c_1 \equiv y_1(\gamma) c_1. \quad (39)$$

If $r_1(\gamma) \ne r_2(\gamma)$, then the general solution to (36) is:

$$a_e(x) = c_1 e^{r1(\gamma)x} + c_2 e^{r_2(\gamma)x} \quad (40a)$$

and

$$b_e(x) = c_3 e^{r1(\gamma)x} + c_4 e^{r_2(\gamma)x}. \quad (40b)$$

Note that the constants c_2 and c_4 satisfy an equation similar to (39):

$$c_4 = -\alpha^{-1} e^{de(r_2(\gamma)+\gamma/v_e)} \cdot \left[r_2(\gamma) + 2\alpha e^{d_e(r_2(\gamma)+\gamma/v_e)} - \alpha + \gamma/v_e\right] c_2 \equiv y_2(\gamma) c_2. \quad (41)$$

Applying the boundary conditions (35) to (40) and using (39) and (41) gives

$$\begin{bmatrix} y_1(\gamma) e^{r_1(\gamma)[w-d_e]} & y_2(\gamma) e^{r_2(\gamma)[w-d_e]} \\ e^{r_1(\gamma)d_h} & e^{r_2(\gamma)d_h} \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (42)$$

Once again, we set the determinant of the above matrix to zero and obtain

$$y_1(\gamma) e^{r_1(\gamma)[w-d_e] + r_2(\gamma)d_h} - y_2(\gamma) e^{r_1(\gamma)d_h + r_2(\gamma)[w-d_e]} = 0. \quad (43)$$

The exponential rate of $i(t)$ is thus the real number γ that makes (43) hold. This number is called the Malthusian parameter associated with the pair of probability density functions h_e and h_h. A similar calculation shows that the second moment $i_2(t)$ decays with the same rate γ.

In the repeated root case, $r_1(\gamma) = r_2(\gamma) = r(\gamma)$, and the general solution to (36) becomes

$$a_e(x) = c_1 e^{r(\gamma)x} + c_2 x e^{r(\gamma)x}, \quad (44a)$$

and

$$b_e(x) = c_3 e^{r(\gamma)x} + c_4 x e^{r(\gamma)x}. \quad (44b)$$

Substituting (44) in (36) produces two homogeneous equations in terms of the constants $c_i, i = 1, \cdots, 4$. By applying the boundary conditions (35) we obtain two additional homogeneous equations. Setting the determinant of the 4 by 4 matrix, representing the four equations, gives the condition for the Malthusian parameter γ. We will not pursue the details of this case any further.

$$\begin{bmatrix} r + 2\alpha e^{de(r+\gamma/v_e)} - \alpha + \gamma/v_e & \alpha e^{de(r+\gamma/v_e)} \\ -\beta e^{dh(-r+\gamma/v_h)} & r - 2\beta e^{dh(-r+\gamma/v_h)} - \beta - \gamma/v_h \end{bmatrix} \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (37)$$

E. Stability

An APD is said to be stable if its mean gain is finite; otherwise, it is said to be unstable. Since the mean gain of the APD is the total area under the mean impulse response $i(t)$, a positive Malthusian parameter γ corresponds to a stable APD and vice versa. Thus the stability condition for a APD can be determined by setting $\gamma = 0$. Substituting 0 for γ in (43) yields

$$y_1(0)e^{r_1(0)[w-d_e]+r_2(0)d_h} - y_2(0)e^{r_1(0)d_h+r_2(0)[w-d_e]} = 0. \tag{45}$$

For the conventional model where d_e and d_h are zero with $\alpha \neq \beta$, we find that $r_1(0) = 0$ and $r_2(0) = -(\alpha - \beta)$; therefore (45) becomes

$$-\beta + \alpha e^{-w(\alpha-\beta)} = 0 \tag{46}$$

which is the classical stability criterion obtained by McIntyre [1]. It can also be shown that when $\alpha = \beta$, for which $r_1(0) = r_2(0) = 0$, the criterion for stability becomes $\alpha w = 1$, which also agrees with McIntyre's result for this case.

