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Electromagnetic Penetrability of Perfectly Conducting Bodies Containing an Aperture

M. I. Sancer and A. D. Varvatsis
Northrop Corporate Laboratories
Pasadena, California

Abstract

Methods are presented to calculate the electromagnetic penetrability due to an aperture in a perfectly conducting sphere and cylinder. The formulations for each geometry are quite different. Dual series equations result from the analysis of the spherical problem, while the analysis of the cylindrical problem leads to an integral equation. These analyses have the potential, for low frequencies, to lead to quantitative error bounds for a class of shielding problems. This report presents the low frequency limit to these two particular problems and outlines the procedure for obtaining low frequency corrections.

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I. Introduction

The general problem treated in this work is the low frequency electromagnetic penetrability of finite structures which contain an aperture. This problem is necessarily difficult because the simplest canonical problem, the infinite perfectly conducting planar sheet with a circular aperture, has a very complex solution [1], [2]. This planar problem has been extensively treated by various approximation techniques [3], [4], [5], [6] and they yield a more tractable solution. We have discussed the planar sheet problem, not only to indicate the limited state of the art, but because the planar solution is used in the approximate analysis of the penetrability of finite structures which contain an aperture. This approximate analysis uses the fact that when the internal Green's function of a structure is known, or equivalently the modes of the closed structure, then one can calculate the fields that penetrate into the structure if one has some limited knowledge of the fields in the aperture. The approximation in this analysis is that the fields in the aperture are the same as those that would be present in that same aperture if it were cut in an infinite planar sheet rather than in the structure of interest. The usefulness of this method is that it is a tractable procedure for obtaining solutions to very difficult problems; however, its limitation is that there is no self-contained procedure for placing quantitative bounds on its accuracy. If one had a cononical solution to compare to the one obtained by the preceding procedure, then one could better assess its domain of validity. Providing solutions whose accuracy is known is one of the two purposes for undertaking this present work. The other purpose is to examine these solutions and infer from them both qualitative and quantitative shielding effectiveness of structures as a function of their size, the aperture size, and wavelength.

In this work we consider a plane wave incident on both a perfectly conducting sphere that has a circular aperture cut in it and on a circular cylinder that contains an infinite slot. For the cylindrical case the incident magnetic field is parallel to the axis of the cylinder. We treat the spherical problem and cylindrical problem by two different methods. We treat the spherical problem by using a method that has previously been used by Sommerfeld in analyzing a scalar cylindrical and a scalar spherical scattering problem [7]. This method allows us to solve for the electromagnetic field that penetrates to the center of the sphere in a manner that allows us to place quantitative bounds on the accuracy of our solution. It suffers, however, from a short coming not mentioned by Sommerfeld. In following Sommerfeld's procedure, there appears a step where it seems a parameter should have been introduced which should be analytically determined. Instead, this parameter has been effectively set equal to unity. In our vector problem we explicitly introduce two unknown parameters and carry through our solution in terms of these parameters. The problem remains to determine these parameters. If we can justify setting these parameters equal to unity then we have our desired procedure and solution. We have set these parameters equal to unity and have considered the static limit. The resulting expressions show that for apertures that are only moderately small, very little of the incident field penetrates into the sphere. The results also show that the shielding is four times more effective for the electric field than it is for the magnetic field.

In analyzing the cylindrical problem we use an integral equation approach that avoids the introduction of the unknown parameters. Using this approach we can show that in the static limit the cylindrical structure allows complete penetration of the incident magnetic field.

II. Scattering by a Sphere With a Circular Aperture

Consider a perfectly conducting sphere with a circular aperture oriented in the coordinate system that will now be described. The origin of the coordinate system is at the center of the sphere described as r = a and the aperture occupies the region $r = a$, $\pi \geq \theta \geq \alpha$. We consider a plane wave incident on this structure given by

$$\underline{E}_i = \hat{a}_x e^{ikz} \quad , \quad \underline{H}_i = Y_o \hat{a}_y e^{ikz} \tag{1}$$

where $Y_o = (\varepsilon_o/\mu_o)^{\frac{1}{2}}$, $k = \omega(\mu_o\varepsilon_o)^{\frac{1}{2}}$, and the suppressed time dependence is $e^{-i\omega t}$. This situation is depicted in figure 1. We will analyze this scattering problem by introducing the Debye potentials u and v. That is

$$\underline{E} = \nabla \times \nabla \times (\underline{r}v) + i\omega\mu_o \nabla \times (\underline{r}u) \tag{2a}$$

and

$$\underline{H} = \nabla \times \nabla \times (\underline{r}u) - i\omega\varepsilon_o \nabla \times (\underline{r}v) \tag{2b}$$

