Circuit and Electromagnetic System Design Notes

Note 38

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Slow Wave Transmission Line Transformers

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Abstract

Development of a periodic structure to couple slow and fast wave transmission lines is described. The slow wave line is a coaxial line with rectangular corrugations on the outer conductor and with smooth metallic walls for the inner conductor. The fast wave structure is a coaxial line with smooth inner and outer metallic conductors. or with outer conductors where the corrugation depth approaches zero. The unit cells, or quarter wave sections, are developed for both binomial and Chebyshev coefficients. The Chebyshev transformer is designed for arbitrary bandwidths from 1.0 to larger numbers, and when the bandwidth equals 1.0, the binomial and Chebyshev transformers are the same.

Chebyshev coefficients are found with unpublished methods. They are then applied in calculating characteristic impedances for prescribed input and output line impedances. The modification here is inclusion of slow/fast wave velocities to the characteristic impedances.

I. Introduction

This report describes a method of design for transformer structures capable of matching slow wave transmission lines with different characteristic impedances for given wavelengths. Design techniques were formerly developed by Seymour Cohn [1] for fast wave lines. This study modifies Cohn's techniques for slow wave lines. The slow wave structure consists of a succession of different characteristic impedances spaced by equal electrical lengths of slow wave lines. The slow wave line may be coaxial, waveguide, or parallel planes. With a given number of steps, the modified method yields the maximum bandwidth for a given VSWR, or a minimum VSWR for a given bandwidth. It is called the Chebyshev transformer because the Chebyshev polynomials are used in its formulation.

II. Quarter Wave Transformers and Small Reflections

Quarter wave transformers are found as intermediate matching sections connecting two transmission lines with different characteristic impedances. When matching two lines over a narrow band of frequencies, one transformer section is adequate. For a wide bandwidth, several intermediate transformer sections are required to achieve the same overall standing wave ratio (VSWR).

The design of a quarter wave transformer section is described by matching a transmission line of real characteristic impedance Z(1) to a resistive load impedance Z(3). The transformer section with a characteristic impedance Z(2) and a quarter wavelength long connects the transmission line and the load Z(3), seen in Fig. 2.1.

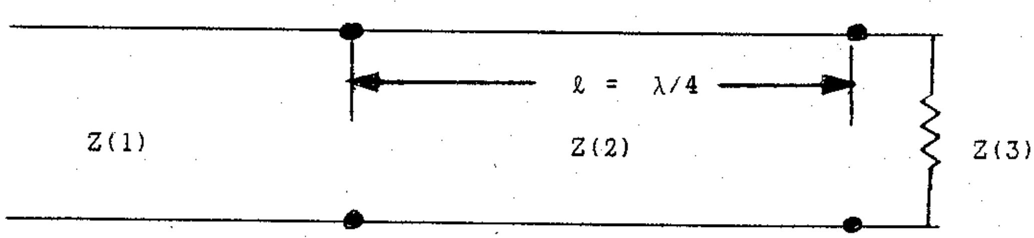

Fig. 2.1 A quarter wave transformer.

The effective load impedance Z seen be the main line when $\ell = \lambda/4$ is

$$Z = Z(2)\,\frac{Z(3)\cos(\beta\ell) + j\,Z(2)\sin(\beta\ell)}{Z(2)\cos(\beta\ell) + j\,Z(3)\sin(\beta\ell)}$$

$$= Z(2)\,\frac{Z(2)}{Z(3)} = \frac{Z(2)^2}{Z(3)} \;, \tag{2.1}$$

where β is the phase constant. If Z(2) is selected to be equal to $\sqrt{Z(1)Z(3)}$, then Z equals Z(1) and the load impedance is matched to the characteristic impedance of the main transmission line. For frequencies or transformer lengths where $\beta\ell$ is not a quarter wavelength, or lengths not equal to $(N\,\lambda/2 + \lambda/4)$, the reflection coefficient is

$$= \frac{Z - Z(1)}{Z + Z(1)} = \frac{Z(3) - Z(1)}{Z(1) + Z(3) + j\,2\sqrt{Z(1)Z(3)}\,\tan(\beta\ell)} \;. \tag{2.2}$$

where Eq. (2.1) and $Z(2)^2$ equals Z(1)Z(3) were used.

When applied to many intermediate transformer sections, the reflection coefficients for several reflecting impedances are developed because they provide simpler approximations for finding the overall reflection coefficient. The approximations are made by considering only first order reflections and dropping terms with multiply reflected waves.

In Fig. 2.2, the reflection coefficients $\Gamma(1)$, $\Gamma(2)$, $\Gamma(3)$, and transmission coefficients T(12), T(21) are

$$\Gamma(1) = \frac{Z(1) - Z(1)}{Z(1) + Z(1)} \;, \qquad \Gamma(2) = -\Gamma(1) \;, \tag{2.3}$$

$$T(12) = 1 + \Gamma(1) = \frac{2Z(2)}{Z(1) + Z(2)} \;, \quad T(12) = 1 + \Gamma(2) = \frac{2Z(1)}{Z(1) + Z(2)} \;, \tag{2.4}$$

$$\Gamma(3) = \frac{Z(3) - Z(2)}{Z(3) + Z(2)} \;. \tag{2.5}$$

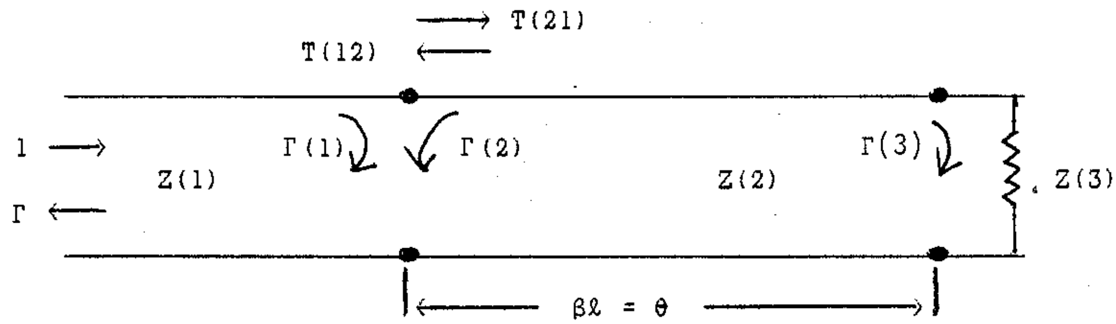

Fig. 2.2. Transformer with two reflecting junctions.

The incident electric field has unit amplitude and the reflected field has a complex amplitude Γ. When the incident field reaches the first junction, the first partial reflected and transmitted waves are Γ(1) and T(21), respectively. When this partial transmitted wave reaches the second junction or load, part of the wave is reflected with a reflection coefficient Γ(3) to create a wave,

$$\Gamma(3)T(21)\,e^{-j\,2\,\theta}\,, \tag{2.6a}$$

incident upon the first junction from the load. The exponent indicates an electrical distance $\beta\ell = \theta$ from first to second junction, and the same distance $\beta\ell = \theta$ from first to second junction, for a total distance $2\beta\ell = 2\,\theta$. Part of the wave is transmitted and becomes part of the overall reflection coefficient,

$$T(12)T(21)\Gamma(3)\,e^{-j\,2\,\theta}\,, \tag{2.6b}$$

and part is reflected back to the load.

$$T(21)\Gamma(2)\Gamma(3)\,e^{-j\,2\,\theta}\,. \tag{2.6c}$$

This reflected wave travels to the load and part of the wave is reflected back to the first junction with amplitude,

$$T(21)\Gamma(2)\Gamma(3)^{2}\,e^{-j\,4\,\theta}\,, \tag{2.6d}$$

and at this junction, the transmitted wave is

$$T(12)T(21)\Gamma(2)\Gamma(3)^{2}\,e^{-j\,4\,\theta}\,, \tag{2.6e}$$

with a wave reflected toward the load,

$$T(21)\Gamma(2)^{2}\Gamma(3)^{2}e^{-j4\theta}, \tag{2.6f}$$

after reflection at the load is incident upon the first junction with amplitude,

$$T(21)\Gamma(2)^{2}\Gamma(3)^{3}e^{-j6\theta}, \tag{2.6g}$$

creating another part of the overall reflection coefficient,

$$T(12)T(21)\Gamma(2)^{2}\Gamma(3)^{3}e^{-j6\theta}. \tag{2.6h}$$

This reflection and transmission continues forever. The infinite sum of reflected waves becomes the overall reflection coefficient,

$$\begin{aligned}\Gamma &= \Gamma(1) + T(12)T(21)\Gamma(3)e^{-j2\theta} + T(12)T(21)\Gamma(3)^{2}\Gamma(2)e^{-j4\theta} \\ &\quad + T(12)T(21)\Gamma(3)^{3}\Gamma(2)^{2}e^{-j6\theta} + T(12)T(21)\Gamma(3)^{4}\Gamma(2)^{3}e^{-j8\theta} \\ &\quad + T(12)T(21)\Gamma(3)\Gamma^{5}(2)^{4}e^{-j10\theta} + \ldots + \\ &= \Gamma(1) + T(12)T(21)\Gamma(3)e^{-j2\theta}\sum_{N=0}^{N=\infty}\Gamma(2)^{N}\Gamma(3)^{N}e^{-j2N\theta},\end{aligned} \tag{2.7}$$

and with the series,

$$\sum_{0}^{N=\infty}X^{N} = (1 - X)^{-1}, \; X < 1, \tag{2.8}$$

the reflection coefficient is

$$\Gamma = \Gamma(1) + \frac{T(12)T(21)\Gamma(3)e^{-j2\theta}}{1 - \Gamma(2)\Gamma(3)e^{-j2\theta}} \tag{2.9}$$

and replacing T(12) by 1 - Γ(1) and T(21) by 1 + Γ(1),

$$\Gamma = \frac{\Gamma(1) + \Gamma(3)e^{-j2\theta}}{1 + \Gamma(1)\Gamma(3)e^{-j2\theta}}. \tag{2.10}$$

If the product of the magnitudes Γ(1) and Γ(3) is much less than 1.0, Γ is approximated by

$$\Gamma = \Gamma(1) + \Gamma(3)\, e^{-j\,2\,\theta} \tag{2.11}$$

so that when reflection magnitudes are less than 0.3 for example, the error in Γ is less than 8 percent.