IV. Numerical Results and Discussion

We have used the recurrence equations obtained in Sections III-B and III-C to determine the mean impulse response function and its standard deviation for a double-carrier multiplication APD with ionization rates α and β and dead space distances d_e and d_h. The mean impulse response function $i(t)$ is determined from (13) once $z_e(t,0)$ and $z_h(t,0)$ are computed. The quantities $z_e(t,x)$, $y_e(t,x)$, $z_h(t,x)$, and $y_h(t,x)$ are evaluated numerically using an iteration method. For instance, the numerical evaluation of the pair $z_e(t,x)$ and $y_e(t,x)$ is carried out as follows: 1) Initially, $z_e(t,x)$ and $y_e(t,x)$ are set to be equal to $u_1(t,x)[1 - H_e(v_e t)]$ and 0, respectively. 2) Equation (10) is discretized, using a suitable mesh size, and then used to generate estimates of $y_e(t,x)$ for $x \in [0,w]$ and $t \geq 0$. 3) Using this estimate of $y_e(t,x)$ in the discrete version of (6), an estimate of $z_e(t,x)$ is generated for $x \in [0,w]$ and $t \geq 0$. 4) An improved estimate of $y_e(t,x)$ is obtained by substituting into the discrete version of (10) the previously calculated estimate of $z_e(t,x)$. 5) Steps 3) and 4) are repeated until convergence is achieved. In a similar manner, the second order quantities $\xi_e(t,x)$, $\varepsilon_e(t,x)$, $\xi_h(t,x)$, $\varepsilon_h(t,x)$, $R_Z(t,x)$, and $R_Y(t,x)$ are numerically evaluated and the standard deviation $\sigma(t)$ is determined using (15).

For a better presentation of the results let us define the normalized time $t^* = t/(w/v_e)$ and the scaled mean current $i^*(t^*) = (w/v_e)i(wt^*/v_e)$. Fig. 1 depicts the behavior of $i^*(t^*)$ for different values of the normalized dead space $d_e/w = d_h/w = d/w$ and different values of the hole-to-electron ionization ratio $k = \beta/\alpha$. The validity of the numerical results has been verified by comparing the mean gain, which is the area under $i^*(t^*)/q$, with known results [15]. When $k = 0$ (as in single-carrier multiplication APD's) our results are in agreement with theory [14].

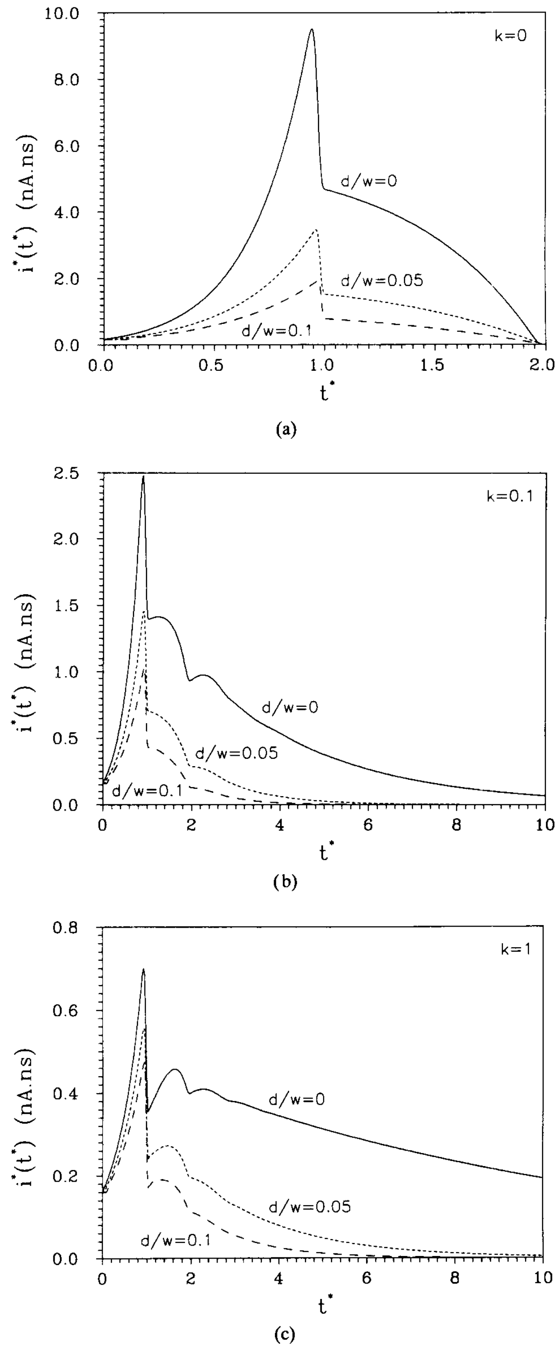

Fig. 1. Scaled mean impulse response function $i^*(t^*) = (w/v_e)i(wt^*/v_e)$ as a function of the normalized time $t^* = t/(w/v_e)$ for an APD with $v_e = v_h$. The value of αw is selected so that the mean gain is 40 in the absence of dead space. The normalized dead space $d/w = d_e/w = d_h/w$ assumes the values 0, 0.05 and 0.1 in the three cases: a) $k = 0$ ($\alpha w = 3.689$); b) $k = 0.1$ ($\alpha w = 2.333$); c) $k = 1.0$ ($\alpha w = 0.975$).

