Interaction Notes
Note 53
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B. MISSILE FLYING THROUGH A HIGH-ALTITUDE SOURCE REGION*

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The input parameters of the problem to be solved and the necessary assumptions to accomplish an approximate solution to this problem are as follows:

Given:

- Exoatmospheric burst
- 20-40 km altitude
- Cylindrical model for the missile geometry (12 meters long)
- Conductivity levels on the order of 10^{-3} mhos/m

The Problem:

- Determine the time history of the current distribution on the surface of the missile.

Assumptions:

- The medium is not perturbed by the presence of the cylindrical scatterer.
- The medium is linear $[\mu_e \neq f(E)]$, isotropic, and homogeneous.

The mathematical formulation, in terms of Maxwell's equations, along with the definition of the various current densities and some specific assumptions concerning their relative magnitudes, are:

*This is a continuation of a previous presentation given on 25 September 1969, at the U.S. Army Mobility Equipment Research and Development Center, Fort Belvoir, Virginia, as part of the Technical Meeting on the Effects of Time-Varying Air Conductivity on EMP Coupling (TMP-67178), Interaction Note 55.

- $\nabla \times \vec{H} = \vec{J}_T + \epsilon_o \dot{\vec{E}}$ (1)
- $\nabla \times \vec{E} = -\mu_o \dot{\vec{H}}$ (2)

where

- $\vec{J}_T = \vec{J}_c + \vec{J}_p + \vec{J}_a$, and (3)

$\vec{J}_c$ = Compton current density in the presence of the scatterer

$\vec{J}_p$ = Plasma current density in the presence of the scatterer

$\vec{J}_a = \vec{J}_{ai} + \vec{J}_{ar}$ = Induced current density on the antenna

With the assumption

- $\vec{J}_{ai} >> \vec{J}_{ar}$,
- $\vec{J}_a \simeq \vec{J}_{ai} = \vec{J}$.

It is to be noted that the problem of γ-flux-antenna interaction has been treated by Taylor, independent of other processes (medium-antenna interaction and medium-EM interaction), to obtain an estimate of the induced antenna currents, $\vec{J}_{ar}$, due to Compton scattering of electrons from the antenna. On the basis of that development, preliminary calculations for a typical situation indicates that these currents ($\vec{J}_{ar}$) are small when compared with those expected to be induced by the EMP fields ($\vec{J}_{ai}$).

The Lorentz model for J_p is:

- $\vec{J}_p = e n_{sec}(t) \mu_e \vec{E}$ (4)
- $\sigma(t) = e n_{sec}(t) \mu_e$ [homogeneous medium] (5)
- $v_c \approx 1.5 \times 10^8/\text{sec}$ [at 20-40 km altitudes] (6)

The Maxwell equations in terms of incident and scattered fields are:

- $\nabla \times \vec{H}_i = \vec{J}_c + \vec{J}_{pi} + \epsilon \dot{\vec{E}}_i$
- $\nabla \times \vec{E}_i = -\mu \dot{\vec{H}}_i$

} EMP fields (7)

$$\nabla \times \vec{H}_s = \vec{J} + \sigma(t)\vec{E}_s + \epsilon\dot{\vec{E}}_s$$
$$\nabla \times \vec{E}_s = -\mu\dot{\vec{H}}_s$$
scattered fields (8)

However, the splitting of Maxwell's equations in terms of incident and scattered fields is valid only under the assumptions (1) that the medium is unperturbed by the presence of the cylindrical scatterer and (2) that the medium is linear. For more details concerning the Lorentz model for J_p, reference is made to the minutes of the previous technical meeting.

The geometry of the chosen analytical model of the antenna and its environment are shown in Figure 1.

