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# Theory of photon coincidence statistics in photon-correlated beams

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## Abstract

The statistics of photon coincidence counting in photon-correlated beams is thoroughly investigated considering the effect of the finite coincidence resolving time. The correlated beams are assumed to be generated using parametric downconversion, and the photon streams in the correlated beams are modeled by two partially correlated Poisson point processes. An exact expression for the mean rate of coincidence registration is developed using techniques from renewal theory. It is shown that the use of the traditional approximate rate, in certain situations, leads to the overestimation of the actual rate. The error between the exact and approximate coincidence rates increases as the coincidence-noise parameter, defined as the mean number of uncorrelated photons detected per coincidence resolving time, increases. The use of the exact statistics of the coincidence becomes crucial when the background noise is high or in cases when high precision measurement of coincidence is required. Such cases arise whenever the coincidence-noise parameter is even slightly in excess of zero. It is also shown that the probability distribution function of the time between consecutive coincidence registration can be well approximated by an exponential distribution function. The well-known and experimentally verified Poissonian model of the coincidence registration process is therefore theoretically justified. The theory is applied to an on-off keying communication system proposed by Mandel which has been shown to perform well in extremely noisy conditions. It is shown that the bit-error rate (BER) predicted by the approximate coincidence-rate theory can be significantly lower than the actual BER obtained using the exact theory. © 1999 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

A source of non-classical light that has generated considerable interest in recent years is photon-correlated beams. The light source takes the form of two beams, the photons of each arrive randomly, but the photons of the two beams are, under ideal conditions, perfectly synchronized in time and space. Pho-

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ton correlated beams can be generated, for example, by spontaneous parametric downconversion [1–5]. This is a nonlinear process in which each of the photons of a pump interacts with a medium exhibiting a second-order nonlinear effect and creates a twin pair of photons called signal and idler. Conservation of momentum ensures that if one photon is observed in one direction, its twin must be present in one and only one matching direction. If the pump is in a coherent state, the statistics of the photons in each of the twin beams obey a Poisson process, but the two processes are, under ideal conditions, completely correlated. Since the joint statistics of the photons of this light source have reduced uncertainty, this light source is squeezed [6–9]. Photon-correlated beams have been proposed for use in a number of applications including optical communications, transmittance estimation, imaging, microscopy, cryptography, tests of the quantum theory of light, and other applications [9–21].

In some applications, pairs of ‘coincident’ photons from photon-correlated beams are used as the information-bearing signal since these coincident photons can be distinguished from noise photons by virtue of their temporal coincidence. A common scheme for the measurement of coincident photons involves the use of two photodetectors and an electronic timer/counter [22,23]. The operation of such a photon coincidence scheme considered in this paper is described as follows [24,25]. The first photodetection by the signal-beam detector (SD) is used to trigger a timer and the first photodetection by the idler-beam detector (ID), following the SD detection, is used to stop the timer.<sup>3</sup> If the time between the start and the stop of the timer is less than a prescribed threshold, called the coincidence resolving time, the counter is incremented by one. In this case, we say that a coincidence event is registered at the time of the photodetection by ID, and the search for subsequent coincidences starts afresh thereafter. On the other hand, if no photons are detected by ID within the coincidence resolving time, the coincidence-counting mechanism starts afresh thereafter

and the timer will be re-triggered as soon as SD detects a photon, and so on. Note that for this coincidence-counting mechanism, if only a single photon is detected by SD within a resolving time and multiple photons are detected by ID within the same resolving time, then only one coincidence is registered (corresponding to the first photodetection by ID following the SD photodetection). In this scheme, the finite width of the resolving time used to register coincident photons allows a fraction of the unwanted uncorrelated photons (resulting from uncorrelated photons) to contribute to the process of coincidence registration. This additional coincidence registration, which is referred to as *accidental coincidence*, becomes a limiting factor in applications when the average number of photons per coincidence resolution time is even slightly greater than zero.

If the photon coincidence property of photon-correlated beams is to be capitalized on in suppressing noise photons (as in the case with the down converted light communication scheme proposed in the pioneering work of Mandel [10] which was also demonstrated by Hong et al. [11]), then knowledge of the effect of accidental coincidence is critical in understanding the performance advantage that photon-correlated beams can offer in comparison to systems using conventional light. The traditional expression for the rate of coincidence registration (the expression used by Mandel [10]) becomes inaccurate when the mean number of photons per resolving time is high (e.g., in excess of 0.1 in our examples). In such cases, our results show that the approximation leads to overestimating the coincidence rate considerably. Developing an exact theory of coincidence statistics is therefore needed to understand the statistics of coincidence counting in cases when the approximate rate is not accurate. In addition, the exact knowledge of the coincidence rate will definitively establish the conditions under which the use of photon-correlated light in various applications can offer a performance advantage over conventional light. The photon correlated light in this paper is assumed to be that generated using spontaneous parametric downconversion; nonetheless, the theory is applicable to other photon-correlated-light situations which can be approximated by our model.

Although the theory is applied to an on-off keying optical communication system, the results are also

<sup>3</sup> In practice, it may be necessary to compensate for the relative delay between the detection times of twin photons due to non-identical signal and idler channel path lengths [11].

applicable to other applications such as transmittance estimation [9], the measurement of quantum efficiency of a detector, microscopy, quantum cryptography [26], and to other radiometric measurement. In addition, the exact theory of coincidence statistics presented in this paper can also be extended to alternative coincidence counting schemes. For example, in positron emission tomography, a coincidence between twin photons (traveling in opposite directions and resulting from the decay of beta particles) is detected if the difference between the time-of-flight of the two photons is within a certain time window [27–29]. Accidental coincidence resulting from non-twin photons can be detrimental to the quality of the reconstructed image [27,30–33]. An exact statistical theory for coincidence can be useful in efforts to reduce the degrading effects of accidental coincidence.

