EXCITATION OF INTERNAL CIRCUITRY BY THE PROPAGATION OF AN ELECTROMAGNETIC FIELD THROUGH A TRANSVERSE SLOT IN A MISSILE SKIN

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SUMMARY

A rocket with removed access plate is simulated by a section of coaxial transmission line with a transverse rectangular slot cut in its sheath. The internal circuit consists of two arbitrary impedances in series with the inner conductor at its ends. The object is to find the currents in these impedances when the cylinder is illuminated from the outside by an electromagnetic field that enters the aperture and excites the internal circuit.

The problem is solved by application of the reciprocal theorem. The current in a dipole antenna is determined when this is in the far field maintained by the slotted coaxial line when driven by a generator in series with one of the load impedances. The reciprocal theorem gives the current in this impedance when the distant dipole is driven. A numerical example is given.

INTRODUCTION

In Fig. 1 is shown a simplified rocket or missile in the form of an aluminum tube with closed ends, radius b and negligible wall thickness which extends from $z = 0$ to $z = s$ along the axis of the cylindrical coordinate system ρ,ϕ,z. The open access door is simulated by a transverse rectangular slot which extends from $z = \ell - W/2$ to $z = \ell + W/2$ in the axial direction and from $\phi = \pi - \Psi/2$ to $\phi = \pi + \Psi/2$ laterally. The center of the slot is at $\rho = b$, $\phi = \pi$, and $z = \ell$. The internal circuit consists simply of a copper coaxial conductor terminated at $z = 0$ in the impedance Z_0, at $z = s$ in the impedance Z_s. The tube is illuminated from the outside by the far field of a distant antenna, which is conveniently taken to be a dipole at a distance r from the center of the slot at the origin of the polar coordinates r,θ,ϕ. The problem is to determine the currents $I_z(0)$ and $I_z(s)$ in the impedances Z_0 and Z_s.

The procedure to be followed is to remove the generator from the center of the distant dipole and connect it in series with the impedance Z_0 (or Z_s) and then to calculate the current I_d at the center of the dipole. According to the reciprocal theorem $I_z(0) = I_d$ (or $I_z(s) = I_d$).

In order to determine I_d it is necessary to obtain the electromagnetic field maintained at the dipole by the driven, slotted, coaxial cylinder. This involves determination of the current in the coaxial line and the field in the slot as intermediate steps.

THE DISTRIBUTION OF CURRENT ALONG THE CENTER CONDUCTOR OF THE COAXIAL SECTION OF TRANSMISSION LINE

When the coaxial line is terminated at $z = s$ in the impedance Z_s and driven by the emf V_0^e in series with the impedance Z_0 at $z = 0$ (as shown in Fig. 1), the current in the center conductor is

$$I_z(z) = V_0^e\, F(w) \tag{1a}$$

with

$$F(w) = \frac{Z_c \cosh \gamma w + Z_s \sinh \gamma w}{Z_c(Z_0 + Z_s) \cosh \gamma s + (Z_c^2 + Z_0 Z_s) \sinh \gamma s} \tag{1b}$$

In (1a,b), $w = s - z$ is the distance from the load Z_s to the point z where the current is determined. It is assumed that the presence of the aperture at $z = \ell$ has a negligible effect on the distribution of current. The complex propagation constant γ and characteristic impedance Z_c of the line are defined by

$$\gamma = \alpha + j\beta = \sqrt{(z^i + j\omega\ell^e)(g + j\omega c)} \tag{2}$$

$$Z_c = \sqrt{(z^i + j\omega\ell^e)/(g + j\omega c)} \tag{3}$$

The internal impedance per loop unit length of the line is

$$z^i = \frac{(1 + j)}{2\pi}\sqrt{\frac{\mu\omega}{2}}\left[\frac{1}{a\sqrt{\sigma_a}} + \frac{1}{b\sqrt{\sigma_b}}\right] \tag{4}$$

where σ_a and σ_b are the conductivities, respectively, of the inner and outer conductors - in this case copper ($\sigma_a = 5.8 \times 10^7$ mhos/m) and aluminum ($\sigma_b = 3.72 \times 10^7$ mhos/m). This formula assumes the frequency to be high enough so that the skin depth $d_s = \sqrt{2/\omega\mu\sigma}$ is small compared with the radius a and the wall thickness of the outer conductor. More general formulas that apply when this is not the case are in the literature [1]. The other line constants have the well-known forms:

$$\ell^e = (\mu/2\pi)\,\ln(b/a) \;;\; g = 2\pi\sigma_d/\ln(b/a) \;;\; c = 2\pi\varepsilon_d/\ln(b/a) \tag{5}$$

where σ_d and $\varepsilon_d = \varepsilon_0\varepsilon_r$ are the conductivity and permittivity of the dielectric in the coaxial line. In this case with air as the dielectric, $\sigma_d \doteq 0$, $\epsilon_r \doteq 1$. As usual, $\mu = \mu_0\mu_r$ where $\mu_0 = 4\pi \times 10^{-7}$ henries/m and $\epsilon_0 = 8.85 \times 10^{-12}$ farads/m. If the dissipation in the line is neglected, $\gamma = jk = \omega\sqrt{\ell^e c}$, $Z_c = \sqrt{\ell^e/c} = (\zeta_0/2\pi)\ln(b/a)$ where $\zeta_0 = \sqrt{\mu_0/\epsilon_0} \doteq 120\pi$ ohms.