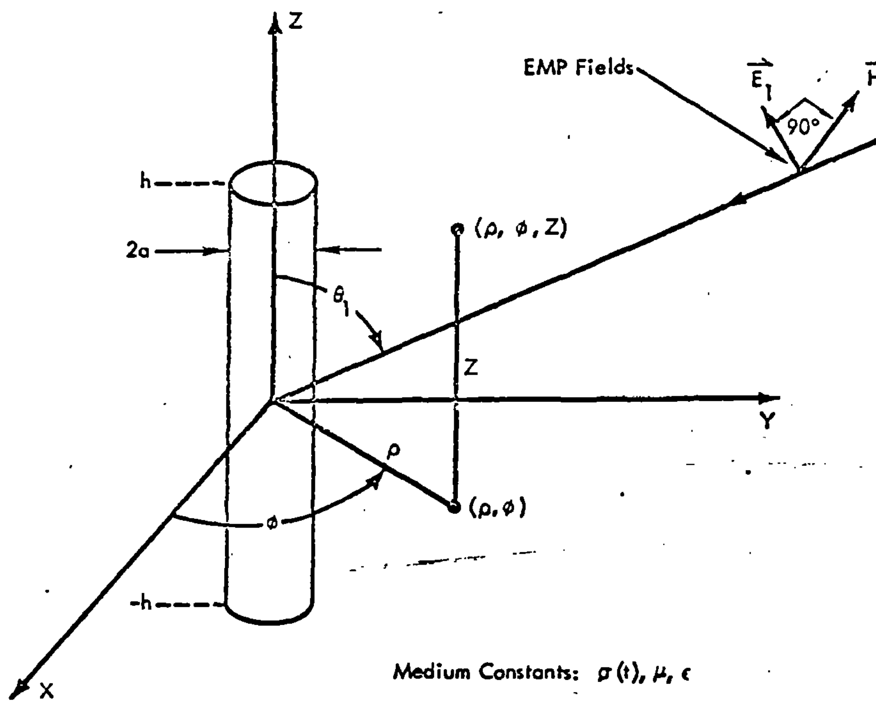

Figure 1. Theoretical model of missile geometry.

The wave equation is:

$$\nabla^2 A - \frac{1}{c^2}\frac{(g\dot{A})'}{g} = -\mu_o J \tag{9}$$

where

$$g(t) = \exp\left[\int_0^t \frac{\sigma(t')}{\epsilon_o}\, dt'\right] \quad (10)$$

Equation (9) has been obtained from Maxwell's equations through conventional EM techniques, i. e., introduction of scalar and vector potentials and the Lorentz gauge. Here the wave equation for the vector potential, A, is given in such a form that the generalized Fourier Series expansion is readily employed as follows:

Let

$$A(r, t) = \sum_n A_n \frac{y_n(t)}{\sqrt{g}} \quad (11)$$

where the set $[y_n(t)]$ is orthonormal in the interval (-b, b), i. e.,

$$\int_{-b}^{b} y_n(t) y_m(t) dt = \delta_{mn}$$

with

$$\delta_{mn} = 0 \quad \text{for} \quad m \neq n$$

$$\delta_{mn} = 1 \quad \text{for} \quad m = n$$

It is seen that the vector potential is expanded in terms of an orthonormal set of functions, $[y_n(t)]$, over a symmetrical interval (-b, b).

On the basis of the orthogonality conditions, the Fourier coefficients are calculated by use of the following equation which may be viewed as a direct integral transform that results in a discrete frequency spectrum:

$$A_n = \int_{-b}^{b} A(r, t) \sqrt{g}\, y_n(t) dt \quad (12)$$

Under this transformation, the wave equation takes the well-known form:

$$\nabla^2 A_n(r) + \frac{\lambda_n}{c^2} A_n(r) = -\mu_o J_n(r) \quad (13)$$

under the conditions:

- $A(r, b) = A(r, -b) = 0$
- $\dot{A}(r, b) = \dot{A}(r, -b) = 0$
- $y''_n + [\lambda_n - q(t)]\, y_n = 0$ [Sturm-Louiville equation]

where

- $q(t) = \dfrac{\dot{\sigma}(t)}{2\epsilon_o} + \left(\dfrac{\sigma}{2\epsilon_o}\right)^2$
- $Y_n(t)$ = eigenfunctions
- λ_n = discrete eigenvalues

It is to be noted that the propagation constant is obtainable in terms of a discrete set of eigenvalues, λ_n, corresponding to a discrete set of eigenfunctions, $[y_n]$, which functions are solutions to the characteristic equation, the Sturm-Louiville equation, in this case. The conditions given are not only physically justifiable but necessary if the wave equation is to take the standard form. Note the dependence of the function $q(t)$ on the time-varying conductivity.

For purposes of mathematical and numerical simplicity in determining an orthogonal set of eigenfunctions, the Sturm-Louiville equation is divided into two equations of the same form under the necessary condition that the $q(t)$ function be even on $(-b, b)$. This condition in no way restricts the generality of this analysis, since, in the present problem, nothing occurs for $t < 0$ provided the time interval of interest is made large enough.

