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~~INTERIM~~ REPORT

RESPONSE OF A MULTICONDUCTOR CABLE TO EXCITATION BY AN ARBITRARY SINGLE-FREQUENCY, CONSTANT-IMPEDANCE SOURCE

BY

SIDNEY FRANKEL

Sidney Frankel and Associates
P. O. Box 126
Menlo Park, California, 94025

for

Sandia Laboratories, Albuquerque

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SUMMARY

The purpose of the phase of the cable study discussed in this interim report has been to develop a formulation relating cable output voltages to total output current as a function of the parameters controlling these quantities, and to determine, if possible, under what conditions this relation might turn out to be particularly simple.

Starting with certain canonical equations of multiconductor line theory, it is a straightforward matter to obtain matrix expressions for terminal voltages and currents in terms of the generalized Thévenin source, a generalized "Thévenin" load, and the various parameters of the line. These results, as usual, are recognizable as generalizations of the conventional two-wire line results. Except for the broadest kinds of statements, hand analysis of these results is impractical for a cable of more than two or three conductors plus shield; for thorough study the model must be referred to a digital computer.

However, to obtain some initial feel for how the cable behaves, a particularly elementary case received special attention. It was assumed, initially, that source- and load- termination matrices were proportional to the line admittance matrix, with independent constants of proportionality for source and load, and that all source open-circuit (Thévenin) emf's were equal. The effect of varying one diagonal element* of the load admittance matrix was then investigated. The ratio of the output voltage on any terminal to the bulk output current was found to be insensitive to (but not entirely independent of) this admittance variation, and, as a corollary, to variation of the line length and operating frequency, only when the load-admittance proportionality factor was high, and when the (common-mode) mismatch of the source to the line was not great, the sensitivity increasing in proportion to the mismatch.

No explicit effect of the number of conductors in the cable was observed.

In addition to the generalized Thévenin's theorem, other generalized theorems were devised and discussed in this report. However, they were not found to be necessary or particularly useful in the present phase of the study.

*Equivalently, one admittance branch connected between a load terminal and ground.

GLOSSARY

Note: numbers in [ ] correspond to reference list, page 54.

A: eq. (31)

$\hat{\underline{A}}$: adjoint of $\underline{A}$ [4]

$a = -j \cot \theta$

a_{ij}: element of i<u>th</u> row, j<u>th</u> column of $\underline{A}$

$\hat{a}_{ij}$: element of i<u>th</u> row, j<u>th</u> column of $\hat{\underline{A}}$

B_{ij}: cofactor of a_{ij} in $\underline{A}$ [4]

$b = j \csc \theta$

b_{jk}, b_{kk}: eqs. (49)

$N^{C}R$: combination of N things taken r at a time
$= N(N-1)(N-2) \cdots (N-r+1)/(r)$.

c_{ik}: eq. (5)

D_A: determinant of $\underline{A}$ [4]

$\underline{I}^i$: line current input matrix, amp. [1]

$\underline{I}^{\circ}$: line output current matrix, amp. [1]

I^s: short-circuit current matrix of equivalent Norton source, amp.

I_j°: element of j<u>th</u> row of $\underline{I}^{\circ}$, amp.

I_T°: output bulk current, amp.

$$I_T^{\circ} = \sum_{j=1}^{N} I_j^{\circ}$$

$\underline{\mathcal{I}}$: unit matrix (NxN)

i: matrix- or summation index

j: (1) - $\sqrt{-1}$

(2) - matrix- or summation index

k_i: source-end termination-admittance proportionality constant

k_o: load-end termination-admittance proportionality constant

K_{ik}: voltage coupling coefficient between ith and jth conductors: eqs. (74)

$\underline{M}$: eqs. (5)

$\underline{N}$: eqs. (5)

N-line: system of N parallel conductors in the vicinity of an (N+1)st reference conductor

$\underline{P}^\circ = \underline{ZY}^\circ$

$\underline{P}^i = \underline{ZY}^i$

$\underline{Q}^i$: inverse of $\underline{P}^i$

u: a parmeter proportional to ΔY^L_{kk}: eqs. (74)

$\underline{V}^g$: generalized Thévenin emf of source, volts [1]

$\underline{V}^i$: line input voltage matrix, volts [1]

$\underline{V}^\circ$: line output voltage matrix, volts [1]

V_g: common N-port source voltage when $V^g_i = V_g$, $i = 1, \cdots, N$

V^g_i: element of the ith row of $\underline{V}^g$

$\underline{Y}$: $\underline{Z}^{-1}$

$\underline{Y}^L$: minimally general load admittance matrix, mhos (section 2.1.1 and reference 1); not equal to $\underline{Y}^\circ$

$\underline{Y}^g$: terminal admittance matrix of Norton source, -source currents set equal to zero.

