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# INTERIM REPORT
# RESPONSE OF A MULTICONDUCTOR CABLE TO EXCITATION AT AN OPEN BREAK IN THE SHIELD

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## 1. Introduction

Report No. FA-162 [1]* gave the results of the analysis of the response of a multiconductor cable to excitation at an open, circumferentially complete, break in the shield. The present report presents the analysis leading to those results. Figure 1 shows schematically the type of situation to be analyzed. A cable containing N conductors runs between two sets of arbitrary terminations. The whole arrangement is shielded continuously except at a circumferential break in the cable shield, assumed uniform in width, and narrow enough compared to a wavelength to be treated as a negligible portion of the total length of the cable. A voltage, $V_g$, (rms) at radian frequency, $\omega$, is assumed imposed across the gap. This voltage causes TEM transmission line currents to travel along the cable (cross-section much less than a wavelength) and local fringing-field currents to flow across the gap. This report is concerned only with the TEM transmission-line currents.

*Numbers in [ ] correspond to Reference list, page 32.

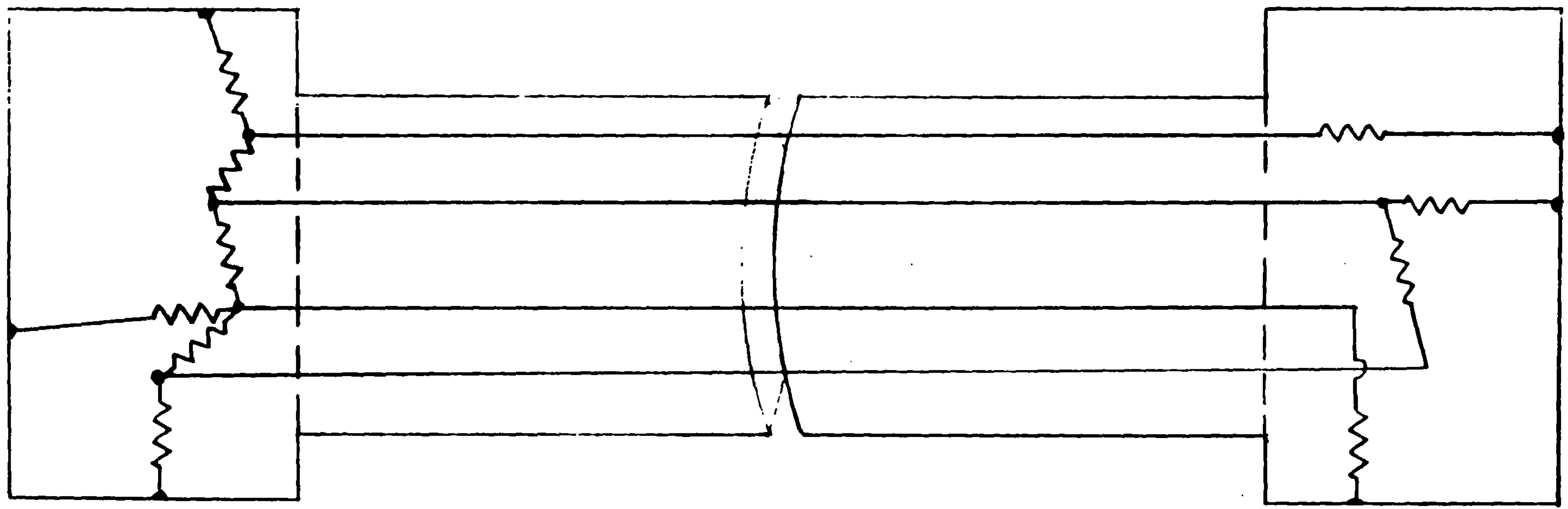

Fig.1. Cable With a Broken Shield

## 2. Mathematical Model

A mathematical model for studying this transmission-line response is obtained by considering the broken cable to be two cables with certain interacting terminal conditions at their junction (the shield break point), and each with its own terminal conditions in the respective terminal boxes of Figure 1. Figure 2 shows a simplified schematic representing this model. For convenience the figure is shown as a conventional two-conductor line, or rather, as two such lines driven in series by a source emf, $V_g$. However, in the diagram, each of the upper traces (±) represents N parallel conductors of arbitrary but unvarying cross-section geometry, while the lower traces (±) represent the broken shield and its impressed emf, $V_g$. Each part of the broken shield is taken to be at reference potential for its associated N conductors. Such an arrangement will be called an N-line.

Bars under the symbols indicate that the symbols are matrices. Thus,

$$\underline{I}_+^i = \text{nx1 matrix, or column vector}$$

$$= \begin{bmatrix} I_1^i \\ I_2^i \\ \cdot \\ \cdot \\ \cdot \\ I_N^i \end{bmatrix} \tag{1a}$$

where $I_m^i$ is the input current to the $m^{th}$ conductor on the right-hand line, $m = 1, \ldots, N$.

Similarly,

$$\underline{I}_-^i = \begin{bmatrix} I_{-1}^i \\ I_{-2}^i \\ \cdot \\ \cdot \\ \cdot \\ I_{-N}^i \end{bmatrix}, \quad \underline{V}_+^i = \begin{bmatrix} V_1^i \\ V_2^i \\ \cdot \\ \cdot \\ \cdot \\ V_N^i \end{bmatrix}, \quad \underline{V}_-^i = \begin{bmatrix} V_{-1}^i \\ V_{-2}^i \\ \cdot \\ \cdot \\ \cdot \\ V_{-N}^i \end{bmatrix} \tag{1b}$$

Similar notations apply to the output quantities $\underline{I}_\pm^o$, $\underline{V}_\pm^o$. The electrical line lengths, $\theta_+$ and $\theta_-$ are given by

