Telegrapher's Equations for Transmission Lines

Telegrapher's equations accurately model the propagation of electrical currents and voltages provided:

- There is a well-defined uniform path for flow of both the signal and return current.
- The conductors are closely spaced in comparison to the wavelength of the signals conveyed.

Assumes the signal and return conductors are insulated from each other, have a uniform cross section along their entire length and are long compared to the spacing.

The equations assume the transmission line may be modeled as a **succession** of small, **independent** elements, each with a transverse-electric-and-magnetic (*TEM*) wave configuration

Lines of electric and magnetic flux are confined to a flat, *perpendicular* plane to the signal flow -- no forward or backward component.

Each element represents a very short length of the line.

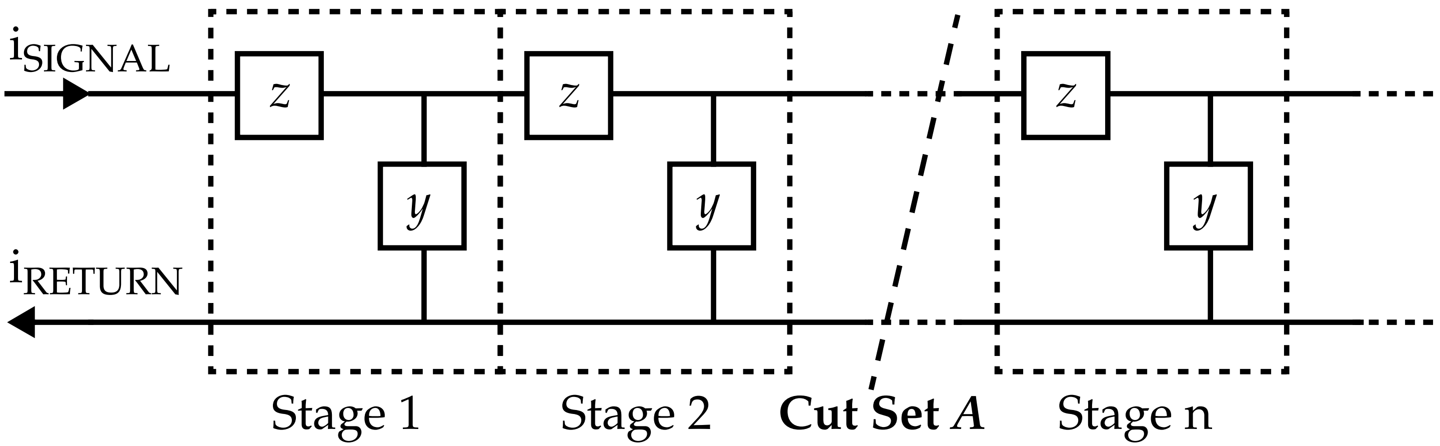
Therefore, it's performance is simple to describe.

Telegrapher's Equations for Transmission Lines

The telegrapher's equations mathematically model the complete line as an infinite cascade of these short elements.

Each element is modeled as:

- An impedance z in series with the signal-and-return current.
- An admittance y shunting the signal conductor to the return conductor.



Impedance z consists of the series resistance of the signal and return conductors and inductance.

Admittance y consists of parasitic capacitance between signal and return conductors, and any DC leakage through the dielectric insulation.

Telegrapher's Equations for Transmission Lines

The series impedance and shunt admittance are defined in units of *ohms-per-unit-length* and *Sieman-per-unit-length*, and vary with frequency.

Kirchoff's current law assumes that for lumped circuit elements, the *sum* of the current *into* and *out* of a device is zero.

No individual device stores current.

This holds for the two conductors crossing the divider labeled *Cut Set A*, i.e., the currents are equal in magnitude and opposite in direction.

