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~~Interim Report~~

ASYMPTOTIC FORMULAS FOR CABLE CAPACITANCE COEFFICIENTS

by

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for

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

1. Introduction

In previous reports [1]* we presented equations describing the steady-state behavior of multiconductor TEM lines subject to general excitation and termination conditions. The dynamic behavior of such lines is a function of two sets of parameters: (1) external parameters represented by terminations and excitations (2) internal parameters represented by the various line admittance (or impedance) coefficients, and the line electrical length. Limitations of the theory presented are that the dielectric be homogeneous and isotropic, that the conductors be lossless, and that the line cross-section geometry be invariant in the direction of wave propagation.

Assuming that these conditions (and the TEM-mode condition) are met, accuracy of behavior prediction depends on the accuracy of the parameters introduced into the model. This report is concerned with some aspects of estimation of the line internal parameters.

For a TEM system it turns out that, in theory, all internal parameters** are determined when the line's electrostatic capacitance coefficients ($C_{ij}, i,j = 1,\ldots,N$, for an N-line), and the permittivity and permeability of the medium are known [2].

The capacitance coefficients, in turn, are determined from a knowledge of the cross-section geometry and the permittivity of the medium. The problem to be solved is typified by the seven-conductor shielded cable shown in cross section in Fig. 1. Three quantities suffice to specify this circularly symmetrical arrangement: a, the common radius of the cable wires; R_c, the locus of the center of the outer six wires; and R_s, the inner radius of the shield. Actually, a small redundancy appears here, since the line coefficients are completely determined by dimensional ratios, rather than absolute dimensions themselves. The ratios used in this report (for the 7-line) are

*Numbers in [ ] correspond to Reference list. Reference 1 contains other related references.

**Except line length.

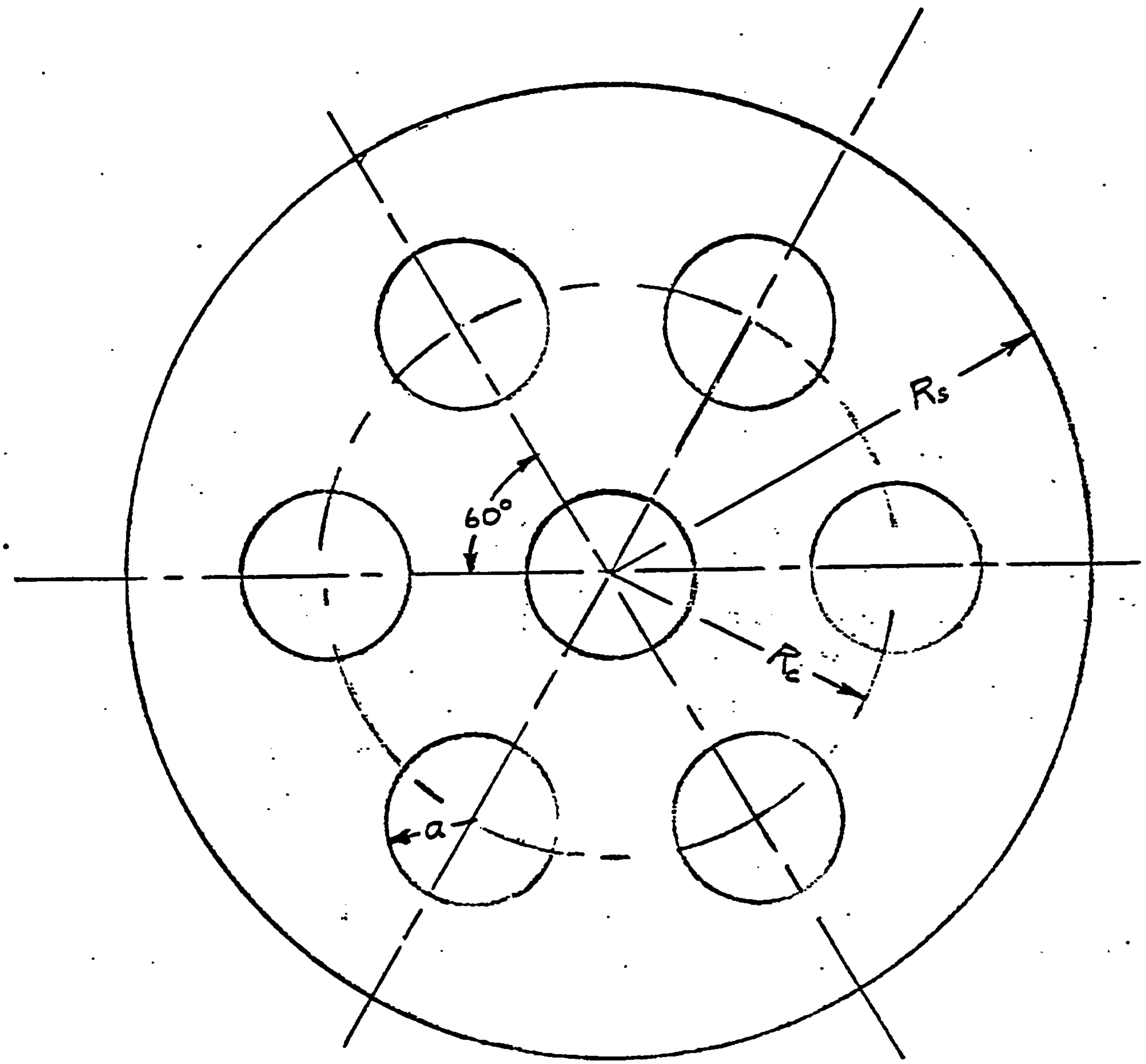

Figure 1. Seven Conductor Cable

$$\left.\begin{aligned} x &= \frac{R_s}{R_c} \\ p &= \frac{R_c}{a} \end{aligned}\right\} \qquad (1)$$

which are more convenient for the purposes of this report than the parameters, ρ and λ, of [3], to which they are related by

$$\left.\begin{aligned} \lambda &= \frac{1}{x} \\ \rho &= px \end{aligned}\right\} \qquad (2)$$