THE AXIAL ELECTRIC FIELD IN THE APERTURE

The axial electric field $E_z(\rho,\phi,z)$ in the interior of a perfectly conducting coaxial line with the current $I_{z1}(z') = 2\pi a K_{z1}(a,z')$ as given by (1a,b) on the inner conductor and an equal and opposite current, $I_{z2}(z') = -I_{z1}(z') = 2\pi b K_{z2}(b,z')$, on the shield vanishes at all points on the surfaces of the conductors and in the dielectric between them.* This zero value is the result of the complete cancellation expressed by the equation $E_{z2}(\rho,\phi,z) = -E_{z1}(\rho,\phi,z)$ where $E_{z1}(\rho,\phi,z)$ is the field due to the current $I_{z1}(z')$ and the associated charge $q_1(z') = (j/\omega)[\partial I_{z1}(z')/\partial z']$ and $E_{z2}(\rho,\phi,z)$ is the field due to $I_{z2}(z')$ and $q_2(z') = (j/\omega)[\partial I_{z2}(z')/\partial z']$.

The axial electric field $E_{z1}(\rho,\phi,z)$ due to the current $I_{z1}(z')$ given in (1a,b) is readily calculated when the conductor is perfect so that $\gamma = jk$. The axial field due to the current

$$I_z(z') = I_m \sin k(h + h' - z') \tag{6}$$

in the range $0 \le z' \le h$ is well known. It is [2]

$$E_z(\rho,\phi,z) = \frac{-j\zeta_0 I_m}{4\pi}\left\{e^{-jkR_1}\left[\frac{\cos kh'}{R_1} + \frac{j(z-h)}{R_1^2}\sin kh' + \frac{z-h}{kR_1^3}\sin kh'\right]\right.$$
$$\left. - e^{-jkR_0}\left[\frac{\cos k(h+h')}{R_0} + \frac{jz}{R_0^2}\sin k(h+h') + \frac{z}{kR_0^3}\sin k(h+h')\right]\right\} \tag{7}$$

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*The currents are located by the primed coordinates ρ',ϕ',z'; the point where the field is calculated in the coaxial line by the unprimed coordinates ρ,ϕ,z.

where $R_0 = \sqrt{\rho^2 + z^2}$ and $R_1 = \sqrt{\rho^2 + (h - z)^2}$. The field due to the current (1a,b) is obtained from (6) as the sum $E_{z1}(\rho,\phi,z) = E_{z1c}(\rho,\phi,z) + E_{z1s}(\rho,\phi,z)$. $E_{z1c}(\rho,\phi,z)$ is given by (7) with the substitution: $I_m = V_0^e Z_c/D$, $kh' = \pi/2$ and $kh = ks$. Similarly, $E_{z1s}(\rho,\phi,z)$ is given by (7) with $I_m = jV_0^e Z_s/D$, $kh' = 0$, $kh = ks$. D is the denominator in (1b). Thus,

$$E_{z1c}(\rho,\phi,z) = \frac{-j\zeta_0 V_0^e Z_c}{4\pi D}\left\{e^{-jkR_1}\left[\frac{-j(s - z)}{R_1^2} - \frac{s - z}{kR_1^3}\right] - e^{-jkR_0}\left[\frac{-\sin ks}{R_0} + \frac{jz}{R_0^2}\cos ks + \frac{z}{kR_0^3}\cos ks\right]\right\} \tag{8a}$$

$$E_{z1s}(\rho,\phi,z) = \frac{\zeta_0 V_0^e Z_s}{4\pi D}\left\{\frac{e^{-jkR_1}}{R_1} - e^{-jkR_0}\left[\frac{\cos ks}{R_0} + \frac{jz}{R_0^2}\sin ks + \frac{z}{kR_0^3}\sin ks\right]\right\} \tag{8b}$$

where $R_1 = \sqrt{\rho^2 + (s - z)^2}$ and $R_0 = \sqrt{\rho^2 + z^2}$. These expressions are independent of ϕ since the field is rotationally symmetrical.

The increments $dE_{z2c}(\rho,\phi,z)$ and $dE_{z2s}(\rho,\phi,z)$ of the field at an arbitrary radius ρ between $\rho = a$ and $\rho = b$ due to the filament of current $K_{z2}(b,\phi',z')bd\phi' = [I_{z2}(z')/2\pi]d\phi'$ on an element $bd\phi'$ of the shield are given by (8a,b) with $R_1(\phi')$ substituted for R_1 and $R_0(\phi')$ for R_0 and with the sign of V_0^e reversed. The new distances are

$$\begin{aligned} R_1(\phi') &= \sqrt{\rho^2 + b^2 - 2\rho b\cos(\phi - \phi') + (s - z)^2} \\ R_0(\phi') &= \sqrt{\rho^2 + b^2 - 2\rho b\cos(\phi - \phi') + z^2} \end{aligned} \tag{9}$$