This paper is organized as follows. In Section 2, we develop a stochastic model for the photon streams in the photon-correlated beams and formally define coincidence events. In Section 3, we develop the exact statistics of the number of coincident photons including the mean rate of coincidence and the probability distribution function of the time between successive coincidence events. The results are compared to the traditional approximate results. In Section 4, we apply the theory to an on-off communication system and provide an assessment of the performance of the system.

## 2. Model

### 2.1. Joint statistics of the signal and idler photons

We adopt a simple stochastic model for which the photon-correlated light beams are regarded as statistically correlated streams of photons [19,20,34,35]. The flux of photons in each beam can therefore be regarded as a point process [36], and these two point processes are statistically correlated. We will assume that the light in each beam is described by a Poisson point process. Under ideal conditions, the signal and idler photons are fully correlated, i.e., the detection of a signal-beam photon at a specific time and location dictates the detection of its twin photon at a

prescribed time and location in the idler beam. In this ideal case, each Poisson process is a copy of the other. Such ideal conditions are achieved, for example, in spontaneous parametric downconversion when the pump is a monochromatic plane wave, the crystal dimensions are infinite, the signal and idler beams are selected by perfectly matched apertures, and ideal detectors are used. In practice, these conditions are not met and the collected signal and idler beams photons are not fully correlated even when matched apertures are used [4,5]. Additionally, the transmission of the signal and idler beams through optical elements results in further reduction of the degree of correlation [16].

To account for the partial correlation of the signal and idler photon numbers, we adopt a simplified model in which the photon streams of the signal and idler beams are the superposition of totally correlated components and totally uncorrelated components [9]. Let  $\lambda_s$  and  $\lambda_i$  denote the total photon flux (mean number of photons per second) of the signal and idler channel, respectively. A fraction  $\beta_s$  of the signal photons are coincident with a fraction  $\beta_i$  of the idler photons so that  $\beta_s \lambda_s = \beta_i \lambda_i$ . For simplicity, we will assume that  $\beta_s = \beta_i = \beta$  and that  $\lambda_s = \lambda_i = \lambda$ . The downconversion parameter  $\beta$  therefore represents the fraction of the correlated photons in the signal and idler beams. We call such signal and idler photons fully-correlated or twin photons. The case  $\beta = 1$  corresponds to full correlation. The remaining fraction of signal photons, arriving at a rate  $(1 - \beta)\lambda$ , are assumed to be totally uncorrelated with all the other photons. Finally, to capture the effect of stray light, photodetector dark current, and other sources of noise, we will assume that the noise photon fluxes  $\mu_s$  and  $\mu_i$  are added to the signal and idler channels, respectively. Unlike the uncorrelated component of the signal and idler photons the noise photons are typically not utilized to bear information [e.g., by modulating the intensity].

### 2.2. Coincidence counting

Consider the coincidence counting scheme described in Section 1, and assume that the quantum efficiencies of the signal-beam and the idler-beam detectors are  $\eta_s$  and  $\eta_i$ , respectively. It is evident

that coincidence events can occur between twin photons, uncorrelated signal photons, and noise photons alike due to the finite resolving time  $h$  and the uncertainty associated with the process of photodetection. Since uncorrelated photons and noise photons contribute to accidental coincidence in similar ways, we group them together as uncorrelated photons. Based on the type of photons (correlated or uncorrelated) contributing to coincidence registration, we can categorize the types of coincidence events as follows:

1. Coincidence between a signal correlated photon and its twin in the idler beam: This situation occurs when both of the twin photons are detected. This type of coincidence is the key in discriminating against background noise.
2. Coincidence registration due to a correlated photon in the signal beam and a non-twin photon in the idler beam: This situation occurs when the signal-channel correlated photon is detected but its twin in the idler channel is not detected. A coincidence may occur in this case if any photon is detected in the idler beam within the resolving time.
3. Coincidence registration due to an uncorrelated signal-beam photon and an idler-beam photon: An uncorrelated signal-beam photon may trigger the counter and cause the registration of a coincidence event if any photon is detected from the idler channel within the resolving time.

The coincidence events of types 2 and 3 above are unwanted coincidence events since they are nonexistent in the ideal case when the detectors are ideal and the resolving time is infinitesimal.

As a result of the Poissonian statistics of the photons in the idler and the signal, the times between consecutive coincidence events are statistically independent. Moreover, since the intensities of the signal and idler beams are assumed constant (as in fully coherent light), the times between successive coincidences have identical statistics.

### 3. Statistics of photon coincidence

We now proceed to develop an exact theory that characterizes the mean rate of coincidence registration. The approach is based on concepts from re-

newal theory [37]. For purposes of comparison, we first give a brief review of the traditional approximate statistics of coincidence [10,11]. For convenience, we denote the total rate of photodetection in each of the signal and idler beams, respectively, by

$$r_s = \eta_s(\lambda + \mu_s) \quad (1)$$

and

$$r_i = \eta_i(\lambda + \mu_i). \quad (2)$$

#### 3.1. Traditional approximate theory of coincidence statistics

The rationale of the traditional approach for finding the rate of coincidence can be stated as follows. The total rate of coincidence has two components: A contribution from twin photons (true coincidence) and a contribution from all other non-twin photons (accidental coincidence), including noise photons, uncorrelated signal and idler photons, and twin photons that are not detected by both detectors. Clearly, the contribution from detected twin photons occur at a rate  $\eta_s \eta_i \beta \lambda$ . The total rate of photodetection from the signal beam, less the detection rate of twin photons, is therefore  $r_s - \eta_s \eta_i \beta \lambda$ . Similarly, the total photodetection rate of photons in the idler beam, less the detection rate of twin photons, is  $r_i - \eta_s \eta_i \beta \lambda$ . The rate of accidental coincidence registrations can be approximately calculated by taking the product of these reduced rates times the coincidence resolving time  $h$ . The approximate total rate of coincidence  $r_{c,\text{approx}}$  is therefore

$$r_{c,\text{approx}} = \eta_s \eta_i \beta \lambda + (r_s - \eta_s \eta_i \beta \lambda)(r_i - \eta_s \eta_i \beta \lambda)h. \quad (3)$$

Clearly, the first term in (3) is due to type 1 coincidence events as described in Section 2. The second term is due to accidental coincidence and it combines coincidence events of types 2 and 3. Using the mean rate formula in (3), we can obtain the mean number of coincidence events,  $\langle N(t) \rangle$ , in an interval  $[0, t]$  by simply taking the product of the rate and the length of the interval:

$$\langle N(t) \rangle = r_{c,\text{approx}} t. \quad (4)$$

The coincidence counting process  $N(t)$  is assumed to have Poisson statistics [10,6].