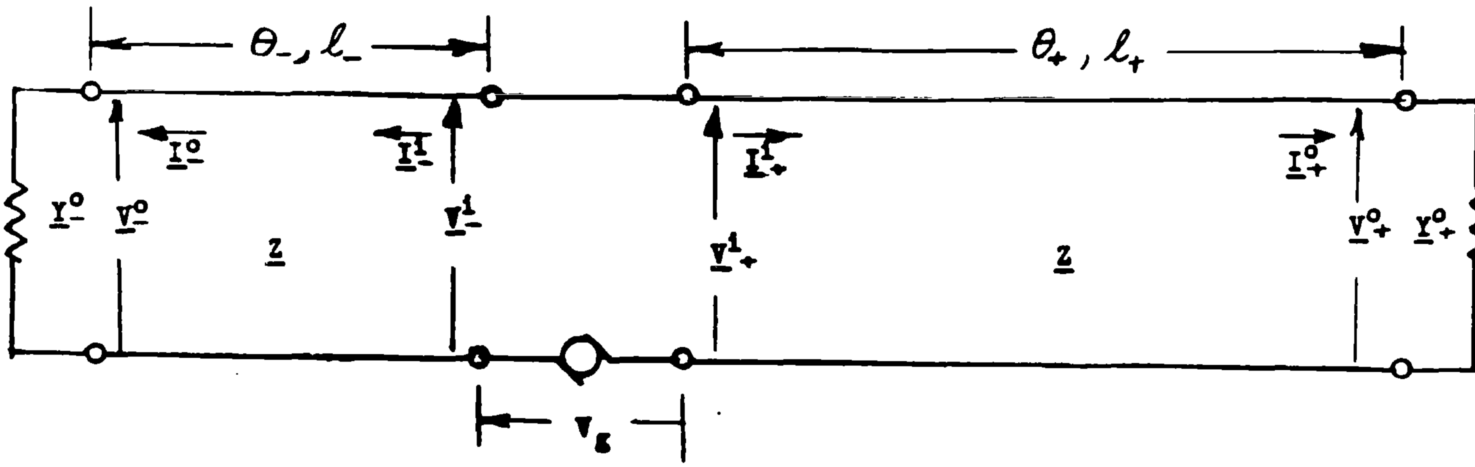


Fig.2. Multiconductor Line Model

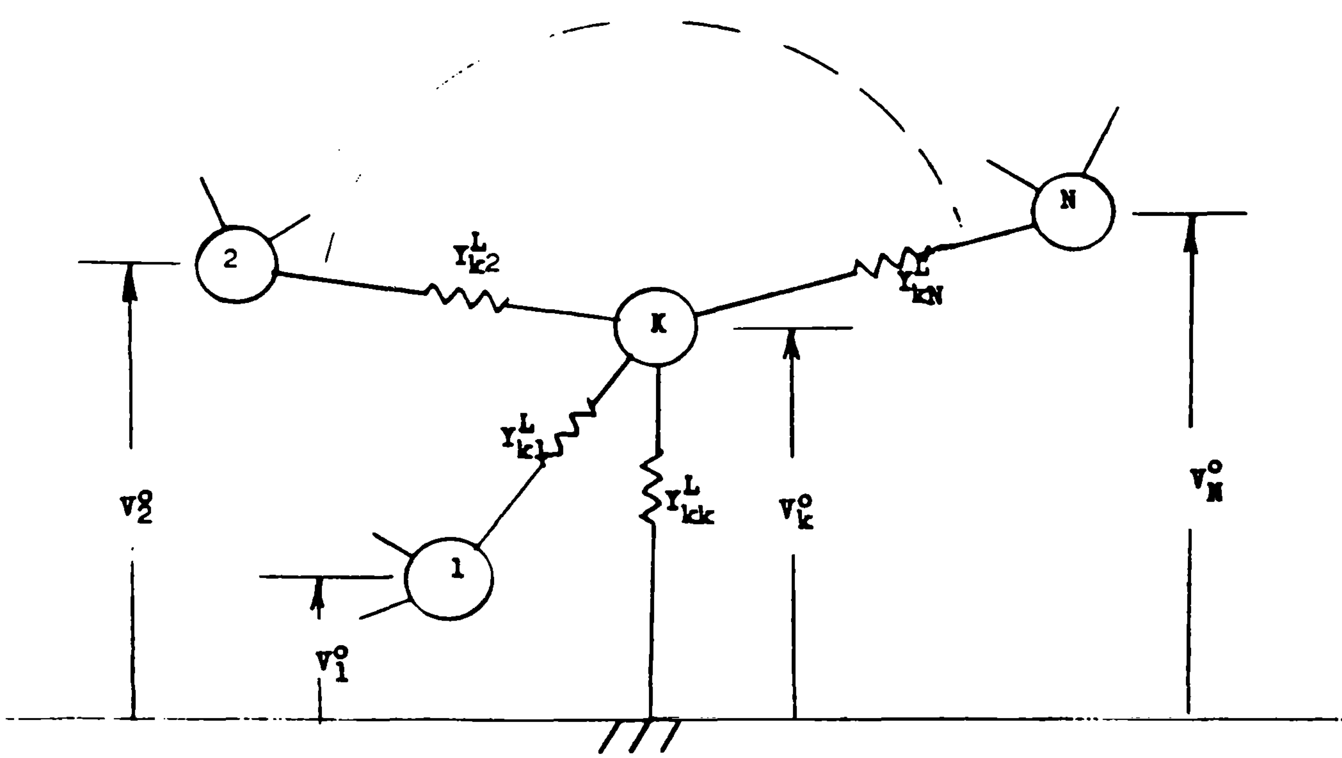


Fig. 3. Output Network Seen at the k$^{th}$ Output Terminal

$$\left.\begin{aligned} \theta_+ &= \frac{\omega \ell_+}{v} \\ \theta_- &= \frac{\omega \ell_-}{v} \end{aligned}\right\} \qquad (2)$$

where v is the velocity of wave propagation on the line:

$$v = (\mu\epsilon)^{-\frac{1}{2}} \qquad (3)$$

and $\omega = 2\pi \times$ frequency

The impedance matrix of the line is

$$\underline{Z} = (Z_{mn})_{N \times N} \qquad (4)$$

where $$Z_{mn} = Z_{nm} = p_{mn}/v, \; m,n = 1, \ldots, N \qquad (5)$$

where the $p_{mn}$ are Maxwell's potential coefficients for the line (LN, Chapter 2).*

The terminal admittance matrices, $\underline{Y}^o_\pm$ , are

$$\left.\begin{aligned} \underline{Y}^o_+ &= (Y^o_{mn}) \\ \underline{Y}^o_- &= (Y^o_{-m,-n}) \end{aligned}\right\} \pm m, \pm n = 1, \ldots, N \qquad (6)$$

The elements $Y^o_{\pm m,\pm n}$ of the terminal admittance matrices are computed from the elements of the terminating network according to the following scheme:

Consult Figure 2, which shows conditions at the $k^{th}$ output terminal of the line of Figure 1. In general, N load admittances are connected to this terminal. $Y^L_{kk}$ is connected between the terminal and the shield (indicated as ground); the remaining admittances $Y^L_{km}$ , $m \neq k$, are connected to the remaining (N-1) output terminals. The output potentials of the various terminals with respect to the shield are also indicated.

*"LN" refers to Lecture Notes for the April 1970 Seminar [2].

The terminal admittance matrices are defined in terms of the terminal voltages and currents by

$$\underline{I}^{o}_{\pm} = \underline{Y}^{o}_{\pm} \ \underline{V}^{o}_{\pm} \tag{7}$$

which yields the typical form

$$I^{o}_{k} = \sum_{m=1}^{N} Y^{o}_{km} V^{o}_{m} \tag{8}$$

On the other hand this current must be the sum of the currents entering the several branches of the termination on the kth terminal in Figure 2, i.e.,

$$I^{o}_{k} = Y^{L}_{kk} V^{o}_{k} + \sum_{\substack{m=1 \\ (k)}}^{N} Y^{L}_{km} (V^{o}_{k} - V^{o}_{m}) \ , \ k = 1, \ldots, N \tag{9}$$

where the symbol (k) under the summation sign indicates that the term corresponding to m = k is excluded from the summation.

Comparison of (8) and (9) yields

$$\left.\begin{aligned} Y^{o}_{km} &= -Y^{L}_{km} = -Y^{L}_{mk} = Y^{o}_{mk} \ , \quad k \neq m \\ Y^{o}_{kk} &= \sum_{m=1}^{N} Y^{L}_{km} \end{aligned}\right\} \tag{10}$$

Equations (10) constitute the connecting link between the load networks and the line termination matrices.

The various terminal conditions are as follows:

At the juncture of the + line and - line:

$$\left.\begin{aligned} \underline{I}^{i}_{-} + \underline{I}^{i}_{+} &= 0 \\ V_{g} \underline{\mathcal{L}} + \underline{V}^{i}_{-} - \underline{V}^{i}_{+} &= 0 \end{aligned}\right. \tag{11}$$

where $\underline{\mathcal{L}}_c$ is the Nx1 unit column vector:

$$\underline{\mathcal{L}}_c = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}_N \tag{12}$$

At the line terminations (loads) the terminal relations have already been stated by Equations (7).

## 3. Derivation of Formulas for $\underline{I}^i_{\pm}$ , $\underline{I}^o_{\pm}$ , $\underline{V}^i_{\pm}$ , $\underline{V}^o_{\pm}$.

We assume the lines to be lossless and write

$$\left.\begin{aligned} a_{\pm} &= -j \cot \theta_{\pm} \\ b_{\pm} &= j \csc \theta_{\pm} \\ j &= \sqrt{-1} \end{aligned}\right\} \tag{13}$$

then [LN, page 2 - 21, Equation (22)]

$$\begin{aligned} \underline{V}^i_+ &= a_+ \underline{Z}\, \underline{I}^i_+ + b_+ \underline{Z}\, \underline{I}^o_+ \\ \underline{V}^o_+ &= -b_+ \underline{Z}\, \underline{I}^i_+ - a_+ \underline{Z}\, \underline{I}^o_+ \\ \underline{V}^i_- &= a_- \underline{Z}\, \underline{I}^i_- + b_- \underline{Z}\, \underline{I}^o_- \\ \underline{V}^o_- &= -b_- \underline{Z}\, \underline{I}^i_- - a_- \underline{Z}\, \underline{I}^o_- \end{aligned} \tag{14}$$

Use Equations (7) and (11) to eliminate $\underline{V}^i_-$ , $\underline{I}^i_-$ , and $\underline{I}^o_{\pm}$ from Equations (14); then writing

$$\underline{P}_{\pm} = \underline{Z}\, \underline{Y}^o_{\pm} \tag{15}$$

and $\underline{\mathcal{I}} =$ N x N unit matrix

$$= \begin{bmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & \cdot & & \\ 0 & & & \cdot & \\ & & & \cdot & 1 \end{bmatrix}_N \tag{16}$$

reduces Equations (14) to the form

$$\left.\begin{array}{llll} \underline{V}_+^i & -b_+ \underline{P}_+ + \underline{V}_+^o & -a_+ \underline{Z}\, \underline{I}_+^i & = 0 \\ & (\underline{\mathcal{I}} + a_+ \underline{P}_+)\underline{V}_+^o & +b_+ \underline{Z}\, \underline{I}_+^i & = 0 \\ \underline{V}_+^i & & +a_- \underline{Z}\, \underline{I}_+^i - b_- \underline{P}_- \underline{V}_-^o & = V_g \underline{\mathcal{I}}_c \\ & & b_- \underline{Z}\, \underline{I}_+^i - (\underline{\mathcal{I}} + a_- \underline{P}_-)\underline{V}_-^o & = 0 \end{array}\right\} \tag{17}$$

Define

$$\left.\begin{array}{l} \underline{M}_\pm = a_\pm \underline{\mathcal{I}} + \underline{P}_\pm \\ \underline{N}_\pm = \underline{\mathcal{I}} + a_\pm \underline{P}_\pm \end{array}\right\} \tag{18}$$

$\underline{N}_\pm$ is used in Equations (17) to simplify algebraic manipulation. $\underline{M}_\pm$ will be required later.