Next we determine u_i and v_i so that when they are substituted into (2) they yield the incident field given by (1). The procedure for obtaining these quantities is as follows. First we note that the radial component of $\underline{E}$ is given in terms of v alone, while the radial component of $\underline{H}$ is given in terms of u alone. That is

$$\underline{r} \cdot \underline{E}_i = r\left(\frac{\partial^2}{\partial r^2} + k^2\right)(rv_i) = r \sin\theta \cos\phi e^{ikr\cos\theta} \tag{3a}$$

and

$$\underline{r} \cdot \underline{H}_i = r\left(\frac{\partial^2}{\partial r^2} + k^2\right)(ru_i) = Y_o r \sin\theta \sin\phi e^{ikr\cos\theta} \tag{3b}$$

Using standard relationships we obtain

$$r\left(\frac{\partial^2}{\partial r^2} + k^2\right)(rv_i) = \frac{1}{ik}\cos\phi \sum_{n=1}^{\infty} (i)^n(2n+1)j_n(kr)P_n^1(\cos\theta) \tag{4a}$$

and

$$r\left(\frac{\partial^2}{\partial r^2} + k^2\right)(ru_i) = \frac{Y_o}{ik} \sin\phi \sum_{n=1}^{\infty} (i)^n (2n+1) j_n(kr) P_n^1(\cos\theta) \tag{4b}$$

From (4) it follows that

$$u_i = \frac{Y_o}{ik} \sin\phi \sum_{n=1}^{\infty} (i)^n \frac{2n+1}{n(n+1)} j_n(kr) P_n^1(\cos\theta) \tag{5a}$$

and

$$v_i = \frac{1}{ik} \cos\phi \sum_{n=1}^{\infty} (i)^n \frac{2n+1}{n(n+1)} j_n(kr) P_n^1(\cos\theta) \tag{5b}$$

We will now argue that the solution to this scattering problem can be obtained by considering two decoupled scalar scattering problems. If u and v satisfy

$$(\nabla^2 + k^2)u = 0 \tag{6a}$$

$$(\nabla^2 + k^2)v = 0 \tag{6b}$$

then $\underline{E}$ and $\underline{H}$ given by (2) satisfy Maxwell's equations. Furthermore, if u satisfies the boundary conditions

$$\left.\begin{array}{l} u = 0 \text{ on conductor} \\ u \text{ and } \frac{\partial u}{\partial r} \text{ continuous through spherical aperture} \end{array}\right\} \tag{7a}$$

and v satisfies the boundary conditions

$$\left.\begin{array}{l} \frac{\partial (rv)}{\partial r} = 0 \text{ on conductor} \\ v \text{ and } \frac{\partial v}{\partial r} \text{ continuous through spherical aperture} \end{array}\right\} \tag{7b}$$

then the tangential component of $\underline{E}$ vanishes on the conducting portion of the sphere and the tangential components of both $\underline{E}$ and $\underline{H}$ are continuous through the aperture. Thus, we can satisfy Maxwell's equations, the perfect conductor boundary conditions, and the continuity boundary conditions by considering (6a) and (7a) and (6b) and (7b) as two uncoupled independent scalar scattering problems with the corresponding incident fields given by (5a) and (5b). If edge conditions were also satisfied then we could be assured that the solution obtained through the analysis of the two uncoupled scalar problems is unique. We will proceed with the analysis of the two scalar problems with the understanding that edge conditions should be verified.

We now consider the problem posed by (5a), (6a) and (7a). Let

$$u = u_i + u_s^< \qquad r \le a \tag{8}$$

and

$$u = u_i + u_s^> \qquad r \ge a \tag{9}$$

where

$$u_s^< = \sum_{n=o}^{\infty} \frac{j_n(kr)}{j_n(ka)} \left\{a_{no} P_n(\cos\theta) + \sum_{m=1}^{n} (a_{nm} \cos m\phi + b_{nm} \sin m\phi) P_n^m(\cos\theta)\right\} \tag{10}$$

and

$$u_s^> = \sum_{n=o}^{\infty} \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \left\{c_{no} P_n(\cos\theta) + \sum_{m=1}^{n} (c_{nm} \cos m\phi + d_{nm} \sin m\phi) P_n^m(\cos\theta)\right\} \;, \tag{11}$$

then application of (7a) leads to

$$\sum_{n=o}^{\infty} \left\{a_{no} P_n(\cos\theta) + \sum_{m=1}^{n} (a_{nm} \cos m\phi + b_{nm} \sin m\phi) P_n^m(\cos\theta) = - u_i(r=a) \qquad 0 < \theta < \alpha \tag{12}$$

and

$$\sum_{n=o}^{\infty} W_n(ka)\{a_{no}P_n(\cos\theta) + \sum_{m=1}^{n} (a_{nm}\cos m\phi + b_{nm}\sin m\phi)P_n^m(\cos\theta) = 0$$

$$\alpha < \theta < \pi \qquad (13)$$

where

$$W'_n(ka) = \frac{kj'_n(ka)}{j_n(ka)} - \frac{kh_n^{(1)'}(ka)}{h_n^{(1)}(ka)} = \frac{ik}{j_n(ka)h_n^{(1)}(ka)(ka)^2} \qquad (14)$$

and the prime associated with the Bessel and Hankel function indicate differentiation with respect to ka.

