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TECHNICAL REPORT

ON THE ATTENUATION OF A PLANE-WAVE ELECTROMAGNETIC FIELD BY AN IMPERFECTLY CONDUCTING PROLATE SPHEROIDAL SHELL

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Synopsis

The shielding ratio, under quasi-static conditions, is determined for an imperfectly conducting prolate spheroidal shell, when the electric field of the incident plane wave is directed parallel to the major axis of the spheroid. General expressions are obtained for the fields within the cavity. It is shown that the shielding ratio of a spheroid having an eccentricity approaching unity is the same as that of the "corresponding" end-capped cylinder.

Introduction

A shield, to be effective, must be constructed in such a manner as to provide a highly conducting path for the flow of surface charge in all directions when the shield is acted upon by an incident electromagnetic field disturbance. A convenient measure of the effectiveness of a shield is the shielding ratio. This quantity is obtained by dividing the field (electric or magnetic) that would be present at a specified point in the absence of the shield by the field at the same point when the shield is present. It is understood that the shield encompasses this point.

A number of writers have directed attention to infinite body shields,[1-5] such as those comprising infinite cylinders, infinite parallel plates, etc. But the question naturally arises as to the precise value of such analyses, when all shields used in practical situations must be of finite dimensions. One point that should be mentioned is that infinite body shielding theory necessarily neglects all resonances that may exist on the exterior of the shield. Nevertheless, such studies are warranted because the general significance of partial reflection and transmission, and skin effect are correctly represented. Furthermore, the phenomenology occurring within a shield with regard to the decay rates of the electric and magnetic fields, their time histories and their spacial distribution when the incident electromagnetic field is a non-repetitive pulse, is of profound theoretical interest. But to avoid blinding the reader to the facts it must be said again that finite shielding theory requires further development before the relevance of infinite body shielding theory to practice becomes known.

The simplest finite shield that can be treated analytically is the imperfectly conducting spherical shell. The shielding ratio for this structure was determined exactly and reported in a recent paper.[6] All possible exterior resonances and interior modes were taken into account in the theory.

The next more complicated finite shield configuration is the spheroidal shell. The prolate spheroid with eccentricity somewhat less than unity is chosen for investigation in the present paper, since it approximates the geometry of a short, fat cylindrical shield with end caps. The long wave or quasi-static approximation is made in the theory because spheroidal wave functions for complex arguments have not been tabulated. Such functions are required for the field representations in the shell region in a completely general rigorous shielding theory. Spheroidal wave functions for complex arguments in the present instance reduce to Legendre polynomials of the first and second kind when the long wave approximation is made. But happily the low frequency case is the one of real interest because at the higher frequencies field penetration into the interior of the shield is restricted by the phenomenon of skin effect. Of course, reflection at the outer boundary of a highly conducting shield is always a most important shielding mechanism.

Theoretical Considerations

It is assumed at the outset that the prolate spheroidal shell is irradiated by a plane-wave electromagnetic field, with the electric field directed parallel to the major axis of the shell, as shown in the figure. Inasmuch as the long wave approximation is made in the theory, it follows that the fields that penetrate the shield are azimuthally symmetric.[7]

Under quasi-static conditions, Maxwell's equations become

$$\text{curl}\ \vec{H} = \sigma\vec{E} \tag{1}$$

$$\text{curl}\ \vec{E} = 0 \tag{2}$$

where the notation used is standard. Expressed in prolate spheroidal coordinates solutions of (1) and (2) in the shell region may be written

$$H_{\phi}^{s}(\eta,\xi) = \sum_{m=1}^{\infty} \left[A_m P_m'(\xi) + B_m Q_m'(\xi)\right] P_m(\eta) \tag{3}$$

$$E_{\eta}^{s}(\eta,\xi) = \frac{-1}{\sigma F\sqrt{\xi^2 - \eta^2}} \sum_{m=1}^{\infty} \frac{d}{d\xi}\left\{\sqrt{\xi^2 - 1}\left[A_m P_m'(\xi) + B_m Q_m'(\xi)\right]\right\} P_m'(\eta) \tag{4}$$

$$E_{\xi}^{s}(\eta,\xi) = \frac{1}{\sigma F\sqrt{\xi^2 - \eta^2}} \sum_{m=1}^{\infty} \left[A_m P_m'(\xi) + B_m Q_m'(\xi)\right] \frac{d}{d\eta}\left\{\sqrt{1 - \eta^2} P_m'(\eta)\right\} \tag{5}$$

The prolate spheroidal coordinates (η,ξ,ϕ) are consistent with the Flammer notation.[8] Here F is the semifocal length of the coordinate system. The P_m' and Q_m' functions are, respectively, associated Legendre functions of the first and second kinds. The superscript s denotes the field components within the shell region.