The entire field $E_{z2}(\rho,\phi,z) = E_{z2c}(\rho,\phi,z) + E_{z2s}(\rho,\phi,z)$ due to the currents in all of the filaments is obtained by integration from $\phi' = -\pi$ to $\phi' = \pi$ of the two components, viz.,

$$E_{z2c}(\rho,\phi,z) = \frac{j\zeta_0 V_0^e Z_c}{4\pi D} \int_{-\pi}^{\pi} \left\{ e^{-jkR_1(\phi')} \left[\frac{-j(s-z)}{R_1^2(\phi')} - \frac{s-z}{kR_1^3(\phi')} \right] \right.$$
$$\left. - e^{-jkR_0(\phi')} \left[\frac{-\sin ks}{R_0(\phi')} + \frac{jz}{R_0^2(\phi')} \cos ks + \frac{z}{kR_0^3(\phi')} \cos ks \right] \right\} \frac{d\phi'}{2\pi} \tag{10a}$$

and

$$E_{z2s}(\rho,\phi,z) = \frac{-\zeta_0 V_0^e Z_s}{4\pi D} \int_{-\pi}^{\pi} \left\{ \frac{e^{-jkR_1(\phi')}}{R_1(\phi')} - e^{-jkR_0(\phi')} \left[\frac{\cos ks}{R_0(\phi')} + \frac{jz}{R_0^2(\phi')} \sin ks \right. \right.$$
$$\left. \left. + \frac{z}{kR_0^3(\phi')} \sin ks \right] \right\} \frac{d\phi'}{2\pi} \tag{10b}$$

Since $E_{z2}(\rho,\phi,z) = -E_{z1}(\rho,\phi,z)$ and $E_{z1}(\rho,\phi,z)$ is given by (8a,b), these integrals do not have to be evaluated. However, they are needed in obtaining the field in the aperture as explained in the following.

When a rectangular aperture is cut in the shield with its center at $\rho = b$, $\phi = \pi$, $z = \ell$, and with the dimensions $b\Psi$ and W which are sufficiently small to satisfy the inequalities,

$$kW \ll 1, \quad W \ll s \quad ; \quad kb\Psi \ll 1, \quad \Psi < \pi \tag{11}$$

it may be assumed that no significant change occurs in the axial distribution of the total currents $I_{z1}(z')$ and $I_{z2}(z')$ on the inner conductor and the inner surface of the sheath. Approximate rotational symmetry in the transverse distribution will continue to be true for the current on the inner conductor and also on the inner surface of the shield except at and quite near the aperture. In the range $\ell - W/2 \leq z' \leq \ell + W/2$, the entire current will be distributed over the conducting sector since it is zero in the aperture. As an approximation it may be assumed that the total current is distributed with a uniform density in each transverse section so that

$$K_{z2}(b,\phi',z') \begin{cases} \doteq I_{z2}(z')/(2\pi - \Psi)b; & -\pi + \Psi/2 \leq \phi' \leq \pi - \Psi/2, \\ & \ell - W/2 \leq z' \leq \ell + W/2 \\ = 0; & \pi - \Psi/2 \leq \phi' \leq -\pi + \Psi/2, \\ & \ell - W/2 \leq z' \leq \ell + W/2 \end{cases} \tag{12}$$

In the conducting strips that bound the aperture at $z = \pm W/2$ and extend to the ends of the coaxial line, the surface current density $K_{z2}(b,\phi',z')$ is quite small near the aperture. Its transverse distribution is relatively unimportant at distances from the aperture that are large compared to $4b^2 \cos^2(\phi'/2)$. It follows that specifically for calculating $E_z(b,\pi,\ell)$ at the center of the aperture, the distribution (12) is a satisfactory approximation provided the aperture is not too near the ends. Note that the total current and its axial distribution are essentially correct and only the transverse distribution is somewhat modified at distances from the slot at which this is of no consequence. It will be required, therefore, that the following inequalities be satisfied:

$$(s - \ell)^2 >> b^2 \quad , \quad \ell^2 >> b^2 \tag{13}$$

With $\rho = b$, $\phi = \pi$, $z = \ell$, the distances $R_1(\phi')$ and $R_0(\phi')$ in (9) become

$$R_1(\phi') = \sqrt{4b^2 \cos^2(\phi'/2) + (s - \ell)^2}$$
$$R_0(\phi') = \sqrt{4b^2 \cos^2(\phi'/2) + \ell^2} \tag{14}$$

Subject to (13), satisfactory approximations are

$$\left. \begin{array}{l} R_1(\phi') \doteq s - \ell \\ R_0(\phi') \doteq \ell \end{array} \right\} \quad \text{in amplitudes,} \tag{15a}$$

$$\left.\begin{aligned} R_1(\phi') &\doteq s - \ell + \frac{2b^2 \cos^2(\phi'/2)}{s - \ell} \\ R_0(\phi') &\doteq \ell + \frac{2b^2 \cos^2(\phi'/2)}{\ell} \end{aligned}\right\} \text{ in phases} \tag{15b}$$