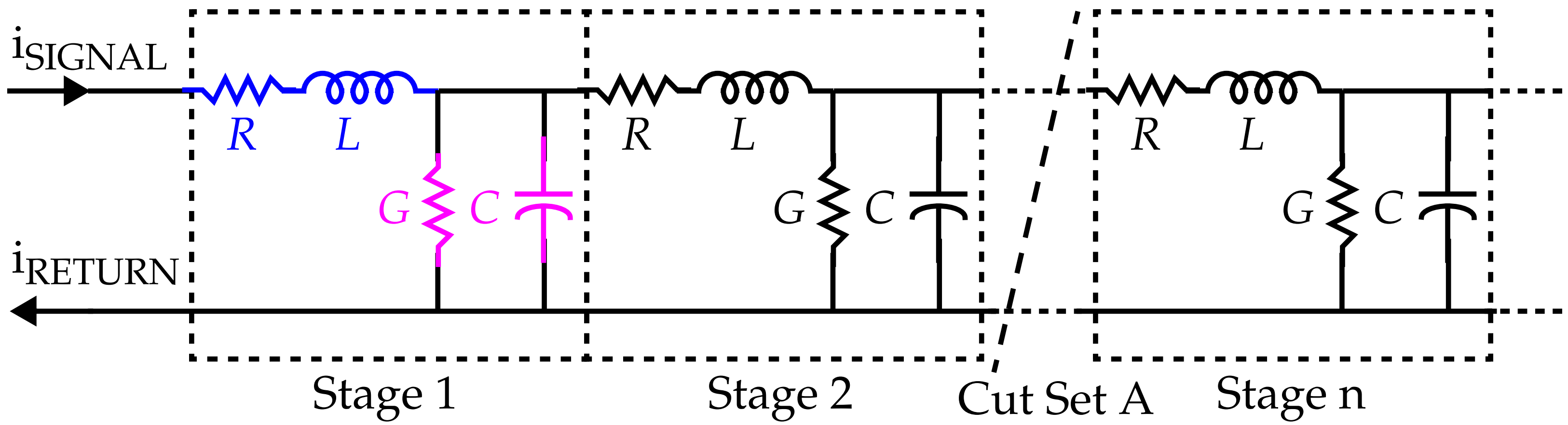
Note that *displacement* current caused by external EM fields was ignored by Kirchoff.

He used the term *lumped-element* to exclude the presence of stray electromagnetic fields -- current flows by direct transport of charged particles.

We can ignore it here too because of the TEM assumption, i.e., TEM propagation precludes direct EM coupling (*leapfrogging*) between stages.

Characteristic Impedance of Transmission Lines

The telegrapher's discrete equivalent circuit model



The series impedance *z* (R and L) and shunt admittance *y* (G and C) are *per unit length* of the transmission line (1 meter in our case).

We will derive the **characteristic impedance**, $Z_C(\omega)$, for the transmission line.

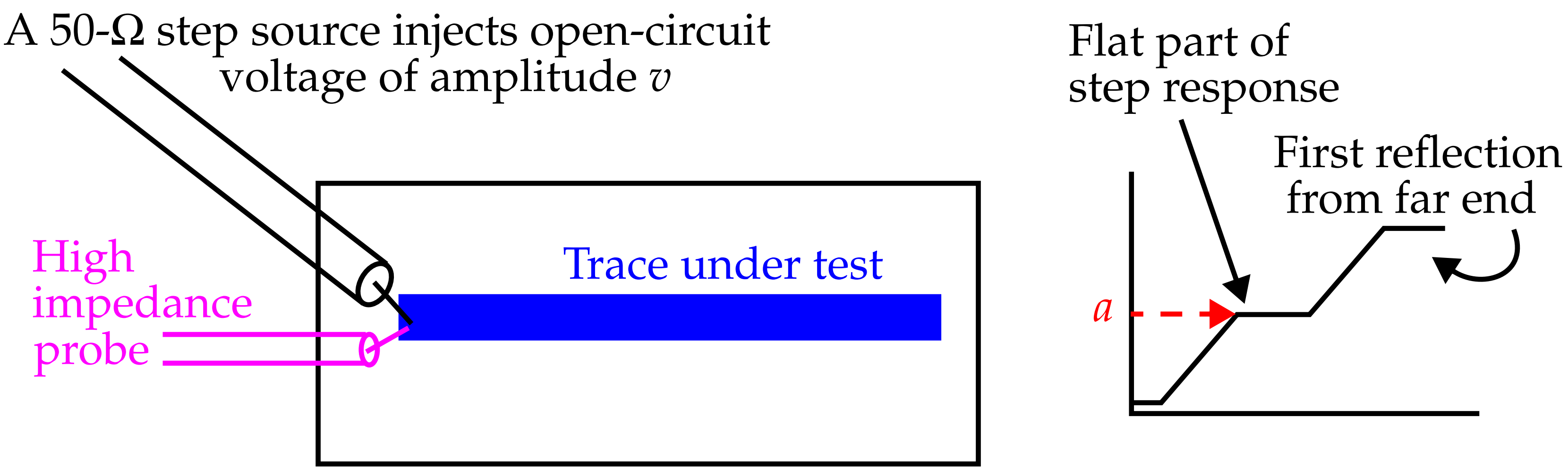
Z_C is the ratio of voltage to current by a signal traveling in *one direction* along the transmission line (it's equal to the input impedance Z_{in}).