Equations (17) become

$$\left.\begin{array}{llll} \underline{V}_+^i & -b_+ \underline{P}_+ \underline{V}_+^o & -a_+ \underline{Z}\, \underline{I}_+^i & = 0 \\ & \underline{N}_+ \underline{V}_+^o & +b_+ \underline{Z}\, \underline{I}_+^i & = 0 \\ \underline{V}_+^i & & +a_- \underline{Z}\, \underline{I}_+^i - b_- \underline{P}_- \underline{V}_-^o & = V_g \underline{\mathcal{I}}_c \\ & & b_- \underline{Z}\, \underline{I}_+^i - \underline{N}_- \underline{V}_-^o & = 0 \end{array}\right\} \tag{19}$$

Solve the second and fourth of these for $\underline{V}_+^o$ and $\underline{V}_-^o$ respectively:

$$\left.\begin{aligned} \underline{V}^o_+ &= -b_+ \, \underline{N}_+^{-1} \, \underline{Z} \, \underline{I}^i_+ \\ \underline{V}^o_- &= b_- \, \underline{N}_-^{-1} \, \underline{Z} \, \underline{I}^i_+ \end{aligned}\right\} \tag{20}$$

Substituting the first of Equations (20) in the first of Equations (19) and solving for $\underline{V}^i_+$:

$$\underline{V}^i_+ = (a_+ - b_+^2 \, \underline{P}_+ \, \underline{N}_+^{-1}) \, \underline{Z} \, \underline{I}^i_+ \tag{21}$$

Substituting Equations (21) and the second of Equations (20) in the third of Equations (19), and collecting terms,

$$(a_+ \underline{\mathcal{U}} - b_+^2 \, \underline{P}_+ \, \underline{N}_+^{-1} - b_-^2 \, \underline{P}_- \, \underline{N}_-^{-1} + a_- \underline{\mathcal{U}}) \; \underline{Z} \, \underline{I}^i_+ = V_g \underline{\mathcal{E}}_c \tag{22}$$

By Equations (13) and elementary trigonometry,

$$b_\pm^2 = -\csc^2 \theta_\pm = -\cot^2 \theta_\pm - 1 = a_\pm^2 - 1 \tag{23}$$

Therefore, with the help of Equations (18) we have

$$\begin{aligned} a_\pm \underline{\mathcal{U}} - b_\pm^2 \, \underline{P}_\pm \, \underline{N}_\pm^{-1} &= [a_\pm \, \underline{N}_\pm + (1 - a_\pm^2) \, \underline{P}_\pm] \, \underline{N}_\pm^{-1} \\ &= [a_\pm \, (\underline{\mathcal{U}} + a_\pm \, \underline{P}_\pm) + (1 - a_\pm^2) \, \underline{P}_\pm] \, \underline{N}_\pm^{-1} \\ &= (a_\pm \underline{\mathcal{U}} + \underline{P}_\pm) \, \underline{N}_\pm^{-1} \\ &= \underline{M}_\pm \, N_\pm^{-1} \end{aligned} \tag{24}$$

Then Equations (24) in (22) yield

$$(\underline{M}_+ \, \underline{N}_+^{-1} + \underline{M}_- \, \underline{N}_-^{-1}) \, \underline{Z} \, \underline{I}^i_+ = V_g \underline{\mathcal{E}}_c \tag{25}$$

Write

$$\underline{\Lambda} = \underline{M}_+ \underline{N}_+^{-1} + \underline{M}_- \underline{N}_-^{-1} \qquad (26)$$

Thus

$$\underline{\Lambda} \; \underline{Z} \; \underline{I}_+^i = V_g \underline{\mathcal{I}_c}$$

and, finally,

$$\underline{I}_+^i = V_g (\underline{\Lambda} \; \underline{Z})^{-1} \underline{\mathcal{I}_c} = -\underline{I}_-^i \qquad (27)$$

by the first of Equations (11).

Using Equations (24) and (27) in (21),

$$\underline{V}_+^i = \underline{M}_+ \; \underline{N}_+^{-1} \underline{Z} \; \underline{I}_+^i$$

$$= V_g \underline{M}_+ \; \underline{N}_+^{-1} \underline{Z} \; (\underline{\Lambda} \; \underline{Z})^{-1} \underline{\mathcal{I}_c}$$

$$= V_g \underline{M}_+ \; \underline{N}_+^{-1} \underline{\Lambda}^{-1} \underline{\mathcal{I}_c}$$

$$= V_g \underline{M}_+ (\underline{\Lambda} \; \underline{N}_+)^{-1} \underline{\mathcal{I}_c} \qquad (28)$$

Using Equation (27) in Equations (20),

$$\underline{V}_\pm^o = \mp b_\pm V_g \underline{N}_\pm^{-1} \underline{Z} \; (\underline{\Lambda} \; \underline{Z})^{-1} \underline{\mathcal{I}_c}$$

$$= \mp b_\pm V_g \underline{N}_\pm^{-1} \underline{\Lambda}^{-1} \underline{\mathcal{I}_c}$$

$$\underline{V}_\pm^o = \mp b_\pm V_g (\underline{\Lambda} \; \underline{N}_\pm)^{-1} \underline{\mathcal{I}_c} \qquad (29)$$

Substituting Equation (28) in the second of Equations (11),

$$\underline{V}_-^i = \underline{V}_+^i - V_g \underline{\mathcal{I}_c}$$

$$= V_g [\underline{M}_+ (\underline{\Lambda}\ \underline{N}_+)^{-1} - \underline{U}] \underline{\mathcal{I}_c}$$

$$= V_g [\underline{M}_+ - \underline{\Lambda}\ \underline{N}_+][\underline{\Lambda}\ \underline{N}_+]^{-1} \underline{\mathcal{I}_c}$$

$$= V_g [\underline{M}_+ - (\underline{M}_+ \underline{N}_+^{-1} + \underline{M}_- \underline{N}_-^{-1}) \underline{N}_+][\underline{\Lambda}\ \underline{N}_+]^{-1} \underline{\mathcal{I}_c}$$

$$= V_g [\underline{M} + \underline{N}_+^{-1} - (\underline{M}_+ \underline{N}_+^{-1} + \underline{M}_- \underline{N}_-^{-1})][\underline{N}_+ (\underline{\Lambda}\ \underline{N}_+)^{-1}] \underline{\mathcal{I}_c}$$

$$= -V_g \underline{M}_- \underline{N}_-^{-1} [(\underline{\Lambda}\ \underline{N}_+) \underline{N}_+^{-1}]^{-1} \underline{\mathcal{I}_c}$$

$$= -V_g \underline{M}_- (\underline{N}_-^{-1} \underline{\Lambda}^{-1}) \underline{\mathcal{I}_c}$$

and thus,

$$\underline{V}_-^i = -V_g \underline{M}_- (\underline{\Lambda}\ \underline{N}_-)^{-1} \underline{\mathcal{I}_c} \tag{30}$$

Finally, Equations (29) in (7) yield

$$\underline{I}_\pm^o = \mp b_\pm V_g \underline{Y}_\pm^o (\underline{\Lambda}\ \underline{N}_\pm)^{-1} \underline{\mathcal{I}_c} \tag{31}$$

For convenience, these results (namely, Equations (27) to (31)) are grouped together as follows:

$$\left.\begin{aligned}
\underline{I}_\pm^i &= \pm V_g (\underline{\Lambda}\ \underline{Z})^{-1} \underline{\mathcal{I}_c} && \text{(a)}\\
\underline{V}_\pm^i &= \pm V_g \underline{M}_\pm (\underline{\Lambda}\ \underline{N}_\pm)^{-1} \underline{\mathcal{I}_c} && \text{(b)}\\
\underline{V}_\pm^o &= \mp b_\pm V_g (\underline{\Lambda}\ \underline{N}_\pm)^{-1} \underline{\mathcal{I}_c} && \text{(c)}\\
\underline{I}_\pm^o &= \mp b_\pm V_g \underline{Y}_\pm^o (\underline{\Lambda}\ \underline{N}_\pm)^{-1} \underline{\mathcal{I}_c} && \text{(d)}
\end{aligned}\right\} \tag{32}$$

$\underline{\Lambda}$ is defined by Equation (26), $\underline{M}_{\pm}$ and $\underline{N}_{\pm}$ are defined by Equations (18), $a_{\pm}$, $b_{\pm}$, and $\underline{P}_{\pm}$ are defined by Equations (13) and (15). $\underline{Y}^{o}_{\pm}$ are calculated from the terminal network admittances using Equations (10).