The approximate field $E_{z2}(b,\pi,\ell)$ can now be obtained from (10a,b) by introducing (15a,b), extending the limits of integration from $-\pi + \Psi/2$ to $\pi - \Psi/2$, and substituting $d\phi'/(2\pi - \Psi)$ for $d\phi'/2\pi$. The field at the center of the aperture due to the currents on the inner surface of the shield is

$$E_{z2}(b,\pi,\ell) = E_{z2c}(b,\pi,\ell) + E_{z2s}(b,\pi,\ell) \tag{16a}$$

where

$$E_{z2c}(b,\pi,\ell) = \frac{-j\zeta_0 V_0^e Z_c}{4\pi D}\left\{e^{-jk(s-\ell)}\left[\frac{-j}{s-\ell} - \frac{1}{k(s-\ell)^2}\right] L(s-\ell,\Psi) - e^{-jk\ell}\left[\frac{je^{jks}}{\ell} + \frac{\cos ks}{k\ell^2}\right] L(\ell,\Psi)\right\} \tag{16b}$$

$$E_{z2s}(b,\pi,\ell) = \frac{\zeta_0 V_0^e Z_s}{4\pi D}\left\{\frac{e^{-jk(s-\ell)}}{s-\ell} L(s-\ell,\Psi) - e^{-jk\ell}\left[\frac{e^{jks}}{\ell} + \frac{\sin ks}{k\ell^2}\right] L(\ell,\Psi)\right\} \tag{16c}$$

In (16b,c)

$$L(x,\Psi) = -\int_0^{\pi-\Psi/2} \exp\left[\frac{-j2kb^2 \cos^2(\phi'/2)}{x}\right] \frac{d\phi'}{\pi - \Psi/2} \tag{16d}$$

with $x = \ell$ or $x = s - \ell$.

The field $E_{z1}(b,\pi,\ell) = E_{z1c}(b,\pi,\ell) + E_{z1s}(b,\pi,\ell)$ due to the currents and charges on the inner conductor is given directly by (8a,b) with $z = \ell$ and $\rho = b$. In order to combine it with (16b,c) the distances R_1 and R_0 have to be approximated in the manner suggested by (15a,b). Since these approximations and those in (15a,b) are quite critical in determining the small difference field in the aperture, an alternative procedure which, in effect, subtracts out the errors introduced by them, is to be preferred. It depends on the statement following (10b) which applies specifically to the field at the surface of the coaxial line when there is no aperture, viz., $E_{z1}(b,\pi,z) = -E_{z2}(b,\pi,z)$. Since the field $E_{z1}(\rho,\phi,z)$ is affected negligibly by the presence of the aperture, it follows that $E_{z1}(b,\pi,z)$ is given by the negative of (16a-d) with $\Psi = 0$. This involves only the substitution of $-L(s - \ell,0)$ and $-L(\ell,0)$ for $L(s - \ell,\Psi)$ and $L(\ell,\Psi)$ where $L(x,0)$ is given by (16d) with $\Psi = 0$. It follows that in the evaluation of the total field in the aperture, viz.,

$$E_z(b,\pi,\ell) = E_{zc}(b,\pi,\ell) + E_{zs}(b,\pi,\ell)$$

$$= E_{z1c}(b,\pi,\ell) + E_{z2c}(b,\pi,\ell) + E_{z1s}(b,\pi,\ell) + E_{z2s}(b,\pi,\ell) \tag{17}$$

the quantity $L(x,\Psi) - L(x,0)$ must be evaluated from (16d) and (17). If it is now recalled that the inequalities $kb << 1$ and $(b/x) < 1$ have been postulated, it follows that the exponents in the integrands must be small enough to permit the retention of only the leading terms in the series expansions of the exponential functions. Thus,

$$L(x,\Psi) - L(x,0) \doteq - \int_0^{\pi-\Psi/2} \left[1 - \frac{j2kb^2 \cos^2(\phi'/2)}{x}\right] \frac{d\phi'}{\pi - \Psi/2} + \int_0^{\pi} \left[1 - \frac{j2kb^2 \cos^2(\phi'/2)}{x}\right] \frac{d\phi'}{\pi} = j\,\frac{kb^2}{x}\,G(\Psi) \tag{18}$$

where

$$G(\Psi) = \frac{\sin(\Psi/2)}{\pi - \Psi/2} \tag{19}$$

and where in (18) $x = s - \ell$ or $x = \ell$.

It follows from (16)-(19) that the total axial field at the center of the aperture is given by (17) with

$$E_{zc}(b,\pi,\ell) = \frac{-\zeta_0 V_0^e k b^2 G(\Psi) Z_c}{4\pi D}\left\{ j\left[\frac{e^{-jk(s-\ell)}}{(s-\ell)^2} + \frac{e^{jk(s-\ell)}}{\ell^2}\right] + \frac{e^{-jk(s-\ell)}}{k(s-\ell)^3} + \frac{e^{-jk\ell}}{k\ell^3}\cos ks \right\} \tag{20a}$$

$$E_{zs}(b,\pi,\ell) = \frac{j\zeta_0 V_0^e k b^2 G(\Psi) Z_s}{4\pi D}\left\{ \frac{e^{-jk(s-\ell)}}{(s-\ell)^2} - \frac{e^{jk(s-\ell)}}{\ell^2} - \frac{e^{-jk\ell}}{k\ell^3}\sin ks \right\} \tag{20b}$$