Let

- $q(-t) = q(t)$ [even function] on $(-b, b)$

Then

$$\left.\begin{aligned} & y''_{en} + [\lambda_n - q(t)]\, y_{en} = 0 \\ & y''_{on} + [\lambda_n - q(t)]\, y_{on} = 0 \end{aligned}\right\} \qquad (14)$$

where

- $y_n = y_{en} - j\, y_{on}$

The split of an eigenfunction, y_n, into an even and an odd part is based on an analogy with the ordinary Fourier cosine and sine series expansion. The fact that y_{en} and y_{on} are even and odd functions, respectively, implies that they are orthogonal with respect to each other but not as yet orthogonal with respect to themselves. The initial conditions sufficient for orthogonality are:

$$\left.\begin{aligned} &\bullet\ y_{on}(b) = 0 \\ &\bullet\ y'_{en}(b) = 0 \end{aligned}\right\} \quad \text{[split of eigenvalues]} \qquad (15)$$

since

- $q(t) > 0$ on $(-b, b)$

and

- $y_{on}(b) = y'_{en}(b) = 0$
- λ_n's are positive real

Equation (15) renders the odd and even eigenfunctions orthogonal and results in a set of odd eigenvalues and even eigenvalues, respectively. These conditions are sufficient but not necessary, for the orthogonality condition given previously is a more general one. The choice of these rather simple initial conditions was based again on an analogy with the Fourier sine and cosine series expansion. As indicated, the eigenvalues (both even and odd) can be shown to be positive real for a monotonically increasing conductivity profile.

For purposes of illustration, the simple case of zero conductivity is:

- $\sigma = 0$
- $g = 1$
- $q = 0$

$$\left.\begin{aligned} &\bullet\ y_{en}(t) = \cos\left(\frac{n\pi}{b}t\right) \\ &\bullet\ y_{on}(t) = \sin\left(\frac{n\pi}{b}t\right) \end{aligned}\right\} \qquad (16)$$

- $y_n(t) = e^{-j\frac{n\pi}{b}t}$

Then

$$A_n(r) = \int_{-b}^{b} A(r,t)\, e^{-j\frac{n\pi}{b}t}\, dt \qquad \text{[Fourier transform]} \qquad (17)$$

Figure 2 shows a typical high-altitude environment described by an electric field pulse and the resulting conductivity profile. It is used as the input to the problem of calculating the time-history of the surface current at the center of a 12-meter missile flying through the source region.

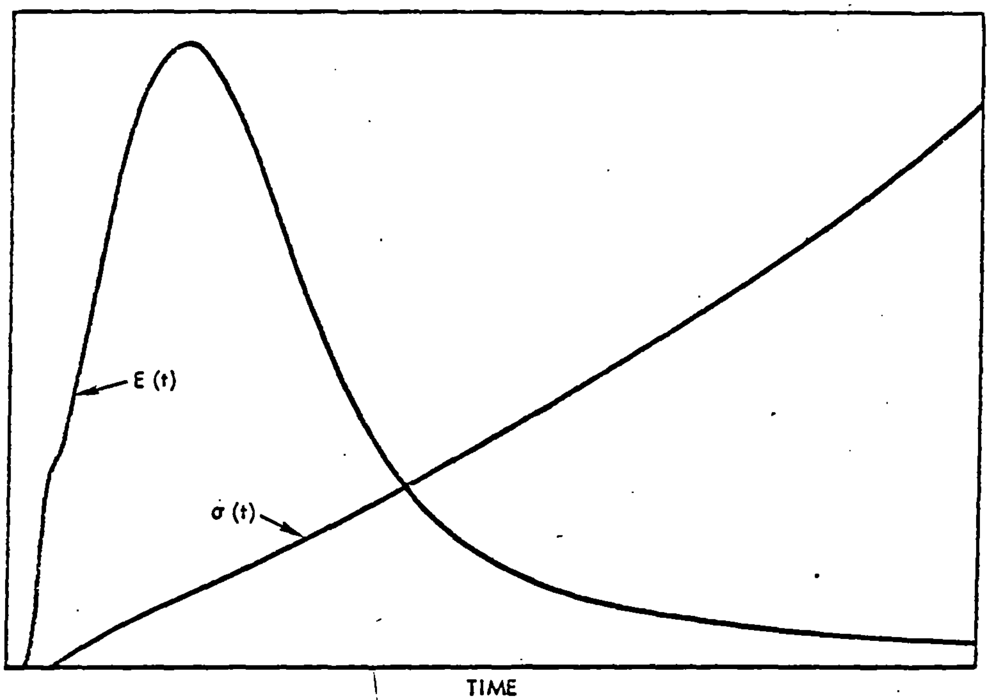

Figure 2. High-altitude EMP environment.