One additional result can be stated at this time. The internal input admittance at the break in the shield is (see Figure 2)

$$Y^{i} = \left(\sum_{m=1}^{N} -I^{i}_{-m}\right) / V_g = \left(\sum_{m=1}^{N} I^{i}_{m}\right) / V_g$$

$$= [(\underline{I}_{+})^{T} \underline{\mathscr{I}_c}]/V_g$$

where $(\underline{I}_{+})^{T}$ is the transpose of $\underline{I}_{+}$ .

4. Discussion of Results

It must be stated immediately that, except for the case of a 1-line, (N = 1), hand computation of the quantities in Equations (32) is quite laborious, and the labor increases exponentially with N.

For instance, for N = 2, the elements of the $\underline{M}_{+}$ and $\underline{N}_{+}$ matrices are, respectively,

$$\left.\begin{aligned}
M_{11} &= a_{+} + Z_{11} Y^{o}_{11} + Z_{12} Y^{o}_{21} \\
M_{12} &= Z_{11} Y^{o}_{12} + Z_{12} Y^{o}_{22} \\
M_{21} &= Z_{21} Y^{o}_{11} + Z_{22} Y^{o}_{21} \\
M_{22} &= a_{+} + Z_{21} Y^{o}_{12} + Z_{22} Y^{o}_{22}
\end{aligned}\right\}$$

and

$$\left.\begin{aligned}
N_{11} &= 1 + a_{+} Z_{11} Y^{o}_{11} + Z_{12} Y^{o}_{21} \\
N_{12} &= a_{+} Z_{11} Y^{o}_{12} + a_{+} Z_{12} Y^{o}_{22} \\
N_{21} &= a_{+} Z_{21} Y^{o}_{12} + a_{+} Z_{22} Y^{o}_{21} \\
N_{22} &= 1 + a_{+} Z_{21} Y^{o}_{12} + a_{+} Z_{22} Y^{o}_{22}
\end{aligned}\right\}$$

In order to compute $\underline{M}_+ \; \underline{N}_+^{-1}$ it is necessary first to invert $\underline{N}_+$. This requires evaluating the determinant

$$D_N = \begin{vmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{vmatrix} = N_{11} N_{22} - N_{12} N_{21}$$

From these we can write

$$\underline{N}_+^{-1} = \frac{1}{D_N} \begin{bmatrix} N_{22} \, , & -N_{21} \\ -N_{12} \, , & N_{11} \end{bmatrix}$$

Having obtained the product $\underline{M}_+ \; \underline{N}_+^{-1}$, the process must be repeated for $\underline{M}_- \; \underline{N}_-^{-1}$. The two products may then be summed to obtain $\underline{\Lambda}$. After $\underline{\Lambda}$ has been multiplied by $\underline{Z}$, the result is again inverted, after which, multiplication by $\underline{\mathcal{Y}}_c$ yields, essentially, the two input currents $I_1^i$ and $I_2^i$.

However, the case of N = 2 is not very interesting for the present investigation. The case of a 7-line is of typical interest. Clearly, solution of Equations (32) by machine computation is indicated.

It is of interest to check the analysis of the response of the system under certain special conditions, namely, with matched loads, and, in general, with the output admittance matrices proportional to the line admittance matrix.

### 4.1 Matched Loads

Sufficient conditions for matched loading will be stated as

$$\underline{P}_{\pm} = \underline{Z} \; \underline{Y}_{\pm}^{o} = \underline{\mathcal{I}} \qquad (33)$$

that is

$$\underline{Y}_{\pm}^{o} = \underline{Z}^{-1} = \underline{Y} \qquad (34)$$

where $\underline{Y}$ is the line admittance matrix.

Under condition (33) Equations (18) become

$$\underline{M}_{\pm} = (a_{\pm} + 1)\, \underline{\ell} = \underline{N}_{\pm}$$

Equation (26) becomes

$$\underline{\Lambda} = 2\, \underline{\ell}$$

Equations (32a - d) become, respectively,

$$\left.\begin{aligned}
\underline{I}^{i}_{\pm} &= \pm\, V_g (2\, \underline{Z})^{-1} \underline{\ell_c} = \pm \tfrac{1}{2}\, V_g\, \underline{Y}\, \underline{\ell_c} && \text{(a)}\\
\underline{V}^{i}_{\pm} &= \pm\, V_g\, (a_{\pm} + 1)\, \underline{\ell} [2(a_{\pm} + 1)\, \underline{\ell}]^{-1} \underline{\ell_c} = \pm \tfrac{1}{2}\, V_g\, \underline{\ell_c} && \text{(b)}\\
\underline{V}^{o}_{\pm} &= \mp\, b_{\pm}\, V_g\, [2(a_{\pm} + 1)]^{-1} \underline{\ell_c} \\
&= \mp \tfrac{1}{2} \frac{b_{\pm}}{a_{\pm} + 1}\, V_g\, \underline{\ell_c} = \pm \tfrac{1}{2}\, V_g\, e^{-j\theta\pm}\, \underline{\ell_c} && \text{(c)}\\
\underline{I}^{o}_{\pm} &= \pm \tfrac{1}{2}\, V_g\, e^{-j\theta\pm}\, \underline{Y}\, \underline{\ell_c} && \text{(d)}
\end{aligned}\right\} \quad (35)$$

Note that if we write

$Y^{c}_{m}$ = common mode characteristic admittance of the $m^{th}$ conductor with respect to ground

$$= \sum_{k=1}^{N} Y_{mk} \qquad (36)$$

then the quantity $\underline{Y}\, \underline{\ell_c}$ appearing in Equations (35a, d) may be written

$$\underline{Y}\, \underline{\ell_c} = \begin{bmatrix} Y^{c}_{1} \\ Y^{c}_{2} \\ \cdot \\ \cdot \\ \cdot \\ Y^{c}_{N} \end{bmatrix} = \underline{Y}^{c} \qquad \text{(say)} \qquad (37)$$

Implications of the results in Equations (35a - d) are:

1. When both ends of the cable are match-terminated, the input voltages to all conductors have the same magnitude, $|\frac{1}{2} V_g|$, but voltages on the opposite sides of the break have opposite signs.

2. The input currents to the two sides of the break satisfy the equations for waves travelling in one direction only, - away from the break.

3. The voltages at the terminations are all equal in magnitude, and have the values of the corresponding input voltages with phase delays equal to the electrical distance from the break to the termination.

These results accord with our usual notions for matched terminations which, in essence, require that no wave be reflected at the termination. Thus Equations (33) and (34) are consistent with normal requirements for matched terminations.

Equations (34) and Equations (10) together imply

$$Y^{o}_{km} = Y_{km} = - Y^{L}_{km} = - Y^{L}_{mk} = Y_{mk} = Y^{o}_{mk}, \ k \neq m \qquad (38a)$$

and, consequently

$$Y^{o}_{kk} = Y_{kk} = \sum_{m=1}^{N} Y^{L}_{km} = Y^{L}_{kk} - \sum_{\substack{m=1 \\ (k)}}^{N} Y_{km}$$

whence

$$Y^{L}_{kk} = \sum_{m=1}^{N} Y_{km} = Y^{c}_{m} \qquad (38b)$$

by Equation (36). Thus for match conditions, the terminating admittance from each terminal to ground is the common-mode characteristic admittance for that conductor. The terminating admittance required between any pair of terminals is apparently the negative of the mutual admittance coefficient for that pair of conductors. However, since by Equation (35c), all terminal voltages at one end of the cable are equal, no current flows in the terminating admittances joining these terminals. Consequently these admittances may have any values, including zero.

4.2 Termination Matrices Proportional to Line Admittance Matrix.

The foregoing is a special case of the more general class of terminations in which the terminal admittance matrices are proportional to the admittance matrix.