The sum of these components gives the final expression for the approximate field at the center of the aperture. It is:

$$E_z(b,\pi,\ell) = \zeta_0 V_0^e k G(\Psi) Y_c H(s,\ell)/4\pi \tag{21a}$$

where $Y_c = 1/Z_c$ and

$$H(s,\ell) = \frac{b^2 Z_c}{D}\left\{ j\,\frac{e^{-jk(s-\ell)}}{(s-\ell)^2}(Z_s - Z_c) - j\,\frac{e^{jk(s-\ell)}}{\ell^2}(Z_s + Z_c) - \frac{e^{-jk(s-\ell)}}{k(s-\ell)^3} Z_c - \frac{e^{-jk\ell}}{k\ell^3}[Z_c \cos ks + jZ_s \sin ks] \right\} \tag{21b}$$

$G(\Psi)$ is given in (19) and

$$D = Z_c(Z_0 + Z_s)\cos ks + j(Z_c^2 + Z_0 Z_s)\sin ks \tag{22}$$

An interesting special case is the completely matched line with both terminations equal to Z_c. With $Z_0 = Z_s = Z_c$, $D = 2Z_c^2 e^{jks}$ and (21b) reduces to

$$H(s,\ell) = -b^2\left\{ j\,\frac{e^{-jk\ell}}{\ell^2} + \frac{e^{-jk(2s-\ell)}}{2k(s-\ell)^3} + \frac{e^{-jk\ell}}{2k\ell^3} \right\} \tag{23}$$

If, in addition, the aperture is located at the center of the line with $s = 2\ell$, the field at its center is

$$H(2\ell,\ell) = -\frac{b^2}{\ell^2}\left\{je^{-jk\ell} + \frac{e^{-j2k\ell}}{k\ell}\cos k\ell\right\} \tag{24}$$

It is significant to note that $H(s,\ell)$ and, hence, $E_z(b,\pi,\ell)$ are proportional to terms with $(b/\ell)^2$ or $b^2/(s-\ell)^2$ as a factor. This means that the axial field in the aperture decreases as the inverse square of the distance from the center of the aperture to the ends of the coaxial line. Unlike $H_\phi(\rho,\phi,z)$ and $E_\rho(\rho,\phi,z)$ which are determined by the local current $I_z(z')$ and charge per unit length $q(z')$, $E_z(\rho,\phi,z)$ in the aperture depends primarily on the distance from the point z to the ends of the line. When the aperture is closed, it is everywhere zero.

THE RADIATION FIELD OF AN INFINITE CYLINDER WITH ELECTRIC FIELD $E_z(b,\pi,\ell)$ IN THE APERTURE

The electric field at points outside and in the radiation zone of a perfectly conducting cylinder that is infinite in length and in which there is an aperture has been determined by Harrington [3]. In general, it has the following components in polar coordinates (r,θ,ϕ) with origin at $\rho = 0$, $z = \ell$, on the axis of the inner conductor opposite the center of the aperture [4].

$$E_\theta^r(r,\theta,\phi) = j\omega\mu\,\frac{e^{-jkr}}{\pi r}\sin\theta\sum_{n=-\infty}^{\infty} e^{jn\phi}\, j^{n+1}\, f_n(-k\cos\theta) \tag{25a}$$

$$E_\phi^r(r,\theta,\phi) = -jk\,\frac{e^{-jkr}}{\pi r}\sin\theta\sum_{n=-\infty}^{\infty} e^{jn\phi}\, j^{n+1}\, g_n(-k\cos\theta) \tag{25b}$$

where the functions $f_n(\xi)$ and $g_n(\xi)$ are given by [5]:

$$f_n(\xi) = j\omega\varepsilon\bar{E}_z(n,\xi)[(k^2 - \xi^2)\, H_n^{(2)}(b\sqrt{k^2 - \xi^2})]^{-1} \tag{26a}$$

$$g_n(\xi) = [\bar{E}_\phi(n,\xi) + \frac{n\xi}{b(k^2 - \xi^2)}\, \bar{E}_z(n,\xi)][\sqrt{k^2 - \xi^2}\, H_n^{(2)\prime}(b\sqrt{k^2 - \xi^2})]^{-1} \tag{26b}$$

In (26a,b), $\bar{E}_z(n,\xi)$ and $\bar{E}_\phi(n,\xi)$ are the Fourier transforms of the electric field in the aperture.

The transforms in (26a,b) are readily evaluated if it is assumed that the electric field in the aperture is approximately equal to its value $E_z(b,\pi,\ell)$ at all points. In this case $E_\phi(b,\pi,\ell) = 0$ and

$$\bar{E}_z(n,\xi) = \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} E_z(b,\phi,z)\, e^{-jn\phi}\, e^{-j\xi z}\, dz$$

$$= \frac{E_z(b,\pi,\ell)}{2\pi} \int_{\pi-\Psi/2}^{\pi+\Psi/2} e^{-jn\phi}\, d\phi \int_{-W/2}^{W/2} e^{-j\xi z}\, dz \tag{27a}$$

The integration yields:

$$\bar{E}_z(n,\xi) = \frac{2E_z(b,\pi,\ell)\, e^{-jn\pi}}{\pi}\; \frac{\sin(n\Psi/2)}{n}\; \frac{\sin(\xi W/2)}{\xi} \tag{27b}$$