III. Multisection Quarter Wave Transformers

For an N-section quarter wave transformer, the reflection coefficients Γ(M) at each junction or step are (M = 1, 2, . . . , N)

$$\Gamma(1) = \frac{Z(3) - Z(2)}{Z(3) + Z(2)} = \rho(1) \; , \tag{3.1a}$$

o

o

o

$$\Gamma(M) = \frac{Z(M+1) - Z(M)}{Z(M+1) + Z(M)} = \rho(M) \; , \tag{3.1b}$$

o

o

o

$$\Gamma(N) = \frac{Z(N+1) - Z(N)}{Z(N+1) + Z(N)} = \rho(N) \; . \tag{3.1c}$$

where Z(1), Z(2), . . . , Z(M), . . . , Z(N+1) are characteristic impedances of the waveguide, stripline or microstrip transmission line, or slow wave structure. These appear in Fig. 3.1,

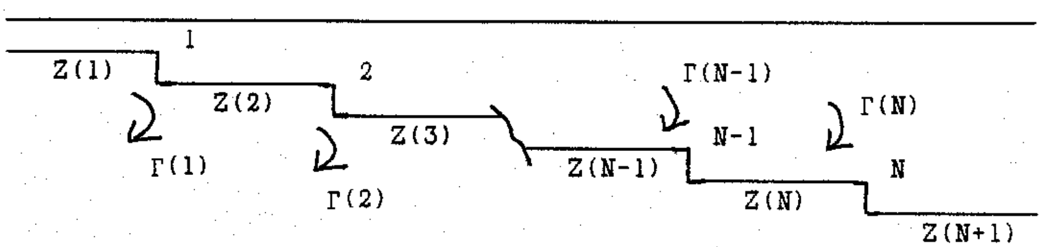

Fig. 3.1. A multisection transformer.

Each section has the same length $\beta\ell = \theta$, where ℓ is a quarter wavelength at the matching frequency. The load impedance Z(N+1) is a resistance or the impedance of the last part of the line. The characteristic impedances are real and

$$Z(N+1) > Z(N) > Z(N-1) > \ldots > Z(M) > \ldots > Z(3) > Z(2) > Z(1), \quad (3.2)$$

so that $\Gamma(M)$ are replaced by $\rho(M)$, the magnitudes of $\Gamma(M)$. If the above impedances are reversed in "greater than" by "less than," the ρ's are negative real terms. We will assume the inequalities above, so that the overall reflection coefficient Γ is

$$\Gamma = \rho(1) + \rho(2)\, e^{-j\,2\,\theta} + \ldots + \rho(M)\, e^{-j\,2\,(M-1)\,\theta} + \ldots + \rho(N)\, e^{-j\,2\,(N-1)\,\theta}, \quad (3.3)$$

where $\exp(-j\,2\,(M-1)\,\theta)$ describes the distances traveled by the Mth partial wave. With symmetry,

$$\rho(1) = \rho(N),\ \rho(2) = \rho(N-1),\ \rho(3) = \rho(N-2),\ \rho(4) = \rho(N-3),\ \ldots, \quad (3.4)$$

so that Γ becomes

$$\Gamma = e^{-j\,(N-1)\,\theta}\,[\,\rho(1)(e^{j\,(N-1)\,\theta} + e^{-j\,(N-1)\,\theta}) + \rho(2)(e^{j\,(N-3)\,\theta} + e^{-j\,(N-3)\,\theta}) + \ldots + \rho([N+1]/2)\,]$$

$$= e^{-j\,(N-1)\,\theta}\,[\sum_{M=1}^{(N-1)/2} \rho(M)(e^{j\,(N+2M+1)\,\theta} + e^{-j\,(N+2M+1)\,\theta}) + \rho([N+1]/2)\,], \quad (3.5a)$$

for odd values of N, and when N is even, the result is

$$\ldots + \rho(N/2)(e^{j\,\theta} + e^{-j\,\theta})\,]$$

$$= e^{-j\,(N-1)\,\theta} \sum_{M=1}^{N/2} \rho(M)(e^{j\,(N-2M+1)\,\theta} + e^{-j\,(N-2M+1)\,\theta}). \quad (3.5b)$$

With the symmetrical transformer, Γ is

$$\Gamma = 2\,e^{-j\,(N-1)\,\theta}\,[\,\rho(1)\cos(N-1)\,\theta + \rho(2)\cos(N-3)\,\theta + \ldots + \rho(M)\cos(N-2M+1)\,\theta + \ldots + 1/2\,\rho([N+1]/2)\,]$$

$$= 2\,e^{-j\,(N-1)\,\theta}\,[\sum_{M=1}^{(N-1)/2} \rho(M)\cos(N-2M+1)\,\theta + 1/2\,\rho([N+1]/2)\,], \quad (3.6a)$$

for odd values of N, and when N is even, the result is

$$+ \ldots + \rho(M) \cos (N-2M+1)\, \theta + \ldots + \rho(N/2) \cos \theta \,]$$

$$= 2\, e^{-j\,(N-1)\,\theta} \sum_{M=1}^{N/2} \rho(M) \cos (N-2M+1)\, \theta\ . \qquad (3.6b)$$

IV. Binomial Transformers

The accepted method for designing transformer structures was Hansen's binomial coefficient design prior to development of the Chebyshev or optimum stepped transformer. The binomial transformer is obtained from the Chebyshev transformers for unity bandwidth ratios.

Maximally flat passband responses are achieved when ρ equals the magnitude of Γ and when the first (N - 1) derivatives with respect to frequency or angle θ vanish at the matching frequency where $\beta\ell = \theta = \pi/2$. These are realized with the function,

$$\Gamma = A\,(1 + e^{-j\,2\,\theta})^{N-1}\ , \qquad (4.1)$$

where ρ is the magnitude of Γ,

$$\rho = |\,\Gamma\,| = |\,A\, 2^{N-1} (\cos \theta)^{N-1}\,|\ , \qquad (4.2)$$

and where N is the number of junctions or steps. The constant A is found at $\theta = 0$ or π, where

$$A(2^{N-1}) = \frac{Z(N+1) - Z(1)}{Z(N+1) + Z(1)}\ , \qquad A = 2^{-(N-1)} \frac{Z(N+1) - Z(1)}{Z(N+1) + Z(1)}\ . \qquad (4.3)$$

With A obtained from Eq. (4.3), Eq. (3.1) is expanded by the Binomial Theorem.

$$= 2^{-(N-1)} \frac{Z(N+1) - Z(1)}{Z(N+1) + Z(1)} (1 + e^{-j\,2\,\theta})^{N-1}$$

$$= 2^{-(N-1)} \frac{Z(N+1) - Z(1)}{Z(N+1) + Z(1)} [1 + (N-1)\, e^{-j\,2\,\theta} + \frac{(N-1)(N-2)}{2!} e^{-j\,4\,\theta} + \ldots$$

$$+ \frac{(N-1)(N-2)\ldots(N-M+1)}{(M-1)!} e^{-j\,2\,(M-1)\,\theta} + \ldots + \frac{(N-1)(N-2)}{2!} e^{-j\,2\,(N-3)\,\theta}$$

$$+ (N-1)\, e^{-j\, 2\, (N-2)\, 0} + e^{-j\, 2\, (N-1)\, 0} \;]$$

$$= 2^{-(N-1)} \frac{Z(N-1) - Z(1)}{Z(N-1) + Z(1)} \; \frac{(N-1)(N-2)\ldots(N-M+1)}{(M-1)!} \, e^{-j\, 2\, (M-1)\, 0}$$

$$= \quad \rho(M)\, e^{-j\, 2\, (M-1)\, 0} \; , \qquad (4.4)$$

where the binomial coefficients $\begin{bmatrix} N-1 \\ M-1 \end{bmatrix}$ are

$$\begin{bmatrix} N-1 \\ M-1 \end{bmatrix} = \frac{(N-1)(N-2)\ldots(N-M+1)}{(M-1)!} = \frac{(N-1)!}{(N-M)!\,(M-1)!} = \begin{bmatrix} N-1 \\ N-M \end{bmatrix} . \qquad (4.5)$$

and some simple terms have the forms or definitions

$$\begin{bmatrix} N-1 \\ 0 \end{bmatrix} = \begin{bmatrix} N-1 \\ N-1 \end{bmatrix} = \frac{(N-1)!}{(N-1)!\,0!} = 1 \; , \; \begin{bmatrix} N-1 \\ 1 \end{bmatrix} = \begin{bmatrix} N-1 \\ N-2 \end{bmatrix} = \frac{(N-1)!}{(N-2)!\,1!} = (N-1) \qquad (4.6)$$

Another simplification is made with the series,

$$\ln X = 2 \frac{X-1}{X+1} + \frac{2}{3} \begin{bmatrix} \frac{X-1}{X+1} \end{bmatrix}^{3} + \ldots + \frac{2}{(2M+1)} \begin{bmatrix} \frac{X-1}{X+1} \end{bmatrix}^{2M+1} + \ldots = 2 \frac{X-1}{X+1} , \qquad (4.7)$$

for M = 1, 2, 3, ..., with small values of X, applied to X equal to Z(N+1)/Z(1) and X equal to Z(M+1)/Z(1). The above approximations are 9 percent in error with X = 3, and 3 percent when X = 2.

