**RC Region (dispersive transmission line)**

RC mode includes all combinations of $\omega$ and $l$ for which the line behaves in a **distributed manner**.

Also, the frequency remains *well below* the point at which the magnitude of $\omega L$ approaches the DC resistance of the line, $R_{DC}$.

The RC region extends from DC up to frequency $\omega_{LC}$ (the LC mode cutoff).

At this point, the reactive component of the propagation coefficient, $\omega L$, becomes equal to the magnitude of the resistive component, $R_{DC}$.

$$\omega_{LC} = \frac{R_{DC}}{L}$$

The length, $l$, of the transmission line where you need to start worrying about RC mode (vs. lumped-element mode) is obtained from

$$l_{LE} \approx \frac{\Delta}{\sqrt{\omega R_{DC}}} \quad \text{for} \ (\omega < R_{DC}/L)$$

Boundary between lumped-element (LE) mode and RC mode

$\Delta = 0.25$
**RC Region**

Substituting $\omega_{LC}$ into this equation yields

$$l_{RC} = \frac{\Delta}{\sqrt{\omega R_{DC} C}} = \frac{\Delta}{\sqrt{\frac{R_{DC}}{L} R_{DC} C}} = \frac{\Delta}{R_{DC}} \sqrt{\frac{L}{C}}$$

$\Delta = 0.25$

L in H/m
C in F/m

Below this length, distributed RC behavior does NOT occur.
**RC Region**

Therefore, we are in RC mode when the total DC resistance of the line, \( l^*R_{DC} \), grows to a value comparable to the high frequency impedance \( \sqrt{L/C} \).

\[
l_{RC}R_{RC} = \Delta \sqrt{\frac{L}{C}}
\]

Note that from the figure in slide 2, we can go directly from lumped-element mode to LC mode, e.g., at 1 meter.

For the PCB trace, its resistance at one meter is only 6.3 \( \Omega \), which is much smaller than the line impedance of 50 \( \Omega \).

For this reason, PCB designers never need to worry about RC mode at the board level.

Telephone lines (24-gauge) will begin exhibiting RC mode around 100 m.

Interesting RC mode **does** occur on-chip, over much smaller wires.

This is due to the larger resistance of the wires, e.g., polysilicon.
RC Region: Input Impedance

The **input impedance** varies strongly with the length of the line and the type of load connected.

\[
Z_{\text{in, loaded}} = Z_C \left( \frac{H^{-1} + H}{2} + \frac{Z_C (H^{-1} - H)}{Z_L \left( \frac{H^{-1} - H}{2} + \frac{Z_C (H^{-1} + H)}{2} \right)} \right)
\]

Recall that line length is incorporated in \( H \)

\[
H(\omega, l) = e^{-l\gamma(\omega)}
\]

This complicates the design of reactive source and load networks needed to establish some target equalization goal in the propagation function.

The problem can be solved by providing end-termination such that \( Z_L = Z_C \).

This *eliminates* reflections and makes the input impedance equal to \( Z_C \), independent of line length.
**RC Region: Input Impedance**

A second solution is for the transmission line to be very long such that $H$ takes on a value significantly less than 1.

Here, the inverse-gain, $H^{-1}$, vastly exceeds $H$, and allows $H$ to be ignored in the input impedance expression, making $Z_{in, \text{loaded}} = Z_C$.

We gave the **characteristic impedance** earlier (ignoring $G$) as

$$Z_C = \sqrt{-\frac{j\omega L + R}{j\omega C}}$$

But in RC mode, we assumed $\omega L$ to be small compared to $R$

$$Z_C = \sqrt{\frac{R}{j\omega C}} = \left(\frac{1-j}{\sqrt{2}}\right)\sqrt{\frac{R}{\omega C}} \quad \text{since} \quad \sqrt{\frac{1}{j}} = \sqrt{-j} = \frac{1-j}{\sqrt{2}}$$

This expression is a complex function of frequency with a phase angle of -45 degrees and a magnitude slope of -10 dB/decade

$$|Z_C| = \sqrt{\frac{R}{\omega C}} \quad \Rightarrow \quad 20\log\left(\sqrt{\frac{1}{10}}\right) \text{dB} = -10 \text{dB}$$

$$\angle Z_C = \tan^{-1}\left(\frac{(-1/\sqrt{2})\sqrt{R/(\omega C)}}{\left(1/\sqrt{2}\right)\sqrt{R/(\omega C)}}\right) = -45^\circ$$
**RC Region: Input Impedance**

As an example, the two wires (AWG 24) running from the central telephone switching office to your phone represent an RC transmission line.