If the cavity region is considered to be a pure dielectric then the fields within the cavity are (for suppressed time dependence $e^{j\omega t}$)

$$H_{\phi}^{c}(\eta,\xi) = \sum_{n=0}^{\infty} C_n R_{1n}^{(1)}(c,\xi) S_{1n}(c,\eta) \tag{6}$$

$$E_\eta^c(\eta,\xi) = j\,\frac{\zeta}{c}\,\frac{1}{\sqrt{\xi^2-\eta^2}}\sum_{n=0}^{\infty} C_n S_{1n}(c,\eta)\,\frac{d}{d\xi}\left[\sqrt{\xi^2-1}\,R_{1n}^{(1)}(c,\xi)\right] \tag{7}$$

$$E_\xi^c(\eta,\xi) = -j\,\frac{\zeta}{c}\,\frac{1}{\sqrt{\xi^2-\eta^2}}\sum_{n=0}^{\infty} C_n R_{1n}^{(1)}(c,\xi)\,\frac{d}{d\eta}\left[\sqrt{1-\eta^2}\,S_{1n}(c,\eta)\right] \tag{8}$$

where $c = k_oF$ and ζ is the characteristic wave impedance of the cavity medium.[9] It should be emphasized that (6)-(8) represent the general solution to Maxwell's equations under the assumed symmetry.

The expansion constants in the foregoing field expressions may be obtained by satisfying the boundary conditions on the fields at the inner and outer surfaces of the shell. From the definition of the $S_{1n}(c,\eta)$ functions it is seen that

$$S_{1n}(c,\eta) \simeq P_n'(\eta) \tag{9}$$

when $c \ll 1$. Matching the tangential electric field across the outer surface of the shell and the tangential electric and magnetic fields across the inner surface yields

$$C_n = -j\,\frac{c}{\zeta}\,\frac{\left(\xi_1^2-1\right)^{-1/2} U_n}{\left[\sqrt{\xi_1^2-1}\,R_{1n}^{(1)}(c,\xi_1)\right]'\left[P_n'(\xi_1)Q_n(\xi_2) - P_n(\xi_2)Q_n'(\xi_1)\right]} \tag{10}$$

where ξ_2 and ξ_1 determine, respectively, the outer and inner surfaces of the shell,

$$\left[\sqrt{\xi_1^2-1}\,R_{1n}^{(1)}(c,\xi_1)\right]' \equiv \frac{d}{d\xi_1}\left[\sqrt{\xi_1^2-1}\,R_{1n}^{(1)}(c,\xi_1)\right] \tag{11}$$

and

$$U_n = \frac{2n+1}{2}\frac{(n-1)!}{(n+1)!}\int_{-1}^{1} d\eta \sqrt{\xi_2^2 - \eta^2}\, P_n'(\eta) E_\eta^t(\eta,\xi_2) \ . \tag{12}$$

Also for convenience $E_\eta^t(\eta,\xi_2)$, the total electric field at the outer surface, is considered to be symmetric in η about $\eta = 0$. Then $U_n = 0$ for even n. In principle the solution for the cavity fields have now been obtained.

It is of most interest to consider prolate spheroidal shell such that $h \geq 10a$, where h is the half-height and a is the midsection radius. Under this imposition the shell looks very much like a "capped" cylindrical tube. Using an internal impedance per unit length, the total tangential electric field may be written

$$E_\eta^t(\eta,\xi_2) = z^i(\eta) I(\eta) \tag{13}$$

where $z^i(\eta)$ is the internal impedance per unit length which is a function of η since the shell varies in cross section and wall thickness. Here $I(\eta)$ represents the total axial current induced in the shell by the incident electromagnetic field.