Bear in mind that signal reflections cause signals to flow in both directions, in which case Z_{in} will not equal Z_C.

Characteristic Impedance of Transmission Lines

You can infer Z_C from measurements of Z_{in} under certain assumptions.

A *time-domain reflectometer* (TDR) is typically used to measure Z_C.



Here, the rise and fall times of the measurement setup complete well before one round-trip delay of the transmission line.

The observed signal using the probe will reach a steady-state value *a* before the arrival of the reflection.

$$Z_C = Z_S\left(\frac{a/v}{1 - a/v}\right)$$

Characteristic Impedance of Transmission Lines

Here, Z_S is the source impedance, v is the open-circuit amplitude of the source and a is the steady-state value of the measured response.

Note the Z_C may change as a function of frequency.

In this case, the step-response waveform will not be *flat*, and it will be difficult to calculate Z_C

Fortunately, impedance changes relatively slowly for most transmission lines (over the relevant frequency range).

To derive Z_C mathematically, write z and y as

$$z = j\omega L + R$$
$$y = j\omega C + G$$

Consider the input impedance of an infinite chain of cascaded blocks.

Adding one more block to the front of the chain won't change the input impedance, $\overline{Z}_C$ of the whole structure.

Characteristic Impedance of Transmission Lines

Mathematically, the addition involves first *combining* the shunt admittance y in parallel with $\overline{Z}_C$ and then *adding* the series impedance z.

$$\tilde{Z}_C = z + \cfrac{1}{\cfrac{1}{\tilde{Z}_C} + y}$$

Multiplying both sides by $(1+y\overline{Z}_C)$

$$\tilde{Z}_C(1 + y\tilde{Z}_C) = z(1 + y\tilde{Z}_C) + \tilde{Z}_C$$

$$y\tilde{Z}_C^2 = z + zy\tilde{Z}_C$$ Cancel the two $\tilde{Z}_C$ terms.

$$\tilde{Z}_C = \sqrt{\frac{z}{y} + z\tilde{Z}_C}$$ Divide both sides by y and take sqrt

This expresses the input impedance of an infinite chain of **discrete** lumped-element blocks.

This only approximates the behavior of a continuous transmission line.

It works better and better as the block size becomes smaller, in the limit, length becomes zero and models the transmission line perfectly.

Characteristic Impedance of Transmission Lines

Splitting each block into a cascade of n blocks, changes the values of R, L, G and C within each block to new values R/n, L/n, etc.

This modifies z and y as z/n and y/n

$$\tilde{Z}_C = \lim_{n \to \infty} \sqrt{\frac{z/n}{y/n} + \frac{z}{n}\tilde{Z}_C}$$

Right-hand term goes to zero

$$\tilde{Z}_C = \sqrt{\frac{z}{y}}$$

Substituting for z and y

$$Z_C(\omega) = \sqrt{\frac{j\omega L + R}{j\omega C + G}}$$

The value of Z_C varies significantly with frequency.

In particular, G hovers near zero in modern transmission lines and R changes noticeably with frequency.

Characteristic Impedance of Transmission Lines

At high frequencies, the terms *R* and *G* are overwhelmed by jωL and jωC, respectively.

Here, impedance remains constant (reaches a plateau).

This feature is of great value for high-speed digital circuits since it makes it possible to terminate transmission lines **with a single resistor**.