We have made use of the fact $u_s^< = u_s^>$ for all θ in order to conclude that $a_{mn} = c_{mn}$ and $b_{mn} = d_{mn}$. Next we use the fact that (12) is valid for all ϕ. Since the ϕ dependence of u_i is $\sin\phi$ we conclude that the only non-zero expansion coefficients are b_{n1}. These are still an infinite set of unknown quantities; however, we will be primarily interested in the electromagnetic field at the center of the sphere and this requires a knowledge of only b_{11}. In order to solve for b_{11} we form the following quadratic form

$$Q_N^u = \int_o^{\alpha} (\sum_{n=1}^{N} b_{n1}\sin\phi\, P_n^1(\cos\theta) + u_i(a))^2 \sin\theta\, d\theta$$

$$+ \int_{\alpha}^{\pi} (\sum_{n=1}^{N} \tau_u W_n b_{n1} P_n^1(\cos\theta))^2 \sin\theta\, d\theta \qquad (15)$$

where

$$W_n = aW'_n = \frac{i}{j_n(ka)h_n^{(1)}(ka)(ka)}$$

We obtain N equations for $b_{11}, \ldots, b_{N1}$ from the requirement that these coefficients should be chosen so that Q_N^u is a minimum. Explicitly these equations are

$$\frac{\partial Q_N^u}{\partial b_{m1}} = 0 \qquad m = 1,\ldots,N \qquad (16)$$

and the approximate solution for b_{11} is a function of N the validity of which is assumed to improve with increasing N. The reasons for considering the minimization of Q_N^u as a criterion for obtaining the b_{n1}'s are the following. If the sphere contained no aperture, $\alpha = \pi$, or if the conducting part of the sphere were removed, $\alpha = 0$, then (16) would yield the correct solutions for the b_{n1}'s for any N. As N approaches infinity we know from (12) and (13) that each integrand in the two integrals comprising Q_N^u approach zero. We also note that each of the terms in (15) is positive so that minimizing the sum guarantees that each term is small. The procedure of introducing Q_N^u for our spherical electromagnetic scattering problem is essentially the same as that used by Sommerfeld in analysis of a scalar cylindrical and a scalar spherical scattering problem [7]. If we followed his analysis more closely, we would set the factor $\tau_u = 1$. His arguments for doing this are not clear and we carry τ_u as an unknown factor until the end of the analysis. Since the edge conditions together with the boundary conditions we have already imposed guarantee that our solution is unique, it is believed that the imposition of the edge conditions should determine τ_u.

Before we perform the operation indicated in (16) we modify (15) by replacing u_i by an expression which is the best possible least square approximation to u_i which contains N terms. It is assumed that no more accurate expression for u_i is justified by our approximate analysis. We now write

$$Q_N^u = \int_0^\alpha \left(\sum_{n=1}^N (b_{n1} + q_n) P_n^1(\cos\theta)\right)^2 \sin\theta\, d\theta + \tau_u^2 \int_\alpha^\pi \left(\sum_{n=1}^N W_n b_{n1} P_n^1(\cos\theta)\right)^2 \sin\theta\, d\theta \quad (17)$$

where

$$u_i \approx \sum_{n=1}^N q_n P_n^1(\cos\theta) \quad , \quad q_n = \frac{Y_o}{ik} (i)^n \frac{2n+1}{n(n+1)} j_n(ka)$$

Performing the differentiation indicated in (16) we obtain

$$\sum_{n=1}^N (b_{n1} + q_n) \int_0^\alpha P_n^1(\cos\theta) P_m^1(\cos\theta) \sin\theta\, d\theta$$

$$+ \tau_u^2 \sum_{n=1}^N W_n W_m b_{n1} \int_\alpha^\pi P_n^1(\cos\theta) P_m^1(\cos\theta) \sin\theta\, d\theta = 0 \quad (18)$$

Next we define

$$h_{nm} = \int_{\alpha}^{\pi} P_n^1(\cos\theta) P_m^1(\cos\theta) \sin\theta d\theta \quad , \tag{19}$$

consequently

$$\int_{0}^{\alpha} P_n^1(\cos\theta) P_m^1(\cos\theta) \sin\theta d\theta = \frac{2}{2m+1} \frac{(m+1)!}{(m-1)!} \delta_{nm} - h_{nm} \tag{20}$$

Substituting (19) and (20) into (18), we obtain

$$\frac{2}{2m+1} \frac{(m+1)!}{(m-1)!} (b_{m1} + q_m) - \sum_{n=1}^{N} (b_{n1} + q_n) h_{nm} + \tau_u^2 \sum_{n=1}^{N} W_n W_m b_{n1} h_{nm} = 0$$

$$m = 1, \ldots, N \tag{21}$$

This is the equation that enables us to algebraically solve for $b_{11}^{(N)}$. Before presenting the computational results for b_{11}, we will derive the equation corresponding to (21) for the Debye potential v.