Symmetry provides the terms,

$$\rho(M) = 2^{-(N-1)} \frac{Z(N+1) - Z(1)}{Z(N+1) + Z(1)} \begin{bmatrix} N-1 \\ M-1 \end{bmatrix} = 2^{-N} \begin{bmatrix} N-1 \\ M-1 \end{bmatrix} \ln \left| \frac{Z(N+1)}{Z(1)} \right|$$

$$\rho(M) = 2^{-N} \begin{bmatrix} N-1 \\ N-M \end{bmatrix} \ln \left| \frac{Z(N+1)}{Z(1)} \right| \qquad = \rho(N-M+1) \; , \qquad (4.8)$$

and with Eq. (4.7),

$$\rho(M) = \frac{Z(M+1) - Z(M)}{Z(M+1) + Z(M)} = \frac{1}{2} \ln \left| \frac{Z(M+1)}{Z(M)} \right| . \qquad (4.9)$$

Combining Eqs. (4.8) and (4.9) yield

$$\rho(M) = 2^{-N} \begin{bmatrix} N-1 \\ N-M \end{bmatrix} \ln \left| \frac{Z(N+1)}{Z(1)} \right| \qquad = \frac{1}{2} \ln \left| \frac{Z(M+1)}{Z(M)} \right| , \qquad (4.10)$$

which is the solution for the logarithmic ratios of the characteristic impedances for adjacent transformer sections M and M+1, where M = 1, 2, 3, ... N. The logarithmic ratios are proportional to the binomial coefficients and a constant given by

$$\begin{bmatrix} N-1 \\ M-1 \end{bmatrix} 2^{-N} \ln\left(\frac{Z(N+1)}{Z(1)}\right) , \qquad (4.11)$$

since Z(1) is the characteristic impedance of the main line at the transformer input, Z(N+1) is the load or characteristic impedance of the line at the end of the transformer, and N is the number of steps or junctions.

The reflection coefficient is expressed in two forms with Eqs. (4.4) and (4.10),

$$\rho = \sum_{M=1}^{N} \rho(M)\, e^{-j\,2\,M\,\theta} = \frac{1}{2} \sum_{M=1}^{N} \ln\left(\frac{Z(M+1)}{Z(M)}\right) e^{-j\,2\,(M-1)\,\theta}$$

$$= 2^{-N} \ln\left(\frac{Z(M+1)}{Z(1)}\right) \sum_{M=1}^{N} \begin{bmatrix} N-1 \\ M-1 \end{bmatrix} e^{-j\,2\,(M-1)\,\theta} . \qquad (4.12)$$

When Eq. (4.10) is divided by Eq. (4.12) for 0 = 0, the result is

$$\frac{\frac{1}{2} \ln\left(\frac{Z(M+1)}{Z(M)}\right)}{\frac{1}{2} \sum \ln\left(\frac{Z(M+1)}{Z(M)}\right)} = \frac{2^{-N} \ln\left(\frac{Z(N+1)}{Z(1)}\right) \begin{bmatrix} N-1 \\ M-1 \end{bmatrix}}{2^{-N} \ln\left(\frac{Z(N+1)}{Z(1)}\right) \sum_{M=1}^{N} \begin{bmatrix} N-1 \\ M-1 \end{bmatrix}} \qquad (4.13)$$

and when like terms are canceled, and with the reduction of a sum of logarithmic terms,

$$\ln\left(\frac{Z(M+1)}{Z(M)}\right) = \ln\left(\frac{Z(2)}{Z(1)}\right) + \ln\left(\frac{Z(3)}{Z(2)}\right) + \ln\left(\frac{Z(4)}{Z(3)}\right) + . . . + \ln\left(\frac{Z(N+1)}{Z(N)}\right)$$

$$= \ln\left(\frac{Z(2)}{Z(1)}\right)\left(\frac{Z(3)}{Z(2)}\right)\left(\frac{Z(4)}{Z(3)}\right) . . . \left(\frac{Z(N+1)}{Z(N)}\right) = \ln\left(\frac{Z(N+1)}{Z(1)}\right) , \qquad (4.14)$$

so that Eq. (4.12) becomes

$$\ln\left(\frac{Z(M+1)}{Z(M)}\right) = \begin{bmatrix} N-1 \\ M-1 \end{bmatrix} \ln\left(\frac{Z(N+1)}{Z(1)}\right) \Big/ \sum_{M=1}^{N} \begin{bmatrix} N-1 \\ M-1 \end{bmatrix} \qquad (4.15)$$

V. Chebyshev Transformers

Hansen's [2] binomial-coefficient design for transmission line transformer sections was described in the preceding section. In his design, the logarithms of the characteristic impedance ratios of adjacent sections were made to be in the ratio of the binomial coefficients.

In this section, the design method calculates the logarithms of the characteristic impedance ratios so that the VSWR has the characteristic "equal ripple" response of a Chebyshev polynomial. Instead of the maximally flat passband characteristic, the Chebyshev transformer provides a variation of reflection coefficient ρ to vary or oscillate between 0 and $\rho(M)$ across the passband. Since the equal ripple response makes ρ behave like a Chebyshev polynomial, it is named the Chebyshev transformer. The reflection is zero at as many different frequencies in the passband as there are transformer sections.

Use of the Chebyshev polynomials can be better understood by looking at the generation and properties of these polynomials. In the differential equation,

$$(1 - X^2)\frac{d^2Y}{dX^2} - X\frac{dY}{dX} + N^2 Y = 0 , \qquad (5.1)$$

where N is an integer, the general solution is a linear combination of the first kind $T_N(x)$, and the second kind $U_N(x)$.

$$Y = C_1 T_N(X) + C_1 U_N(X) , \qquad (5.2)$$

where we will use the first kind $T_N(X)$. The functional forms of the Chebyshev polynomials are

$$T_N(X) = \cos(N \cos^{-1} X) = X^N - \begin{bmatrix} N \\ 2 \end{bmatrix} X^{N-2} (1 - X^2) + \begin{bmatrix} N \\ 4 \end{bmatrix} X^{N-4} (1 - X^2)^2 +$$

$$\binom{N}{6} X^{N-6}(1-X^2)^3 - \binom{N}{8} X^{N-8}(1-X^2)^4 + \binom{N}{10} X^{N-10}(1-X^2)^5 +\ldots+$$

$$(-1)^M \binom{N}{2M} X^{N-2M}(1-X^2)^M +\ldots+ (-1)^{N/2}\binom{N}{N}(1-X^2)^{N/2} \quad . \ N \text{ even.}$$

$$\text{and } +\ldots+ (-1)^{(N-1)/2}\binom{N}{N-1}(1-X^2)^{(N-1)/2} \quad . \quad N \text{ odd.} \qquad (5.3)$$

$$U_N(X) = \sin(N \sin^{-1} X) = \sqrt{1-X^2}\ [\ \binom{N}{1} X^{N-1} - \binom{N}{3} X^{N-3}(1-X^2) +$$

$$\binom{N}{5} X^{N-5}(1-X^2)^2 - \binom{N}{7} X^{N-7}(1-X^2)^3 + \binom{N}{9} X^{N-9}(1-X^2)^4 +\ldots+$$

$$(-1)^{M-1}\binom{N}{2M-1} X^{N-2M+1}(1-X^2)^{M-1} +\ldots+$$

$$(-1)^{(N-2)/2}\binom{N}{N-1}(1-X^2)^{(N-2)/2} \quad . \ N \text{ even. and for } N \text{ odd.}$$

$$+\ldots+ (-1)^{(N-1)/2}\binom{N}{N}(1-X^2)^{(N-1)/2} \quad . \qquad (5.4)$$

If $X = \cos(\theta)$ in Eq. (5.3) and $X = \sin\theta$ in Eq. (5.4). other forms for $T_N(X)$ and $U_N(X)$ are

$$T_N(\cos(\theta)) = \cos(N\theta) \quad . \qquad (5.5)$$

$$U_N(\cos(\theta)) = \sin(N\theta) \quad . \qquad (5.6)$$

As functions of X. the first few terms of $T_M(X)$ and $U_M(X)$ are

$$T_0(X) = 1, \quad T_1(X) = X, \quad T_2(X) = 2X^2 - 1, \quad T_3(X) = 4X^3 - 3X.$$

$$T_4(X) = 8X^4 - 8X^2 + 1, \quad T_5(X) = 16X^5 - 20X^3 + 5X, \ldots,$$

$$T_M(X) = 2X\,T_{M-1}(X) - T_{M-2}(X) \quad . \qquad (5.7)$$

$$U_0(X) = 0, \quad U_1(X) = \sqrt{1 - X^2}, \quad U_2(X) = 2X\sqrt{1 - X^2},$$

$$U_3(X) = [4X^2 - 1]\sqrt{1 - X^2}, \quad U_4(X) = [8X^3 - 4X]\sqrt{1 - X^2},$$

$$U_M(X) = 2XU_{M-1}(X) - U_{M-2}(X) \;, \tag{5.8}$$

where a few properties of the Chebyshev functions may be noted. When X = 1, or 0 = 0,

$$T_1(1) = T_2(1) = T_3(1) = . . . = T_M(1) = 1 \;, \tag{5.9}$$

and when X = -1, or 0 = , for odd orders,

$$T_1(-1) = T_{2M+1}(-1) = -1 \;, \tag{5.10}$$

and for even orders,

$$T_2(-1) = T_{2M}(-1) = +1 \;, \tag{5.11}$$

resulting in an increasing number of oscillations between -1 and +1 (or 0 and 180 degrees) as M increases, while the absolute value of $T_M(X)$ increases asymptotically with increasing values of X (greater than 1) as

$$T_M(X) \longrightarrow 2^{M-1} X^M \;, \tag{5.12}$$

so that "equal ripple" is encountered!

With U (X), the function is not defined for the absolute value of X greater than one. For X = +1 or -1 (θ = 0 or 180 degrees),

$$U_M(+1 \text{ or } -1) = U_M(+1 \text{ or } -1) = 0, \; M = 0, 1, 2, 3, . . . \tag{5.13}$$

and for X = 0,

$$U_{4M-1}(0) = 1 \;, \qquad U_{4M-1}(0) = -1, \; M = 1, 2, 3, . . . \tag{5.14}$$

where $U_M(X)$ oscillates between +1 and -1, with oscillations increasing with M. It is not suitable for equal-ripple filters, antennas, or transformers because of the region outside of X equal to +1 and -1.

