Interaction Notes

Note 77

August 1971

Scattering by a Spherical Shell With a Small Circular Aperture

by

B. Enander*
Northrop Corporate Laboratories
and
California Institute of Technology
Pasadena, California

Abstract

The electromagnetic field inside a spherical shell with a small circular hole is studied. The shell, which is perfectly conducting and infinitely thin, is illuminated by a plane wave.

The electric field in a small hole in a spherical shell is assumed to have the same shape as that in a small hole in a plane screen. The shape of this field determines the scattered fields inside and outside the sphere except for the amplitudes. The amplitudes are then determined by the "coupling" to the incident field.

*On sabbatical leave from The Royal Institute of Technology, Stockholm, Sweden.

1. Introduction

The problem treated in this paper is the penetration of an electromagnetic wave into a spherical, metallic shell with an aperture. There probably does not exist any simple solution to this problem because the problem of an aperture in an infinite conducting screen has a very complex solution.

For low frequencies our problem has been solved by formulating the problem as a Fredholm integral equation of the first kind (Thomas 1963). An integral equation approach could be used also in the case of higher frequencies but does not seem particularly convenient. A method for solving the present problem for arbitrary large holes at arbitrary frequencies has been given by Chang and Senior (Chang 1969). They used the method to compute the back-scattering from spheres with large holes. Their results agree very well with experiments. The method consists in approximating the scattered field by a finite number of terms from a general infinite series expression for the field and then applying the boundary conditions together with the method of least square error. Sancer and Varvatsis have used essentially the same method (Sancer 1970), but obtained two simpler problems through the use of Debye potentials. However, the method given in (Chang 1969) gives better results because they have taken the field behavior at the edge of the aperture into account. A modified formulation of (Chang 1969) using the Debye potentials would probably be the most convenient method.

In this paper we will give another approximate method, which can be used when the aperture is small. It is believed to be simpler in use than the method just described. Our method is equivalent with one used in a similar accoustic problem in (Morse 1953).

2. Analysis

Consider an infinitely thin, perfectly conducting spherical shell with its center at the origin. The sphere has a radius equal to a and an aperture occuping the region $r = a$, $0 \le \theta < \theta_o$, as shown in Fig. 1. The incident plane wave is

$$\underline{E}^i = \hat{x}e^{jkz} \tag{1}$$

$$\underline{H}_i = -\hat{y}Y_o e^{jkz} \tag{2}$$

where k is the propagation constant and Y_o the admittance of free space. The suppressed time dependence is $e^{j\omega t}$.

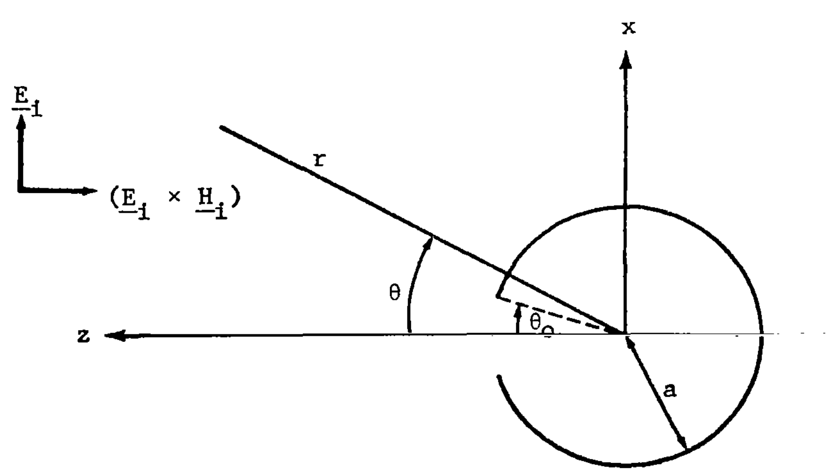

Figure 1. Plane wave scattered from sphere with aperture.