Let

$$v = v_i + v_s^{<} \qquad r \leq a \tag{22}$$

and

$$v = v_i + v_s^{>} \qquad r \geq a \tag{23}$$

where

$$v_s^{<} = \sum_{n=o}^{\infty} \frac{j_n(kr)}{j_n(ka)} \left\{ e_{no} P_n(\cos\theta) + \sum_{m=1}^{n} (e_{nm} \cos m\phi + f_{nm} \sin m\phi) P_n^m(\cos\theta) \right\} \tag{24}$$

and

$$v_s^{>} = \sum_{n=o}^{\infty} \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \left\{ g_{no} P_n(\cos\theta) + \sum_{m=1}^{n} (g_{nm} \cos m\phi + h_{nm} \sin m\phi) P_n^m(\cos\theta) \right\} \tag{25}$$

Application of the boundary conditions (7b) leads to the equations

$$\sum_{n=0}^{\infty} r_n \{e_{no} P_n(\cos\theta) + \sum_{m=1}^{n} (e_{nm} \cos m\phi + f_{nm} \sin m\phi) P_n^m(\cos\theta)$$

$$= [- v_i - a \frac{\partial v_i}{\partial r}]_{r=a} \qquad 0 < \theta < \alpha \tag{26}$$

and

$$\sum_{n=0}^{\infty} \beta_n \{e_{no} P_n(\cos\theta) + \sum_{m=1}^{n} (e_{nm} \cos m\phi + f_{nm} \sin m\phi) P_n^m(\cos\theta) = 0$$

$$\alpha < \theta < \pi \tag{27}$$

It is not necessary to separately solve for g_{nm} and h_{nm} as they are simply related to e_{nm} and f_{nm} by

$$r_n e_{nm} = s_n g_{nm} \quad , \quad r_n f_{nm} = s_n h_{nm} \tag{28}$$

The quantities remaining to be defined in (26) through (28) are

$$r_n = 1 + \frac{ka}{j_n(ka)} \frac{dj_n(ka)}{d(ka)} \tag{29a}$$

$$s_n = 1 + \frac{ka}{h_n^{(1)}(ka)} \frac{dh_n^{(1)}(ka)}{d(ka)} \tag{29b}$$

$$\beta_n = 1 - \frac{r_n}{s_n} \tag{29c}$$

We now note that for our particular incident field, the only ϕ dependence in v_i is $\cos\phi$, so that from the fact that (26) is valid for all ϕ we can conclude that the only non-zero expansion coefficients to be determined are the e_{n1}'s.

Finally we form

$$Q_N^v = \int_0^{\alpha} \left(\sum_{n=1}^{N} (r_n e_{n1} + t_n) P_n^1(\cos\theta)\right)^2 \sin\theta d\theta + \tau_v^2 \int_{\alpha}^{\pi} \left(\sum_{n=1}^{N} \beta_n e_{n1} P_n^1(\cos\theta)\right)^2 \sin\theta d\theta \tag{30}$$

where

$$\left[v_i + a \frac{\partial v_i}{\partial r}\right]_{r=a} = \cos\phi \sum_{n=1}^{\infty} t_n P_n^1(\cos\theta)$$

so that

$$t_n = \frac{1}{ik} (i)^n \frac{2n+1}{n(n+1)} \left(j_n(ka) + ka \frac{dj_n(ka)}{d(ka)}\right) \tag{31}$$

Again τ_v^2 serves the same role as did τ_u^2. Choosing e_{m1} to minimize (30) and using (19) and (20) we obtain

$$\frac{2}{2m+1} \frac{(m+1)!}{(m-1)!} (r_m^2 e_{m1} + r_m t_m) - \sum_{n=1}^{N} (r_n e_{n1} + t_n) r_m h_{nm} + \tau_v^2 \sum_{n=1}^{N} \beta_n \beta_m e_{n1} h_{nm} = 0$$

$$m = 1,\ldots,N \tag{32}$$

This equation serves the role for the v potential that (21) serves for the u potential. As mentioned before it is only the b_{11} and e_{11} terms which contribute to the scattered fields at the center of the sphere. For this computation

$$u_s^< = b_{11} \frac{j_1(kr)}{j_1(ka)} \sin\phi \sin\theta + \text{non-contributing terms} \tag{33}$$

$$v_s^< = e_{11} \frac{j_1(kr)}{j_1(ka)} \cos\phi \sin\theta + \text{non-contributing terms} \tag{34}$$

For $r \to 0$, we rewrite (2) as

$$H_{rs} \sim \frac{\partial^2}{\partial r^2} (r u_s^<) \;,\; H_{\theta s} \sim \frac{1}{r} \frac{\partial^2}{\partial r \partial\theta} (r u_s^<) \;,\; H_{\phi s} \sim \frac{1}{r \sin\theta} \frac{\partial^2}{\partial r \partial\phi} (r u_s^<) \tag{35}$$

$$E_{rs} \sim \frac{\partial^2}{\partial r^2} (r v_s^<) \;,\; E_{\theta s} \sim \frac{1}{r} \frac{\partial^2}{\partial r \partial\theta} (r v_s^<) \;,\; E_{\phi s} \sim \frac{1}{r \sin\theta} \frac{\partial^2}{\partial r \partial\phi} (r v_s^<) \tag{36}$$