Write

$$\underline{Y}^{o}_{\pm} = k_{\pm}\,\underline{Y} \tag{39}$$

where $k_{\pm}$ are scalar constants. Then

$$\left.\begin{aligned} \underline{P}_{\pm} &= k_{\pm}\,\underline{\mathscr{I}} \\ \underline{M}_{\pm} &= (a_{\pm} + k_{\pm})\,\underline{\mathscr{I}} \\ \underline{N}_{\pm} &= (1 + a_{\pm}\,k_{\pm})\,\underline{\mathscr{I}} \end{aligned}\right\} \tag{40}$$

Write

$$\left.\begin{aligned} L_{\pm} &= \frac{a_{\pm} + k_{\pm}}{1 + a_{\pm}\,k_{\pm}} \\ L &= L_{+} + L_{-} \end{aligned}\right\} \tag{41}$$

Thus

$$\left.\begin{aligned} \underline{M}_{\pm}\,\underline{N}_{\pm}^{-1} &= L_{\pm}\,\underline{\mathscr{I}} \\ \underline{\Lambda} &= L\,\underline{\mathscr{I}} \\ \underline{\Lambda}\,\underline{Z} &= L\,\underline{Z} \\ \underline{\Lambda}\,\underline{N}_{\pm} &= L(1 + a_{\pm}\,k_{\pm})\,\underline{\mathscr{I}} \end{aligned}\right\} \tag{42}$$

Equations (32) become, respectively,

$$\begin{aligned}
&\underline{I}^{i}_{\pm} = \pm \frac{V_g}{L}\, \underline{Y}^{c} && \text{(a)}\\
&\underline{V}^{i}_{\pm} = \pm V_g \frac{L_{\pm}}{L_{+} + L_{-}}\, \underline{\mathcal{I}c} && \text{(b)}\\
&\underline{V}^{o}_{\pm} = \pm \frac{V_g}{L} \frac{-b_{\pm}}{1 + a_{\pm}\, k_{\pm}}\, \underline{\mathcal{I}c} && \text{(c)}\\
&\underline{I}^{o}_{\pm} = \pm k_{\pm} \frac{V_g}{L} \frac{-b_{\pm}}{1 + a_{\pm}\, k_{\pm}}\, \underline{Y}^{c} && \text{(d)}
\end{aligned} \qquad (43)$$

For matched conditions, $k_{\pm} = 1$. Note that again, for the proportional-termination case, the whole line acts in the common mode, with all conductor potentials equal.* Each conductor carries a current proportional to its common-mode characteristic admittance. Only terminal admittances to ground need meet the proportionality requirement.

Operation in the common mode implies that all conductors (except the shield) operate in parallel, so that the system is, in effect, a 1-line with characteristic admittance

$$Y^{c}_{o} = \sum_{k=1}^{N} Y^{c}_{k} = \sum_{k=1}^{N} \sum_{j=1}^{N} Y_{kj} \qquad (44)$$

by Equation (36). Then $k_{\pm}$ are the VSWR's (or their reciprocals) corresponding to the total load admittances at either end of the cable.

From the first of Equations (41),

$$L_{\pm} = \frac{-j \cot \theta_{\pm} + \dfrac{Y^{o}_{\pm}}{Y^{c}_{o}}}{1 - j\left(\dfrac{Y^{o}_{\pm}}{Y^{c}_{o}}\right) \cot \theta_{\pm}} \qquad (45)$$

where $Y^{o}_{\pm}$ are the total output admittances in parallel at either end. Equation (45) is recognized as the usual normalized mapping of the load impedance of a

*At any one line cross-section.

1-line to its input. [Reference 3, page 22-4.] The quantity, L (Equations (41)) is therefore the sum of these normalized impedances in series. The admittance seen by the voltage source at the break is then $L^{-1}$ multiplied by $Y_o^c$ (cf. Equation (43a)). Equation (43b) then states that the driving voltages on opposite sides of the break divide in proportion to these input-impedances.

## 4.3 Interpretation of $\underline{M}_{\pm}\ \underline{N}_{\pm}^{-1}$ and $\underline{\Lambda}$

The interpretations above suggest that in the general case, (Equation (39) not satisfied), the quantities $\underline{M}_{\pm}\ \underline{N}_{\pm}^{-1}$ are to be interpreted as normalized mappings of the load impedance matrices to the input (break-point) terminals, $\underline{\Lambda}$ is then the sum of these normalized impedances, while $\underline{\Lambda}\ \underline{Z}$ is the total actual input impedance matrix, since the line impedance matrix, $\underline{Z}$, is just the factor required to neutralize the normalization. The quantity

$$\underline{Q}_{\pm} = \underline{M}_{\pm}\ (\underline{\Lambda}\ \underline{N}_{\pm})^{-1}$$

which appears in Equation (32b) is easily transformed to

$$\underline{Q}_{\pm} = (\underline{M}_{\pm}\ \underline{N}_{\pm})(\underline{M}_{+}\ \underline{N}_{+}^{-1} + \underline{M}_{-}\ \underline{N}_{-}^{-1})^{-1}$$

which expresses the input voltage division factors in proportion to the input matrix impedances, and so on.

Note that $k_{\pm} = 0$ implies an open-circuited termination admittance matrix, while $k_{\pm} = \infty$ implies a short-circuited admittance matrix.

## 5. Impedance Matrix For a Seven-Conductor Cable

The configuration to be analyzed is shown in Figure 4. Results of this analysis were previously submitted as a memorandum identified as FA-163 [4].

The analysis is approximate, based on the assumption that conductor radii, a, are small compared to distances between their centers and from their centers to the outer shield. The method is explained, with examples, in LN, Chapter 4.*

*A relatively easy analog check on limits of accuracy may be made by modelling the configuration with Teledeltos (resistance) paper. See Appendices A and B of Reference 5.

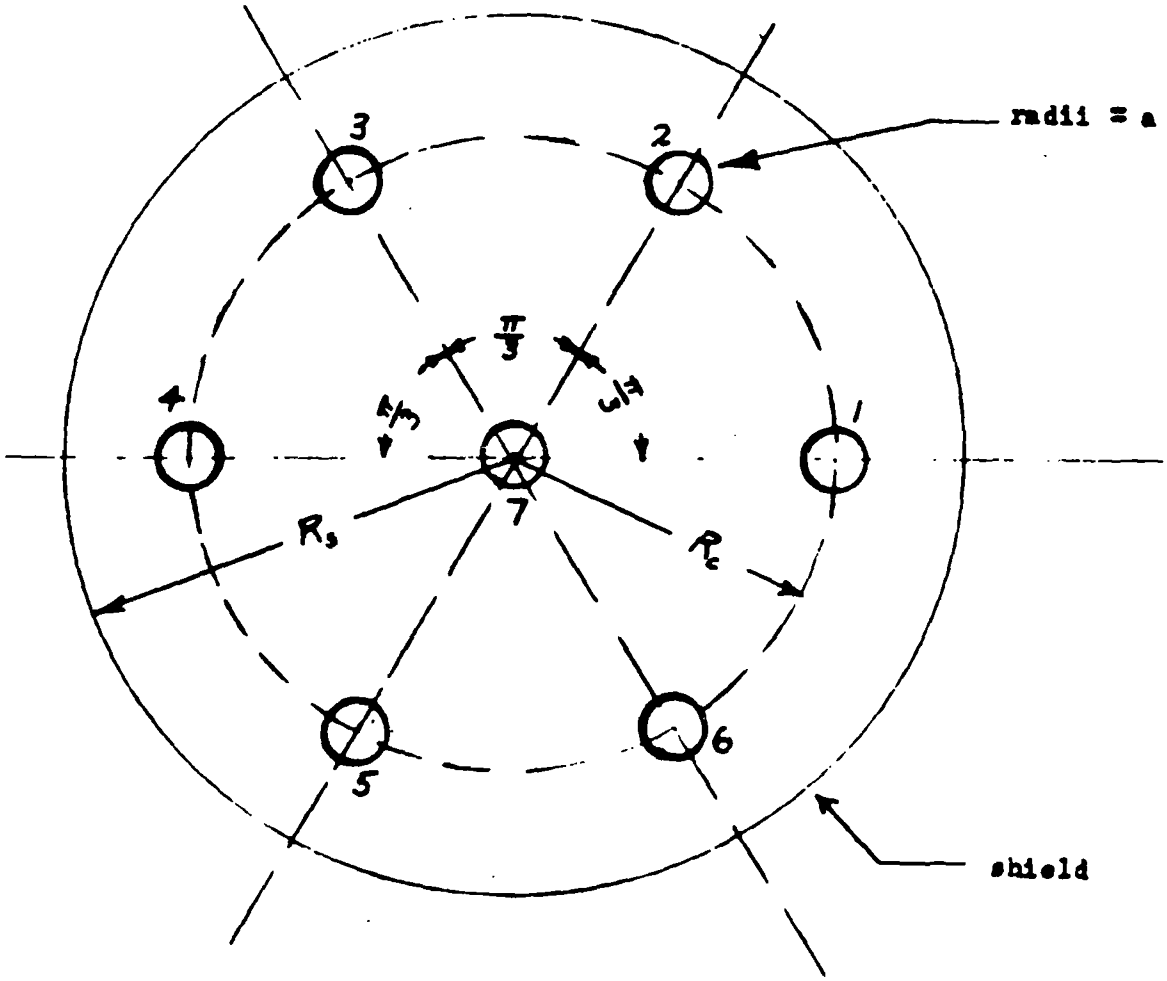


Fig. 4. Cross-Section of Seven-Conductor Shielded Cable

(Numbers adjacent to conductors correspond to subscripts of impedance coefficients, $Z_{ij}$.)