To obtain an approximate solution for the scattered field, valid when the hole is small ($ka \sin \theta_o << 1$ and $\theta_o << 1$) we shall proceed in the following way. First we introduce the Debye potentials. This transforms the original vector problem into two decoupled scalar problems. We then assume that the shape of the electric field in the hole is the same as in a small circular hole in a plane screen for the case of normal incidence. This enables us to write down the shape (θ-dependence) for the Debye potentials in the aperture. The potentials (and fields) are then known except for an amplitude constant. This is determined by the coupling in the aperture to the incident field.

To obtain a scalar problem we represent the field by the Debye potentials u and v (Born 1970, Harrington 1961, Sancer 1970). u has to satisfy

$$\nabla^2 u + k^2 u = 0 \tag{3}$$

$$\left.\begin{aligned} &u = 0 \text{ on conductor} \\ &u \text{ and } \frac{\partial u}{\partial r} \text{ continuous through aperture} \end{aligned}\right. \tag{4}$$

The incident field corresponds to

$$u_i = -\frac{Y_o}{ik} \sin \phi \sum_{n=1}^{\infty} (i)^n (2n + 1) j_n(kr) P_n^1(\cos \theta) \tag{5}$$

v has to satisfy

$$\nabla^2 v + k^2 v = 0 \tag{6}$$

$$\frac{\partial(rv)}{\partial r} = 0 \text{ on conductor}$$

$$v \text{ and } \frac{\partial v}{\partial r} \text{ continuous through aperture} \tag{7}$$

Further

$$v_i = \frac{1}{ik} \cos\phi \sum_{n=1}^{\infty} (i)^n \frac{2n+1}{n(n+1)} j_n(kr) P_n^1(\cos\theta) \tag{8}$$

In this manner the scattering problem has been transformed into two decoupled scalar problems. When u and v satisfying (3) to (8) have been found, the electric and magnetic fields are obtained from (Born 1970)

$$E_r = \frac{\partial^2(rv)}{\partial r^2} + k_o^2(rv)$$

$$E_\theta = \frac{1}{r}\frac{\partial^2(rv)}{\partial r\partial\theta} - j\omega\mu_o \frac{1}{r\sin\theta}\frac{\partial(ru)}{\partial\phi} \tag{9}$$

$$E_\phi = \frac{1}{r\sin\theta}\frac{\partial^2(rv)}{\partial r\partial\phi} + j\omega\mu_o \frac{1}{r}\frac{\partial(ru)}{\partial\theta}$$

$$H_r = \frac{\partial^2(ru)}{\partial r^2} + k_o^2(ru)$$

$$H_\theta = \frac{1}{r}\frac{\partial^2(ru)}{\partial r\partial\theta} + j\omega\varepsilon_o \frac{1}{r\sin\theta}\frac{\partial(rv)}{\partial\phi} \tag{10}$$

$$H_\phi = \frac{1}{r\sin\theta}\frac{\partial^2(ru)}{\partial r\partial\phi} - j\omega\varepsilon_o \frac{1}{r}\frac{\partial(rv)}{\partial\theta}$$

Using (3) and the spherical coordinates introduced in Fig. 1, u inside the sphere can be written

$$u = \sum_{m=1}^{\infty} b_n \frac{j_n(kr)}{j_n(ka)} P_n^1(\cos\theta)\sin\phi \tag{11}$$

Outside the sphere we write, using (4)

$$u = u_i + u_r + \sum_{m=1}^{\infty} b_n \frac{h_n^{(2)}(kr)}{h_n^{(2)}(ka)} P_n^1(\cos\theta)\sin\phi \tag{12}$$

where u_i corresponds to the incident field. u_r corresponds to the scattered field when there is no aperture in the sphere. Similarly for v

$$v = \sum_{n=1}^{\infty} e_n \frac{j_n(kr)}{j_n(ka)} P_n^1(\cos\theta)\cos\phi \tag{13}$$

and

$$v = v_i + v_r + \sum_{n=1}^{\infty} e_n \frac{h_n^{(2)}(kr)}{h_n^{(2)}(ka)} P_n^1(\cos\theta)\cos\phi \tag{14}$$