The wires are twisted, yielding the following L, C and R values:

- \( R = 0.165 \, \Omega/m \)
- \( L = 400 \, nH/m \)
- \( C = 40 \, pF/m \)
- \( \omega = 1600 \, Hz \times 2\pi = 10053 \, rad/s \)

\[
|Z_C| = \sqrt{\frac{0.0272 + 1.6 \times 10^{-5}}{4.02 \times 10^{-7}}} = 640.6 \, \Omega
\]

\[
\angle Z_C = \sqrt{\tan^{-1}\left(\frac{0.00402}{0.165}\right) \div 90^\circ} = -44.3^\circ
\]

\[
|Z_C| = \sqrt{\frac{0.165}{j4.02 \times 10^{-7}}} = 640.7 \, \Omega
\]

\[
\angle Z_C = \sqrt{\frac{0^\circ}{90^\circ}} = -45^\circ
\]

Telephone lines have a characteristic impedance of 600 \( \Omega \) in the voice band, but at high frequencies, it reduces to 100 \( \Omega \).

Note that characteristic impedance varies markedly with frequency.
RC Region Propagation Function

For the best results, the termination must match \( Z_C \) over the entire frequency range spanned from \( \omega_{LE} \) and \( \omega_{LC} \).

Consider the propagation function of a unit-sized RC transmission line

The response is shown for several cases: open-circuit and three loading conditions.
**RC Region Propagation Function**

Open-circuit response shows the *least overall loss* at high frequencies, and is the most common configuration.

The *matched-end terminator* curve is the response when the transmission line is configured with a matched end-termination impedance $Z_C(\omega)$.

The matched-end configuration degrades the line’s response in two ways.

- It reduces the available signal at the end of the line.
- It introduces a *tilt* to the propagation function.

  The *tilt* introduces significant amounts of intersymbol interference, which can cause **bit errors**.

Binary signals tolerate *tilt* of no more than approximately 3 dB (at most 6 dB).

Therefore, although match-end termination makes the input impedance independent of line length, it causes **severe** degradation in the transfer response.
**RC Region Propagation Function**

The remaining curves are from *resistive load* configurations, equal to 1/2 and 1/10, respectively of the aggregate series resistance of the transmission line.

Although the signal attenuation is higher than the open-circuit configuration, the overall attenuation curve is **flattened**.

The *flattening* occurs up through higher frequencies than either of the previous cases, making it possible to send binary data at higher bandwidths.

These resistive termination schemes show a classic **gain-bandwidth** tradeoff. You can *improve* the *bandwidth* at the expense of *reducing signal amplitude*.

The upper limit of the achievable bandwidth is defined by the onset of the LC mode of operation, i.e., when $j\omega L$ exceeds $R$.

In LC mode, a resistance of $Z_0$ is best for termination (to be discussed).

Using $Z_0$ eliminates reflections in LC mode while simultaneously providing a relatively flat propagation function in RC mode.
**RC Region Propagation Coefficient**

You can derive the propagation coefficient starting with

\[
\gamma = \sqrt{(j\omega L + R)(j\omega C + G)}
\]

In this region, ignore \(j\omega L\) and high freq. dependencies of \(R\) and \(C\)

\[
H(\omega, l) = e^{-l\gamma(\omega)}
\]

And simplifying for RC mode.

\[
\gamma = \sqrt{R(j\omega C)}
\]

\[
H(\omega, l) = e^{-l\sqrt{R(j\omega C)}}
\]

Now substitute for \(H\) in

\[
G = \frac{v_3}{v_1} = \frac{1}{\left[\left(\frac{H^{-1} + H}{2}\right)\left(1 + \frac{Z_S}{Z_L}\right) + \left(\frac{H^{-1} - H}{2}\right)\left(\frac{Z_S}{Z_C} + \frac{Z_C}{Z_L}\right)\right]}
\]

And analyze under various assignments/assumptions for \(Z_S, Z_C\) and \(Z_L\), e.g., \(Z_S = 0\) and \(Z_L = \infty\) yields

\[
G(\omega) = \frac{2}{H^{-1} + H}
\]
The normalized step response of a unit-sized RC transmission line

The degraded risetime is obvious for the case of the matching end-termination impedance equal to $Z_C(\omega)$.

The resistive termination shows superior risetime, at the cost of reduced signal amplitude --> in this case the DC gain is $1/3$ given $Z_L = (l*R)/2$. 