The suitable function for equal ripple is the Chebyshev polynomial of the first kind $T_M(X)$, with X replaced by cos (θ) in Eq. (5.5). As θ varies from 0 to 180 degrees, the range of X varies from +1 to -1. If the bandwidth or equal ripple characteristic is chosen from θ(1) to 180 degrees - θ(1), we must choose

$$T_M\left(\frac{\cos(\theta)}{\cos(\theta(1))}\right) = \cos M\left(\cos^{-1}\frac{\cos(\theta)}{\cos(\theta(1))}\right) , \qquad (5.15)$$

where the argument is unity when θ = θ(1), and less than unity for $\theta(1) < 0 < \pi - \theta(1)$. This defines the bandwidth B by

$$B = \frac{f(2)}{f(1)} = \frac{\theta(2)}{\theta(1)} = \frac{180^{o} - \theta(1)}{\theta(1)} \quad . \qquad (5.16)$$

in coaxial or two conductor transmission lines, and in waveguides by

$$B = \frac{f(2)}{f(1)} = \frac{\theta(2)}{\theta(1)} = \frac{\lambda\ (\text{guide } 1)}{\lambda\ (\text{guide } 2)} \quad . \qquad (5.15)$$

For a given value of B, θ(1) is found with

$$\theta(1) = \frac{180^{o}}{1 + B} \quad . \qquad (5.18)$$

When B = 1.0, then θ(1) = θ (2), and both are 90^{o}. But with B = 2.0, f(2) = 2 f(1), satisfied by typical ranges of 200 and 400 MHz, and 3 and 6 GHz. With a narrower B, such as 1.2, we have 3 and 3.6 GHz, and 299 and 240 MHz. The frequency band for a given bandwidth ratio B is

$$f = f(2) - f(1) = (B - 1)\ f(1) \ , \qquad (5.19)$$

or with a center frequency f(θ),

$$f(\theta) = (f(1) + f(2))/2 = f(1)\ (1 + B)/2 = f(2)\ (1 + B)/2\ B \ . \qquad (5.20)$$

With the above descriptions of Chebyshev polynomials and bandwidths, we return to the overall reflection in Eq. (3.3) for all multisection quarter wave transformers with small reflections at each step,

$$\Gamma = \rho(1) + \rho(2)\, e^{-j\,2\,\theta} + \rho(3)\, e^{-j\,4\,\theta} + \ldots + \rho(M)\, e^{-j\,2\,(M-1)\,\theta}$$

$$+ \ldots + \rho(N)\, e^{-j\,2\,(N-1)\,\theta}$$

$$= e^{-j\,(N-1)\,\theta} \left[\rho(1)\, e^{-j\,(N-1)\,\theta} + \rho(2)\, e^{-j\,(N-3)\,\theta} + \ldots + \right.$$

$$\left. \rho(M)\, e^{-j\,(N-2M+1)\,\theta} + \ldots + \rho(N)\, e^{-j\,(N-1)\,\theta} \right] , \qquad (5.21)$$

and the magnitude of the reflection coefficient is

$$\rho = |\,\Gamma\,| = \rho(1)\, e^{-j\,(N-1)\,\theta} + \rho(2)\, e^{-j\,(N-3)\,\theta} + \ldots + \rho(M)\, e^{-j\,(N-2M+1)\,\theta}$$

$$+ \ldots + \rho(N)\, e^{-j\,(N-1)\,\theta} , \qquad (5.22)$$

and with the same symmetry as that expressed in Eq. (3.4) the above expression is

$$\rho = 2\,\rho(1) \cos\,(N-1)\,\theta + 2\,\rho(2) \cos\,(N-3)\,\theta + \ldots + 2\,\rho(M) \cos\,(N-2M+1)\,\theta$$

$$+ \ldots + \rho([N+1]/2) \text{ , for odd N, 1, 3, 5, . . ., and for N even} \qquad (5.23)$$

$$= 2\,\rho(1) \cos\,(N-1)\,\theta + \ldots + 2\,\rho(N/2) \cos\,\theta , \qquad (5.24)$$

In order to have the Chebyshev form of equal ripple across the bandwidth of the slow wave structure, we first adjust the bandwidth ratio B with

$$Y = \cos\,(\theta) = X \cos\,(\theta(1)) \qquad (5.25)$$

so that the cosine functions become

$$\cos\,(\theta) = Y = X \cos\,(\theta(1)) , \qquad (5.26a)$$

$$\cos\,(2\theta) = 2\,Y^2 - 1 = 2\,X^2 \cos^2\,(\theta(1)) - 1 , \qquad (5.26b)$$

$$\cos\,(3\theta) = 4\,Y^3 - 3\,Y = 4\,X^3 \cos^3\,(0(1)) - 3\,X \cos\,(\theta(1)) , \qquad (5.26c)$$

$$\cos\,(4\theta) = 8\,Y^4 - 8\,Y^2 + 1 = 8\,X^4 \cos^4\,(\theta(1)) + 1 , \qquad (5.26d)$$

$$\cos(5\theta) = 16Y^5 - 20Y^3 + 5Y = 16X^5\cos^5(\theta(1)) - 20X^3\cos^3(\theta(1)) + 5X\cos(\theta(1)) \,, \tag{5.26e}$$

$$\cos(6\theta) = 32Y^6 - 48Y^4 + 18Y^2 - 1 = 32X^6\cos^6(\theta(1)) - 48X^4\cos^4(\theta(1)) + 18X^2\cos^2(\theta(1)) - 1 \,, \tag{5.26f}$$

et cetera, et cetera, et cetera. We now match powers of X in Eq. (5.26a) to (5.26f) et cetera, et cetera, et cetera with powers of X in $\alpha(N)$ in $T_{N-1}(X)$ to evaluate the arbitrary constants $\rho(1)$, $\rho(2)$, . . ., $\rho(N)$. The term $\alpha(N)$ is given by

$$\alpha(N) = \frac{\frac{1}{2}\ln\left(\frac{Z(N+1)}{Z(1)}\right)}{T_{N-1}\left(\frac{1}{\cos(\theta(1))}\right)} \,. \tag{5.27}$$

When N = 1, $\rho(M)$ are found with Eq. (5.23),

$$\rho = \rho(1) = \alpha(1)\,T_0(X) = \frac{\frac{1}{2}\ln\left(\frac{Z(2)}{Z(1)}\right)}{T_0\left(\frac{1}{\cos(\theta(1))}\right)}\,T(X)_0 = \frac{1}{2}\ln\left(\frac{Z(2)}{Z(1)}\right) \,. \tag{5.28}$$

and when N = 2, $\rho(M)$ are found with Eq. (5.24),

$$\rho = 2\,\rho(1)\cos(\theta) = 2\,\rho(1)\,X\cos(\theta(1) = \alpha(2)\,T_1(X)$$

$$\rho = \frac{\frac{1}{2}\ln\left(\frac{Z(2)}{Z(1)}\right)}{T_1\left(\frac{1}{\cos(\theta(1))}\right)}\,X \,, \tag{5.29}$$

so that coefficients of X are equal to each other.

$$\rho(1) = \frac{1}{2\cos(\theta(1))} \; \frac{\frac{1}{2}\ln\left(\frac{Z(3)}{Z(1)}\right)}{T_1\left(\frac{1}{\cos(\theta(1))}\right)}$$

$$= \frac{\ln\left(\frac{Z(3)}{Z(1)}\right)}{4\cos(\theta(1))\, T_1\left(\frac{1}{\cos(\theta(1))}\right)} = \frac{1}{4}\ln\left(\frac{Z(3)}{Z(1)}\right) . \qquad (5.30)$$

from $T_1(B) = B = 1.0/\cos(\theta(1))$. With $N = 3$, $\rho(M)$ are found with Eq. (5.23)

$$\rho = 2\,\rho(1)\cos(2\theta) + \rho(2) = \rho(1)\left[\,(4\cos^2\theta(1))\,X^2 - 2\,\right] + \rho(2)$$

$$= \alpha(3)\, T_2(X) = \frac{\frac{1}{2}\ln\left(\frac{Z(4)}{Z(1)}\right)}{T_2\left(\frac{1}{\cos(\theta(1))}\right)}\,(2X^2 - 1) , \qquad (5.31)$$

and when coefficients of powers of X are equated,

$$\rho(1) = \frac{1}{4\cos^2(\theta(1))} \; \frac{\ln\left(\frac{Z(4)}{Z(1)}\right)}{T_2(B)} = \frac{B^2\ln\left(\frac{Z(4)}{Z(1)}\right)}{4\,T_2(B)} . \qquad (5.32a)$$

$$\rho(2) = \frac{\ln\left(\frac{Z(4)}{Z(1)}\right)}{2\,T_2(B)}\left[\,B^2 - 1\,\right] . \qquad (5.32b)$$

When $B = 2.0$, the reflection coefficients are

$$\rho(1) = \frac{B^2\ln\left(\frac{Z(4)}{Z(1)}\right)}{8\,T_2(B)} = \frac{\ln\left(\frac{Z(4)}{Z(1)}\right)}{2\left[\,2B^2 - 1\,\right]} = \frac{\ln\left(\frac{Z(4)}{Z(1)}\right)}{14} , \qquad (5.33a)$$

$$\rho(2) = \ln \left(\frac{Z(4)}{Z(1)}\right) \left(\frac{3}{2}\right) , \qquad (5.33b)$$

and the ratios of reflection coefficients are

$$\rho(1):\ \rho(2):\ \rho(3) = 1.0:\ 1.5:\ 1.0 . \qquad (5.34)$$

With four steps, the ratios of reflection coefficients become

$$\rho(1):\ \rho(2):\ \rho(3):\ \rho(4) = 1.0:\ 2.25:\ 2.25:\ 1.0 . \qquad (5.34)$$

Derivation of subsequent reflection coefficients becomes increasingly tedious but logically straightforward. A simplified method for calculation these coefficients was developed for antenna arrays by Ross E. Graves, Stanford University, in an unpublished report.