We now consider the tangential electric field in the aperture, when the aperture is small ($ka \sin\theta_o \ll 1$ and $\theta_o \ll 1$). This is assumed to be the same as the E-field in a small circular aperture in an infinite plane conducting screen for the case of normal incidence of a plane wave. According to (Bouwkamp 1950) this gives for $r = a$ and $0 \le \theta \le \theta_o$

$$E_\phi \approx K\sqrt{2(\cos\theta - \cos\theta_o)}\,\sin\phi$$

$$E_\theta \approx -\left\{K\sqrt{2(\cos\theta - \cos\theta_o)} + \frac{K}{2}\,\frac{\sin^2\theta}{\sqrt{2\cos\theta - \cos\theta_o}}\right\}\cos\phi \tag{15}$$

where K is an unknown constant. We want to translate (15) into expressions for u and v in the aperture. Put

$$\frac{1}{a}\left\{\frac{\partial(rv)}{\partial r}\right\}_{r=a} = f(\theta)\cos\phi$$

$$j\omega\mu_o u(r=a) = g(\theta)\sin\phi \tag{16}$$

(9) can then be written

$$\frac{df}{d\theta} - \frac{g}{\sin\theta} = \frac{E_\theta}{\cos\phi} \tag{17}$$

$$-\frac{f}{\sin\theta} + \frac{dg}{d\theta} = \frac{E_\phi}{\sin\phi} \tag{18}$$

where E_θ and E_ϕ now denotes fields in the aperture. Equations (17) and (18) yield

$$\frac{d}{d\theta}\left(\sin\theta\,\frac{dg}{d\theta}\right) - \frac{g}{\sin\theta} = E_\theta + \frac{\partial}{\partial\theta}\left(E_\phi \sin\theta\right) \tag{19}$$

and

$$\frac{d}{d\theta}\left(\sin\theta\,\frac{df}{d\theta}\right) - \frac{f}{\sin\theta} = E_\theta + \frac{\partial}{\partial\theta}\left(E_\theta \sin\theta\right) \tag{20}$$

From (19) we obtain, using (11) and (16) and the differential equation which $P_n^1(\cos\theta)$ satisfies

$$- j\omega\mu_o \sum_{n=1}^{\infty} n(n+1) b_n P_n^1(\cos\theta)\sin\theta = E_\theta + \frac{\partial}{\partial\theta}(E_\phi \sin\theta) \qquad (21)$$

Inserting the approximate expressions for the tangential electric field in the aperture we finally get

$$j\omega\mu_o \sum_{n=1}^{\infty} n(n+1) b_n P_n^1(\cos\theta)\sin\theta = \begin{cases} \dfrac{3K}{2} \dfrac{\sin^2\theta}{\sqrt{2(\cos\theta - \cos\theta_o)}} & 0 \le \theta < \theta_o \\ 0 & \theta_o < \theta < \pi \end{cases} \qquad (22)$$

and using the orthogonality relations for P_n^1

$$b_n = \frac{K_1(2n+1)}{n^2(n+1)^2} \int_o^{\theta_o} \frac{P_n(\cos\theta)\sin^2\theta d\theta}{\sqrt{2(\cos\theta - \cos\theta_o)}} \qquad (23)$$

where a new unknown constant, K_1, has been introduced. Analytic expressions for the integral are given in an appendix. We are now in the position that we can calculate all b_n, u, v, and the fields except for the common unknown amplitude constant K_1. To calculate K_1 we use the condition (4) that $\frac{\partial u}{\partial r}$ is continuous through the aperture. This condition together with (11) and (12) gives

$$\sum_{n=1}^{\infty} b_n k\left(\frac{j_n'(ka)}{j_n(ka)} - \frac{h_n'(ka)}{h_n(ka)}\right) P_n^1(\cos\theta) = \frac{\partial}{\partial r}(u_i + u_r)_{r=a} \qquad (24)$$

u_r is the reflected field for the case of a sphere with no aperture and is obtained from (4) and (5) and the radiation condition

$$u_r = + \frac{Y_o}{ik} \sin \phi \sum_{n=1}^{\infty} i^n (2n+1) \frac{h_n^{(2)}(kr)}{h_n^{(2)}(ka)} j_n(ka) P_n^1(\cos \theta) \tag{25}$$

We can now compute K_1 from (24) putting $\theta = \theta_o$ and using expressions for b_n, u_i and u_r according to (5), (23) and (25). When u is known v is then obtained from (18) and the fields from (9) and (10).