Dead space causes $i^*(t^*)$ to die out faster, resulting in a narrower mean impulse response and a reduced mean gain. From the logarithmic plot of $i^*(t^*)$ in Fig. 2, we see that the tail of $i^*(t^*)$ decays exponentially which confirms our asymptotic

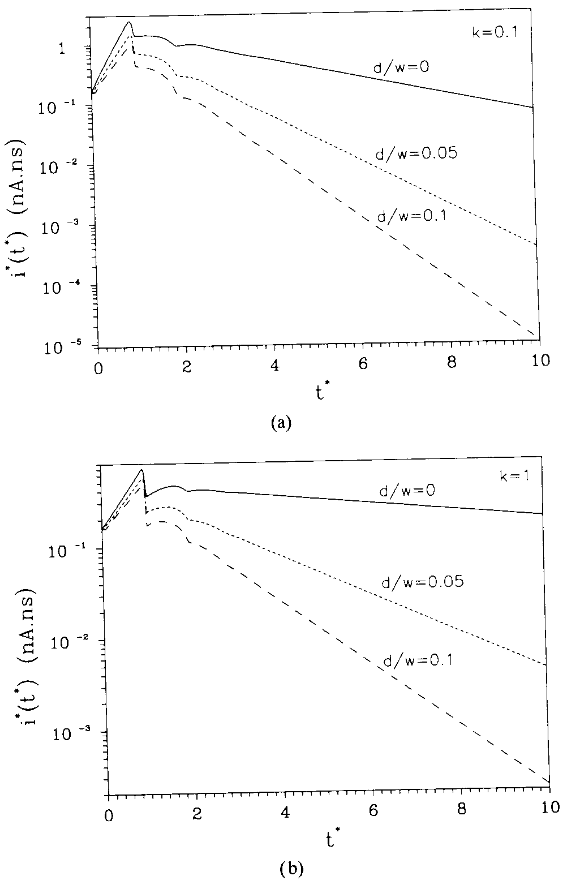

Fig. 2. Logarithmic plots of the scaled mean impulse response function $i^*(t^*) = (w/v_e)i(wt^*/v_e)$ as a function of the normalized time $t^* = t/(w/v_e)$ for an APD with $v_e = v_h$. The value of αw is selected so that the mean gain is 40 in the absence of dead space. These plots demonstrate the exponentially decaying tails of the mean impulse response function. The normalized dead space $d/w = d_e/w = d_h/w$ assumes the values 0, 0.05 and 0.1 in the two cases: a) $k = 0.1$ ($\alpha w = 2.333$); c) $k = 1.0$ ($\alpha w = 0.975$).

results that were established analytically. Furthermore, dead space tends to increase this exponential decay rate. From these plots it can be observed that this increase in the scaled decay rate $\gamma^* = \gamma w/v_e$ becomes more significant as k increases. To demonstrate this effect further, (43) is solved numerically to generate Fig. 3 which shows the dependence of the scaled decay rate γ^* on the normalized dead space d/w.

The multiplication noise of the impulse response function is demonstrated in Fig. 4 in which $i^*(t^*) + \sigma^*(t^*)$ is displayed for different values of d/w where $\sigma^*(t^*) = (w/v_e)\sigma(wt^*/v_e)$ is the scaled standard deviation. As t^* increases, $\sigma^*(t^*)$ decreases to zero at a rate $\sqrt{\gamma^*}$. Dead space reduces the noise $\sigma^*(t^*)$ for all t^* when all other parameters are fixed. The fluctuation in $I(t)$ about its mean value is, therefore, reduced as a result of the presence of dead space. This result is expected since in an earlier analysis of the gain [15] it was shown that dead space reduces the excess noise factor which is also a measure of multiplication noise.