Problems of this class, involving round wires in circular sheaths, cannot generally be solved analytically in closed form. Accurate results must be sought by numerical solution of Laplace's equation or a Green's theorem integral-equation formulation. Analog methods using resistance cards, resistance networks, or electrolytic tanks, are also available. Asymptotic solutions for certain limiting values of the parameters are possible in closed form; the generation of such solutions is the subject of this report. The two parameters, p and x, yield four asymptotic regions:

A. $x, p \to \infty$. Qualitatively this region is described as that for which the wire radius is much less than the distance between wires, and much less than the distance from any wire to the shield.

B. $p \to 2$; $x \to (1 + \frac{1}{p}) \to 3/2$. Qualitatively this region is that for which the wires are so large and the shield so small that the gap between wires (g_c) and the gap between any outer wire and the shield (g_s) are much less than the wire radius.

C. $p \to 2$; $x \to \infty$. For this case the wires are bunched closely compared to their radii, but the shield radius is much greater than the radius of the locus of the centers of the outer six conductors.

D. $p \to \infty$, $x \to (1 + \frac{1}{p}) \to 1$. The wire radii are small compared to the shield diameter, but the gap between the outer six wires and the shield is much less than the wire radius.

The D region appears to be of academic interest only. The C region does not appear to be of practical interest either. However, one of the experiments at Sandia used parameters falling within its scope. Thus, one set of parameters was, in inches

$$\left.\begin{aligned} a &= 0.001 \\ R_c &= 0.0025 \\ R_s &= 40 \end{aligned}\right\}$$

yielding

$$\left.\begin{aligned} p &= 2.5 \\ x &= 16{,}000 \end{aligned}\right\}$$

2. Analysis: The Electrostatic Problem

Assume the wires carry charges q_i and potentials V_i ($i = 1, \ldots, 7$). The potential of the shield is zero and the total charge on its inner surface is $-\sum_{i=1}^{7} q_i$. Then the capacitance coefficients are defined by

$$\underline{Q} = \underline{C}\,\underline{V} \tag{3}$$

where $\underline{Q}$, $\underline{C}$, and $\underline{V}$ are matrices:

$$\underline{Q} = \begin{bmatrix} q_1 \\ \vdots \\ q_7 \end{bmatrix} ; \underline{V} = \begin{bmatrix} V_1 \\ \vdots \\ V_7 \end{bmatrix}$$

$$\underline{C} = \begin{bmatrix} C_{11}, & C_{12}, & \ldots, & C_{17} \\ C_{21}, & C_{22}, & \ldots, & C_{27} \\ \cdot & \cdot & \cdot & \cdot \\ C_{71}, & C_{72}, & \ldots, & C_{77} \end{bmatrix} \tag{4}$$

In $\underline{C}$, $C_{ij} = C_{ji}$ for every i, j. Furthermore the symmetry of Fig. 1 suggests that

$$\left.\begin{aligned} C_{11} &= C_{22} = \ldots = C_{66} \\ C_{12} &= C_{23} = \ldots = C_{61} \\ C_{13} &= C_{24} = \ldots = C_{51} \\ C_{17} &= C_{27} = \ldots = C_{67} \end{aligned}\right\} \tag{5}$$

If we include C_{14} and C_{77} in the list of Equations (5), we note that only six different capacitances need to be determined.

From (3) and (4) the defining equation for any individual C_{ij} is

$$C_{ij} = \left.\frac{q_i}{V_j}\right|_{V_k = 0,\ k \neq j} \qquad \text{(a)}$$

or

$$C_{ij} = \left.\frac{q_j}{V_i}\right|_{V_k = 0,\ k \neq i} \qquad \text{(b)} \tag{6}$$

From (3)

$$\underline{V} = \underline{C}^{-1} \underline{Q} = \underline{P} \underline{Q} \tag{7}$$

where

$$\underline{P} = \begin{bmatrix} p_{11}, p_{12}, \cdot \cdot \cdot \cdot, p_{17} \\ p_{21}, p_{22}, \cdot \cdot \cdot \cdot, p_{27} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ p_{71}, p_{72}, \cdot \cdot \cdot \cdot, p_{77} \end{bmatrix} \tag{8}$$

is the potential-coefficient matrix. The defining equation for an individual p_{ij} is

$$p_{ij} = \left.\frac{V_i}{q_j}\right|_{q_k = 0,\ k \neq j} \qquad \text{(a)}$$

or

$$p_{ij} = \left.\frac{V_j}{q_i}\right|_{q_k = 0,\ k \neq i} \qquad \text{(b)} \tag{9}$$

Sometimes the p_{ij} are more conveniently determined than the C_{ij} [3, 4].