Since $\bar{E}_\phi(n,\xi) = 0$, it follows from (26a,b) with (27) that

$$f_n(\xi) = E_z(b,\pi,\ell)\left\{\frac{j2\omega\varepsilon\, \sin(n\Psi/2)\, \sin(\xi W/2)\, e^{-jn\pi}}{\pi n\xi(k^2 - \xi^2) H_n^{(2)}(b\sqrt{k^2 - \xi^2})}\right\} \tag{28a}$$

$$g_n(\xi) = E_z(b,\pi,\ell)\left\{\frac{2\, \sin(n\Psi/2)\, \sin(\xi W/2)\, e^{-jn\pi}}{\pi b(k^2 - \xi^2)^{3/2}\, H_n^{(2)\prime}(b\sqrt{k^2 - \xi^2})}\right\} \tag{28b}$$

The substitution of $\xi = -k\cos\theta$ in (28a,b), combined with the condition $kW \ll 1$ from (11), gives:

$$f_n(-k\cos\theta) = jE_z(b,\pi,\ell)\left\{\frac{\omega\epsilon W \sin(n\Psi/2)\, e^{-jn\pi}}{\pi k^2 \sin^2\theta\, H_n^{(2)}(kb\sin\theta)}\right\} \tag{29a}$$

$$g_n(-k\cos\theta) = -E_z(b,\pi,\ell)\left\{\frac{W\cos\theta\,\sin(n\Psi/2)\, e^{-jn\pi}}{\pi b k^2 \sin^3\theta\, H_n^{(2)\prime}(kb\sin\theta)}\right\} \tag{29b}$$

These expressions can be inserted into (25a,b) to obtain the components of the radiation field. They are

$$E_\theta^r(r,\theta,\phi) = -E_z(b,\pi,\ell)\,\frac{W\, e^{-jkr}}{\pi^2 r\sin\theta}\, F_\theta(\Psi,\theta,\phi) \tag{30a}$$

$$E_\phi^r(r,\theta,\phi) = jE_z(b,\pi,\ell)\,\frac{W\, e^{-jkr}}{\pi^2 r\sin\theta}\, F_\phi(\Psi,\theta,\phi) \tag{30b}$$

where

$$F_\theta(\Psi,\theta,\phi) = \sum_{n=-\infty}^{\infty} \frac{j^{n+1}\sin(n\Psi/2)\, e^{-jn(\pi-\phi)}}{n\, H_n^{(2)}(kb\sin\theta)} \tag{31a}$$

$$F_\phi(\Psi,\theta,\phi) = \frac{\cot\theta}{kb}\sum_{n=-\infty}^{\infty} \frac{j^{n+1}\sin(n\Psi/2)\, e^{-jn(\pi-\phi)}}{H_n^{(2)\prime}(kb\sin\theta)} \tag{31b}$$

Alternatively, with $H_{-n}^{(2)}(x) = (-1)^n H_n^{(2)}(x)$ it follows that

$$F_\theta(\Psi,\theta,\phi) = \frac{j\Psi}{2H_0^{(2)}(kb\sin\theta)} + 2\sum_{n=1}^{\infty} \frac{j^{n+1}\sin(n\Psi/2)\cos n(\pi-\phi)}{n\, H_n^{(2)}(kb\sin\theta)} \tag{31c}$$

$$F_\phi(\Psi,\theta,\phi) = \frac{2\cot\theta}{kb}\sum_{n=1}^{\infty} \frac{j^{n}\sin(n\Psi/2)\sin n(\pi-\phi)}{H_n^{(2)\prime}(kb\sin\theta)} \tag{31d}$$

Note that the field $E_z(b,\pi,\ell)$ in the aperture excites a horizontally polarized ϕ-component as well as a vertically polarized θ-component. This is a consequence of the transverse currents excited on the cylinder near the aperture. However, in the principal plane $\theta = \pi/2$, $E^r_\phi(r,\pi/2,\phi) = 0$; in the plane $\phi = \pi$, $E^r_\phi(r,\theta,\pi) = 0$. In (30a,b) the field $E_z(b,\pi,\ell)$ in the aperture is given by (21). In special cases, $H(s,\ell)$ in (21a) is given by (23) or (24).

Since the coaxial line is electrically small in radius as required in (11), the following approximate forms of the Hankel functions with small arguments ($x^2 << 1$) may be used:

$$H_0^{(2)}(x) \doteq 1 - j\frac{2}{\pi}\left[\ell n \frac{x}{2} + 0.5772\right] \quad ; \quad H_0^{(2)'}(x) \doteq -\left(j\frac{2}{\pi x} + \frac{x}{2}\right) \tag{32a}$$

$$H_1^{(2)}(x) \doteq j\frac{2}{\pi x} + \frac{x}{2} \quad ; \quad H_1^{(2)'}(x) \doteq -j\frac{2}{\pi x^2} + \frac{1}{2} \tag{32b}$$

With these values, the leading first two terms in the sum are

$$F_\theta(\Psi,\theta,\phi) \doteq \frac{j\Psi}{2 + j(4/\pi)\ell n(2/\gamma kb \sin\theta)} - j\pi kb \sin\theta \sin(\Psi/2) \cos\phi \tag{33a}$$

$$F_\phi(\Psi,\theta,\phi) \doteq -\pi kb \cos\theta \sin\theta \sin(\Psi/2) \sin\phi \tag{33b}$$

In (33a), $\ell n\, \gamma = 0.5772$.