As a first step, the configuration under study is located in the complex z-plane with the outer shield tangent to the y-axis and the x-axis coinciding with its diameter (Figure 5). This configuration is next mapped onto the w-plane by means of the transformation

$$w = 1/z \tag{46}$$

(LN, Chapter 4, p. 45). Figure 6 shows the result of this operation. The shield circle has been mapped into the plane (line) parallel to the v-axis with equation

$$u = \frac{1}{2R_s} \tag{47}$$

The centers of the seven small conductors, which, in the z-plane were located at

$$\left.\begin{aligned} z_k &= R_s + R_c\, e^{j(k-1)\frac{\pi}{3}},\ k = 1, \ldots, 6 \\ z_7 &= R_s \end{aligned}\right\} \tag{48}$$

are located in the w-plane at

$$w_k = \frac{1}{z_k} \quad , k = 1, \ldots, 7$$

For sufficiently small raddi, a, in the z-plane, the traces of the conductors in the w-plane are well-approximated by circles with centers at $w_k$ (LN, Chapter 4, pp. 47-51). However, the radii of these circles are different from their common radius in the z-plane, and, in fact, are generally different from one another. If $a_w^{(k)}$ is the radius of the $k^{th}$ conductor in the w-plane,

$$a_w^{(k)} \approx \left|\frac{dw}{dz}\right|_{(k)} a = \frac{a}{|z|_k^2} = a\,|w_k|^2 \tag{49}$$