The value of the **characteristic impedance** at the plateau is called Z_0

$$Z_0 \overset{\Delta}{=} \lim_{\omega \to \infty} Z_C(\omega) = \sqrt{\frac{L}{C}}$$

Note that for VERY high frequencies, this does not hold because the circuit becomes overwhelmed with multiple **non-TEM** modes of propagation.

Also known as **waveguide modes**.

Therefore, this expression is bound by frequency on both ends.

More on this later...

Propagation Coefficient of Transmission Lines

Signals propagating along a transmission line are *attenuated* by a certain factor H as they pass through each unit length of the line.

The signal amplitude *decays* **exponentially with distance**.

The per-unit-length attenuation factor H is called the **propagation function** of the transmission line, and it varies with frequency, e.g., $H(\omega)$.

Let $H(\omega)$ represent the curve of attenuation vs. frequency ω in a unit-length segment, and $H(\omega, l)$ the curve for a line of length l.

An exponential relationship describes the relationship

$$H(\omega, l) = [H(\omega)]^l$$

The complex logarithm of $H(\omega)$ is appropriate for exponentials because the response scales linearly with l.

$$\gamma(\omega) \overset{\Delta}{=} -\ln H(\omega)$$

Negative indicates *attenuating*

$$H(\omega) = e^{-\gamma(\omega)} \longrightarrow H(\omega, l) = e^{-l\gamma(\omega)}$$

Propagation Coefficient of Transmission Lines

The *negative natural logarithm* of the per-unit-length propagation function H is called the **propagation coefficient**.

Units are in *complex nepers per meter*.

We use α and β to represent the real and imaginary parts as

$$\alpha(\omega) \overset{\Delta}{=} Re[\gamma(\omega)] = Re[-\ln(H(\omega))]$$ Attenuation induced by H

$$\beta(\omega) \overset{\Delta}{=} Im[\gamma(\omega)] = Im[-\ln(H(\omega))]$$ Phase delay

α is expressed in units of *nepers per unit length*.

An attenuation of 1 neper per unit length (α = 1) equals -8.6858896 dB of gain per unit length

$$20\log\left(\frac{1}{e}\right) = -8.6858896 \text{ dB}$$

So a value of α = 1 scales a signal by 1/e = 0.367879 as it passes through each unit length of the transmission line.

Propagation Coefficient of Transmission Lines

β is expressed in units of *radians per unit length*.

A phase delay of one radian per unit length ($\beta = 1$), equals **-57.295779** degrees of phase shift per unit length.

Remember, α, β and γ all vary with frequency, even if not shown with their frequency arguments.

Alternative representation of α and β

$$|H(\omega)| = e^{-\alpha}$$

$$\angle H(\omega) = -\beta$$

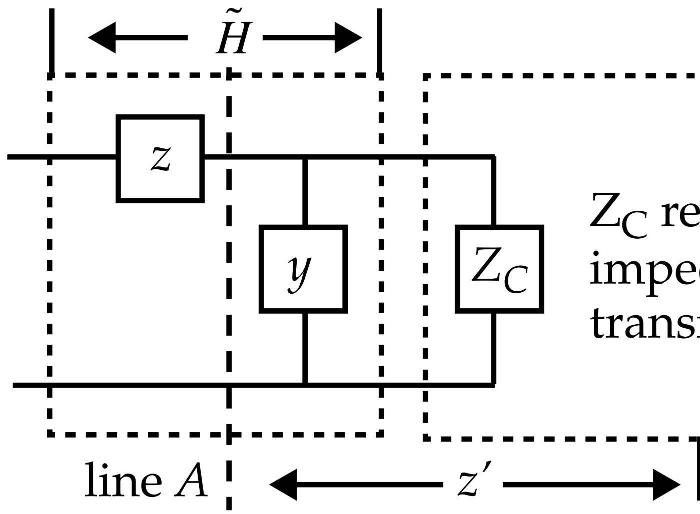
Bringing l back, yields

$$H(\omega, l) = e^{-l\gamma(\omega)} = e^{-l(\alpha + j\beta)}$$

Important point to remember is that signals propagating on a transmission line **decay exponentially with distance**.

Relating Propagation Coefficient with Transmission Line Parameters

Adding one unit-size discrete transmission block to the head of a transmission line with input impedance Z_C.



Z_C represents the input impedance of a continuous transmission line.