In the analysis to determine $E^r_\theta(r,\theta,\phi)$ and $E^r_\phi(r,\theta,\phi)$ the conducting cylinder has been treated as if infinitely long, whereas it is actually only very long compared to the radius b of the cylinder and the axial length W of the aperture. Since the currents excited on the outside of the cylinder by the field in the electrically small slot are highly localized, an increase in its length beyond a certain limit is unimportant in its effect on the far field except in directions near $\theta = 0$ near the axis of the cylinder. In terms of the reciprocal theorem, this is confirmed by the recent work of Kao [6] in which it is shown that the currents far from the ends of a long but finite cylinder differ negligibly from those on an infinitely long cylinder with the same radius and orientation relative to the incident field. Note, however, that the internal length of the coaxial section is important in determining the circuit properties of the terminated transmission line.

CURRENT IN THE DIPOLE; APPLICATION OF RECIPROCAL THEOREM

Let a dipole antenna be placed in the far field of the slotted cylinder with its axis parallel to $E_\theta^r(r,\theta,\phi)$. The current at the center of the dipole is [7]

$$I_d(0) = -2h_{e\theta}(\pi/2)\ Y_A\ E_\theta^r(r,\theta,\phi) \tag{34}$$

where $2h_{e\theta}(\pi/2)$ is the complex effective length when the antenna is parallel to the incident component $E_\theta^r(r,\theta,\phi)$, which is given by (30a) with (31a), and Y_A is the admittance of the dipole.

According to the Rayleigh-Carson reciprocal theorem [8] the impedanceless generator V_0^e may be moved to the center of the dipole from its position in series with the impedance Z_0 at $z = 0$ in the coaxial line, and the current $I_z(0)$ will then equal the current $I_d(0)$ given by (34). Hence, with (30a) and (31a)

$$I_z(0) = \frac{2h_{e\theta}(\pi/2)\ Y_A\ E_z(b,\pi,\ell)\ W}{\pi^2 \sin\theta}\ \frac{e^{-jkr}}{r}\ F_\theta(\Psi,\theta,\phi) \tag{35a}$$

where $E_z(b,\pi,\ell)$ is given by (21). More explicitly, the current in the load of the slotted coaxial receiving system is

$$I_z(0) = \frac{2h_{e\theta}(\pi/2)\ \zeta_0 V_0^e Wk}{4Z_c Z_A \pi^3 \sin\theta}\ \frac{e^{-jkr}}{r}\ H(s,\ell)\ G(\Psi)\ F_\theta(\Psi,\theta,\phi) \tag{35b}$$

Note that V_0^e in (35b) is the emf driving the dipole antenna; $Z_c = 1/Y_c$.

The electromagnetic field maintained by the dipole at the center of the aperture in the cylinder is

$$E^{inc} = j \frac{\zeta_0 V_0^e}{2\pi} \frac{e^{-jkr}}{r} \frac{kh_{e\theta}(\pi/2)}{Z_A} \tag{36}$$

since the electrical effective length is the same as the far field factor by the reciprocal theorem. If (36) is used to eliminate V_0^e in (35b), the following final general formula is obtained:

$$I_z(0) = \frac{-jE^{inc}W}{\pi^2 Z_c \sin\theta} H(s,\ell)\, G(\Psi)\, F_\theta(\Psi,\theta,\phi) \tag{37}$$

$H(s,\ell)$ is defined in (21b) or (23) or (24), $G(\Psi)$ is given by (19), and $F_\theta(\Psi,\theta,\phi)$ is in (31a) or (33a). This formula gives the desired solution for the current in the impedance Z_0 as a result of the excitation of the coaxial line by an axially directed incident field E^{inc} in the aperture. The current $I_z(s)$ in Z_s is obtained simply by interchanging ends. (The current resulting from a transverse incident field can be derived in a similar manner with a distant dipole oriented parallel to $E_\phi^r(r,\theta,\phi)$. Its maximum possible value is always smaller than the maximum possible value of (37).)

SPECIAL CASE AND ILLUSTRATIVE EXAMPLE

When the terminations at both ends are matched so that $Z_0 = Z_s = Z_c = (\zeta_0/2\pi)\ln(b/a)$, and the aperture is at the center of the coaxial section so that $s = 2\ell$, (24) applies. That is

$$H(2\ell,\ell) = -\frac{b^2}{\ell^2}\left\{ je^{-jk\ell} + \frac{e^{-j2k\ell}}{k\ell}\cos k\ell \right\}$$

For normal incidence, $\theta = \pi/2$, $\phi = \pi$, so that from (31a)

$$F_\theta(\Psi,\pi/2,\pi) = \sum_{n=-\infty}^{\infty} \frac{j^{n+1}\sin(n\Psi/2)}{n\, H_n^{(2)}(kb)}$$

or, from (33a),

$$F_\theta(\Psi,\pi/2,\pi) \doteq \frac{j\Psi}{2 + j(4/\pi)\ell n(2/\gamma kb)} + j\pi kb \sin(\Psi/2)$$

Illustrative Example:

Let f = 1 MHz so that λ = 300 m and k = $2\pi/\lambda$ = 0.0209; W = 0.25 m, Ψ = 0.5 radian, b = 1.0 m, and a = 1 mm = 10^{-3} m. Then kb = 0.2094 × 10^{-3}; $\ell n(b/a)$ = 6.908, $\sin(\Psi/2)$ = 0.2474. Also let s = 15 m, ℓ = 7.5 m so that $k\ell$ = 0.1568. It follows that H(15,7.5) = -0.1094 + j0.0168; G(.5) = 0.0856; $F_\theta(.5,\pi/2,\pi)$ = (4.43 + j0.829) × 10^{-2}; Z_c = 414.48 ohms. With these values

$$\left| \frac{I_z(0)}{E^{inc}} \right| = 0.026\ \mu A$$

If E^{inc} is 10^5 volts/m, $|I_z(0)|$ = 2.6 mA. In this example the dimensions of the slot are $b\Psi$ = 0.5 m (19.69") along the circumference and W = 0.25 m (9.84") along the axial direction. Note that kW = 0.00525 << 1 and $kb\Psi$ = 0.0105 << 1.

CONCLUSION

A missile with its access plate removed has been approximated by a large coaxial cable with an arbitrarily located transverse slot of rectangular shape cut in the sheath. The inner circuit is represented by the inner conductor with a load impedance in series at each end. Formulas have been derived for the currents in the load impedances in terms of the magnitude of a plane-wave field incident on the slot.

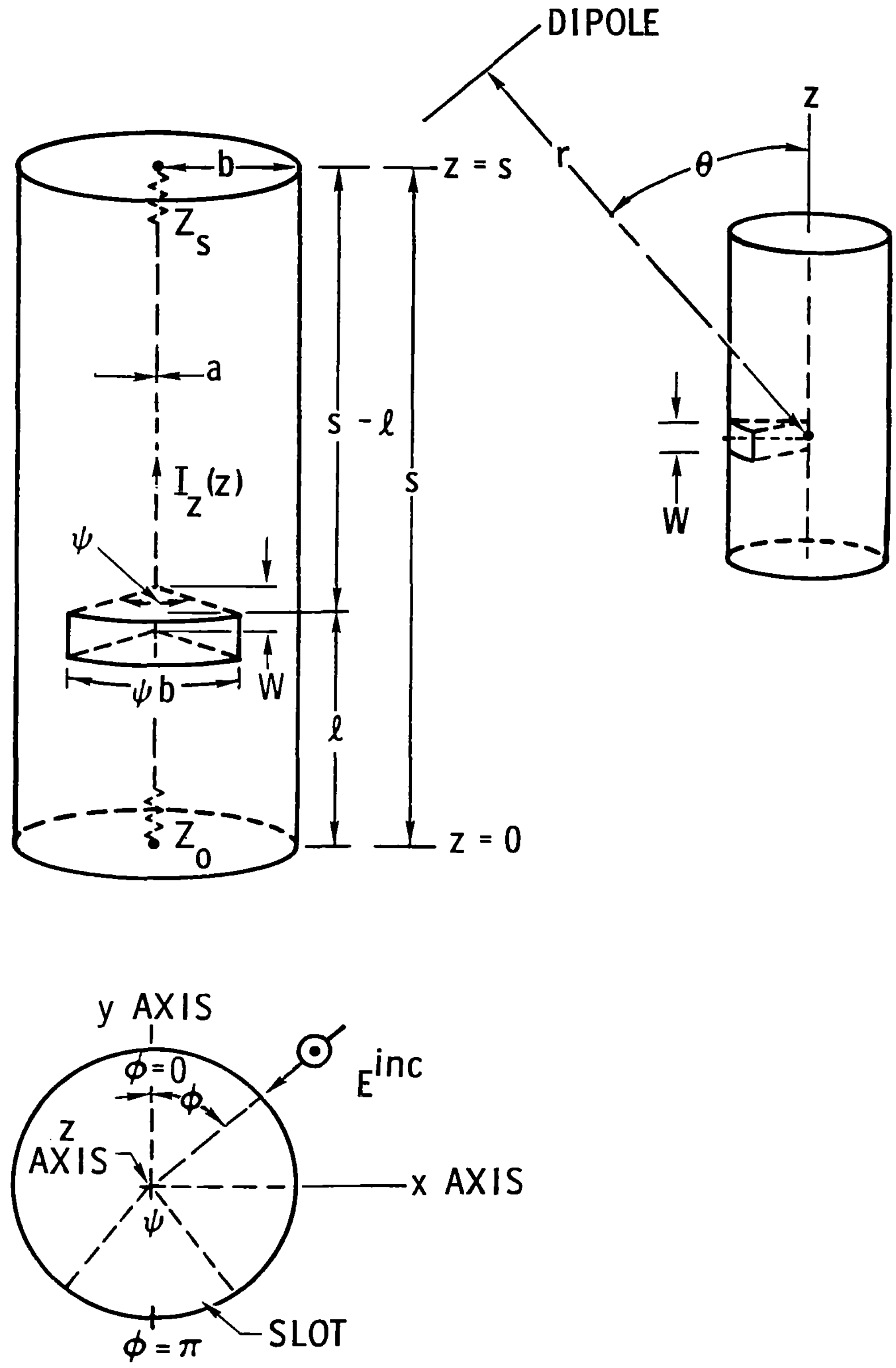

Figure 1. Coordinates and Parameters for Internally Loaded Cylinder with Aperture

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[5] Ibid., p. 247, Eq. (5-149).

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