Appendix

Evaluation of $\int_0^{\theta_o} \frac{P_n^1(\cos\theta)\sin^2\theta d\theta}{\sqrt{2(\cos\theta - \cos\theta_o)}}$

In this appendix we shall derive an analytic expression for this integral. Using notations from (Mac Robert 1947) we write

$$I = \int_0^{\theta_o} \frac{P_n^1(\cos\theta)\sin^2\theta d\theta}{\sqrt{2(\cos\theta-\cos\theta_o)}} = n(n+1)\int_0^{\theta_o} \frac{T_n^{-1}(\cos\theta)\sin^2\theta d\theta}{\sqrt{2(\cos\theta-\cos\theta_o)}}$$

Since (Mac Robert 1947, p. 335)

$$T_{n+1}^{-1}(\cos\theta) = \frac{\sin\theta}{2} F(-n, n+3, 2, \tfrac{1}{2}(1-\cos\theta))$$

we have

$$I = \frac{n(n+1)}{2}\sum_{r=0}^{n} \frac{(-n)_r (n+3)_r}{2^r (2)_r \Gamma(r+1)} \int_0^{\theta_o} \frac{(1-\cos\theta)^r \sin\theta^3 d\theta}{(\cos\theta-\cos\theta_o)^{\frac{1}{2}}}$$

where $(d)_r = \Gamma(d+r)/\Gamma(r)$. Making the substitution

$$\cos\theta = 1 - t(1-\cos\theta_o)$$

we find

$$\int_0^{\theta_o} (\cos\theta - \cos\theta_o)^{-\frac{1}{2}}(1 - \cos\theta)^r \sin^3\theta d\theta$$

$$= 2(1 - \cos\theta_o)^{r+3/2} \int_0^1 t^{r+1}(1 - t)^{-\frac{1}{2}} dt - (1 - \cos\theta_o)^{r+5/2} \int_0^1 t^{r+2}(1 - t)^{-\frac{1}{2}} dt$$

$$= \frac{\Gamma(r+2)\Gamma(\frac{1}{2})}{\Gamma(r + \frac{5}{2})} 2(1 - \cos\theta_o)^{r+3/2} - \frac{\Gamma(r+3)\Gamma(\frac{1}{2})}{\Gamma(r+3\frac{1}{2})} (1 - \cos\theta_o)^{r+5/2}$$

Divide I into two parts

$$I = I_1 + I_2$$

where

$$I_1 = n(n + 1)(1 - \cos\theta_o)^{3/2} \Gamma(\tfrac{1}{2}) \sum_{r=0}^{n} \frac{(-n)_r (n+3)_r \Gamma(r+2)}{2^r (2)_r \Gamma(r+1)\Gamma(r + \frac{5}{2})} (1 - \cos\theta_o)^r$$

and

$$I_2 = -\frac{n(n+1)\Gamma(\frac{1}{2})}{2} (1 - \cos\theta_o)^{5/2} \sum_{r=0}^{n} \frac{(-n)_r (n+3)_r \Gamma(r+3)}{2^r (2)_r \Gamma(r+1)\Gamma(r + \frac{7}{2})} (1 - \cos\theta_o)^r$$

To evaluate I_1 we write

$$I_1 = \frac{4}{3} n(n + 1)(1 - \cos\theta_o)^{3/2} \sum_{r=0}^{n} \frac{(-n)_r (n+3)_r}{2^r \Gamma(r+1) (\frac{5}{2})_r} (1 - \cos\theta_o)^r$$

$$= \frac{4}{3} n(n + 1)(1 - \cos\theta_o)^{3/2} F(-n, n + 3, \tfrac{5}{2}, \tfrac{1}{2}(1 - \cos\theta_o))$$