R = 1 $\Omega$/m
$C = 1F/m$
$l = 1m$
**RC Region Step Response**

In the RC region, risetime scales with the square of length. Therefore, doubling length quadruples risetime.

Also, the speed of operation achievable scales inversely with the square of transmission line length.

In this case, you must wait longer (slower operational speed) for the signal to reach the same level of magnitude for longer length lines.

W.C. Elmore described a way to estimate the delay of RC circuit that is used (in variations) to validate on-chip timing.

His technique works only with well-damped circuits composed of any number of series resistance and shunt capacitances.

It does not work with circuits involving inductance, resonance, overshoot or any form of poorly damped or non-monotonic behavior.

It can be used to quickly compute a reasonable upper bound on the delay of complicated tree and bus structures used on-chip.
LC Region

The LC region is characterized by the growth of inductive reactance to the point where it exceeds the magnitude of the DC resistance.

At $\omega_{LC}$ point, $\omega L$ equals $R$

$$\omega_{LC} = \frac{R_{DC}}{L}$$
LC Region

Recalling our earlier analysis, you are in the lumped-element region if

\[ l_{LE} \approx \frac{\Delta}{\omega \sqrt{LC}} \quad \text{for} \ (\omega > (R_{DC}/L)) \]

An interesting feature of LC mode is that the attenuation does vary much with frequency.

In most digital applications, the LC region is fairly narrow (and can be non-existent).

Since dB is already a logarithmic unit, this is a double log

![Graph showing attenuation vs frequency in different regions](image-url)
**LC Region Characteristic Impedance**

We have already derived characteristic impedance in the LC region from

\[
Z_C(\omega) = \sqrt{\frac{j\omega L + R}{j\omega C}} \quad \Rightarrow \quad Z_0 = \sqrt{\frac{L}{C}}
\]

The difference in \(Z_C\) and \(Z_0\) at 3 times \(\omega_{LC}\) is on order of 5%, at 10x its 0.5%.

However, near \(\omega_{LC}\), \(Z_C\) is significantly different.

\[
Z_C(\omega) = \sqrt{\frac{R}{j\omega C}} \quad \Rightarrow \quad Z_0 = \sqrt{\frac{L}{C}}
\]

The impedance at \(\omega = 0\) is **infinity** and decreases with higher frequencies.
**LC Region Propagation Coefficient**

At frequencies below $\omega_{LC}$ (approximately 3 MHz), both the real and imaginary components decrease at a rate proportional to the inverse square root of frequency.

Above $\omega_{LC}$, the imaginary part goes to zero and the overall impedance flattens out to 50 $\Omega$.

The Propagation Coefficient for this region graphically:

\[
\gamma = \sqrt{R(j\omega C)}
\]

\[
\gamma = \frac{(1 + j)}{2} \sqrt{R(\omega C)}
\]

- Imaginary part (linear phase): $\angle\gamma(\omega) \rightarrow j\omega \sqrt{LC} \rightarrow 90^\circ$
- Real part (constant loss): $|\gamma(\omega)| \rightarrow \frac{R_{DC}}{2\sqrt{L/C}}$ (derived next slide)

PCB trace: fixed DC resistance and no skin effect
**LC Region Propagation Coefficient**

Below $\omega_{LC}$ in the RC region, both the real part of the propagation coefficient (log of attenuation) and the imaginary part (phase in radians) rise together in proportion to the *square root of frequency*.

Above $\omega_{LC}$ in the LC region, attenuation and phase become de-coupled.

Here, the imaginary part grows linearly while the real part stays fixed.

Starting with the propagation coefficient, factor out a $j\omega$ term.

$$\gamma(\omega) = \sqrt{(j\omega L + R)(j\omega C)}$$

$$\gamma(\omega) = j\omega \sqrt{LC} \sqrt{1 + \frac{R_{DC}}{j\omega L}}$$

The square root on the left can be approximated (valid for $\omega >> \omega_{LC}$)

$$\gamma(\omega) = j\omega \sqrt{LC} \left(1 + \frac{1}{2} \frac{R_{DC}}{j\omega L}\right) = j\omega \sqrt{LC} + \frac{R_{DC}}{2 \sqrt{L/C}}$$

Then substitute $Z_0$ for $\sqrt{L/C}$. 
**LC Region Propagation Coefficient**

This expression shows a linear-phase ramp

\[ \text{Im}(\gamma) \rightarrow \omega \sqrt{LC} \]

and a steady-state value

\[ \text{Re}(\gamma) \rightarrow \frac{R_{DC}}{2Z_0} \]

The linear phase indicates that the one-way propagation function \( H \) of an LC-mode transmission line acts like a large **time-delay element**.

\[ t_p \triangleq \frac{1}{v_0} = \sqrt{LC} = \frac{\sqrt{\varepsilon\rho}}{c} \text{ s/m} \]

The **delay** varies in proportion to the length of the transmission line, where doubling the length doubles the delay.