The approximate expression in (3) is accurate only if the mean number of photons per coincidence resolving time is much less than unity. To understand its limitations, consider the following scenario: Suppose that the signal-beam photon flux rate is moderately high so that it is likely to have more than one photon detection per resolving time. Now suppose that a photon triggers the coincidence counter, and further assume that other signal photons are detected within the resolving time following the triggering, then these additional signal photons will not contribute to coincidence events since the counter is not responsive to them during the resolving time. The approximate rate Eq. (3) does not take this factor into account and therefore overestimates the actual coincidence rate. However, if the resolving time is sufficiently small, so that the likelihood of detecting more than one photon is negligible, then the above situation will not have a significant impact and the approximation becomes accurate. The exact theory, developed in the next subsection, will provide a simple exact formula for the coincidence rate for any  $h$ .

### 3.2. Exact rate of coincidence registration: A renewal-theory approach

In this subsection, we develop an exact expression for the mean rate of coincidence registrations. To our knowledge, this is the first time that this exact rate is reported. We first determine the average time between consecutive coincidence registration events and then take its reciprocal to obtain the desired coincidence rate.

We now derive a set of renewal (or recurrence) equations for the mean of the first coincidence random variable  $C$ . The mean rate of coincidence is then  $1/\langle C \rangle$ . Without loss of generality, assume that the coincidence counter starts at time  $t = 0$  and that the first photon detection by SD occurs at time  $\xi$ . Since the photon arrival in each channel is modeled by a Poisson process, the random time,  $X$ , to the first photon detection by SD is an exponentially-distributed random variable with mean  $1/r_s$  [36]. Thus, the probability density function of the random variable  $X$  is

$$f_X(\xi) = r_s e^{-r_s \xi}, \quad \xi \geq 0. \quad (5)$$

Further, given that a photon is detected by SD at time  $\xi$ , the conditional probability that it is actually one of twin photons is simply the ratio between the detection rate of correlated photons by SD to the total detection rate by SD. This conditional probability is therefore

$$\frac{\beta\lambda}{\lambda + \mu_s},$$

and the conditional probability that the photon is not one of the correlated photons is of course

$$1 - \frac{\beta\lambda}{\lambda + \mu_s} = \frac{(1 - \beta)\lambda + \mu_s}{\lambda + \mu_s}.$$

In addition, we will later use the fact that the probability that a photon is detected by ID in any time increment  $[\xi, \xi + d\xi]$  is  $r_i e^{-r_i \xi} d\xi$ .

Given the condition that the first photon detection by SD occurs at time  $\xi$ , the events that follow can be decomposed, in a mutually exclusive way, into certain useful events that will facilitate the derivation of the expected length  $\langle C \rangle$ . These mutually exclusive events are described below.

1. Define  $A_1$  as the event that the first photodetection by SD (at time  $\xi$ ) is indeed a correlated photon *and* that its twin is also detected by ID. Observe that if  $A_1$  occurs, then a coincidence registration occurs at time  $\xi$ . Note that

$$P(A_1) = \frac{\beta\lambda\eta_i}{\lambda + \mu_s}.$$

2. Define  $A_2$  as the event that the first photodetection by SD (at time  $\xi$ ) is a correlated photon, that its twin is *not* detected by ID, *and* that a photon is detected by ID in the interval  $[\tau, \tau + d\tau]$  within  $h$  units of time following  $\xi$ . Here, a coincidence registration occurs at time  $\xi + \tau$  if the event  $A_2$  occurs. Note that

$$P(A_2) = \frac{\beta\lambda(1 - \eta_i)}{\lambda + \mu_s} r_i e^{-r_i \tau} d\tau.$$

3. Define  $A_3$  as the event that the first photodetection by SD (at time  $\xi$ ) is a correlated photon, that its twin is *not* detected by ID, *and* that no photons are detected by ID in the interval  $[\xi, \xi +$

$h]$ . The occurrence of  $A_3$  implies that no coincidence event has been registered up to time  $\xi + h$ , and that the counter will therefore start afresh thereafter in search of coincidence events. Using the time  $\xi + h$  as a starting point, the time to the first coincidence registration is a random variable  $\tilde{C}$  that has an identical probability distribution as that of  $C$ . Since the probability of not detecting any photons (by ID) in the interval  $[\xi, \xi + h]$  is  $1 - \int_0^h r_i e^{-r_i \tau} d\tau$ , the probability of the event  $A_3$  is

$$P(A_3) = \frac{\beta\lambda(1 - \eta_i)}{\lambda + \mu_s} e^{-r_i h}.$$

- Define  $A_4$  as the event that the first photodetection by SD (at time  $\xi$ ) is not a correlated photon and that a photon is detected by ID in the interval  $[\tau, \tau + d\tau]$  within  $h$  units of time following  $\xi$ . In this case, a coincidence is registered at time  $\tau + d\tau$  if the event  $A_4$  occurs. Note that the probability of the event  $A_4$  is

$$P(A_4) = \left(1 - \frac{\beta\lambda}{\lambda + \mu_s}\right) (1 - \eta_i) r_i e^{-r_i \tau} d\tau.$$