In the Appendix an expression is derived for the internal impedance per unit length of a very thin cylindrical tube. It is

$$z^i = \frac{1}{\pi}\frac{1}{\sigma\left(a_2^2 - a_1^2\right)} \tag{14}$$

where a_1 and a_2 are the inner and outer radii, respectively, of the tube. Since the corresponding radii for the prolate spheroidal shell vary with η, then (14) yields

$$z^i(\eta) = z^i(0)(1 - \eta^2)^{-1} \tag{15}$$

where

$$z^i(0) = \frac{1}{\pi} \frac{1}{\sigma\left(\xi_2^2 - \xi_1^2\right)} \tag{16}$$

provided (16) is not large as is the case for a highly conducting shield, i.e., $\left|z^i(0)I_\eta(0)\right| \ll E_z^{inc}$, the current induced in the shell is essentially the same as would be induced in a perfectly conducting structure. According to Taylor[10] the current distribution induced in a perfectly conducting prolate spheroid is

$$I(\eta) = -j\pi \frac{E_z^{inc}(1 - \eta^2)^{1/2}}{k_o\zeta} \sum_{\ell=1,3,5,\ldots}^{\infty} \frac{A_{1\ell}S_{1\ell}(c,\eta)}{\left[\left(\xi_2^2 - 1\right)^{1/2} R_{1\ell}^{(4)}(c,\xi_2)\right]'} \tag{17}$$

where for odd ℓ

$$A_{1\ell} = j^\ell \frac{S_{1\ell}(c,0)}{\sum_{n=0,2,\ldots}^{\infty} \frac{(2 + n)!}{n!(2n + 3)!}\left[d_n^{1\ell}(c)\right]^2} . \tag{18}$$

Therefore for $c \ll 1$ and $\xi_2 \simeq 1$

$$U_n = -j\pi \frac{E_z^{inc} z^i(0)}{k_o\zeta} \frac{A_{1n}}{\left[\sqrt{\xi_2^2 - 1} R_{1n}^{(4)}(c,\xi_2)\right]'} . \tag{19}$$

With the use of (19) in (10) one may readily determine the fields throughout the cavity within the shield.

It is of particular interest to obtain the "on axis" fields. Since $R_{1n}^{(1)}(c,1) = 0$ for $n > 0$, then

$$E_\xi^c(\eta,1) = H_\phi^c(\eta,1) = 0 \ . \tag{20}$$

The other component of the electric field may be obtained using

$$P_n^1(\xi_1)Q_n(\xi_2) - P_n(\xi_2)Q_n'(\xi_1) \simeq \left(\xi_1^2 - 1\right)^{-1/2} \tag{21}$$

and $\xi_1 \simeq 1$ in (10). It is

$$E^c(\eta,1) \simeq z^i(\eta)I(\eta) \ . \tag{22}$$

This result confirms those of Harrison and King,[11,12] who actually assume that (22) is true.

Conclusion

The shielding ratio afforded by highly conducting prolate spheroidal shells may be determined easily when the long wave approximation is imposed. It is fortunate that the quasi-static case is of most interest, because at higher frequencies skin effect becomes an important shielding mechanism. It is of interest to observe that the results of Harrison and King[11,12] who employed antenna theory in part to solve the problem of the isolated cylindrical shield of finite length are consistent with the results obtained in this paper.

Recently the authors succeeded in obtaining precisely the current in a rather fat cylindrical receiving and scattering antenna irradiated by a plane-wave field, using both the Fourier series method, and an open-ended scheme for iterating the appropriate integral equation for the current. It is hoped that from this knowledge it will be possible to obtain with accuracy the interior fields in the tube. Discussion of this interesting subject is reserved for a later paper.

APPENDIX

Internal Impedance of a Thin Cylinder

The following unrestricted formula for the internal impedance per unit length of a cylindrical tube was derived by King:[7,13]

$$z^i = \frac{\gamma}{2\pi a_2 \sigma}\left[\frac{J_o(\gamma a_2)N_1(\gamma a_1) - N_o(\gamma a_2)J_1(\gamma a_1)}{J_1(\gamma a_2)N_1(\gamma a_1) - N_1(\gamma a_2)J_1(\gamma a_1)}\right]$$

where $\gamma = \frac{1}{\delta}(1 - j)$ with δ as the skin depth, J_n is the usual Bessel function, and N_n is the Neuman function. Consider that the cylindrical tube is thin, i.e., γa_1, $\gamma a_2 \ll 1$, then

$$J_o(\gamma a_2)N_1(\gamma a_1) - N_o(\gamma a_2)J_1(\gamma a_1) \simeq -\frac{2}{\pi}(\gamma a_1)^{-1}$$

and

$$J_1(\gamma a_2)N_1(\gamma a_1) - N_1(\gamma a_2)J_1(\gamma a_1) \simeq -\frac{1}{\pi}\left(\frac{a_2}{a_1} - \frac{a_1}{a_2}\right).$$

Using the above formulas in the expression for z^i yields

$$z^i = \frac{1}{\pi}\frac{1}{\sigma\left(a_2^2 - a_1^2\right)}.$$

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