Using (35) and (36) at the origin we find that

$$\overline{E}_T = \overline{E}_i + \overline{E}_s = (1 + \frac{2}{3} \frac{ke_{11}}{j_1(ka)})\overline{E}_i \tag{37}$$

and

$$\overline{H}_T = \overline{H}_i + \overline{H}_s = (1 + \frac{2}{3} Z_o \frac{kb_{11}}{j_1(ka)})\overline{H}_i \tag{38}$$

Equations (37) and (38) are exact but we must use approximations to determine b_{11} and e_{11}. To approximate these quantities we return to (21) and (32). If we set N = 1 in each of these equations then we obtain our first approximations for these quantities to be

$$b_{11}^{(1)} = \frac{3Y_o j_1(ka)}{2k(-1 + \tau_u^2\gamma)} \tag{39}$$

and

$$e_{11}^{(1)} = -\frac{3j_1(ka)}{2k(1 + \tau_v^2\delta)} \tag{40}$$

where

$$\gamma = \frac{W_1^2 h_{11}}{h_{11} - 4/3} \tag{41}$$

and

$$\delta = \frac{\beta_1^2 h_{11}}{r_1^2(4/3 - h_{11})} \tag{42}$$

The quantities W_1, r_1, β_1 and h_{11} have been defined in (15), (29a), (29c) and (19). Substituting (39) and (40) into (37) and (38) we find

$$\overline{E}_T^{(1)} = \frac{\tau_v^2\delta}{1 + \tau_v^2\delta}\overline{E}_i \tag{43}$$

and

$$\overline{H}_T^{(1)} = \frac{\tau_u^2\gamma}{-1 + \tau_u^2\gamma}\overline{H}_i \tag{44}$$

If we could justify setting $\tau_u = \tau_v = 1$, then (43) and (44) would be our desired result. For the sake of presenting some numerical results that are easy to obtain and are of potential usefulness, we set τ_n and τ_v equal to unity as suggested by Sommerfeld's analysi [7] and consider the limit $ka \to 0$, to obtain

$$\overline{E}_T^{(1)} \sim \frac{9}{16}\, \varepsilon^4 \overline{E}_i \tag{45}$$

and

$$\overline{H}_T^{(1)} \sim \frac{9}{4}\, \varepsilon^4 \overline{H}_i \tag{46}$$

where $\varepsilon = \pi - \alpha$ and ε is assumed to be small for the validity of (45) and (46). This shows that the shielding of the electric field is four times greater than that of the magnetic field.

In future work it is intended to obtain the next corrections to (45) and (46) by letting $N = 2$.

III. Scattering by a Cylinder With an Axial Aperture

We consider a plane wave incident on a cylinder with the magnetic field directed along the axis of the cylinder. We introduce the natural coordinate system (r, θ, z) with the origin at the center of the cylinder so that it is described as $r = a$. The cylinder is perfectly conducting but it contains an aperture corresponding to the region $\theta_1 \le \theta \le \theta_2$. This scattering problem is depicted in figure 2. The incident field is

$$H_{zi} = H_i e^{ikx}$$

and the total magnetic field satisfies the equation

$$(\nabla^2 + k^2)H_z = 0$$

where

$$\nabla^2 = L_r + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \quad , \quad L_r = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right)$$

with the requirement that the scattered magnetic field satisfies the radiation condition at infinity. We now introduce the two Green's function appropriate to this problem. The external Green's function G_e satisfies the equation

$$(\nabla^2 + k^2)G_{e,I} = -\frac{1}{r'}\delta(r - r')\delta(\theta - \theta') = -\delta(\underline{r} - \underline{r}') \qquad \begin{matrix} e \to r' \ge a \\ I \to r' \le a \end{matrix}$$

the boundary condition $(\partial G_e)/(\partial r) = 0$ on the entire cylindrical surface and the radiation condition at infinity. The internal Green's function G_I also satisfies the preceeding equation and the boundary condition $(\partial G_I)/(\partial r) = 0$, but the additional boundary condition is that it should be finite at the origin. It is straightforward using standard procedures to solve for G_e and G_I. The resulting expressions are

$$G_e(\underline{r}|\underline{r}') = \frac{1}{4i}\sum_{n=-\infty}^{\infty} \frac{H_n^{(1)}(k_o r')}{H_n^{(1)'}(ka)} \{J_n'(kaH_n^{(1)}(kr) - J_n(kr)H_n^{(1)'}(ka)\} e^{in(\theta-\theta')}$$

$$a \le r \le r'$$

$$G_e(\underline{r}|\underline{r}') = \frac{1}{4i}\sum_{n=-\infty}^{\infty} \frac{H_n^{(1)}(kr)}{H_n^{(1)'}(ka)} \{J_n'(ka)H_n^{(1)}(kr') - J_n(kr')H_n^{(1)'}(ka)\} e^{in(\theta-\theta')}$$