- Define  $A_5$  as the event that the first photodetection by SD (at time  $\xi$ ) is not a correlated photon and that no photon is detected by ID in the interval  $[\xi, \xi + h]$ . Similarly to  $A_3$ , the occurrence of  $A_5$  implies that no coincidence event is registered up to time  $\xi + h$ , and that the counter will therefore start afresh in search of coincidence events at time  $t = \xi + h$ . Using the time  $\xi + h$  as a starting point, the time to the first coincidence registration is a random variable  $\tilde{C}$  that has an identical probability distribution as that of  $C$ . In this case,

$$P(A_5) = \left(1 - \frac{\beta\lambda}{\lambda + \mu_s}\right) (1 - \eta_i) e^{-r_i h}.$$

We now use the above events to observe that under the condition that the first photon detection by SD occurs at time  $\xi$ , the random variable  $C$  can be analyzed as follows:

$$C = \begin{cases} \xi, & \text{if event } A_1 \text{ occurs,} \\ \xi + \tau, & \text{if either event } A_2 \text{ or event } A_4 \text{ occur,} \\ \xi + h + \tilde{C} & \text{if either event } A_3 \text{ or event } A_5 \text{ occur.} \end{cases} \quad (6)$$

By averaging  $C$  over the mutually exclusive events  $A_1$  through  $A_5$ , and by using the fact that  $\langle C \rangle = \langle \tilde{C} \rangle$ , we obtain an expression for the conditional mean of the time to the first coincidence registration given that the first photon detection by SD is at  $\xi$ . This conditional mean can be shown to be

$$\begin{aligned} E[C|X = \xi] &= \frac{\beta\lambda}{\lambda + \mu_s} \left\{ \eta_i \xi + (1 - \eta_i) (\langle C \rangle + \xi + h) e^{-r_i h} \right. \\ &\quad \left. + \int_0^h (1 - \eta_i) (\xi + \tau) r_i e^{-r_i \tau} d\tau \right\} \\ &\quad + \frac{(1 - \beta)\lambda + \mu_s}{\lambda + \mu_s} \left\{ (\langle C \rangle + \xi + h) e^{-r_i h} \right. \\ &\quad \left. + \int_0^h (\xi + \tau) r_i e^{-r_i \tau} d\tau \right\}. \end{aligned} \quad (7)$$

Finally, to obtain the mean  $\langle C \rangle$ , we remove the conditioning in  $E[C|X = \xi]$  by averaging over all possible  $X$ , i.e.,

$$\langle C \rangle = \int_0^\infty E[C|X = \xi] f_X(\xi) d\xi. \quad (8)$$

Upon substituting (7) and (5) in (8) and carrying out the algebra, we obtain an expression for the mean rate of coincidence,  $r_c = 1/\langle C \rangle$ , given by

$$r_c = \frac{1 - e^{-r_i h} \left(1 - \frac{\eta_i \beta\lambda}{\lambda + \mu_s}\right)}{\frac{1}{r_s} + \frac{1}{r_i} (1 - e^{-r_i h}) \left(1 - \frac{\eta_i \beta\lambda}{\lambda + \mu_s}\right)}. \quad (9)$$

The rate equation (9) can be expressed in terms of a key parameter  $\rho_c = ((1 - \beta)\lambda + \mu_i)\eta_i h$ , called the *coincidence-noise parameter*, which represents the mean number of detected uncorrelated photons per coincidence resolving time in the idler beam. Using this parameter, we obtain

$$r_c = \frac{1 - e^{-(\eta_i \lambda \beta h + \rho_c)} \left(1 - \frac{\eta_i \beta\lambda}{\lambda + \mu_s}\right)}{\frac{1}{r_s} + \frac{1}{r_i} (1 - e^{-(\eta_i \lambda \beta h + \rho_c)}) \left(1 - \frac{\eta_i \beta\lambda}{\lambda + \mu_s}\right)}. \quad (10)$$

As expected, in the case of ideal coincidence counting (i.e.,  $h = 0$  and in which case  $\rho_c = 0$ ), the formula for  $r_c$  reduces to  $\eta_s \eta_i \beta \lambda$  which is simply the rate of the simultaneous detection of fully correlated photons. Furthermore, an expansion of (9) in terms of the parameter  $h$  shows that the traditional formula given in (3) is a first-order approximation of the exact rate. Eq. (10) reveals exactly how the rate is dependent on the two factors that govern accidental coincidence, i.e., the accidental noise parameter  $\rho_c$  and the coincidence resolving time  $h$ . (Note that varying  $h$  alone has a different effect on  $r_c$  than varying  $\rho_c$  while holding  $h$  fixed.)

### 3.3. Probability distribution function of the time between coincidence events

The knowledge of the mean rate of coincidence alone is not sufficient to describe the statistics of the number of coincidence events in a given time interval. What is required is knowledge of the probability density function of the random time  $C$  between successive coincidence counts. If this probability density function is exponential, then the coincidence registration process is Poissonian [35,36] (i.e., the number of coincidence registrations in any time interval is a Poisson random variable with a mean value which can be determined by taking the product of the time interval and the coincidence rate given in (9)). We will show that the probability density function of the time between successive coincidence registrations is not exactly exponential but can be approximated accurately by an exponential probability density function.