The Sturm-Louiville differential equation has been solved using numerical techniques and a high-speed digital computer. More specifically, the SCEPTRE computer code has been utilized to calculate the transient response of a circuit whose behavior is described by the Sturm-Louiville second order differential equation with appropriate initial conditions on the circuit inductors and capacitors. Figure 3 shows an example of an eigenfunction of order 4 over the interval of interest (0, 500 ns).

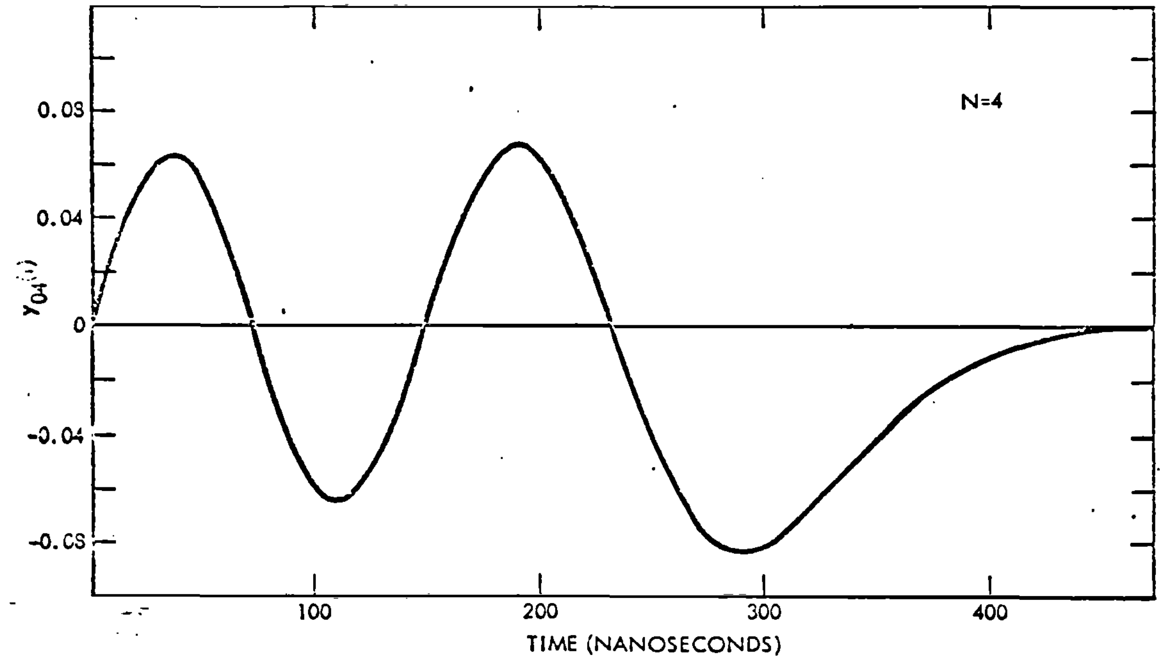

Figure 3. Fourth-order odd eigenfunction.

Note the sine-like variation of this and, if fact, all the odd eigenfunctions. The choice of the interval (-500, 500 ns) was based on the assumption that the most significant part of the system response would occur during the first 100 ns and that at 500 ns the response would be negligibly small. This turned out to be the case. Figure 4 shows another example of a higher order odd eigenfunction (y_{08}).