The line admittance - and impedance coefficients are then simply

$$\left.\begin{aligned} Y_{ij} &= v\, C_{ij} \\ Z_{ij} &= v^{-1}\, p_{ij} \end{aligned}\right\} \quad i, j = 1, \ldots, 7 \tag{10}$$

where

$$v = (\mu \epsilon)^{-\frac{1}{2}} \tag{11}$$

μ = permeability of medium, H/meter

ϵ = permittivity of medium, F/meter

2.1 Discussion of Case A: $x, p \to \infty$

This case has been covered in [3]. An upper bound to errors incurred in departing from the asymptotic conditions is given in Fig. 2, (adapted from Ref. 2, page 4-13). This is a "worst case" situation, which assumes that the charge distribution on the conductor for which a coefficient is being evaluated has been distorted by having all of the opposite charge on a single conductor of the same diameter. Clearly, this is generally not true. Any other arrangement of conductors that reduces the charge distribution distortion will also reduce the error. As an interesting example, consider the case of a round wire in a square shield (Fig. 3). The asymptotic characteristic impedance for this configuration is shown in Fig. 4. For D/a = 4, Fig. 2 suggests an upper bound for the error of about 5%. On the other hand, a recent numerical solution by D. H. Sinnott/K. M. Harvey [5] shows the error to be less than one part in 10,000! Even for the case D/a = 2.2, which corresponds roughly to p = 2.2 in the notation of Section 1, the error was only 7%, whereas Fig. 2 suggests an upper bound of the order of 70%.

Thus, if we set an error limit of about 5%, then the A-asymptote should prove useful at least for $(p \geq 4,\ x \geq \frac{4}{p} + 1)$ for all admittance coefficients, and possibly for $p \geq 2.2$ for some.

On the other hand, justification for using Fig. 2 as an approximate error upper bound for coefficients of a system of more than two conductors rests on the assumption that a coefficient between a pair of conductors is not seriously affected by the presence of other conductors. This is approximately true when potential coefficients are determined, since the procedure requires the extraneous conductors to float. Thus the net charge on any extraneous conductor is zero; its effect on the field distribution is a second-order dipole effect resulting from redistribution of its neutral charge.