Using various formulas for hypergeometric functions (Erdelyi 1953, pp. 101, 102, 105) we find

$$F(-n, n+3, \tfrac{5}{2}, \tfrac{1}{2}(1-\cos\theta_o))$$

$$= -\frac{5}{2}\frac{n+2}{n+1}\frac{1}{\sin\theta_o}\frac{d}{d\theta_o}\left\{(1+\cos\theta_o)F(\tfrac{5}{2}+n, -\tfrac{1}{2}-n, \tfrac{1}{2}, \frac{1-\cos\theta_o}{2})\right\}$$

$$= -\frac{5}{2}\frac{n+2}{n+1}\frac{1}{\sin\theta_o}\frac{d}{d\theta_o}\left\{(1+\cos\theta_o)\frac{2\sin(2n+3)\frac{\theta_o}{2}}{(2n+3)\sin\theta_o}\right\}$$

which gives

$$I_1 = -\frac{10}{3}n(n+2)\frac{(1-\cos\theta_o)^{\frac{1}{2}}}{\sin\theta_o}\left\{\sin\frac{\theta_o}{2}\cos(n+\tfrac{3}{2})\theta_o - \frac{\cos\frac{\theta_o}{2}\sin(n+\frac{3}{2})\theta_o}{2(n+\frac{3}{2})}\right\}$$

The evaluation of the term I_2 is done in a very similar manner. We obtain

$$\int_o^{\theta_o}\frac{P_n^1(\cos\theta)\sin^2\theta\, d\theta}{\sqrt{2(\cos\theta-\cos\theta_o)}}$$

$$= -\frac{5}{3}\frac{n(n+2)}{n+\frac{3}{2}}\frac{(1-\cos\theta_o)^2(1+\cos\theta_o)}{\sin\theta_o}\left\{(n+\tfrac{3}{2})\sin\frac{\theta_o}{2}\cos(n+\tfrac{3}{2})\theta_o\right.$$

$$\left.-\frac{1}{2}\cos\frac{\theta_o}{2}\sin(n+\tfrac{3}{2})\theta_o\right\} + \frac{n(1-\cos\theta_o)^{5/2}}{2(n+\frac{5}{2})(n+\frac{3}{2})(n+2)}\left\{(3n^3+5n^2+n+2)\tan^{-1}\frac{\theta_o}{2}\sin n\theta_o\right.$$

$$+ n(n+1)(n+2)\cos n\theta_o + n(3n^2+2n+1)\tan^{-2}\frac{\theta_o}{2}\sin n\theta_o - n^3\tan^{-3}\frac{\theta_o}{2}\sin n\theta_o$$

$$-\frac{3n^2}{2}(1+\tan^2\frac{\theta_o}{2})\tan^{-4}\frac{\theta_o}{2}\cos n\theta_o - \frac{3n^2+n+1}{2}(1+\tan^2\frac{\theta_o}{2})\tan^{-3}\frac{\theta_o}{2}\sin n\theta_o$$

$$\left.+\frac{3n}{4}(1+\tan^2\frac{\theta_o}{2})^2\tan^{-5}\frac{\theta_o}{2}\sin n\theta_o\right\}$$

Acknowledgement

I wish to thank Drs. K. Lee, L. Marin, C. H. Papas and M. Sancer for helpfull discussions.

References

Born, M. and Wolf, E., Principles of Optics, Pergamon Press, 1970.

Bouwkamp, C. J., Philips Res. Repts., 5, pp. 401-422, 1950.

Chang, S. and Senior, T., "Scattering by a Spherical Shell with a Circular Aperture," Radiation Laboratory, Univ. of Michigan, April 1969, Scientific Rep. No. 5.

Erdely, A. et al., Higher Transcendental Functions, Vol. I, New York, 1953.

MacRobert, T. M., Spherical Harmonics, London, 1947.

Morse, P. M. and Feshbach, H., Methods of Theoretical Physics, New York, 1953.

Sancer, M. I. and Varvatsis, A. D. "Electromagnetic Penetrability of Perfectly Conducting Bodies Containing an Aperture," EMP Interaction Note Number 49, August 1970.

Thomas, D. P., Proc. Camb. Phil. Soc., 59, pp. 197-209, 1963.