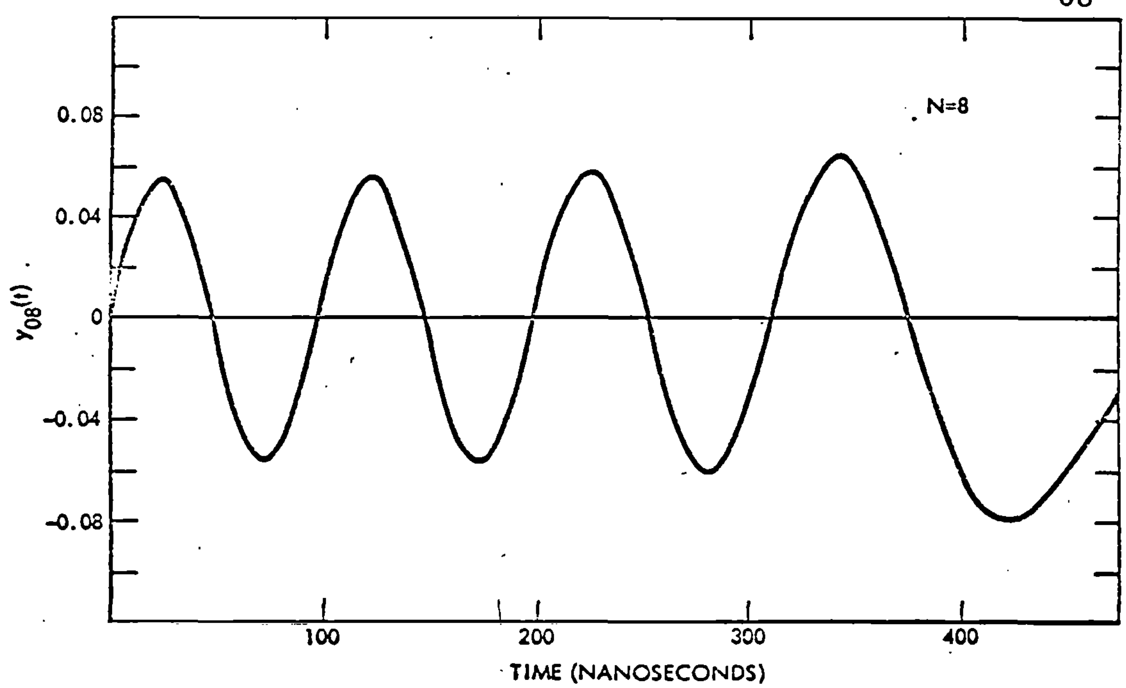

Figure 4. Eighth-order odd eigenfunction.

Note that unlike the sine and cosine eigenfunctions, the envelope of this waveform is slowly increasing with time over the interval. Figure 5 shows the eigenvalues, both even and odd, versus their respective index.