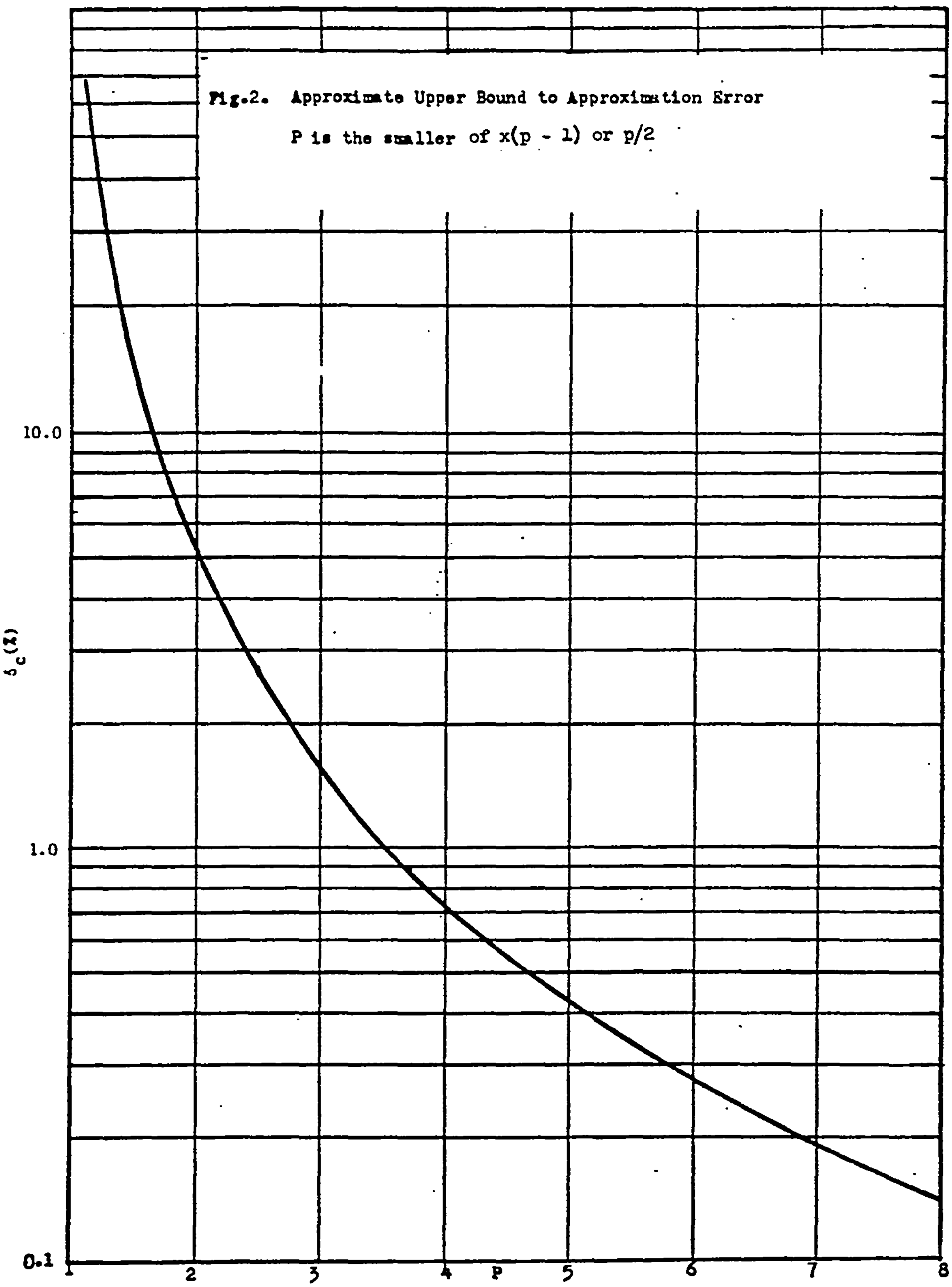

Fig.2. Approximate Upper Bound to Approximation Error

P is the smaller of x(p - 1) or p/2

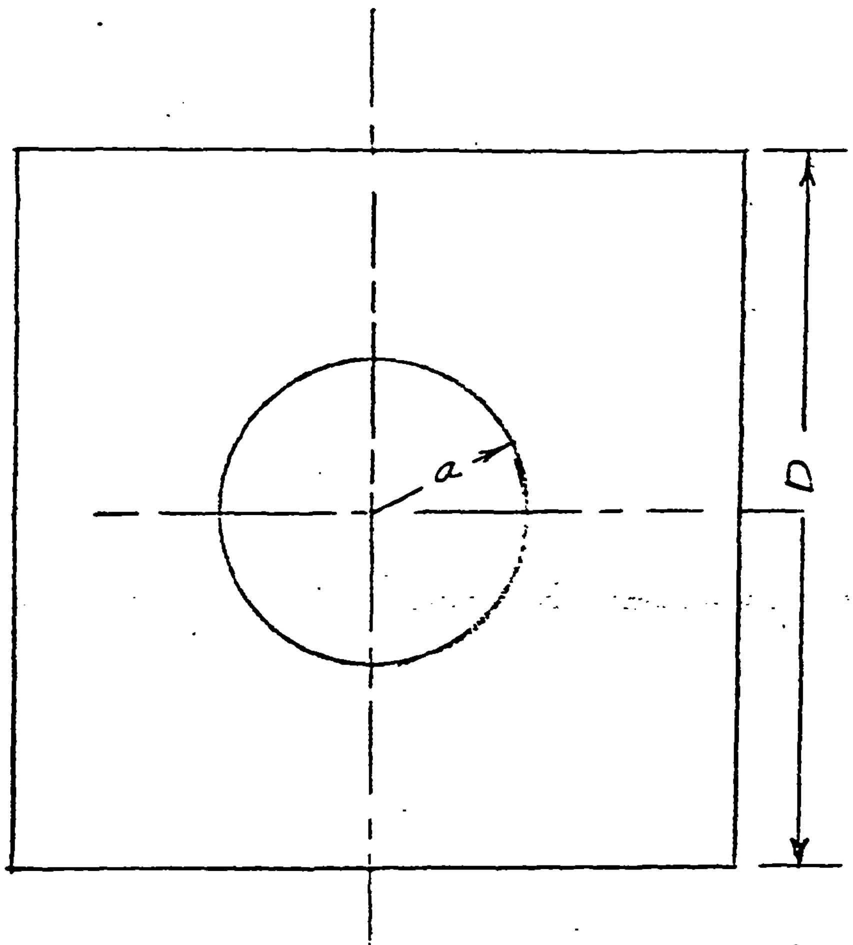

Figure 3. Round Wire in Square Shield

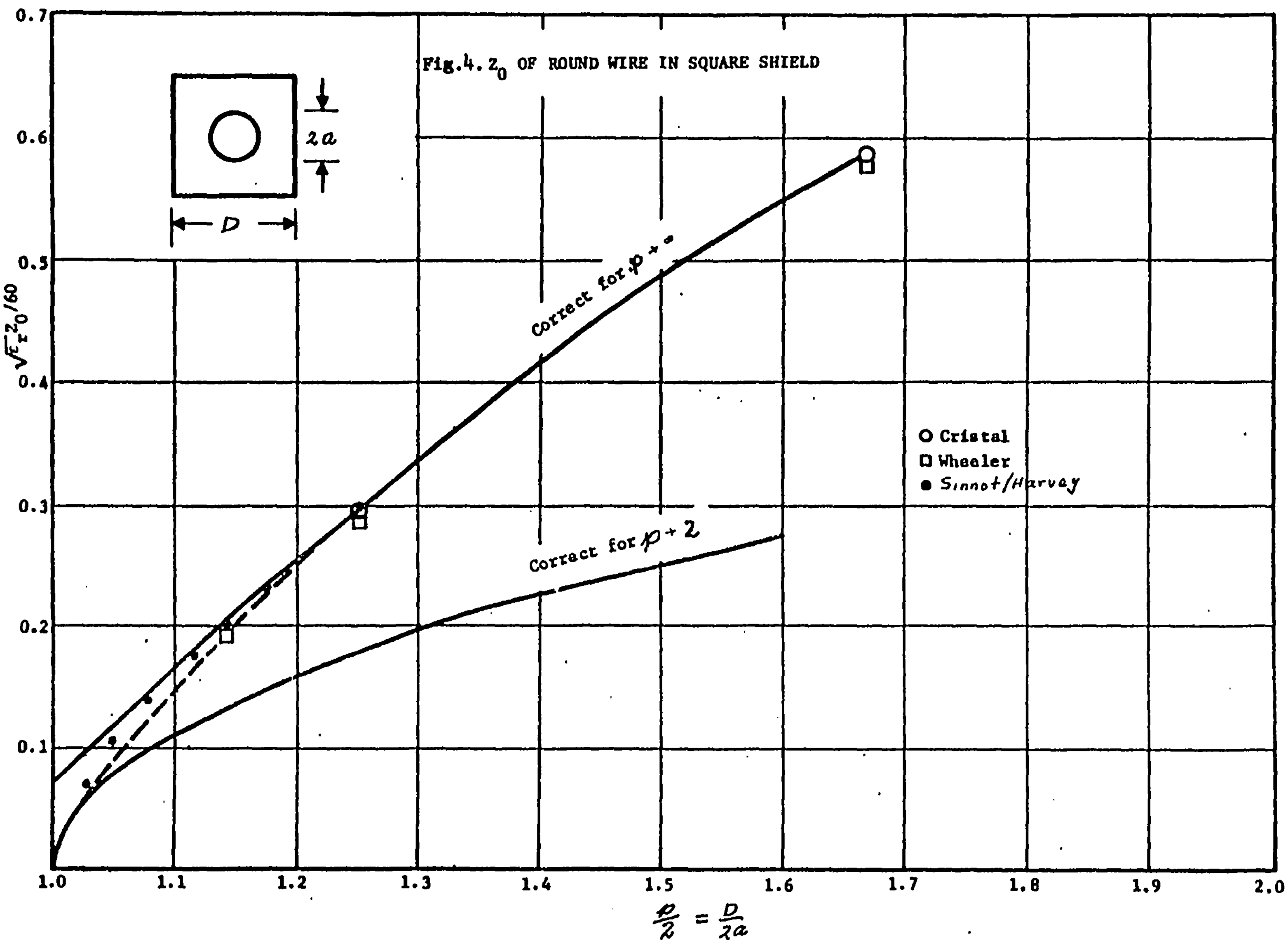

Fig.4. Z_0 OF ROUND WIRE IN SQUARE SHIELD
2a
D
Correct for p → ∞
Correct for p → 2
O Cristal
□ Wheeler
● Sinnot/Harvey
$\sqrt{\epsilon_r} Z_0/60$
$\frac{p}{2} = \frac{D}{2a}$
0.7
0.6
0.5
0.4
0.3
0.2
0.1
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0

But in a capacitance determination the extraneous conductors are grounded. This means they carry net charges required to cancel the potentials induced by the pertinent pair of conductors. The centroids of such charges will deviate from the conductor centers when the conductors are closely spaced, thus introducing additional error not accounted for by Fig. 2.

A hybrid situation occurs in determining Y_{77} of Fig. 1. This coefficient is determined as the ratio of charge on conductor No. 7 to its potential when all other conductors are grounded. When the conductors are closely spaced, intuition suggests that the determination is essentially that of the characteristic admittance of a cage transmission line. The inner conductor will have a relatively uniform charge distribution, corresponding to the case of Figs. 3 and 4, while the conductors of the outer cage will have the severe shift of centroid of charge corresponding to the error curve of Fig. 2. For instance, for an error of 5% for the cage, use the 10% point of Fig. 2, corresponding to $p = 3.33$.

Because of the relative uniformity of charge in this case, it should be expected that the A region would extend approximately to this value of p.