$$r' \le r < \infty$$

$$G_I(\underline{r}|\underline{r}') = \frac{1}{4i}\sum_{n=-\infty}^{\infty} \frac{J_n(kr)}{J_n'(ka)} \{H_n^{(1)'}(ka)J_n(kr') - H_n^{(1)}(kr')J_n'(ka)\} e^{in(\theta-\theta')}$$

$$0 \le r \le r'$$

$$G_I(\underline{r}|\underline{r}') = \frac{1}{4i}\sum_{n=-\infty}^{\infty} \frac{J_n(kr')}{J_n'(ka)} \{H_n^{(1)'}(ka)J_n(kr) - H_n^{(1)}(kr)J_n'(ka)\} e^{in(\theta-\theta')}$$

$$r' \le r \le a$$

where prime denotes differentiation with respect to the argument ka. In order to derive our integral equation we now use the Green's theorem

$$H_z(\nabla^2 + k)G(\underline{r}|\underline{r}') - G(\underline{r}|\underline{r}')(\nabla^2 + k^2)H_z = \nabla \cdot (H_z \nabla G - G \nabla H_z)$$

and integrate this equation over the internal and external regions. The integration over the internal region yields

$$- H_z(\underline{r}') = \int_0^{2\pi} \hat{a}_r \cdot (H_z \nabla G_I - G_I \nabla H_z) a^2 d\theta \tag{47}$$

Using the facts that $\hat{a}_r \cdot \nabla G_I = 0$ at $r = a$ for all θ and $\hat{a}_r \cdot \nabla H_z = 0$ on the conducting portion of the cylinder (47) becomes

$$H_z(\underline{r}') = \int_{\theta_1}^{\theta_2} \frac{\partial H_z}{\partial r} G_I(\underline{r}|\underline{r}') a d\theta \tag{48}$$

The integration over the external region yields

$$-H_z(\underline{r}') = \lim_{r\to\infty} \int_0^{2\pi} \hat{a}_r \cdot (H_z\nabla G_e - G_e\nabla H_z) r\,d\theta - \int_0^{2\pi} \hat{a}_r \cdot (H_z\nabla G_e - G_e\nabla H_z) a^2 d\theta \qquad (49)$$

The first integral on the right hand side of (49) is equal to the negative of the total field, H_{zT}, that would be present if the cylinder contained no aperture. One could prove this by substituting the explicit expression for G_e and noting that at infinity it is only the incident part of the total H_z that could contribute to this integral at infinity. Another way to see that this is true is to note that the second integral in (49) vanishes if there is no aperture. In fact it is only the integral over the aperture in the second integral that is non-zero. Using these facts (49) becomes

$$H_z(\underline{r}') = H_{zT} - \int_{\theta_1}^{\theta_2} \frac{\partial H_z}{\partial r} G_e(\underline{r}|\underline{r}') a\,d\theta \qquad (50)$$

We now combine (48) and (50) using the boundary conditions that H_z and $(\partial H_z)/(\partial r)$ are continuous through the aperture to obtain

$$H_T(\underline{r}') = \int_{\theta_1}^{\theta_2} \frac{\partial H_z}{\partial r} (G_I + G_e) a\,d\theta \qquad (51)$$

In order to use more standard notation we interchange the role of θ and θ' and define the known quantity $H_T(r) \equiv f(\theta)$ and the unknown quantity $(\partial H_z)/(\partial r') = \Phi(\theta')$. Equation (51) then becomes

$$\int_{\theta_1}^{\theta_2} \Phi(\theta') K(\theta,\theta') d\theta' = f(\theta) \qquad (52a)$$

where

$$K(\theta,\theta') = \sum_{n=-\infty}^{\infty} \frac{i e^{in(\theta-\theta')} a}{\pi^2 (ka)^2 J_n'(ka) H_n^{(1)'}(ka)} \qquad (52b)$$

and

$$f(\theta) = \frac{-2iH_i}{\pi ka} \sum_{n=-\infty}^{\infty} \frac{(i)^n e^{-in\theta}}{H_n^{(1)'}(ka)} \qquad (52c)$$

We now use the asymptotic forms for J_n and $H_n^{(1)}$ for small arguments

$$H_o^{(1)'}(u) \sim \frac{2i}{\pi u} \quad , \quad H_n^{(1)'}(u) \sim \frac{n!}{\pi}\left(\frac{u}{2}\right)^{-n-1} \quad n \geq 1$$

$$J_o'(u) \sim -\frac{u}{2} \quad , \quad J_n'(u) \sim \frac{1}{(n-1)!}\left(\frac{u}{2}\right)^{n-1} \quad n \geq 1$$

where u = ka.