Grave's method is similar to that employed with Pascal's triangle for binomial coefficients. Pascal's triangle is formed by always inserting a number 1.0 in the first row. In the second row, another number 1.0 is always placed in the left of the element above and the same number 1.0 to the right of the element above. This ritual for element placement is described above because a similar ritual is followed with Grave's pyramid. In successive rows, elements are formed by adding the two elements on the left and right of the calculated element in the row above. When elements are absent, they are assumed to be zero. Pascal's triangle appears in Table I for several rows.

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

Table I. Pascal's Triangle for Binomial Coefficients.

Grave's pyramid is formed by always inserting a number 2.0 in the first row. In the second row another number B, representing the bandwidth described earlier, is always placed to the left of the element above and the same number B is always placed to the right of the number above. In successive rows, elements are formed by adding the two elements on the left and right in the row above, multiplying the sum by B, and subtracting the element in the second row above the entry being calculated from this product. When elements are absent, they are assumed to be zero. Grave's pyramid appears in Table II as a function of B.

			2			
		B		B		
	B**2		2*B**2-2		B**2	
B**3		3*B**3-3*B		3*B**3-3*B		B**3

Table II. Grave's Pyramid for Chebyshev Coefficients.

When B = 2.0, Grave's Pyramid has the values seen in Table III.

						2						
					2		2					
				4		6		4				
			8		16		16		8			
		16		48		66		48		16		
	32		120		210		210		120		32	
64		288		612		774		612		288		64

Table III. Grave's Pyramid for Chebyshev Coefficients, B = 2.0.

When the elements of Table III are normalized with respect to the elements at the ends, we have the reflection coefficient ratios seen below in Table IV.

						1.0						
					1.0		1.0					
				1.0		1.5		1.0				
			1.0		2.25		2.25		1.0			
		1.0		3.0		4.125		3.0		1.0		
	1.0		3.75		6.5675		6.5675		3.75		1.0	
1.0		4.5		9.5625		12.09375		9.5625		4.51		1.0

Table IV. Grave's Pyramid for Normalized Chebyshev Coefficients, B = 2.0.

VI. Matching of Periodic Structures

The binomial coefficients and the Chebyshev coefficients described in Sections 4.0 and 5.0 are used to match one transmission line or waveguide to another transmission line or waveguide with different characteristic impedances. The first unloaded line or guide may be a slow wave periodic structure, a fast wave structure, or a two conductor transmission line such as coaxial lines, parallel wire lines, shielded pairs, parallel plates, or parallel bars. The input guide or loaded line may be any of the above.

Slow wave structures are waveguides and transmission lines loaded at periodic intervals with identical obstacles such as a reactive element like a diaphragm. The diaphragm may be a rectangular serration in the form of a thin fin or a thick rectangular or square shape. This type of waveguide structure has two important properties, which are passband-stopband characteristics and support of electromagnetic waves with phase velocities much slower than the velocity of light. An excellent physical picture of slow waves is found in the serpentine or sinusoidal serrated slow wave structure, where the periodic obstacles look like a roller coaster track. The electromagnetic waves follow the serpentine path with the velocity of light, but their velocities along the axis of the guide are much less. Another example is the helical wire in a traveling wave tube (TWT), where the waves travel along the surface of the wire with the velocity of light, but their axial velocities are much slower.

The passband-stopband characteristic is the presence of frequency bands in which waves propagate with negligible attenuation separated by frequency bands in which the waves are cut off and do not propagate. The former is a passband, and the latter is a stopband. The passband-stopband is useful for frequency filtering.

The property of periodic structures to support slow waves with phase velocities slower than that of light, is applied in linear magnetron type (M type) TWTs with magnetically focused electron beams wand a slow wave structure of periodic annular cavities, ordinary (O type) TWTs with a slow wave structure in the form of a helix, resistance wall amplifiers with a slow wave structure made with resistive linings, and related structures. Other applications are found in transmission lines coupling TWTs to slow waves, where the periodic structure continues from source to antenna.

Metallic waveguides are one example of fast wave structures. In waveguides, modal waves are obliquely incident on the guide walls as they propagate along the guide. As frequency decreases toward cutoff, the angle of incidence approaches zero. With frequencies increasing above cutoff, the angle of incidence increases. If we observe the velocity of the wave front parallel to the direction of propagation, the phase velocity is greater than the velocity measured normal to the wave front. It will be greater in any oblique direction. As the angle of incidence approaches zero and the frequency approaches the cutoff frequency for the propagating mode, the phase velocity measured in the direction of propagation approaches infinity. For this reason, the guide is a "fast wave" guide. The same phenomena occurs in dielectric and insular waveguides.

In most designs for matching slow to fast wave guides, the fast wave guide is identical to the slow wave guide when the obstacles are removed. In this Section we consider a slow wave coaxial line with square wave serrations on the outer conductor, while the smooth coaxial line has a phase velocity equal to that of light.

The inner and outer conductor radii of the coaxial line are R(I) and R(O), respectively, with R(O) = 6.5 cm and R(I) = 8.2 cm. The characteristic impedance Z(0) of the smooth coaxial line is

$$Z(0) = \frac{\eta}{2\pi} \ln \left(\frac{R(O)}{R(I)}\right) = 60 \ln \left(\frac{R(O)}{R(I)}\right) , \tag{6.1}$$

where η is the intrinsic impedance of free space or air, 120 π ohms. The characteristic impedance $Z_S(0)$ of the slow wave coaxial line is

$$Z_S(0) = \left(\frac{C}{V}\right) \frac{\eta}{2\pi} \ln \left(\frac{R(O)}{R(I)}\right) = \left(\frac{C}{V}\right) 60 \ln \left(\frac{R(O)}{R(I)}\right) , \tag{6.2}$$

where (C/V) is the ratio of the phase velocity of light C to that of the slow wave structure V. Figure 6.1 depicts the slow wave and fast wave guides to be matched.

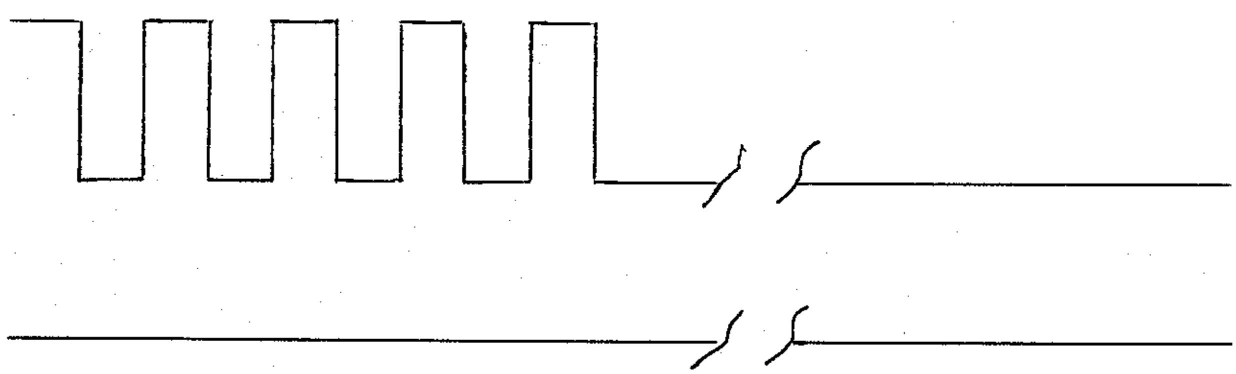

Fig. 6.1. Slow and fast wave structures.

A more detailed picture of the slow wave structure is seen in Fig. 6.2, which describes the coordinate system. The teeth are rectangular or square. The inner conductor is located at Y = - h, where h is the distance between the inner conductor R(I) and the outer conductor R(O) of the coaxial line.

Although the actual slow wave structure is a coaxial line with the outer corrugated surface radius R(0) of 8.2 cm to the teeth surfaces, a planar model yields similar results when the corrugation parameters are the same because of the relatively close spacing between inner and outer conductors. Although d = 1.4 cm, b = 0.7 cm, and ℓ = 1.8 cm, the results are valid for any combination of these parameters.