On the other hand, for the B region, Fig. 3 suggests that the result would not be too useful for $p > 2.05$. The question then concerns the magnitude of the uncertainty in the range $2.05 < p < 3.33$. This is discussed in Section 3.

For the remaining coefficients we have to accept the error suggested by Fig. 2.

Before going on to Case B we rewrite the A-case formulas for the Z_{ij} in terms of (x, p):

$$
\left.
\begin{aligned}
&Z_{11} = Z_{22} = Z_{33} = Z_{44} = Z_{55} = Z_{66} = \zeta \ln\left[p\,\frac{x^2-1}{x}\right] \\
&Z_{12} = Z_{23} = Z_{34} = Z_{45} = Z_{56} = Z_{61} = \tfrac{1}{2}\,\zeta \ln\left[\frac{x^4-x^2+1}{x^2}\right] \\
&Z_{13} = Z_{24} = Z_{35} = Z_{46} = Z_{51} = Z_{62} = \tfrac{1}{2}\,\zeta \ln\left[\frac{x^4+x^2+1}{3x^2}\right] \\
&Z_{14} = Z_{25} = Z_{36} = \zeta \ln\left[\frac{x^2+1}{2x}\right] \\
&Z_{17} = Z_{27} = Z_{37} = Z_{47} = Z_{57} = Z_{67} = \zeta \ln x \\
&Z_{77} = \zeta \ln (px) \\
&\qquad \zeta = 60/\sqrt{\epsilon_r} \\
&\qquad \epsilon_r = \text{relative dielectric constant}
\end{aligned}
\right\} \quad (12)
$$

2.2 Discussion of Case B: $p \to 2;\ x \to 1 + \frac{1}{p} \to \frac{3}{2}$

The method for obtaining coefficients when all conductors are almost touching is adapted from Wheeler [6]. Consider Fig. 5, which shows a portion of a seven-conductor cable of large conductors with close-fitting shield.

To determine Y_{11} and Y_{12} we place a potential on conductor No. 1 and note the total resulting charge on that conductor (for Y_{11}), or the total resulting charge on conductor No. 2 (for Y_{12}).

But when the conductors and shield are very closely spaced, most of the electric field and, therefore, most of the charge, will be concentrated in the four regions marked "g", and there will be essentially no charge in between. Thus the four regions can be treated as independent capacitances between pairs of conductors and their effects combined by simple addition. Furthermore, there