To derive an expression for the probability distribution function (PDF),  $F_C(t)$ , defined as  $P\{C \leq t\}$ , we follow the same technique used to derive (7) and (8). In particular, we first evaluate the conditional PDF under the occurrence of each of the events  $A_1$  to  $A_5$ , and then we take the average over these events. We omit the details of the derivation and only present the final result. For  $t \leq h$ ,

$$F_C(t) = 1 - e^{-r_s t} \left( \frac{r_s e^{(r_s - r_i)t} - r_i}{r_s - r_i} \right), \quad (11)$$

and for  $t > h$ ,

$$\begin{aligned} F_C(t) &= \frac{\beta \lambda \eta_i}{\lambda + \mu_s} (1 - e^{-r_s t}) + \left( 1 - \frac{\beta \lambda \eta_i}{\lambda + \mu_s} \right) \\ &\times \left\{ (1 - e^{-r_i h})(1 - e^{-r_s(t-h)}) \right. \\ &+ e^{-r_s t} \left[ e^{r_s h} - 1 - \frac{r_s}{r_s - r_i} (e^{(r_s - r_i)h} - 1) \right] \\ &\left. + e^{-r_i h} \int_0^{t-h} r_s e^{-r_s s} F_C(t-s-h) ds \right\}, \end{aligned}$$

where  $r_s$  and  $r_i$  are given by (1) and (2), respectively. The above integral equation is solved numerically (using numerical integration and the initial values given in (11)) and the results are presented in the next subsection.

### 3.4. Comparison between the exact and approximate statistics of coincidence

The discrepancy between the exact results and the traditional approximation becomes insignificant when the quantity  $r_i h$  is very small. This condition occurs when either the coincidence-noise parameter  $\rho_c$  or the quantity  $\eta_i \lambda \beta h$  are not ‘close’ to zero. The results obtained from our examples indicate that if  $\rho_c > 0.1$ , a noticeable error is observed in the approximation. In our examples, we assume that  $h = 0.1$  ns and that the downconversion parameter  $\beta$  is 0.5 [10]. We illustrate the effect of the parameter  $\rho_c$  by plotting the rate of coincidence as a function of  $\rho_c$  in two cases of high and low signal-to-noise ratio (SNR). We take the signal and idler photon fluxes as  $\lambda = 1.8 \times 10^8$  photons/s [i.e., light in the ten pico Watt range], and the quantum efficiencies  $\eta_s$  and  $\eta_i$  both assume the value 0.1. For the high SNR case, as depicted in Fig. 1, the background-noise flux is varied so that the coincidence-noise parameter  $\rho_c$  is increased to 0.2 (in this case, the minimum value for  $\lambda/\mu_s$  is 1/10). For the low SNR case, the background-noise flux is varied so that the coincidence-noise parameter  $\rho_c$  is increased up to 0.4 (the minimum value for  $\lambda/\mu_s$  is 1/20). It is seen from Fig. 1

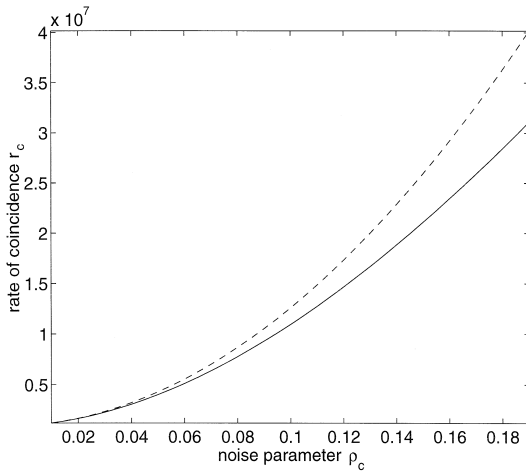


Fig. 1. The rate of coincidence registration as a function of the accidental noise parameter  $\rho_c$ . Solid and dashed lines represent the exact and approximate results, respectively. The following set of parameters are used:  $\eta_s = \eta_i = 0.1$ ,  $\beta = 0.5$ ,  $h = 1$  ns, and  $\lambda = 1.8 \times 10^8$  photons/s. The noise flux is varied so that the parameter  $\rho_c$  ranges from 0 to 0.2.

that the error in the coincidence rate ranges from being negligible in the case  $\rho_c = 0$  to approximately 12% for the case  $\rho_c = 0.1$ , and up to 32% when  $\rho_c = 0.2$ . The error becomes much larger as the noise parameter increases (as seen from Fig. 2) reaching a value of 64% when  $\rho_c = 0.4$ . Our results also indicate that the error in the approximation is negligible

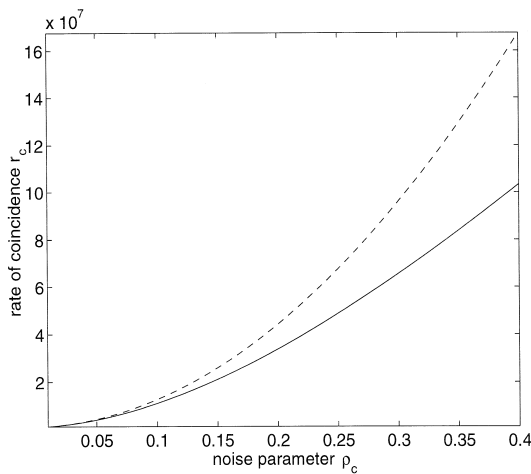


Fig. 2. Same as Fig. 1 but in this case the noise flux is varied so that the parameter  $\rho_c$  ranges from 0 to 0.4.

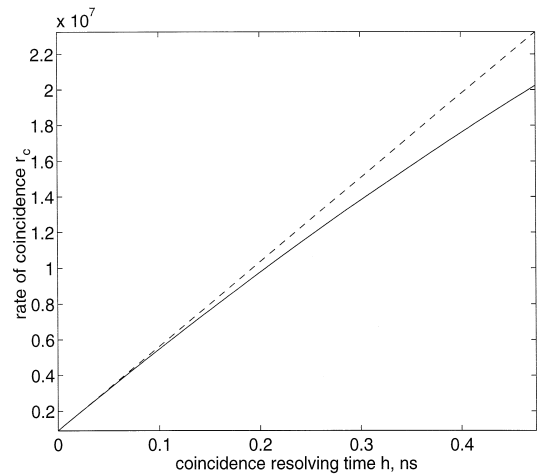


Fig. 3. The rate of coincidence registration as a function of the coincidence resolving time  $h$ . Solid and dashed lines represent the exact and approximate results, respectively. The following set of parameters are used:  $\eta_s = \eta_i = 0.1$ ,  $\beta = 0.5$ ,  $\lambda = 1.8 \times 10^8$  photons/s and  $\mu_s = \mu_i = 2 \times 10^9$  photon/s.

if the noise parameter is below 0.04 (corresponding to an error of 2%).