Using the small argument asymptotic form

$$K(\theta,\theta') \sim -\frac{ia}{\pi^2 u^2}\left\{i\pi + \pi\left(\frac{u}{2}\right)^2 2 \sum_{n=1}^{\alpha} \frac{\cos n(\theta-\theta')}{n}\right\} \tag{53}$$

The sum that appears in (53) can be expressed in closed form as

$$\sum_{n=1}^{\infty} \frac{\cos n(\theta-\theta')}{n} = \ln\left|2\sin\left(\frac{\theta-\theta'}{2}\right)\right| \tag{54}$$

Substituting (53) and (54) into (52a) we obtain

$$-\int_{\theta_1}^{\theta_2} \frac{\Phi(\theta')a}{\pi u^2}\, d\theta' + \frac{i}{2\pi}\int_{\theta_1}^{\theta_2} \Phi(\theta')\ln\left|2\sin\left(\frac{\theta-\theta'}{2}\right)\right| a\, d\theta' = f(\theta) \sim H_i(1 + 2iu\cos\theta) \tag{55}$$

For $(\theta-\theta')/2 << 1$, $\ln|2\sin((\theta-\theta')/2)| \approx \ln|\theta-\theta'|$, (55) is a Carleman integral equation and it can be solved exactly[8]. It is not necessary to solve (55) to get the low-frequency limit for the field that penetrates to the center of the cylinder. Returning to (48) we can write the field at the center of the cylinder

$$H_z(0) = \frac{1}{4iJ_o'(ka)}\left\{H_o^{(1)'}(ka)J_o(ka) - H_o^{(1)}(ka)J_o'(ka)\right\}\int_{\theta_1}^{\theta_2} \Phi(\theta')a\,d\theta'$$

$$H_z(0) = \frac{1}{J_o'(ka)(ka)2\pi}\int_{\theta_1}^{\theta_2} \Phi(\theta')a\,d\theta' \tag{56}$$

In the limit ka → 0, u → 0, we find from (55) that

$$\int_{\theta_1}^{\theta_2} \Phi(\theta')a d\theta' \sim - H_i \pi (ka)^2 \tag{57}$$

substituting this into (56), we find that

$$H_z(0) \sim - \frac{\pi (ka)^2 H_i}{2\pi (ka) J_o'(ka)} \sim H_i \tag{58}$$

This is true no matter where the aperture is located.

Summary of Results

A method for calculating the electromagnetic field that penetrates a perfectly conducting spherical shell containing a circular aperture is presented. This method has the potential for allowing the determination of quantitative error bounds on this shielding problem. It also allows one to readily obtain the low frequency limit. In this limit the electric field is reduced four times as much as the magnetic field.

The effect of an axial aperture in a perfectly conducting cylinder is studied in a manner that would allow low frequency corrections. The static limit for the field that penetrates the structure is found when the incident magnetic field is axially directed. For this case their is no shielding and the entire field penetrates through the aperture no matter where it is located relative to the direction of the incident field.