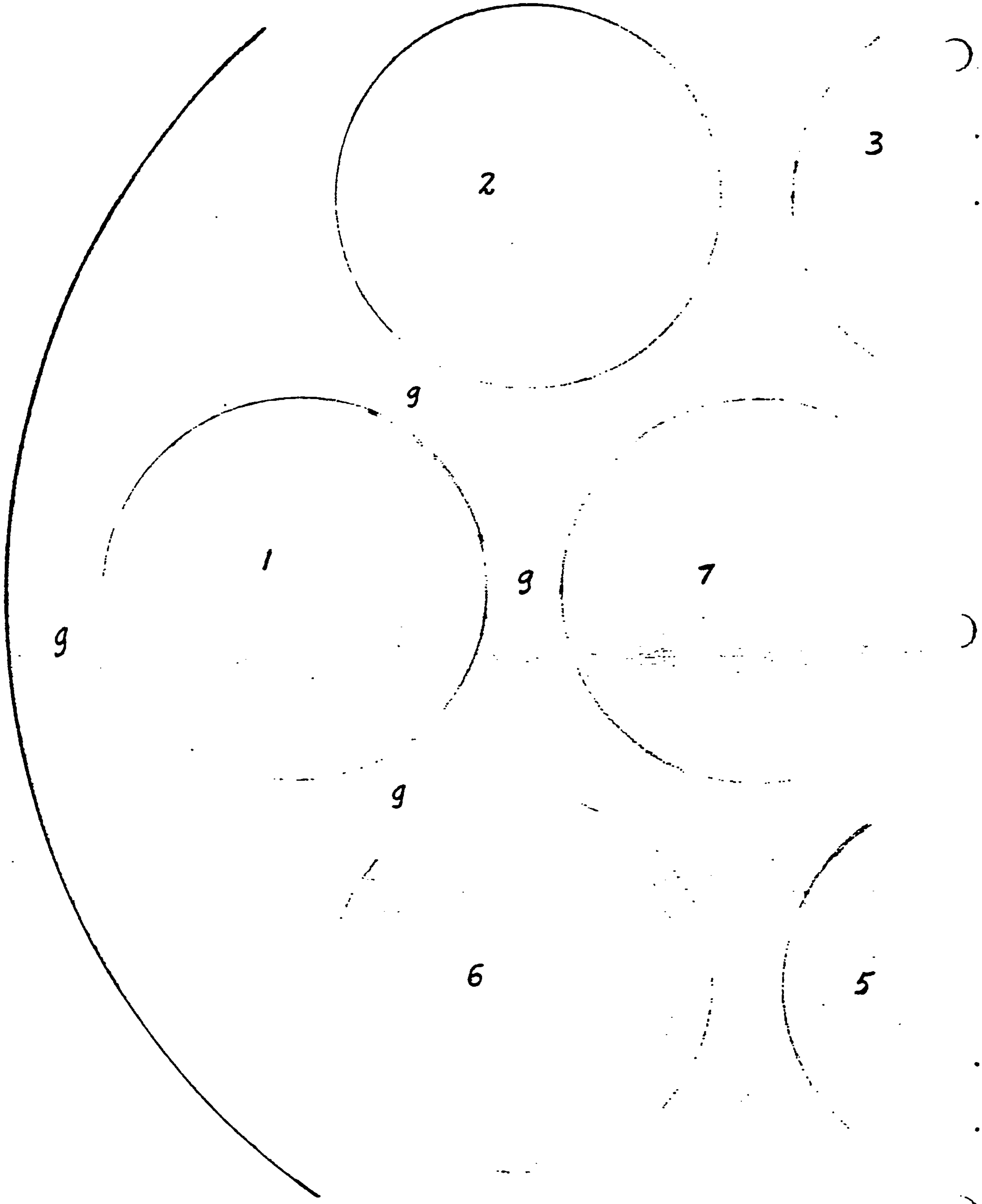

Figure 5. Cable of Large Conductors with Close-Fitting Shield

are only two independent capacitance configurations represented in Figs. 6a and 6b, respectively. Here for the sake of complete generality we indicate conductors of different radii at (a), although for our specific example they are equal. At (b) the smaller circle represents one of the outer six conductors, while the outer circle represents the shield.

A single formula expresses the capacitance (per meter) for both of these configurations. It is, in MKS units [7],

$$C = 2\pi\varepsilon \left[\cosh^{-1}\left(\pm \frac{R_1^2 + R_2^2 - D^2}{2R_1R_2}\right)\right]^{-1} \tag{13}$$

where the upper sign is used for (b), the lower for (a).

For the asymptotic situations under study these expressions can be simplified. First, for case (a), we have

$$R_1 = R_2 = a$$

$$D = R_c = 2a + g_c, \; g_c \ll a$$

therefore

$$C_a = 2\pi\varepsilon \left\{\cosh^{-1}\left[\frac{1}{2}\left(\frac{R_c}{a}\right)^2 - 1\right]\right\}^{-1}$$

$$= 2\pi\varepsilon \left\{\cosh^{-1}\left[\frac{1}{2} p^2 - 1\right]\right\}^{-1} \tag{14}$$

Elementary hyperbolic transformations yield

$$\cosh^{-1}\left(\frac{1}{2} p^2 - 1\right) = 2 \cosh^{-1}\left(\frac{p}{2}\right) \tag{15}$$