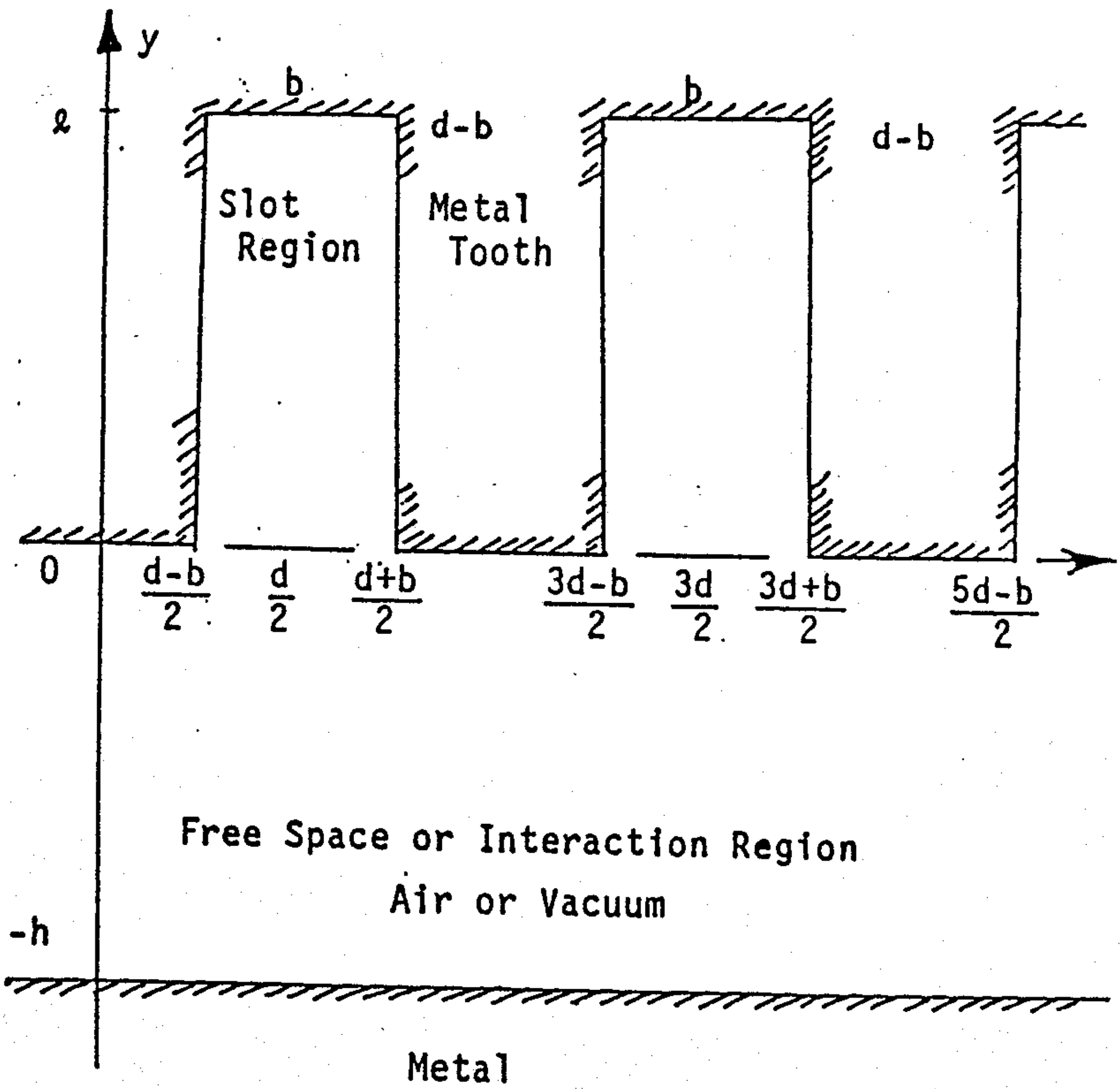

Fig. 6.2. Coordinate system for the slow wave structure.

In designing the transformer sections, the characteristic impedance Z(1) of the first section is equal to the slow wave impedance in Eq. (6.2), while the characteristic impedance of the last section Z(N+1) is equal to the characteristic impedance of the smooth coaxial line in Eq. (6.1). Since coaxial lines have no cutoff frequencies for TEM modes, they are only fast wave guides for modes higher than TEM modes.

Binomial and Chebyshev transformers will be developed in subsequent paragraphs to match the characteristic impedance Z(1) to match the characteristic impedance Z(N+1) of the output or load section (N+1) in an N-step transformer. In order to find Z(1) as a function of slow wave guide geometry, the ratio of phase velocity of light in an infinite dielectric space to slow wave phase velocity in the dielectric material between coaxial conductors must be found in section (1) of the transformer. These ratios are C/V(M), where M = 1,. . ., N+1.

C/V(1) is calculated with equating slow wave and transmission line impedances in the drifts pace for electron beam and electromagnetic wave interactions. If a slow wave transmission line with phase velocity ratio C/V(M) is introduced, the series impedance per unit length, $Z_{SE}(M)$, is

$$Z_{SE}(M) = j \left(\frac{C}{V(M)}\right)^2 \frac{\omega\mu}{2\pi} \ln\left(\frac{R(O)}{R(I)}\right) \text{ ohms per meter ,} \tag{6.3}$$

while the parallel susceptance $Y_{PA}(M)$ per unit length is

$$Y_{PA}(M) = j \frac{2\pi\omega\varepsilon}{\ln\left(\frac{R(O)}{R(I)}\right)} \text{ mhos per meter .} \tag{6.4}$$

Equations (6.3) and (6.4) can be combined to obtain the characteristic impedance in Eq. (6.2),

$$Z(M) = \sqrt{Z_{SE}(M) / Y_{PA}(M)} = \frac{C}{V(M)} \frac{\eta}{2\pi} \ln\left(\frac{R(O)}{R(I)}\right) \text{ ohms ,} \tag{6.5}$$

and the phase velocity V(M) is

$$V(M) = \frac{1}{\sqrt{Z_{SE}(M)\, Y_{PA}(M)}} = \frac{1}{\sqrt{(C/V(M))\; Z_{SE}(M)\, Y_{PA}(M)}} = \frac{C}{C/V(M)} = V(M) \quad . \quad (6.6)$$

For the short circuited radial transmission line, formed by circular, parallel, infinitely conducting planes or tooth walls, with radial length $\ell(M)$ and width B(M), the input impedance Z(slot) is

$$Z_{slot}(M) = j\, Z_0(M) \tan k\,\ell(M) = j \frac{D(M)}{2\pi R(0)} \tan k\,\ell(M) \quad , \qquad (6.7)$$

where D(M) is the width of a unit cell (M) or section (M).

The series impedance per unit length for a unit cell with $\ell(M) = 0.0$ (in a smooth coaxial line) is

$$Z_{SE}(M) = j \frac{\omega\mu}{2\pi} \ln\left(\frac{R(0)}{R(I)}\right) \text{ ohms per meter} \quad , \qquad (6.8)$$

so that the slow wave impedance due to radial cavities are the difference between Eqs. (6.3) and (6.8), multiplied by the cavity width B(M),

$$Z_{slot}(M) = j\left[\left(\frac{C}{V(M)}\right)^2 - 1\right] \frac{\omega\mu}{2\pi} B(M) \ln\left(\frac{R(0)}{R(I)}\right) \quad , \qquad (6.9)$$

and when Eqs. (6.7) and (6.9) are equated,

$$j \frac{D(M)}{2\,R(0)} \eta \tan k\,\ell(M) = j\left[\left(\frac{C}{V(M)}\right)^2 - 1\right] \frac{\omega\mu}{2\pi} B(M) \ln\left(\frac{R(0)}{R(I)}\right) \quad , \quad (6.10)$$

and with $\omega\mu = k\,\eta$, and canceling terms,

$$\frac{D(M)}{R(M)} \tan k\,\ell(M) = \left[\left(\frac{C}{V(M)}\right)^2 - 1\right] k\, B(M) \ln\left(\frac{R(0)}{R(I)}\right) \quad ,$$

$$\tan k\,\ell(M) = \left[\left(\frac{C}{V(M)}\right)^2 - 1\right] \frac{B(M)}{D(M)} k\, R(0) \ln\left(\frac{R(0)}{R(I)}\right) \quad ,$$

$$\ell(M) = \frac{1}{k} \arctan\left[\left(\frac{C}{V(M)}\right)^2 - 1\right] \frac{B(M)}{D(M)} k\, R(0) \ln\left(\frac{R(0)}{R(I)}\right) \quad , \qquad (6.11)$$

$$\frac{C^2}{V(M)^2} = 1 + \left[\frac{D(M)}{k\ D(M)\ R(O)\ \ln\left(\frac{R(O)}{R(I)}\right)} \right] \tan k\ell\ (M) \ . \qquad (6.12)$$

where B(M)/D(M) = B/D in the first section (1). When the expression C/V(1) is obtained for the input section (1) of the slow wave guide, the slow wave characteristic impedance Z(1) is found with Eq. (6.5).

With D(1) = 1.40 cm, B(1) = 0.70 cm, ℓ (1) = 1.80 cm, λ = 8.12 cm (frequency F = 3.69 GHz), the ratio V(1)/C is 0.342. If the ratio V(1)/C is chosen to be 0.34 (based on computer simulation results) ℓ(1) is 1.8032 cm. This correlation with simulation results indicates a good approximation to the dispersion curve with Eqs. (6.11) and (6.12).

The characteristic impedances of the N-step or N-junction transformers are calculated with Eqs. (4.10) and (4.14),

$$\ln\left(\frac{Z(M+1)}{Z(M)}\right) = 2\ \rho(M) = 2\ \rho(M)\ \frac{\ln\left(\frac{Z(N+1)}{Z(1)}\right)}{\ln\left(\frac{Z(N+1)}{Z(1)}\right)} = \rho\ (M)\ \frac{\ln\left(\frac{Z(N+1)}{Z(1)}\right)}{\frac{1}{2}\ln\left(\frac{Z(M+1)}{Z(M)}\right)}$$

$$= \rho(M)\ \frac{\ln\left(\frac{Z(N+1)}{Z(1)}\right)}{\rho(1) + \rho(2) + \rho(3) + \ .\ .\ . + \rho(N)} \ . \qquad (6.13)$$

With ρ(M) known from Pascal's Triangle and Grave's Pyramid, the characteristic impedances can be obtained for the binomial and Chebyshev transformers. Since Z(1) is known from the ratio C/V(1), Z(2) is calculated. After Z(2) is found, Z(3) is next calculated. This procedure continues until the remaining Z(M) are found, with M = 4, 5, . . ., N. The procedure for finding binomial and Chebyshev coefficients simplifies calculating characteristic impedances for both sets of sections.

With the values of Z(M) calculated, the subsequent values of the C/V(M) are then found with

$$\frac{C}{V(M)} = Z(M) / 60 \ln\left(\frac{R(O)}{R(I)}\right), \quad M = 2, 3, 4, \ldots, N, N+1 .$$

The widths D(M) of the unit cells are equal to a half wavelength of the slow waves,

$$D(M) = \frac{C}{2 F \, C/V(M)} , \tag{6.15}$$

which are found with C/V(M) ratios. The cavity widths B(M) are proportional to D(M) with

$$B(M) = \frac{B}{D} = D(M) , \tag{6.16}$$

when we do not have square waves (teeth may be thin fins or fat teeth with negligible gaps).