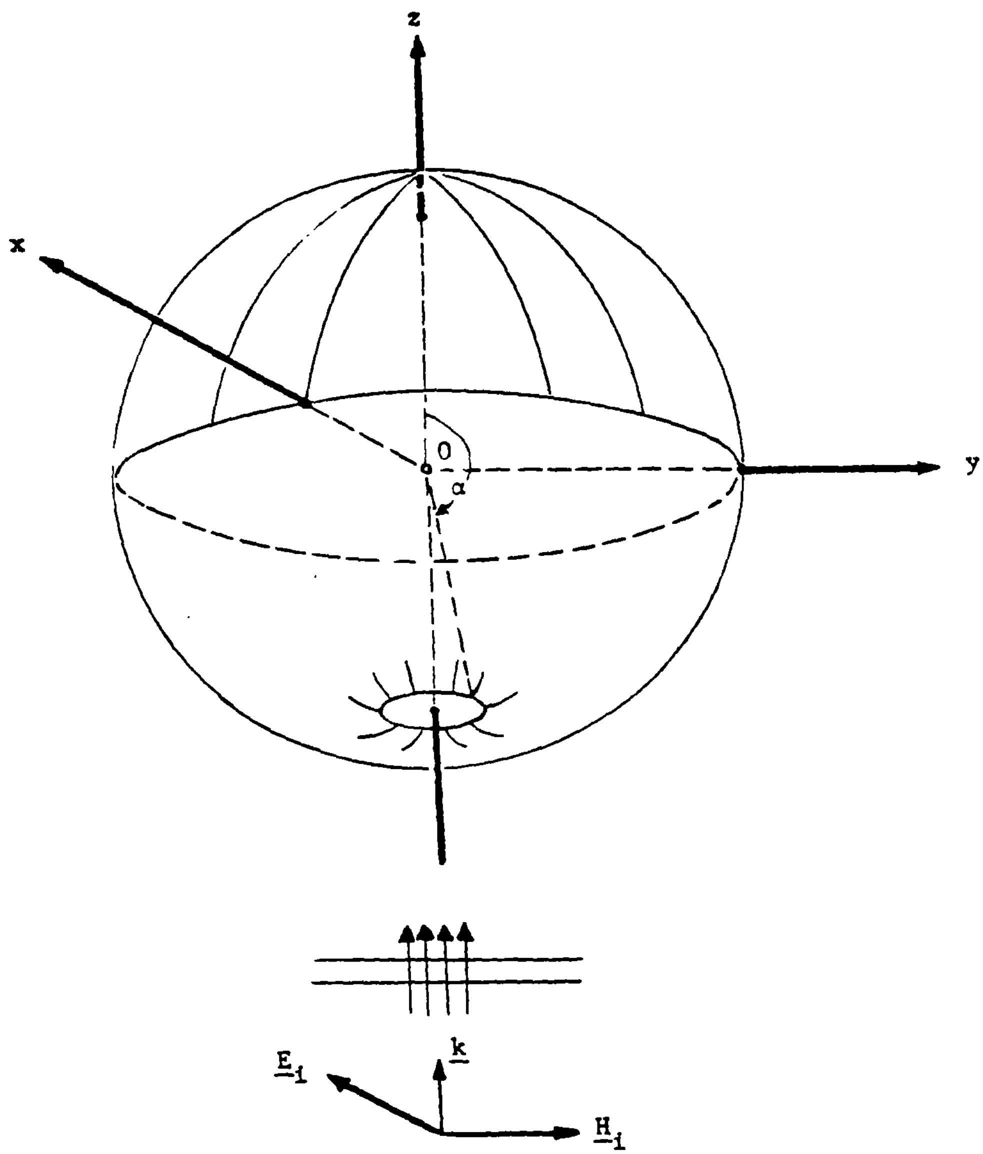

Figure 1. Spherical scattering geometry.

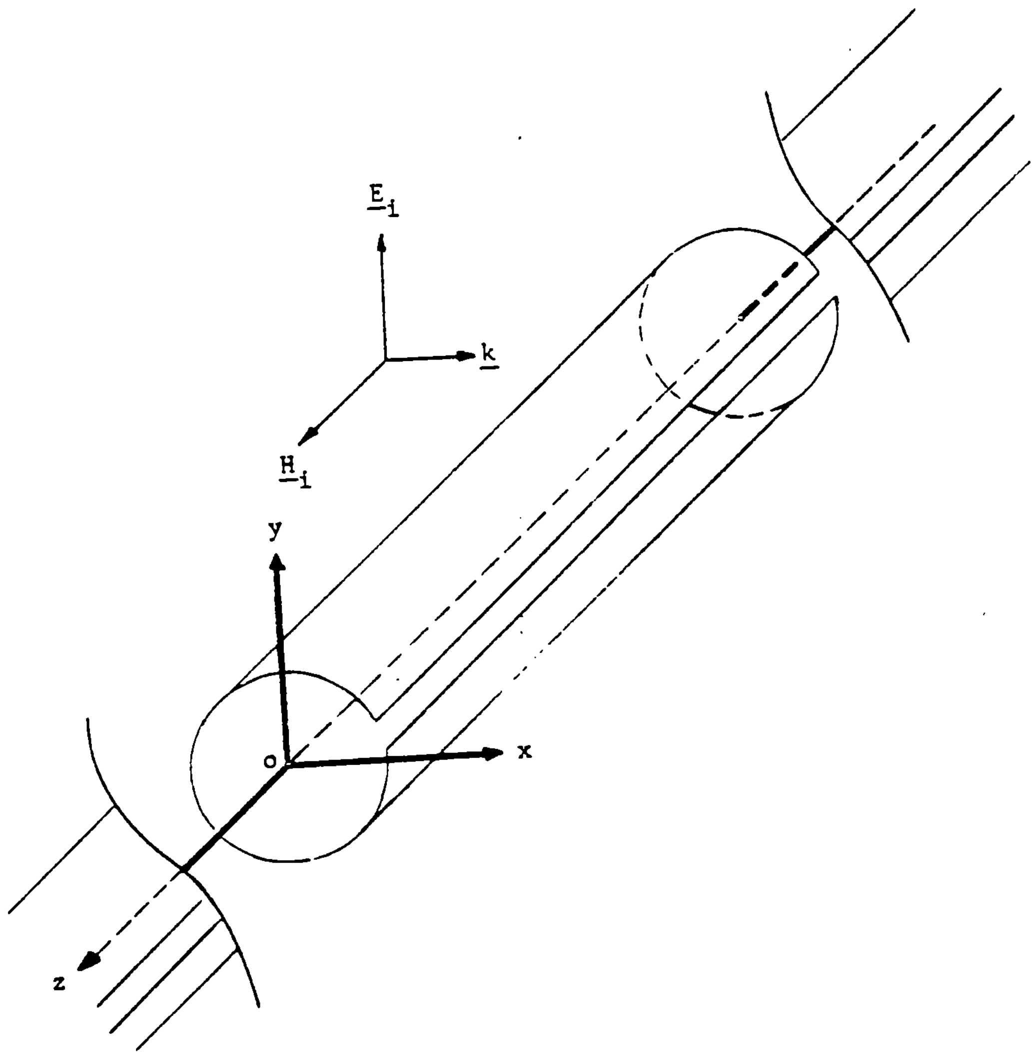

Figure 2. Cylindrical scattering geometry.

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