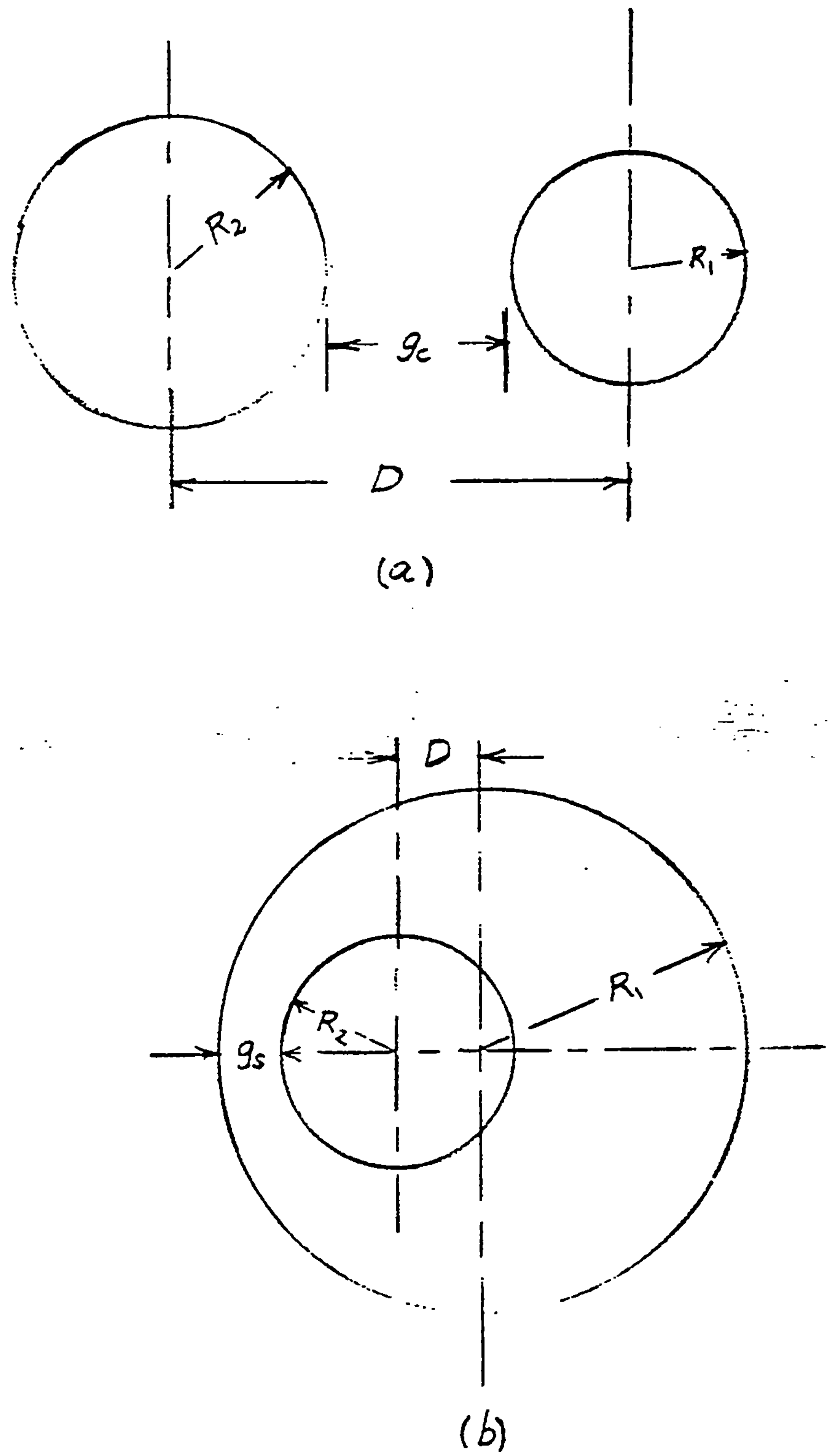

Figure 6. Capacitance Between Circular Cylinders

But as $g_c \to 0$, $R_c \to 2_a$, so $\frac{p}{2} \to 1$, and $\cosh^{-1}\left(\frac{p}{2}\right) \to 0$. Let

$$u = \cosh^{-1}\left(\frac{p}{2}\right) \to 0$$

then

$$\frac{p}{2} = \cosh u \approx 1 + \tfrac{1}{2} u^2$$

whence

$$u \approx (p - 2)^{\frac{1}{2}} \tag{16}$$

Using (16) in (14) via (15),

$$C_a \approx 2\pi\varepsilon\left\{2(p - 2)^{\frac{1}{2}}\right\}^{-1}$$

$$= \frac{\pi\varepsilon}{(p - 2)^{\frac{1}{2}}} \tag{17}$$

Then, as usual

$$Y_a = vC = \frac{1}{2\zeta(p - 2)^{\frac{1}{2}}} \tag{18}$$

For the (b) case of Fig. 6 we have

$$\left.\begin{aligned} R_1 &= R_s \\ R_2 &= a \\ D &= R_c \end{aligned}\right\}$$

$$C_b = 2\pi\varepsilon \left[\cosh^{-1} \frac{R_s^2 + a^2 - R_c^2}{2aR_s}\right]^{-1}$$

$$= 2\pi\varepsilon \left\{\cosh^{-1} \left[\frac{p^2(x^2 - 1) + 1}{2px}\right]\right\}^{-1} \qquad (19)$$

Again writing

$$u = \cosh^{-1}\left[\frac{p^2(x^2 - 1) + 1}{2px}\right]$$

we have, approximately

$$\cosh u = 1 + \tfrac{1}{2} u^2 = \frac{p^2(x^2 - 1) + 1}{2px}$$

whence

$$u = \left[\frac{(px - 1)^2 - p^2}{px}\right]^{\frac{1}{2}} \qquad (20)$$

and (19) is

$$C_b \rightarrow 2\pi\varepsilon \left[\frac{px}{(px - 1)^2 - p^2}\right]^{\frac{1}{2}} \qquad (21)$$

and

$$Y_b \rightarrow \frac{1}{\zeta} \left[\frac{px}{(px - 1)^2 - p^2}\right]^{\frac{1}{2}} \qquad (22)$$

Then we have, immediately, the asymptotic values:

$$\left.\begin{aligned} Y_{11} &= 3\,Y_a + Y_b \\ Y_{12} &= -\,Y_a \\ Y_{13} &= \ldots = Y_{14} = \ldots = Y_{15} = \ldots = 0 \\ Y_{17} &= Y_{27} = \ldots = Y_{67} = -\,Y_a \\ Y_{77} &= 6\,Y_a \end{aligned}\right\} \qquad (23)$$

3. Behavior of Y_{77}

Region-A and Region-B curves for Y_{77} are given in Fig. 7. Actually the ordinates are

$$y_{77}^{-1} = (\zeta\, Y_{77})^{-1}$$

This manner of display yields a zero at $p = 2$ instead of an infinity for the fat-wire asymptote.