Define z′ as the impedance looking to the right of line A.

The transmission coefficient is defined by the resistor-divider theorem.

$$\tilde{H} = \frac{z'}{z + z'} \qquad \text{with} \quad Z_C = \sqrt{\frac{z}{y}}$$

Where z′ is the parallel combination of admittance y and impedance Z_C.

Relating Propagation Coefficient with Transmission Line Parameters

Substituting

$$\tilde{H} = \frac{\frac{1}{y + \sqrt{y/z}}}{z + \frac{1}{y + \sqrt{y/z}}} = \frac{1}{zy + \sqrt{zy} + 1}$$

Multiply left by $(y + \sqrt{y/z})$

This expresses the transfer function of one discrete block of unit size.

Splitting the unit sized block into a succession of n blocks, each of length $1/n$ and taking the limit

$$\tilde{H} = \lim_{n \to \infty} \left[\frac{1}{(z/n)(y/n) + \sqrt{(z/n)(y/n)} + 1} \right]^n$$

The combined response of cascade of n blocks equals the response of an individual block of size $1/n$ **raised** to the *nth power.*

$$\tilde{H} = \lim_{n \to \infty} \left[\frac{zy/n + \sqrt{zy}}{n} + 1 \right]^{-n}$$

Relating Propagation Coefficient with Transmission Line Parameters

Using the fact that

$$\lim_{n \to \infty} [(a/n) + 1]^{-n} = e^{-a} \qquad \text{where} \qquad a = (zy/n) + \sqrt{zy}$$

But zy/n goes to zero, so only the right term of a survives.

$$H = e^{-\sqrt{zy}} \qquad \text{and therefore,} \qquad \gamma = \sqrt{zy}$$

Finally, substituting for y and z

$$\gamma = \sqrt{(j\omega L + R)(j\omega C + G)}$$ **Propagation coefficient**

The telegrapher's equation predicts the amplitude and phase response for a single mode of propagation on a transmission line given R, L, G and C

$$H(\omega, l) = e^{-l\sqrt{(j\omega L + R)(j\omega C + G)}}$$

Lossless Transmission Line

We indicated earlier (ideal) lossless transmission lines propagate signals with no distortion or attenuation.

This requires R = G = 0, yielding

$$Z_C = \sqrt{\frac{j\omega C}{j\omega L}} = \sqrt{\frac{L}{C}} \qquad \text{(derived earlier)}$$

$$\gamma(\omega) = \sqrt{(j\omega L)(j\omega C)} = j\omega\sqrt{LC} \qquad \text{(assumed earlier)}$$

The real and imaginary parts of γ give the attenuation in units of nepers/m and phase delay in units of rad/m, respectively.

For a unit length of an ideal transmission line, the transfer function is a **simple linear-phase delay**

$$H(j\omega) = e^{-j\omega\sqrt{LC}}$$

Note the real part of propagation coefficient is zero, i.e., no loss, while the imaginary part is a constant times ω.

Lossless Transmission Line

The delay per unit length (and velocity) equals

$$\text{delay} = \sqrt{LC}$$

$$\text{velocity} = \frac{1}{\sqrt{LC}}$$

(This is the assumption we started with last slide set - EM theory...)

Note if L and C are given in H/in. and L/in., then delay is in s/in. Scale appropriately, e.g., by 10^{-12} to get in ps/in.

Also, we talked about propagation velocity in the introduction slide set, where we indicated that it depended on the dielectric.

The general form of the relationship is given by

$$v = \frac{c}{\sqrt{\varepsilon_r \mu_r}}$$

c is the velocity of light in a vacuum
ε_r equal to the permittivity (dielectric constant)
μ_r equal to the permeability

Since most dielectric insulating materials are *non-magnetic*, $\mu_r = 1$.

Lossless Transmission Line

Given our new expression, then the following must hold

$$\frac{c}{\sqrt{\varepsilon_r \mu_r}} = \frac{1}{\sqrt{LC}}$$

This indicates that for a given transmission line configuration (with a constant ε), changing L or C necessarily changes the other variable.

For example, if a stripline trace is widened, C increases and the L decreases, i.e., the product remains constant.

Current flow/behavior of a pulse traveling down a transmission line