The dependence of the mean coincidence rate on the resolving time is depicted in Figs. 3 and 4. Fig. 3 corresponds to resolving times in the range 0 ~ 0.5 ns, and the noise levels  $\mu_s$  and  $\mu_i$  are both  $\lambda/2 = 5 \times 10^7$  photons/s. All the other parameters are as

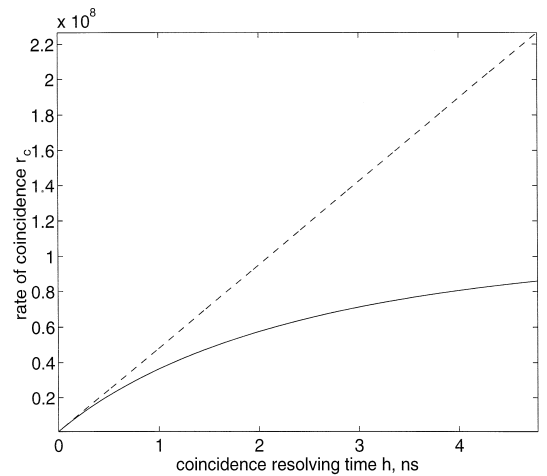


Fig. 4. Same as Fig. 3 but in this case the coincidence resolving time  $h$  ranges from 0 to 4 ns.



before. The exact rate of coincidence reveals that the accidental coincidence rate is a nonlinear function of the resolving time  $h$ . The nonlinearity becomes more severe as  $h$  increases, as seen from Fig. 4. We emphasize that the traditional approximate rate of coincidence yields the well-known linear dependence of the accidental coincidence on  $h$ . The knowledge of such nonlinear dependence of the accidental coincidence can be very useful in applications which are known to have a high level of such undesired accidental coincidence [27–33].

The discrepancy between exact and approximate results is also manifested in the probability distribution function of the random time between successive coincidence registrations. Figs. 5 and 6 show a comparison between the exact and approximate PDF of the time between coincidence registrations for the cases  $\rho_c = 0.1$  and  $\rho_c = 0.2$ , respectively, where the correlated signal photon flux is taken as  $\lambda = 1.8 \times 10^8$  photons/s. It is clear from (11) that the PDF of  $C$  is not an exponential PDF in the initial phase of the distribution where  $t \leq h$ . The intuitive reason for this behavior is that when a finite coincidence counting resolving time is used, the process of registering coincident events is no longer memoryless. For example, it is impossible to register two consecutive coincidences within a counting time  $h$ , and this is a manifestation of ‘memory.’ The plots of the exact

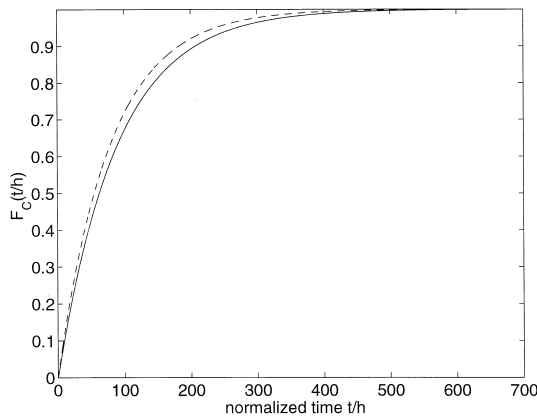


Fig. 5. The probability distribution function, as a function of the normalized time  $t/h$ , of the time between successive coincidence events. Solid and dashed lines represent the exact and approximate results, respectively. The following set of parameters are used:  $\eta_s = \eta_i = 0.1$ ,  $\beta = 0.5$ ,  $h = 1$  ns, and  $\lambda = 1.8 \times 10^8$  photons/s. The noise parameter  $\rho_c$  is 0.1.

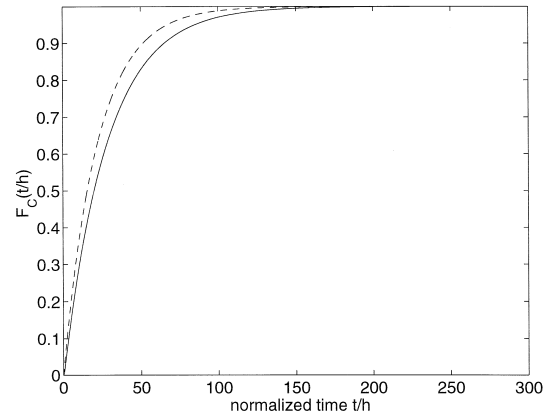


Fig. 6. Same as Fig. 5 but the coincidence parameter is increased to 0.2.

PDF indicate, nonetheless, that for values of the normalized time  $t/h$  in excess of unity, the PDF of  $C$  can be well approximated by an exponential PDF. This observation justifies approximating the number of coincidence events in a given interval by a Poisson random variable which has been verified experimentally [6]. Equivalently, the coincidence registration, as a point process, can be approximated by a Poisson process. The key issue here is that the exact theory presented here enables us to predict the *correct* rate of this approximately Poisson process.