If we compare two APD's with equal mean gains, one with dead space and one with no dead space (ideal), then

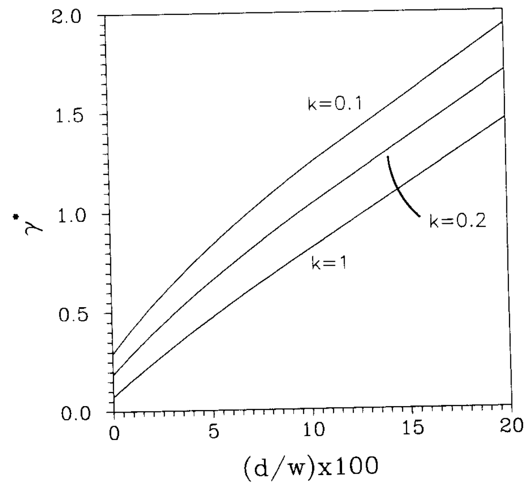

Fig. 3. The scaled exponential decay rate $\gamma^* = w\gamma/v_e$ as a function of the normalized dead space $d/w = d_e/w = d_h/w$. Values of the hole-to-electron ionization ratio are taken to be $k = 0.1$ ($\alpha w = 2.333$), $k = 0.2$ ($\alpha w = 1.893$), and $k = 1.0$ ($\alpha w = 0.975$).

our results show that dead space causes the mean impulse response function to die out slower. This is expected since dead space, on the average, tends to decrease the multiplication rate and hence a longer time is required to produce the same total number of multiplications. This effect is depicted in Fig. 5(a). The noise about the mean impulse response, on the other hand, is less for the dead space case. This can be seen by considering the signal-to-noise ratio of the impulse response function defined by $\mathrm{SNR}(t) = i(t)/\sigma(t)$. As shown in Fig. 5(b), $\mathrm{SNR}(t^*)$ in the dead space case is higher for all t^*. In particular, the decay rate of $\mathrm{SNR}(t^*)$ (which is equal to $\sqrt{\gamma^*}$.) becomes less as a result of dead space and therefore, the tail of the impulse response function becomes less noisy.

V. Conclusion

We presented a theory, based on recurrence equations, which allowed us to compute the mean and the standard deviation of the impulse response function, of double-carrier multiplication avalanche photodiodes, as functions of time following photo-excitation. This theory took into account the ages of carriers. Therefore, it accommodated for the effect of dead space, which is the minimum distance a newly generated carrier must travel before it can impact ionize, on the statistics of the impulse response function. Recurrence equations were also derived for the probability distribution function of the impulse response function.

We have found that the presence of dead space results in a reduction in both the mean and the standard deviation of the impulse response for all time. We have shown analytically that the mean impulse response function exhibits an exponentially decaying tail with a rate that increases with the increase in dead space. Dead space, therefore, tends to reduce the effect of feedback caused by double-carrier multiplication which is responsible for the extended tail of the impulse response function. Keeping the mean gain fixed, dead space