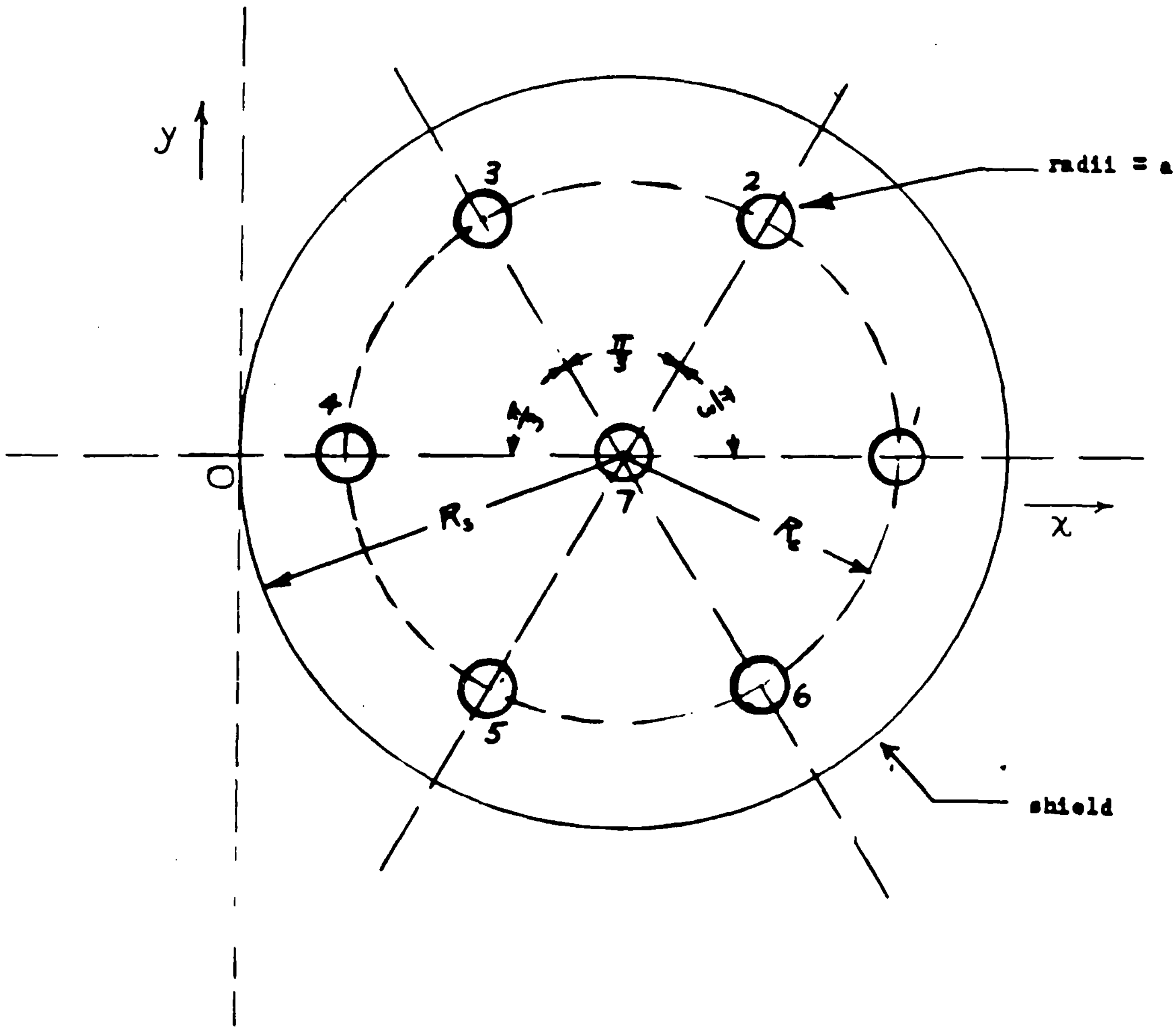


Fig. 5. Seven-Conductor Cable Cross-Section Situated in the Complex z-Plane

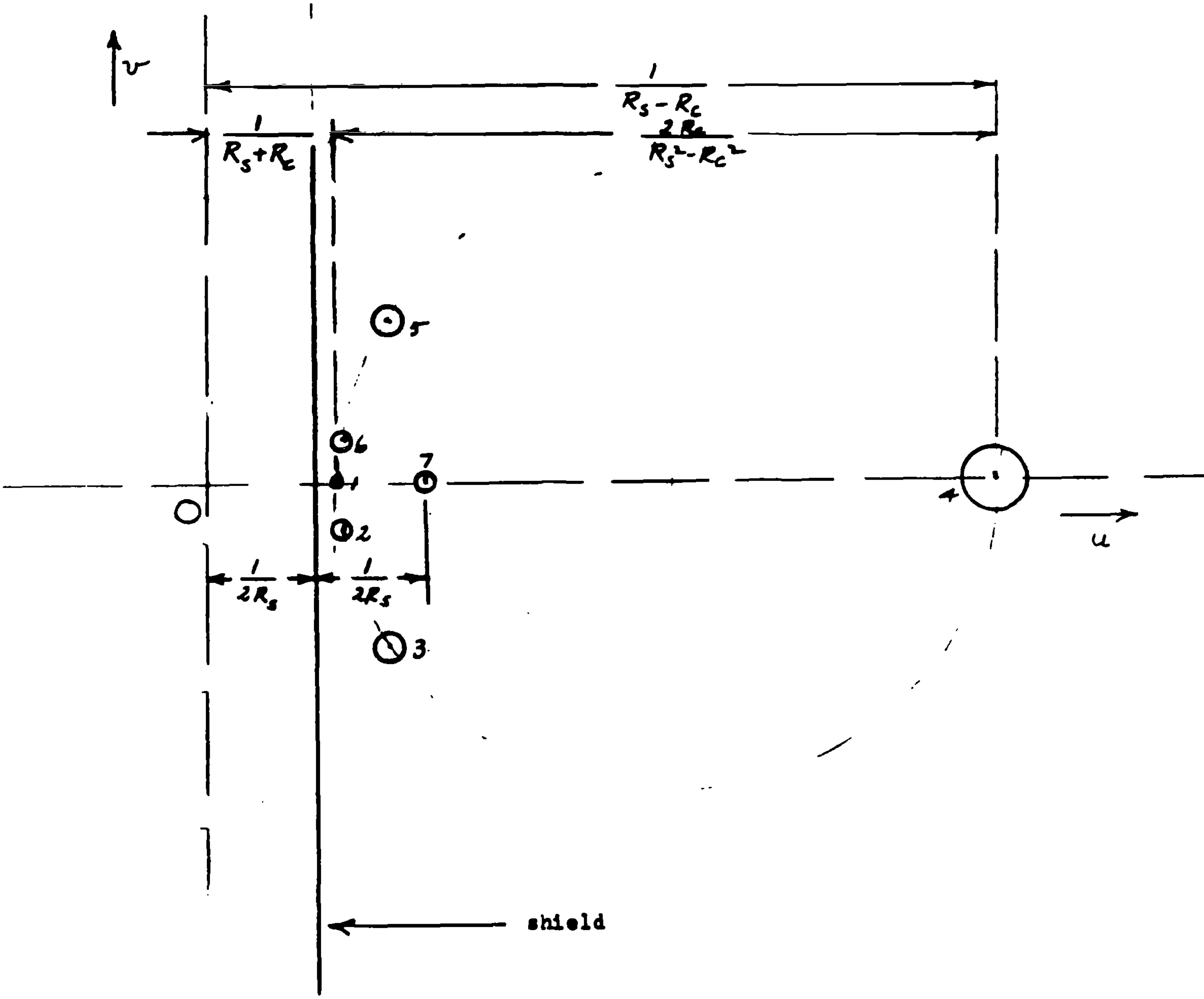


Fig. 6. Seven-Conductor Cable Mapped onto the w-Plane by Means of the Transformation $w = 1/z$

where $\left|\frac{dw}{dz}\right|_k$ is the derivative of the transformation evaluated at the $k^{th}$ conductor (LN, ibid.).

Since electrostatic potential coefficients remain invariant under a conformal transformation, the problem has been changed to finding the coefficients of a number of small circular conductors above a ground plane. This problem is discussed with examples in LN, Chapter 4, pp. 14 - 21. In general, the impedance coefficients are given by

$$\left.\begin{aligned} Z_{jk} &= \frac{p_{jk}}{v} = \sqrt{\mu\epsilon}\, p_{jk} = \frac{60}{\sqrt{\epsilon_r}} \ln \left| \frac{w_k - \bar{w}_j}{w_k - w_j} \right| , \; k \neq j \\ Z_{kk} &= \frac{60}{\sqrt{\epsilon_r}} \ln \left| \frac{w_k - \bar{w}_k}{a_w^{(k)}} \right| \end{aligned}\right\} \tag{50}$$

where

$p_{jk}$ = potential coefficient between $j^{th}$ and $k^{th}$ conductor

$v$ = velocity of propagation, m/s

$\mu$ = permeability, H/m

$\epsilon$ = permittivity, F/m

$\epsilon_r$ = relative dielectric constant

$|w_k - w_j|$ = distance between centers of $j^{th}$ and $k^{th}$ conductors in w-plane, $j \neq k$

$|w_k - \bar{w}_j|$ = distance between $k^{th}$ conductor and image of $j^{th}$ conductor, $j \neq k$

$|w_k - \bar{w}_k|$ = distance between $k^{th}$ conductor and its own image

$a_w^{(k)}$ is given by Equation (49).

A seven-conductor cable has $7^2$ or 49 impedance coefficients. However, by virtue of the fact that $Z_{ij} = Z_{ji}$ for every i, j, the number of independent coefficients reduces to $\frac{1}{2}(7x8)$, or 28. Furthermore, in the present instance, because of the symmetries involved, many of the independent coefficients are equal. In fact, the arrangement of Figure 4 yields only six different values for the various coefficients.

By inspection of Figure 4 one can readily determine which coefficients have a common value. Thus,

(1) $Z_{11} = Z_{22} = Z_{33} = Z_{44} = Z_{55} = Z_{66}$

(2) $Z_{12} = Z_{23} = Z_{34} = Z_{45} = Z_{56} = Z_{61}$

(3) $Z_{13} = Z_{24} = Z_{35} = Z_{46} = Z_{51} = Z_{62}$

(4) $Z_{14} = Z_{25} = Z_{36}$

(5) $Z_{17} = Z_{27} = Z_{37} = Z_{47} = Z_{57} = Z_{67}$

(6) $Z_{77}$

All other coefficients of the $\underline{Z}$ matrix

$$\underline{Z} = \begin{bmatrix} Z_{11}, Z_{12}, \ldots, Z_{17} \\ Z_{21}, Z_{22}, \ldots, Z_{27} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ Z_{71}, Z_{72}, \ldots, Z_{77} \end{bmatrix}$$

are obtained by using $Z_{ij} = Z_{ji}$, i, j = 1, ... , 7.

Details of calculation of the various coefficients follow.*

*It is recommended that future coefficient calculations for N > 7 be handled by computer, with numerical substitution directly in Equations (50).

(1). <u>$Z_{11}$, etc.</u>

$$z_1 = R_s + R_c \; ; \; w_1 = \frac{1}{R_s + R_c} = u_1 + j0$$

$$\bar{w}_1 = \bar{u}_1 = \frac{1}{R_s} - \frac{1}{R_s + R_c} = \frac{R_c}{R_s(R_s + R_c)}$$

$$w_1 - \bar{w}_1 = \frac{R_s - R_c}{R_s(R_s + R_c)}$$

$$a_w^{(1)} = a\,|w_1|^2 = \frac{a}{(R_s + R_c)^2}$$

$$Z_{11} = \frac{60}{\sqrt{\epsilon_r}} \ln \left| \frac{w_1 - \bar{w}_1}{a_w^{(1)}} \right|$$

$$= \frac{60}{\sqrt{\epsilon_r}} \ln \left\{ \left[ \frac{R_s - R_c}{R_s(R_s + R_c)} \right] \left[ \frac{(R_s + R_c)^2}{a} \right] \right\}$$

$$= \frac{60}{\sqrt{\epsilon_r}} \ln \left\{ \frac{R_s^2 - R_c^2}{aR_s} \right\}$$

Write

$$\rho = R_s/a$$

$$\lambda = R_c/R_s \qquad (51)$$

then

$$Z_{11} = \frac{60}{\sqrt{\epsilon_r}} \ln [\rho(1 - \lambda^2)] \qquad (52)$$

(2). $\underline{Z_{12}}$, etc.

$$z_1 = R_s + R_c = x_1 + j0$$

$$z_2 = R_s + R_c\, e^{j\frac{\pi}{3}} = (R_s + \tfrac{1}{2}R_c) + j\,\tfrac{1}{2}\sqrt{3}\,R_c$$

$$= x_2 + j\,y_2$$

$$w_1 = \frac{1}{z_1} = \frac{1}{x_1}$$

$$w_2 = \frac{1}{z_2} = \frac{1}{x_2 + j\,y_2} = \frac{x_2}{r_2^2} - j\,\frac{y_2}{r_2^2} = u_2 + jv_2$$

where

$$r_2^2 = x_2^2 + y_2^2 = R_s^2 + R_s\,R_c + R_c^2$$

$$\bar{w}_2 = \frac{1}{R_s} - u_2 + jv_2 = \frac{1}{R_s} - \frac{x_2}{r_2^2} + j\,\frac{y_2}{r_2^2}$$

$$w_1 - w_2 = \frac{1}{x_1} - \frac{x_2}{r_2^2} + j\,\frac{y_2}{r_2^2} = \frac{(r_2^2 - x_1\,x_2) + j\,x_1\,y_2}{x_1\,r_2^2}$$

$$w_1 - \bar{w}_2 = \frac{1}{x_1} - \frac{1}{R_s} + \frac{x_2}{r_2^2} - j\,\frac{y_2}{r_2^2}$$

$$= \frac{(R_s\,r_2^2 - x_1\,r_2^2 + x_1\,R_s\,x_2) - j\,x_1\,R_s\,y_2}{x_1\,R_s\,r_2^2}$$

$$\left|\frac{w_1 - w_2}{w_1 - \bar{w}_2}\right|^2 = R_s^2 \left\{\frac{(r_2^2 - x_1 x_2)^2 + x_1^2 y_2^2}{(R_s r_2^2 - x_1 r_2^2 + x_1 R_s x_2)^2 + x_1^2 R_s^2 y_2^2}\right\}$$

$$= R_s^2 \left\{\frac{r_2^2(r_2^2 - 2x_1 x_2 + x_1^2)}{r_2^2(r_2^2 R_c^2 - 2R_c x_1 R_s x_2 + x_1^2 R_s^2)}\right\}$$

$$= R_s^2 \left\{\frac{(R_s^2 + R_s R_c + R_c^2) - (R_s + R_c)(2R_s + R_c) + (R_s + R_c)^2}{R_c^2(R_s^2 + R_s R_c + R_c^2) - R_c R_s(R_s + R_c)(2R_s + R_c) + R_s^2(R_s + R_c)^2}\right\}$$

$$= R_s^2 \frac{R_c^2}{R_s^4 - R_s^2 R_c^2 + R_c^4} = \frac{\lambda^2}{1 - \lambda^2 + \lambda^4}$$

$$Z_{12} = \frac{60}{\sqrt{\epsilon_r}} \ln \left|\frac{w_1 - \bar{w}_2}{w_1 - w_2}\right|$$

$$= \frac{30}{\sqrt{\epsilon_r}} \ln \left[\frac{1 - \lambda^2 + \lambda^4}{\lambda^2}\right]$$