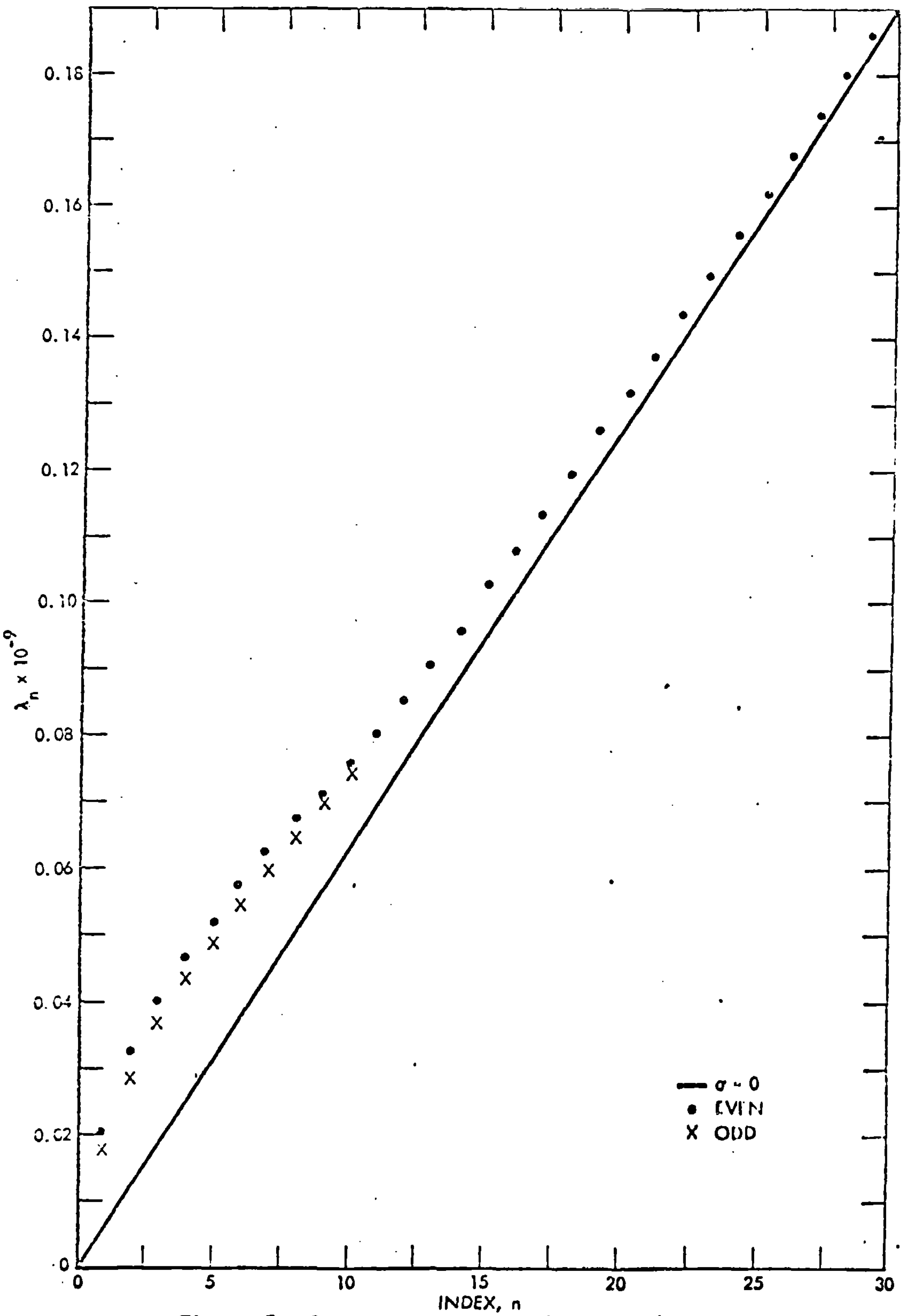

Figure 5. Discrete spectrum of eigenvalues.

The solid straight line represents the locus of the eigenvalues for zero conductivity. Note the splitting of even and odd eigenvalues which is more apparent for small indices or, equivalently, small frequencies. The separation between even and odd eigenvalues diminishes for higher n values and, as expected, they converge asymptotically towards the zero-conductivity eigenvalues, for at high frequencies the medium becomes transparent. For purposes of comparison, the transient response of a 12-meter missile for various conductivity profiles is presented in Figure 6.

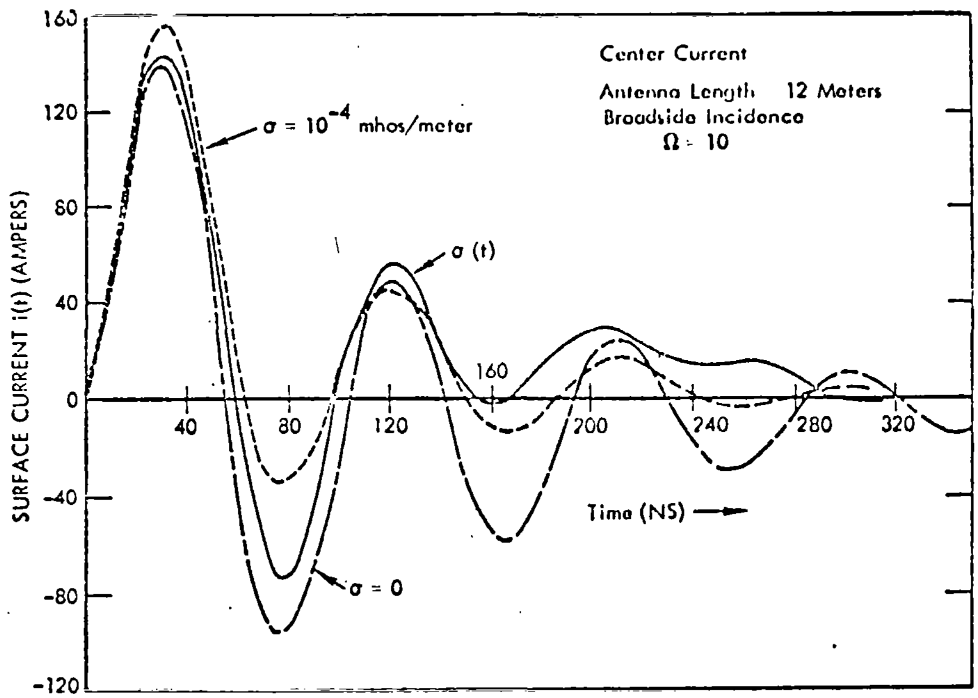

Figure 6. Antenna transient response for various conductivity profiles.

It is noted that at early times ($0 \leq t \leq 100$ ns), the $\sigma(t)$ response is bounded by the zero-conductivity and the 10^{-4} mhos/meter conductivity response, with the fundamental period being approximately the same. However, for later times the above observation is not valid. The $\sigma(t)$ conductivity response not only has a different periodic behavior but it also has a higher DC level.

The above preliminary conclusions are based on a particular conductivity profile and EMP environment and as such cannot, at this time, be generalized. Final conclusions would have to be established on the basis of an extensive parametric study with a number of typical EMP environments and missile dimensions.