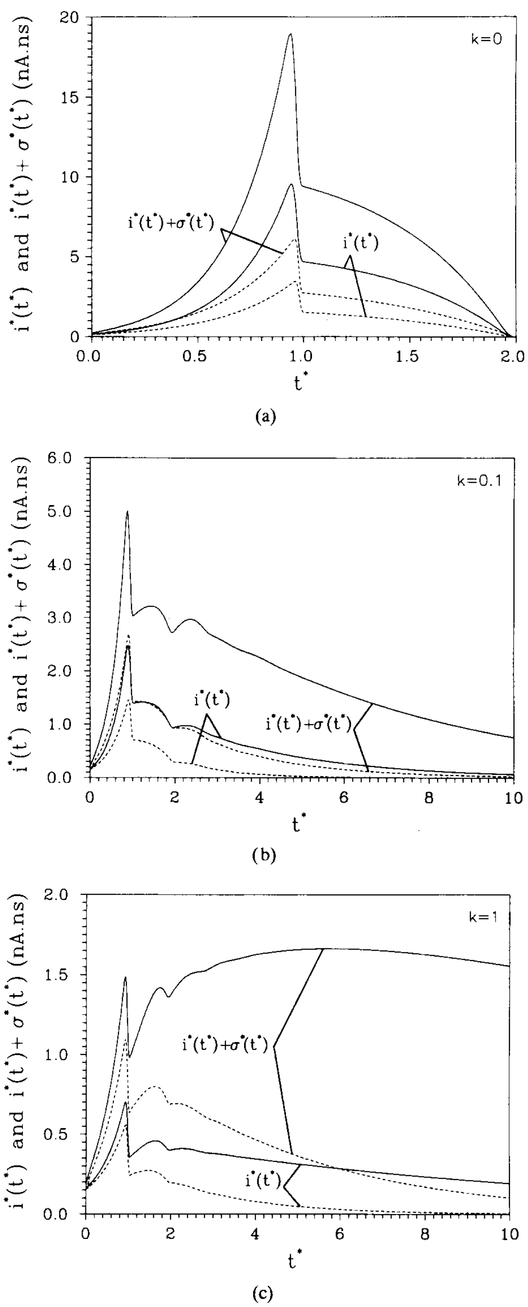

Fig. 4. The scaled mean impulse response function plus one scaled standard deviation $i^*(t^*) + \sigma^*(t^*)$ is compared with $i^*(t^*)$ assuming $v_e = v_h$. The normalized dead space $d/w = d_e/w = d_h/w$ assumes the values 0 and 0.05 in the three cases: a) $k = 0$ ($\alpha w = 3.689$); b) $k = 0.1$ ($\alpha w = 2.333$); c) $k = 1.0$ ($\alpha w = 0.975$).

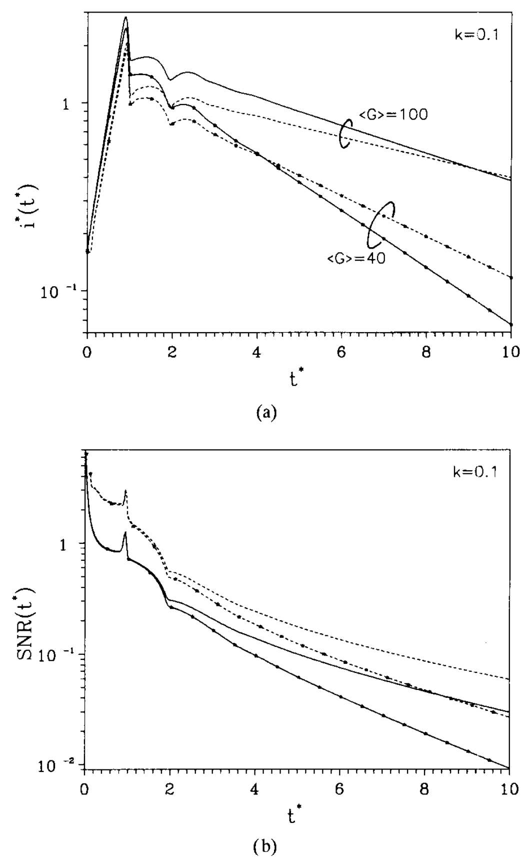

Fig. 5. Comparison between two APD's with equal mean gains $\langle G \rangle$, one with dead space (dashed) with $d/w = d_e/w = d_h/w = 0.1$, and the other with no dead space (solid). Two cases are considered, $\langle G \rangle = 40$ ($\alpha w = 2.333$, $d/w = 0$; $\alpha w = 3.734$, $d/w = 0.1$) represented by starred graphs, and $\langle G \rangle = 100$ ($\alpha w = 2.456$, $d/w = 0$; $\alpha w = 3.957$, $d/w = 0.1$) represented by plain graphs. The hole-to-electron ionization ratio $k = 0.1$ and $v_e = v_h$. These plots represent: a) The scaled mean impulse response $i^*(t^*)$ demonstrating that dead space results in longer tails; b) The signal-to-noise ratio SNR(t^*) demonstrating that dead space enhances the SNR for all time.

results in a longer response time which may degrade the bit-error rate of the optical communication system due to intersymbol-interference. At the same time, dead space enhances the signal-to-noise ratio of the impulse response for all time which will tend to enhance the bit-error rate. It is therefore important to include dead space in the performance analysis of receivers in optical communication systems. We are currently in the process of determining the effect of dead space on the performance of optical receivers and our preliminary results indicate that dead space can enhance the performance.

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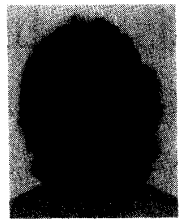

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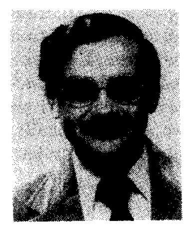

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