(3). <u>$Z_{13}$, etc.</u>

Same as $Z_{26}$

$$z_2 = (R_s + \tfrac{1}{2} R_c) + j\tfrac{1}{2}\sqrt{3}\, R_c = x_2 + jy_2$$

$$z_6 = z_2^* = x_2 - jy_2$$

$$w_2 = \frac{1}{z_2} = \frac{x_2}{r_2^2} - j\,\frac{y_2}{r_2^2}$$

$$w_6 = w_2^* = \frac{x_2}{r_2^2} + j\,\frac{y_2}{r_2^2}$$

$$\bar{w}_6 = \frac{1}{R_s} - w_6^* = \frac{1}{R_s} - w_2 = \frac{1}{R_s} - \frac{x_2}{r_2^2} + j\,\frac{y_2}{r_2^2}$$

$$w_2 - w_6 = -j\,\frac{2y_2}{r_2^2}$$

$$w_2 - \bar{w}_6 = \frac{x_2}{r_2^2} - j\,\frac{y_2}{r_2^2} - \frac{1}{R_s} + \frac{x_2}{r_2^2} - j\,\frac{y_2}{r_2^2}$$

$$= \left(\frac{2x_2}{r_2^2} - \frac{1}{R_s}\right) - j\,\frac{2y_2}{r_2^2}$$

$$\left|\frac{w_2 - \bar{w}_6}{w_2 - w_6}\right|^2 = \frac{1}{R_s^2} \cdot \frac{4x_2^2\,R_s^2 - 4x_2\,R_s\,r_2^2 + r_2^4 + 4y_2^2\,R_s^2}{4y_2^2}$$

$$= \frac{4r_2^2\,R_s^2 - 4x_2\,R_s\,r_2^2 + r_2^4}{4y_2^2\,R_s^2}$$

$$= \frac{r_2^2}{4y_2^2\,R_s^2}\left[4R_s^2 - 2R_s(2R_s + R_c) + (R_s^2 + R_s\,R_c + R_c^2)\right]$$

$$= \frac{r_2^2}{4y_2^2\,R_s^2}\,(R_s^2 - R_s\,R_c + R_c^2)$$

$$= \frac{(R_s^2 + R_s\,R_c + R_c^2)(R_s^2 - R_s\,R_c + R_c^2)}{3R_s^2\,R_c^2}$$

$$= \frac{(1 + \lambda + \lambda^2)(1 - \lambda + \lambda^2)}{3\lambda^2}$$

$$= \frac{1 + \lambda^2 + \lambda^4}{3\lambda^2}$$

$$Z_{13} = Z_{26} = \frac{30}{\sqrt{\epsilon_r}} \ln \left[\frac{1 + \lambda^2 + \lambda^4}{3\lambda^2}\right] \qquad (54)$$

(4). $Z_{14}$, etc.

$$z_1 = x_1 = R_s + R_c$$

$$z_4 = x_4 = R_s - R_c$$

$$w_1 = \frac{1}{x_1} \; ; \; w_4 = \frac{1}{x_4} \; ; \; \bar{w}_4 = \frac{1}{R_s} - w_4^* = \frac{1}{R_s} - \frac{1}{x_4} = \frac{x_4 - R_s}{R_s \, x_4}$$

$$\left|\frac{w_1 - \bar{w}_4}{w_1 - w_4}\right| = \left|\frac{\frac{1}{x_1} - \frac{x_4 - R_s}{R_s \, x_4}}{\frac{1}{x_1} - \frac{1}{x_4}}\right|$$

$$= \frac{1}{R_s} \left|\frac{R_s \, x_4 - x_1 \, x_4 + x_1 \, R_s}{x_4 - x_1}\right|$$

$$= \frac{1}{R_s} \left|\frac{R_s^2 + R_c^2}{-2R_c}\right| = \frac{1 + \lambda^2}{2\lambda}$$

$$Z_{14} = \frac{60}{\sqrt{\epsilon_r}} \ln \left| \frac{w_1 - \bar{w}_4}{w_1 - w_4} \right| = \frac{60}{\sqrt{\epsilon_r}} \ln \left[ \frac{1 + \lambda^2}{2\lambda} \right] \tag{55}$$

(5). <u>$Z_{17}$, etc.</u>

$$z_1 = R_s + R_c = x_1$$

$$z_7 = R_s = x_7$$

$$w_1 = \frac{1}{x_1} \;;\; \bar{w}_1 = \frac{1}{R_s} - \frac{1}{x_1} = \frac{x_1 - R_s}{R_s\, x_1} = \frac{R_c}{R_s\, x_1}$$

$$w_7 = \frac{1}{x_7} = \frac{1}{R_s}$$

$$\left| \frac{w_7 - \bar{w}_1}{w_7 - w_1} \right| = \left| \frac{\frac{1}{R_s} - \frac{R_c}{R_s\, x_1}}{\frac{1}{R_s} - \frac{1}{x_1}} \right|$$

$$= \left| \frac{x_1 - R_c}{x_1 - R_s} \right| = \frac{R_s}{R_c} = \frac{1}{\lambda}$$

$$Z_{17} = \frac{60}{\sqrt{\epsilon_r}} \ln \left| \frac{w_7 - \bar{w}_1}{w_7 - w_1} \right| = \frac{60}{\sqrt{\epsilon_r}} \ln \left( \frac{1}{\lambda} \right) \tag{56}$$

(6). $\underline{Z_{77}}$

$$z_7 = R_s \ ; \ w_7 = \frac{1}{R_s} \ ; \ \bar{w}_7 = \frac{1}{R_s} - \frac{1}{R_s} = 0$$

$$a_w^{(7)} = |w_7|^2 a = w_7^2 a$$

$$Z_{77} = \frac{60}{\sqrt{\epsilon_r}} \ln \left|\frac{w_7 - \bar{w}_7}{a_w^{(7)}}\right| = \frac{60}{\sqrt{\epsilon_r}} \ln \left|\frac{1}{w_7 a}\right|$$

$$= \frac{60}{\sqrt{\epsilon_r}} \ln \frac{R_s}{a} = \frac{60}{\sqrt{\epsilon_r}} \ln \rho \qquad (57)$$

For convenience, Equations (52 - 57) for the 28 independent coefficients are collected below:

$$\left.\begin{aligned}
&Z_{11} = Z_{22} = Z_{33} = Z_{44} = Z_{55} = Z_{66} = \zeta \ln [\rho(1 - \lambda^2)] \\
&Z_{12} = Z_{23} = Z_{34} = Z_{45} = Z_{56} = Z_{61} = \tfrac{1}{2} \zeta \ln \left[\frac{1 - \lambda^2 + \lambda^4}{\lambda^2}\right] \\
&Z_{13} = Z_{24} = Z_{35} = Z_{46} = Z_{51} = Z_{62} = \tfrac{1}{2} \zeta \ln \left[\frac{1 + \lambda^2 + \lambda^4}{3\lambda^2}\right] \\
&Z_{14} = Z_{25} = Z_{36} = \zeta \ln \left[\frac{1 + \lambda^2}{2\lambda}\right] \\
&Z_{17} = Z_{27} = Z_{37} = Z_{47} = Z_{57} = Z_{67} = \zeta \ln \left(\frac{1}{\lambda}\right) \\
&Z_{77} = \zeta \ln \rho \\
&Z_{ji} = Z_{ij}, \ i, j = 1, \ldots, 7 \\
&\zeta = 60/\sqrt{\epsilon_r}
\end{aligned}\right\} \qquad (58)$$

References

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