#### 4. Application: Performance of an on-off keying communication system using Photon Correlated Beams

We now consider the optical communication system first proposed by Mandel [10] as shown schematically in Fig. 7. When the message ‘1’ is transmitted (hypothesis  $H_1$ ), the photon fluxes of the signal and idler beams are  $\lambda + \mu_s$  and  $\lambda + \mu_i$ , respectively, and when the message ‘0’ is transmitted (hypothesis  $H_0$ ), the signal and idler photon fluxes are respectively  $\mu_s$  and  $\mu_i$ . The coincidence counter counts coincident photons in a bit of duration  $T$ , and this coincidence count is used to determine the transmitted message. Let  $N_0$  and  $N_1$  denote the number of coincident photons in per bit under hypotheses  $H_0$  and  $H_1$ , respectively. Using the Poisson-process model for the coincidence registration processes (as justified in 3.4), the measured quantities  $N_0$  and  $N_1$

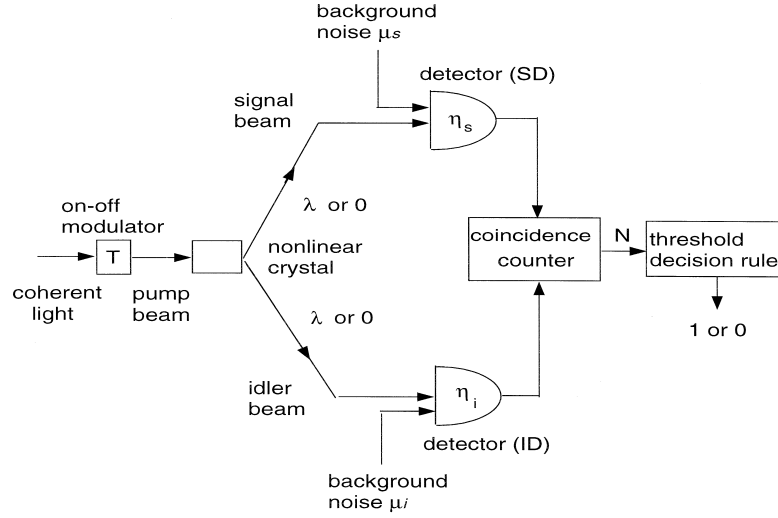


Fig. 7. Schematic diagram of an on-off keying communication system using correlated-photons beams.

are modeled by Poisson random variables with means which can be computed using the theory of Section 3. These average registration counts can be computed exactly using (9) or approximately using (3) with the appropriate photon fluxes under each hypothesis. In particular, the exact averages are

$$\langle N_0 \rangle_{\text{exact}} = \frac{1 - e^{-r_{i0}h}}{\frac{1}{r_{s0}} + \frac{1}{r_{i0}}(1 - e^{-r_{i0}h})} T, \quad (12)$$

and

$$\langle N_1 \rangle_{\text{exact}} = r_c T, \quad (13)$$

where  $r_{s0} = \eta_s \mu_s$ ,  $r_{i0} = \eta_i \mu_i$ , and  $r_c$  is given by (9). On the other hand, the approximate mean coincidence counts are

$$\langle N_0 \rangle_{\text{approx}} = \eta_s \eta_i \mu_s \mu_i h T \quad (14)$$

and

$$\langle N_1 \rangle_{\text{approx}} = r_{c,\text{approx}} T, \quad (15)$$

where  $r_{c,\text{approx}}$  is given by (3).

We employ a threshold decision rule which announces  $H_0$  if the observed coincidence registration count is below the prescribed threshold, and announces of  $H_1$  otherwise. From standard decision theory [38], the optimal threshold for the decision rule, which will minimize the bit error rate (BER), can be computed from the Poissonian distributions of

$N_0$  and  $N_1$ . The optimal threshold rule is computed by selecting the threshold  $\theta$  as the point of intersection of the conditional probability mass functions of  $N_0$  and  $N_1$ , and can be shown to be

$$\theta = \frac{\langle N_1 \rangle - \langle N_0 \rangle}{\log \langle N_1 \rangle - \log \langle N_0 \rangle}.$$

We now compare the performance of the communication system when the exact and approximate

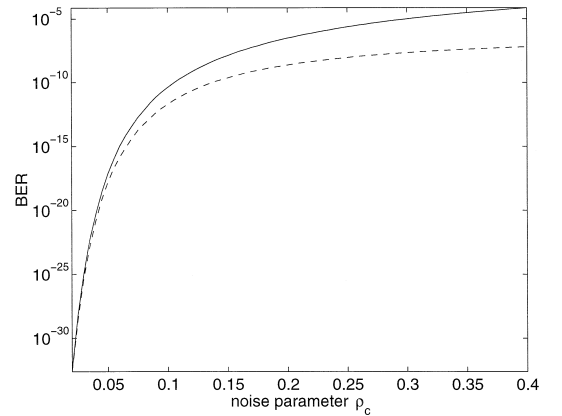


Fig. 8. The bit-error rate (BER) as a function of the coincidence-noise parameter  $\rho_c$ . Solid and dashed lines represent the exact and approximate results, respectively. The following set of parameters are used:  $\eta_s = \eta_i = 0.1$ ,  $\beta = 0.5$ ,  $h = 1$  ns, and  $\lambda = 1.8 \times 10^8$  photons/s. The coincidence parameter is varied by changing the background noise flux  $\mu_i$ .

coincidence statistics are used. Fig. 8 shows the BER as a function of the noise-parameter  $\rho_c$ . The coincidence parameter is varied by changing the background-noise photon flux  $\mu_s$ . First, this curve shows the range of values of the accidental coincidence noise parameter  $\rho_c$  where exact and approximate results are similar. In particular, the exact and approximate BER are almost equal for values of  $\rho_c$  below 0.05 photons. However, the exact BER becomes higher than the approximate BER by a factor of 200 when  $\rho_c$  reaches 0.25. For noise parameters greater than 0.25, the BER obtained using the exact coincidence theory is greater than the ones obtained using the approximate theory by a factor ranging from 200 to  $10^3$ . For example, when  $\rho_c = 0.25$  photons, the approximate BER is  $8.72 \times 10^{-9}$  while the exact BER is  $2.28 \times 10^{-6}$ . The performance is therefore more sensitive to  $\rho_c$  than what had been originally predicted and reported in [10]. The dependence of the BER on the bit duration  $T$  is depicted in Fig. 9. It is seen that the BER computed using the exact theory is significantly greater than the results obtained from the approximate theory.