The inductance L(M) and capacitance C(M) per unit lengths are relatively simpler,

$$L(M) = \left(\frac{C}{V(M)}\right)^2 \frac{\mu}{2\pi} \ln\left(\frac{R(O)}{R(I)}\right) \text{ henries per meter} . \tag{6.17}$$

$$C(M) = \frac{2 \pi \varepsilon}{\ln\left(\frac{R(O)}{R(I)}\right)} \text{ farads per meter} . \tag{6.18}$$

Adjustments for compatibility between teeth periods B(1), phase velocities V(1)/C, and cavity or tooth lengths (1), and frequency F create minor departures from simulated results. These departures are only 1-2 percent, but simulation results are only models and depend only upon the interpretation of the simulation interpreter. The following Tables indicate the slow wave structure design for a Chebyshev transformer with 11 or less sections, and 10 or less steps. Table V is Grave's Pyramid for the successive Tables with 11 to 2 sections.

1.0				
1.0				
1.5	1.0			
2.25	1.0			
4.125	3.0	1.0		
6.5625	3.75	1.0		
12.0938	9.5625	4.51	1.0	
20.0156	13.125	5.25	1.0	
37.0078	30.75	17.25	6.0	1.0
62.7539	44.7188	21.9375	6.75	1.0

Table V. Grave's Pyramid for 10 Steps and Normalized Chebyshev Coefficients, with Bandwidth = 2.0.

In the following sequence, Table IV starts with the transition summary for 11 sections and 10 steps and continues to Table XV, transition summary for 2 sections and 1 step. Bandwidth is 2.0, and columns indicate section (M), characteristic impedance Z(M,N) for steps (N) in the transformer, the step (M), ratio of slow wave phase velocity to velocity of light (V(M)/C), tooth period D(M) and height H(M), cavity reactance X(M), and inductance L(M) and capacitance C(M) per unit cells.

Sect	Z(M.N) ohms	V/C	Tooth Period	X (Slot) ohms	Tooth Ht. cm	L/Cell nH	C/Cell pF
1	40.9998	0.3400	1.3804	113.91	1.8032	5.549	3.301
2	40.8388	0.3413	1.3858	113.35	1.8012	5.527	3.314
3	39.7690	0.3505	1.4231	109.59	1.7873	5.382	3.403
4	36.4819	0.3821	1.5513	97.88	1.7355	4.937	3.710
5	30.5985	0.4556	1.8496	76.18	1.5888	4.141	4.423
6	23.9068	0.5831	2.3674	49.57	1.2417	3.235	5.661
7	18.6785	0.7463	3.0300	26.00	0.6853	2.528	7.245
8	15.6662	0.8898	3.6126	10.25	0.2475	2.120	8.638
9	14.3713	0.9700	3.9381	2.67	0.0598	1.945	9.417
10	13.9948	0.9961	4.0441	0.34	0.0075	1.894	9.670
11	13.9399	1.0000	4.0600	0.00	0.0000	1.886	9.708

Table VI. Transition Summary for 11 Sections and 10 Steps.

Sect	Z(M.N) ohms	V/C	Tooth Period	X (Slot) ohms	Tooth Ht. cm	L/Cell nH	C/Cell pF
1	40.9998	0.3400	1.3804	113.91	1.8032	5.549	3.301
2	40.7000	0.3425	1.3906	112.86	1.7995	5.508	3.325
3	38.9468	0.3579	1.4532	106.68	1.7757	5.271	3.475
4	34.3159	0.4062	1.6493	90.02	1.6915	4.644	3.944
5	27.3838	0.5091	2.0668	63.73	1.4574	3.706	4.942
6	20.8712	0.6679	2.7717	36.32	0.9578	2.825	6.484
7	16.6551	0.8370	3.3981	15.67	0.3945	2.254	8.126
8	14.6747	0.9499	3.8567	4.50	0.1029	1.986	9.222
9	14.0426	0.9927	4.0303	0.64	0.0598	1.900	9.637
10	13.9399	1.0000	4.0600	0.00	0.0000	1.886	9.708

Table VII. Transition Summary for 10 Sections and 9 Steps.

Sect	Z(M.N) ohms	V/C	Tooth Period	X (Slot) ohms	Tooth Ht. cm	L/Cell nH	C/Cell pF
1	40.9998	0.3400	1.3804	113.91	1.8032	5.549	3.301
2	40.4421	0.3447	1.3994	111.95	1.7962	5.473	3.346
3	37.6337	0.3704	1.5037	102.02	1.7554	5.094	3.596
4	31.4453	0.4433	1.7998	79.39	1.6159	4.256	4.304
5	23.9068	0.5831	2.3674	49.57	1.2417	3.235	5.661
6	18.1755	0.7670	3.1139	22.51	0.6156	2.440	7.446
7	15.1855	0.9180	3.7270	7.51	0.1767	2.055	8.912
8	14.1321	0.9864	4.0048	1.20	0.0204	1.912	9.576
9	13.9399	1.0000	4.0600	0.00	0.0000	1.886	9.708

Table VIII. Transition Summary for 9 Sections and 8 Steps.

Sect	Z(M.N) ohms	V/C	Tooth Period	X (Slot) ohms	Tooth Ht. cm	L/Cell nH	C/Cell pF
1	40.9998	0.3400	1.3804	113.91	1.8032	5.549	3.301
2	39.9654	0.3488	1.4161	110.28	1.7899	5.409	3.386
3	35.6242	0.3913	1.5887	94.78	1.7191	4.821	3.799
4	27.9014	0.4996	2.0284	65.77	1.4821	3.776	4.850
5	20.4848	0.6805	2.7629	34.55	0.9138	2.772	6.607
6	16.0434	0.8689	3.5277	12.35	0.3035	2.171	8.436
7	14.3007	0.9748	3.9576	2.23	0.0499	1.935	9.463
8	13.9399	1.0000	4.0600	0.00	0.0000	1.886	9.708

Table IX. Transition Summary for 8 Sections and 7 Steps.

Sect	Z(M.N) ohms	V/C	Tooth Period	X (Slot) ohms	Tooth Ht. cm	L/Cell nH	C/Cell pF
1	40.9998	0.3400	1.3804	113.91	1.8032	5.549	3.301
2	39.0907	0.3566	1.4478	107.19	1.7778	5.290	3.462
3	32.6902	0.4264	1.7313	84.02	1.6515	4.424	4.140
4	23.9068	0.5831	2.3674	49.77	1.2417	3.235	5.661
5	17.4833	0.7973	3.2371	20.01	0.5164	2.366	7.741
6	14.6207	0.9534	3.8710	4.18	0.0951	1.979	9.256
7	13.9399	1.0000	4.0600	0.00	0.0000	1.886	9.708

Table X. Transition Summary for 7 Sections and 6 Steps.

Sect	Z(M.N) ohms	V/C	Tooth Period	X (Slot) ohms	Tooth Ht. cm	L/Cell nH	C/Cell pF
1	40.9998	0.3400	1.3804	113.91	1.8032	5.549	3.301
2	37.5094	0.3716	1.5088	101.56	1.7533	5.076	3.608
3	28.7222	0.4853	1.9705	69.98	1.5183	3.887	4.712
4	19.8987	0.7005	2.8442	31.83	0.8438	2.693	6.801
5	15.2371	0.9149	3.7144	7.80	0.1843	2.062	8.882
6	13.9399	1.0000	4.0600	0.00	0.0000	1.886	9.708

Table XI. Transition Summary for 6 Sections and 5 Steps.

Sect	Z(M.N) ohms	V/C	Tooth Period	X (Slot) ohms	Tooth Ht. cm	L/Cell nH	C/Cell pF
1	40.9998	0.3400	1.3804	113.91	1.8032	5.549	3.301
2	34.7297	0.4041	1.6296	91.56	1.7006	4.700	3.897
3	23.9068	0.5831	2.3674	49.57	1.2417	3.235	5.661
4	16.4566	0.8471	3.4391	14.60	0.3650	2.227	8.224
5	13.9399	1.0000	4.0600	0.00	0.0000	1.886	9.708

Table XII. Transition Summary for 5 Sections and 4 Steps.

Sect	Z(M.N) ohms	V/C	Tooth Period	X (Slot) ohms	Tooth Ht. cm	L/Cell nH	C/Cell pF
1	40.9998	0.3400	1.3804	113.91	1.8032	5.549	3.301
2	30.1244	0.4627	1.8787	74.37	1.5724	4.077	4.492
3	18.9724	0.7347	2.9831	27.43	0.7249	2.568	7.133
4	13.9399	1.0000	4.0600	0.00	0.0000	1.886	9.708

Table XIII. Transition Summary for 4 Sections and 3 Steps.

Sect	Z(M,N) ohms	V/C	Tooth Period	X (Slot) ohms	Tooth Ht. cm	L/Cell nH	C/Cell pF
1	40.9998	0.3400	1.3804	113.91	1.8032	5.549	3.301
2	23.9068	0.5831	2.3674	49.57	1.2417	3.235	5.661
3	13.9399	1.0000	4.0600	0.00	0.0000	1.886	9.708

Table XIV. Transition Summary for 3 Sections and 2 Steps.

Sect	Z(M,N) ohms	V/C	Tooth Period	X (Slot) ohms	Tooth Ht. cm	L/Cell nH	C/Cell pF
1	40.9998	0.3400	1.3804	113.91	1.8032	5.549	3.301
2	13.9399	1.0000	4.0600	0.00	0.0000	1.886	9.708

Table XV. Transition Summary for 2 Sections and 1 Step.

The following Tables indicate the slow wave structure design for a Binomial transformer with 11 or less sections, and 10 or less steps. Table XVI is Pascal's Triangle for the successive Tables with 11 to 2 sections.

In the following sequence, Table XVII starts with the transition summary for 11 sections and 10 steps and continues to Table XXVI, the transition summary for 2 sections and 1 step. Bandwidth is 1.0, and columns indicate section (M), characteristic impedances Z(M,N) for steps (N) in the the transformer, the step (M), the ratio of slow wave phase velocity to velocity of light (V/C), tooth period D(M) and Height H(M), cavity reactance X(M), and inductance L(M) and Capacitance C(M) per unit cells.