As discussed in Section 2.1, the true curve should be close to the fat asymptote for $p < 2.05$ and close to the thin asymptote for $p > 3.33$.* Between these values a more-or-less gradual transition presumably exists. The true value has an error less than half the difference between these curves. Whether or not this is good enough depends on the specific problem requirements. In the event it is not, recourse must be had to numerical solution of the electrostatic field problem.

*Although Y_{77} is, generally, a function of both x and p, computations of the thin asymptote show that its value varies only about 1% in the range $1.43 < x < 3.33$, for $p = 4$. Presumably, this variation decreases with p.

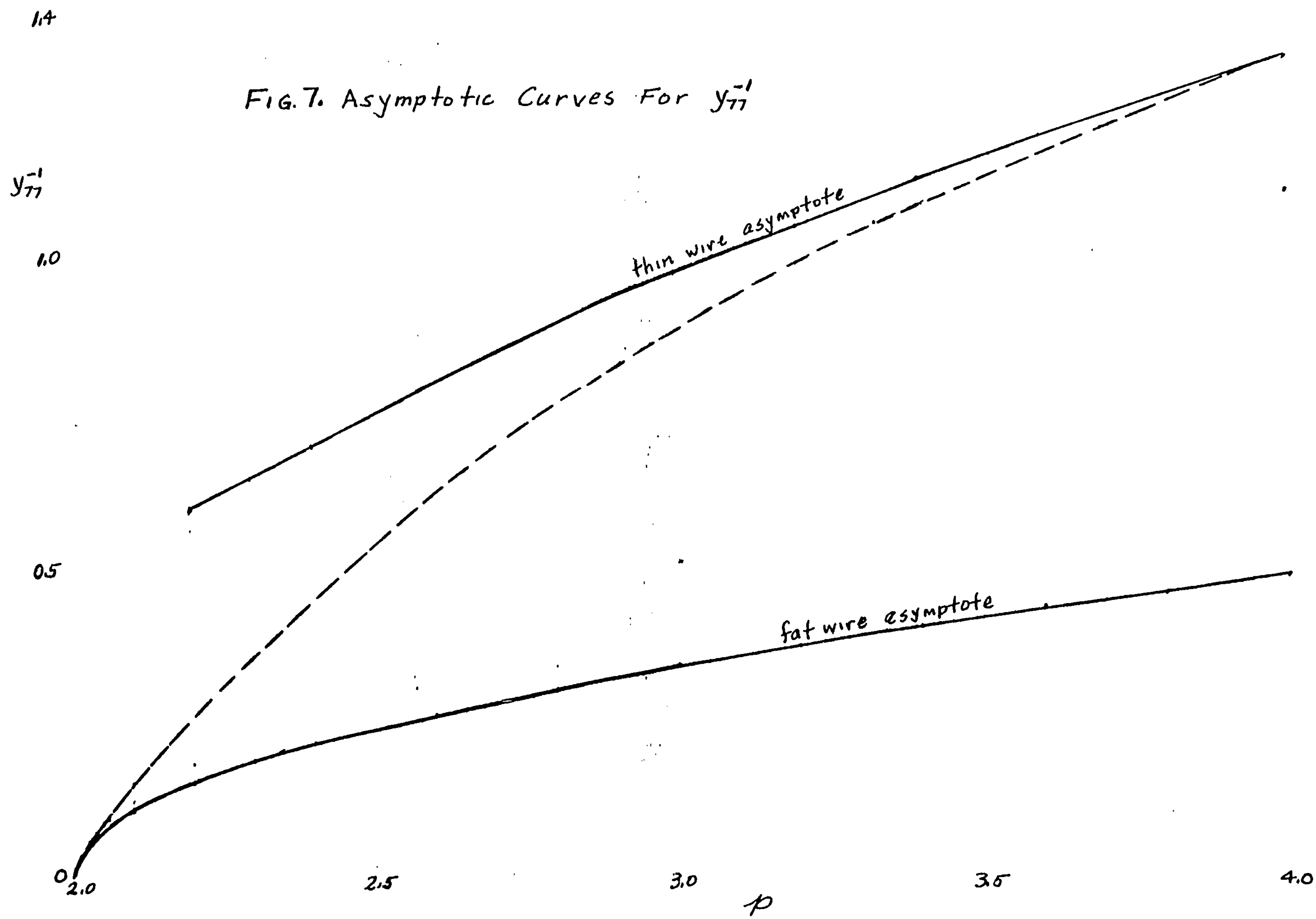

FIG. 7. Asymptotic Curves For y_{77}^{-1}

The dashed curve of Fig. 7 represents our best estimate of the true curve of y_{77}^{-1}, obtained by a method similar to that of Ref. 2 (pp. 4-66 ff.).

Pending discussions with Sandia regarding the usefulness of this result, we have not considered it advisable to devote further effort to analysis of the remaining coefficients.

4. Conclusion

The asymptotic data are generally useful outside the range $2.05 < p < 4$, except that the upper limit of the excluded range may be reduced to $p = 3.33$ for the coefficient Y_{77}. For accurate results in the excluded range, numerical methods must be used.

5. Acknowledgment.

Data for the thin wire asymptote were furnished by George Steigerwald, Division 2627, Sandia Laboratories.

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