Another factor that governs the performance of the system is the duration of the coincidence resolving window  $h$ . It is interesting to note that both the exact and approximate BER decrease as  $h$  increases, as seen from Fig. 10. However, the exact BER decreases with  $h$  at slower rate than the approximate

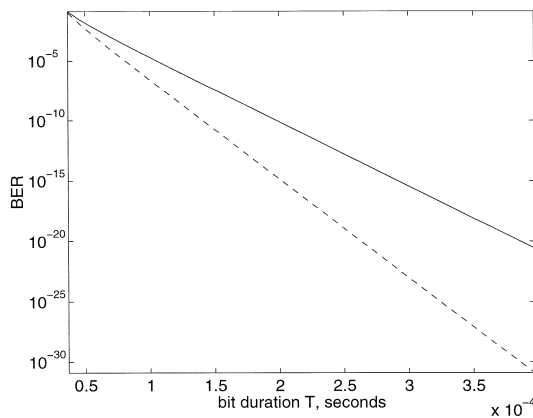


Fig. 9. The bit-error rate (BER) as a function of the bit duration  $T$ . Solid and dashed lines represent exact and approximate results, respectively. The following set of parameters are used:  $\eta_s = \eta_i = 0.1$ ,  $\beta = 0.5$ ,  $\mu_i = 2.0 \times 10^9$  photons/s, and  $\lambda = 1.8 \times 10^8$  photons/s.

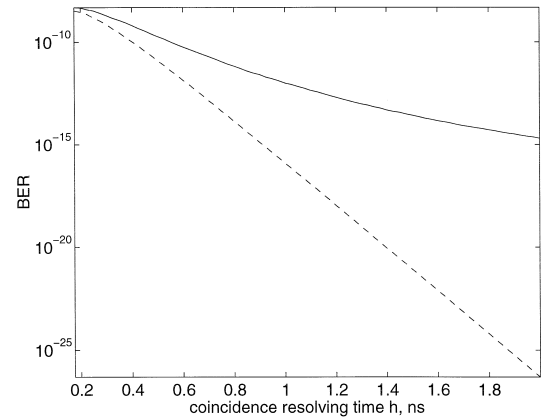


Fig. 10. The bit-error rate (BER) as a function of the coincidence resolving time  $h$ . Solid and dashed lines represent results obtained using the exact and approximate expressions, respectively, for the rate of coincidence. The following set of parameters are used:  $\eta_s = \eta_i = 0.1$ ,  $\beta = 0.5$ ,  $\mu_i = 2.0 \times 10^9$  photons/s, and  $\lambda = 1.8 \times 10^8$  photons/s. The coincidence-noise parameter  $\rho_c$  is varied by changing coincidence resolving time  $h$ .

BER, and more importantly, the exact BER eventually increases, as seen in Fig. 11. For the parameters used in our example, the exact BER decrease for  $h$  in the range 0.4 ns  $\sim$  5 ns, and it increase for values of  $h$  in excess of 8 ns. For  $h$  in the range 6 ns  $\sim$  7 ns, the exact BER is almost constant. This behavior of the BER cannot be predicted at all within the confines of the approximate theory of coincidence

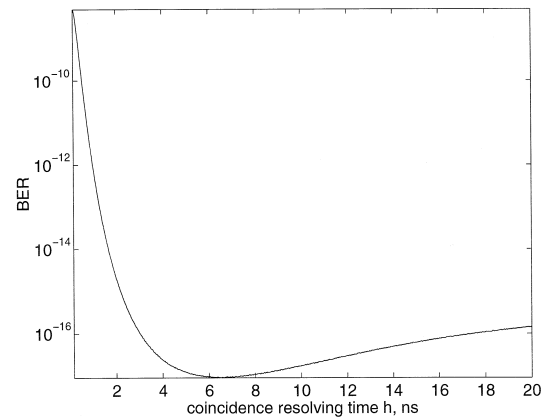


Fig. 11. Exact bit-error rate (BER) as a function of the coincidence resolving time  $h$ . Other parameters are the same as those corresponding to Fig. 10.

which yields an exponential decay of the BER as a function of  $h$ .

## 5. Conclusion

We have developed an exact theory for the statistics of coincidence in photon-correlated beams of coherent light taking into account the finite width of the coincidence resolving time. Our results extend the traditional approximate model for the rate of coincidence to cases when the mean number of detected photons per coincidence resolving time is not negligibly small. For example, it is seen from the cases considered that if the mean number of detected photons per coincidence resolving time is 0.1, the exact coincidence rate is approximately 12% lower than the approximate rate. It is shown that the dependence of the rate of coincidence on the coincidence resolving time is nonlinear, and the rate becomes progressively less than the traditional approximate prediction as the coincidence resolving time increases. Furthermore, an exact evaluation of the probability distribution function of the time between successive coincidence registrations is carried out providing a theoretical justification for the experimentally verified Poissonian statistics of the coincidence registration process. As an application to the theory, we considered the two-channel on-off communication scheme proposed by Mandel [10] and showed that the traditional approximation leads to overemphasizing the advantage of the communication scheme in situations when the background noise level is high. The theory presented can also be of benefit in efforts to reduce the degrading effect of accidental coincidence in positron emission tomography [33]. The derived expression for the exact rate of coincidence is simple and the technique can be modified and extended to the case of partially coherent light where the photon flux is no longer deterministic. This can be done by first conditioning on a specific realization of the random photon flux and applying the current theory to determine the conditional rate of coincidence. We then could average the conditional rate over all possible realizations of the random photon flux to obtain the average rate of coincidence. The renewal-theory technique presented in this paper can also be modified to generalize the

conventional photon correlation theory reported in [23] to cases when photon-correlated beams are used in place of conventional light.

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