1									
	1								
2		1							
	3		1						
6		4		1					
	10		5		1				
20		15		6		1			
	35		21		7		1		
70		56		28		8		1	
	126		84		36		9		1

Table XVI. Pascal's Triagle or Graves' Pyramid for 10 Steps and Normalized Binomial Coefficients with Bandwidth = 1.0.

Sect	Z(M,N) ohms	V/C	Tooth Period	X (Slot) ohms	Tooth Ht. cm	L/Cell nH	C/Cell pF
1	40.9998	0.3400	1.3804	113.91	1.8032	5.549	3.301
2	40.9135	0.3407	1.3833	113.61	1.8021	5.537	3.308
3	40.1448	0.3472	1.4098	110.91	1.7923	5.433	3.371
4	37.2122	0.3746	1.5209	100.50	1.7483	5.036	3.637
5	31.1759	0.4471	1.8154	78.36	1.6076	4.219	4.341
6	23.9068	0.5831	2.3674	49.57	1.2417	3.235	5.661
7	18.3325	0.7404	3.0872	24.29	0.6376	2.481	7.382
8	15.3587	0.9076	3.6849	8.50	0.2021	2.078	8.812
9	14.2368	0.9791	3.9753	1.85	0.0410	1.927	9.506
10	13.9693	0.9979	4.0515	0.18	0.0040	1.890	9.688
11	13.9399	1.0000	4.0600	0.00	0.0000	1.886	9.708

Table XVII. Transition Summary for 11 Sections and 10 Steps.

Sect	Z(M,N) ohms	V/C	Tooth Period	X (Slot) ohms	Tooth Ht. cm	L/Cell nH	C/Cell pF
1	40.9998	0.3400	1.3804	113.91	1.8032	5.549	3.301
2	40.8273	0.3414	1.3862	113.31	1.8011	5.525	3.315
3	39.4738	0.3531	1.4338	108.54	1.7832	5.342	3.428
4	35.0803	0.3974	1.6133	92.81	1.7081	4.748	3.858
5	27.7061	0.5031	2.0427	65.01	1.4730	3.750	4.885
6	20.6284	0.6758	2.7436	35.21	0.9304	2.792	6.560
7	16.2921	0.8556	3.4738	13.71	0.3405	2.205	8.307
8	14.4788	0.9628	3.9089	3.32	0.0750	1.960	9.347
9	13.9988	0.9958	4.0429	0.37	0.0081	1.894	9.668
10	13.9399	1.0000	4.0600	0.00	0.0000	1.886	9.708

Table XVIII. Transition Summary for 10 Sections and 9 Steps.

Sect	Z(M,N) ohms	V/C	Tooth Period	X (Slot) ohms	Tooth Ht. cm	L/Cell nH	C/Cell pF
1	40.9998	0.3400	1.3804	113.91	1.8032	5.549	3.301
2	40.6556	0.3429	1.3921	112.71	1.7989	5.502	3.329
3	38.3263	0.3637	1.4767	104.48	1.7664	5.189	3.531
4	32.1092	0.4341	1.7626	81.86	1.6355	4.346	4.215
5	23.9068	0.5831	2.3674	49.57	1.2417	3.235	5.661
6	17.7997	0.7832	3.1796	32.62	0.5622	2.409	7.603
7	14.9123	0.9348	3.7953	5.91	0.1370	2.018	9.075
8	14.0579	0.9916	4.0259	0.74	0.0162	1.902	9.627
9	13.9399	1.0000	4.0600	0.00	0.0000	1.886	9.708

Table XIX. Transition Summary for 9 Sections and 8 Steps.

Sect	Z(M,N) ohms	V/C	Tooth Period	X (Slot) ohms	Tooth Ht. cm	L/Cell nH	C/Cell pF
1	40.9998	0.3400	1.3804	113.91	1.8032	5.549	3.301
2	40.3144	0.3458	1.4039	111.51	1.7945	5.456	3.357
3	36.4363	0.3826	1.5533	97.71	1.7346	4.931	3.714
4	28.2961	0.4926	2.0001	67.32	1.5000	3.829	4.783
5	20.1983	0.6902	2.8020	33.23	0.8802	2.734	6.700
6	15.6858	0.8887	3.6081	10.34	0.2504	2.123	8.628
7	14.1769	0.9933	3.9921	1.48	0.0327	1.919	9.546
8	13.9399	1.0000	4.0600	0.00	0.0000	1.886	9.708

Table XX. Transition Summary for 8 Sections and 7 Steps.

Sect	Z(M,N) ohms	V/C	Tooth Period	X (Slot) ohms	Tooth Ht. cm	L/Cell nH	C/Cell pF
1	40.9998	0.3400	1.3804	113.91	1.8032	5.549	3.301
2	39.6405	0.3517	1.4277	109.13	1.7855	5.365	3.414
3	33.4913	0.4162	1.6899	86.99	1.6720	4.532	4.041
4	23.9068	0.5831	2.3674	49.77	1.2417	3.235	5.661
5	17.0652	0.8169	3.3165	17.84	0.4552	2.310	7.930
6	14.4179	0.9668	3.9254	2.95	0.0664	1.951	9.386
7	13.9399	1.0000	4.0600	0.00	0.0000	1.886	9.708

Table XXI. Transition Summary for 7 Sections and 6 Steps.

Sect	Z(M,N) ohms	V/C	Tooth Period	X (Slot) ohms	Tooth Ht. cm	L/Cell nH	C/Cell pF
1	40.9998	0.3400	1.3804	113.91	1.8032	5.549	3.301
2	38.3263	0.3637	1.4767	104.48	1.7664	5.189	3.531
3	29.2662	0.4763	1.9338	71.08	1.5403	3.961	4.624
4	19.5288	0.7138	2.8981	30.09	0.7975	2.643	6.930
5	14.9123	0.9348	3.7953	5.91	0.1370	2.018	9.075
6	13.9399	1.0000	4.0600	0.00	0.0000	1.886	9.708

Table XXII. Transition Summary for 6 Sections and 5 Steps.

Sect	Z(M,N) ohms	V/C	Tooth Period	X (Slot) ohms	Tooth Ht. cm	L/Cell nH	C/Cell pF
1	40.9998	0.3400	1.3804	113.91	1.8032	5.549	3.301
2	35.8273	0.3981	1.5797	95.52	1.7231	4.849	3.777
3	23.9068	0.5831	2.3674	49.57	1.2417	3.235	5.661
4	15.9525	0.8738	3.5478	11.85	0.2899	2.159	8.484
5	13.9399	1.0000	4.0600	0.00	0.0000	1.886	9.708

Table XXIII. Transition Summary for 5 Sections and 4 Steps.

Sect	Z(M,N) ohms	V/C	Tooth Period	X (Slot) ohms	Tooth Ht. cm	L/Cell nH	C/Cell pF
1	40.9998	0.3400	1.3804	113.91	1.8032	5.549	3.301
2	31.3074	0.4453	1.8078	78.86	1.6117	4.237	4 323
3	18.2555	0.7636	3.1002	23.91	0.6268	2.471	7.413
4	13.9399	1.0000	4.0600	0.00	0.0000	1.886	9.708

Table XXIV. Transition Summary for 4 Sections and 3 Steps.

Sect	Z(M,N) ohms	V/C	Tooth Period	X (Slot) ohms	Tooth Ht. cm	L/Cell nH	C/Cell pF
1	40.9998	0.3400	1.3804	113.91	1.8032	5.549	3.301
2	23.9068	0.5831	2.3674	49.57	1.2417	3.235	5.661
3	13.9399	1.0000	4.0600	0.00	0.0000	1.886	9.708

Table XXV. Transition Summary for 3 Sections and 2 Steps.

Sect	Z(M,N) ohms	V/C	Tooth Period	X (Slot) ohms	Tooth Ht. cm	L/Cell nH	C/Cell pF
1	40.9998	0.3400	1.3804	113.91	1.8032	5.549	3.301
2	13.9399	1.0000	4.0600	0.00	0.0000	1.886	9.708

Table XXVI. Transition Summary for 2 Sections and 1 Step.

VII. Conclusions

This paper describes the use of binomial coefficients and Chebyshev coefficients in the design of multisection quarter wave transformers. To reduce the sidelobe level of linear in-phase broadside antenna arrays, John Stone Stone [1] proposed that the sources have amplitudes proportional to the coefficients of binomial coefficients. After John Stone Stone's work, Charles Dolph [2] then applied Chebyshev coefficients to sidelobe reduction in antenna arrays in 1946, and Seymour Cohn [3] applied Dolph's results to electromagnetic filters in 1955. Examples with both binomial and Chebyshev coefficients are presented in slow to faster wave guides or transmission lines. The modifications to the work by Stone Stone, Dolph, and Cohn is derivation of characteristic impedances Z(M) for each unit cell (M) with binomial and Chebyshev coefficients, and then equating these Z(M) to the product of the ratio of the phase velocity of light C to the slow wave phase velocity V(M) in unit cell (M), C/V(M), given by

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$$Z(M) = Z(N+1)*C/V(M), \qquad (7.1)$$

where Z(N+1) is the characteristic impedance of the fast wave guide or line at the transformer output.

Although the slow wave velocity V(1) was found from circuit concepts for the input line, unit cell (1), more exact results will be developed with methods based on matching boundary conditions between wave in the radial cavities or teeth slots and the guiding region between inner smooth conductor and outer serrated conductor.

VIII. Acknowledgments

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IX. References

[1] John Stone Stone, US patents 1,643,323 and 1,715,433.

[2] Charles L. Dolph, "A Current Distribution for Broadside Arrays which Optimizes the Relationship between Beam Width and Side-Lobe Level," Proc. IRE, vol. 34, pp. 335-348, June 1946.

[3] Seymour N. Cohn, "Optimum Design of Stepped Transmission Line Transformers," IRE Trans. Microwave Theory & Techniques, vol. MTT-3, pp. 16-21, April 1955.