The transfer loss in nepers per meter or **resistive loss coefficient** (does NOT account for skin-effect)

\[ \alpha_{r, DC} \triangleq \text{Re}[\gamma(\omega)] = \frac{1}{2} \frac{R_{DC}}{Z_0} \text{ neper/m} \]
**LC Region Propagation Coefficient**

In dB

\[ \alpha_r,_{DC} \Delta = \text{Re}[\gamma(\omega)] = 4.34 \frac{R_{DC}}{Z_0} \text{ dB/m} \]

The magnitude of \( H \) is given by the real part of the propagation function

\[ |H(\omega, l)| = e^{-l\frac{1}{2}\frac{R_{DC}}{Z_0}} \]

Doubling the length, doubles the loss.

The property that signals in the LC region have substantial phase delay and low attenuation indicates they may act as **high-Q** resonant circuits.

- PCB trace (no skin effect, etc)
- Open circuited at far end
LC Region Terminations
All LC transmission lines exhibit similar resonant peaks when driven by a source impedance less than $Z_0$.

Controlling $Z_L$ and $Z_S$ can resolve this problem.

There are three classical ways of stabilizing an LC transmission line, i.e., eliminating the resonance.

Each of these uses a resistive termination to provide a circuit gain that is proportional (and desirable) to the propagation function $H(\omega)$.

This strategy works well for PCB traces, which are relatively short in length, producing a propagation function $H$ that is nearly flat with linear phase.

For PCB traces, the line acts like nothing more than a time-delay element with a small amount of attenuation.
LC Region Terminations

End Termination

Low-impedance driver:

\[ Z_S << Z_C \]

High-impedance load:

\[ Z_L >> Z_C \]

Source Termination

\[ Z_S = Z_C \]

Both-ends Termination

\[ Z_S = Z_C \]


**LC Region Terminations**

For **end termination**, assuming that $Z_L$ is close to $Z_C$ and $Z_S$ is much less than $Z_C$.

$$G = \frac{1}{\left(\frac{H^{-1} + H}{2}\right)\left(1 + \frac{Z_S}{Z_L}\right) + \left(\frac{H^{-1} - H}{2}\right)\left(\frac{Z_S}{Z_C} + \frac{Z_C}{Z_L}\right)}$$

Substituting 1 for the $Z_C/Z_L$ terms and 0 for $Z_S/Z_C$ yields

$$G \approx \frac{1}{\left(\frac{H^{-1} + H}{2}\right)(1 + 0) + \left(\frac{H^{-1} - H}{2}\right)(0 + 1)} = H$$

For **source termination**, assuming $Z_S$ is close to $Z_C$ and $Z_L$ is much larger than $Z_C$ yields

$$G \approx \frac{1}{\left(\frac{H^{-1} + H}{2}\right)(1 + \frac{1}{\infty}) + \left(\frac{H^{-1} - H}{2}\right)(1 + \frac{1}{\infty})} = H$$
LC Region Terminations

For **both-ends termination**, assuming $Z_S = Z_C = Z_L$ yields

$$G = \frac{1}{\left[ \left( \frac{H^{-1} + H}{2} \right)(1 + 1) + \left( \frac{H^{-1} - H}{2} \right)(1 + 1) \right]} = \frac{H}{2}$$

Note that unlike RC mode, attenuation in LC mode does **not** vary with frequency, therefore the speed of operation is not directly **limited** by length.

Since LC, skin-effect and dielectric-loss-limited regions all share the same asymptotic high-frequency value of $Z_0$, the same termination schemes work.

For the LC region, the propagation function $H$ is flat while it is **not flat** in the skin-effect and dielectric-loss-limited regions.

For these regions, $H$ acts as a low-pass filter, attenuating and dispersing the edges of signals.
**Mixed-Mode Operation**

System **A** is typical of PCB traces and consists of a transmission line with length less than $l_{RC}$ and operates at frequencies over 0 - 20 MHz.

It spans only the LC and lumped-element regions so terminating at $Z_0$ should work well, assuming you don’t have a reactive load.

System **B** operates in three modes, where the response in RC mode is a strong function of frequency, requiring a *frequency-varying* termination network.