$\underline{Y}^i$: source-end termination admittance matrix, mhos [1]

$\underline{Y}^\circ$: load-end termination admittance matrix, mhos [1] not equal to $\underline{Y}^L$

$\Delta \underline{Y}^L$: increment in $\underline{Y}^L$

ΔY^L_{ij}: increment in Y^L_{ij}

$\Delta \underline{Y}^\circ$: increment in $\underline{Y}^\circ$

Y^e_j: common-mode,- or even-mode characteristic admittance of j<u>th</u> conductor: eq. (59)

Y^e_0: common-mode,- or even-mode characteristic admittance of whole line; $Y^e_0 = \sum_{j=1}^{N} Y^e_j$

Y^L_{jk}: element of j<u>th</u> row, k<u>th</u> column of $\underline{Y}^L$

Y^g_{jk}: element of j<u>th</u> row, k<u>th</u> column of $\underline{Y}^g$

Y^i_{jk}: element of j<u>th</u> row, k<u>th</u> column of $\underline{Y}^i$

Y°_{jk}: element of j<u>th</u> row, k<u>th</u> column of $\underline{Y}^\circ$

$\underline{Z}$: line impedance matrix: $\underline{Y}^{-1}$

$\underline{Z}^i$: $(\underline{Y}^i)^{-1}$

$\underline{Z}^\circ$: $(Y^\circ)^{-1}$

β: eq. (27)

γ: eq. (20)

δ^j_k: Kronecker delta: eq. (60) ff.

σ_i: ratio of i<u>th</u> conductor output voltage to output bulk current: eq. (63)

θ: Electrical length of line, radians.

INTERIM REPORT

RESPONSE OF A MULTICONDUCTOR CABLE TO EXCITATION

BY AN ARBITRARY SINGLE-FREQUENCY, CONSTANT-IMPEDANCE SOURCE

1. Introduction

This is the second interim report submitted in partial fulfillment of the requirements of Contract No. 11-1756. The first interim report dealt with the response of a cable to excitation by a single-frequency source at an open break in the shield [1]*. The present report deals with the response of a cable to a single-frequency, constant-impedance, but otherwise arbitrary source at one end of the cable. The source is assumed to have N terminals (plus ground), at any, or all, of which source potentials may exist.

In their essential physical behavior the two configurations differ only in that, for the broken-shield problem, the model is that of two lines in series with a single independent excitation voltage from which all conductor potentials are determined, while, in the present instance, only a single line is involved, but each of the N conductors may be subjected to an independent excitation potential derived from fixed emf's within an arbitrary source network.

This report deals with the response of an N-line** when subjected to such an arbitrary set of potentials. Expressions are derived for the line response in terms of these potentials, the line parameters, and a sufficiently general class of line terminations (including source impedances).

In order to avoid unnecessary generality of terminal conditions, generalizations of Thévenin's and Norton's theorems are presented. (The assumptions of constant source, load, and line parameters justify the use of the superposition theorem.) A generalization of the compensation theorem was also obtained in the expectation that it might prove useful for some aspects of analysis. However, no applications of this theorem are used in the present report.

In an attempt to gain some insight into the effect of variation of the termination, results were obtained for terminal voltages and bulk terminal current in terms of a load admittance matrix equal to the sum of a matrix

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*Numbers in [] correspond to Reference List, page 44.

**See reference 1.

proportional to the line admittance matrix and an incremental load admittance matrix. Behavior for various values of the proportionality constant was investigated.

2. Basic Model: Line Schematic

Sandia's Request for Proposal on Transmission Line Analysis, attached to D. E. Merewether's letter of June 17, 1970, to Sidney Frankel and Associates, indicates the need for analysis of the behavior of a shielded line and of an unshielded line excited at one end in the interior of one of the terminal boxes of the cable (Figs. 3 and 4 of the Request for Proposal, respectively). To the extent that both of these configurations can be simulated by TEM structures, there is no essential difference in their method of analysis. This assumption is made in the subsequent discussion. Thus, a common schematic for the two configurations is shown in Fig. 1. As in the previous report, [1], the N conductors above ground are indicated by a single line, with currents, voltages, terminal admittances, and line impedance parameters presented as matrices. The previous report introduced the concepts of a line impedance matrix, $\underline{Z}$, and a load admittance matrix $\underline{Y}^L$. In that report, $\underline{Y}^L$ was chosen, for an N line, as a set of $\frac{1}{2}$ N(N + 1) admittances interconnecting all possible pairs ${}_{(N+1)}C_2$ of N terminals. More complicated terminating networks are possible; but insofar as line response (currents and voltages) is concerned, a network of $\left[{}_{(N+1)}C_2\right]$ admittances is a sufficient termination, and all linear passive networks with (N + 1) accessible terminals are externally equivalent to it.

The source in Fig. 1 is a generalized Thévenin source. Generalized Thévenin - , Norton - and compensation theorems are discussed in the next section, and in sections 4.1 and 4.2.

2.1 Generalized Thévenin's Theorem

We first state the theorem in its usual scalar form: [Ref. 2, Chapter II, Section 11].

If an impedance, Z_L, be connected between any two points of a circuit, the resulting (single-frequency) current, I, through the impedance is the ratio of the potential difference, V_g, between the two points, prior to the connection, and the sum of the values of (1), the connected impedance, Z_L, and (2) the impedance, Z_g, of the circuit, measured between the two points, when $V_g = 0$. That is,

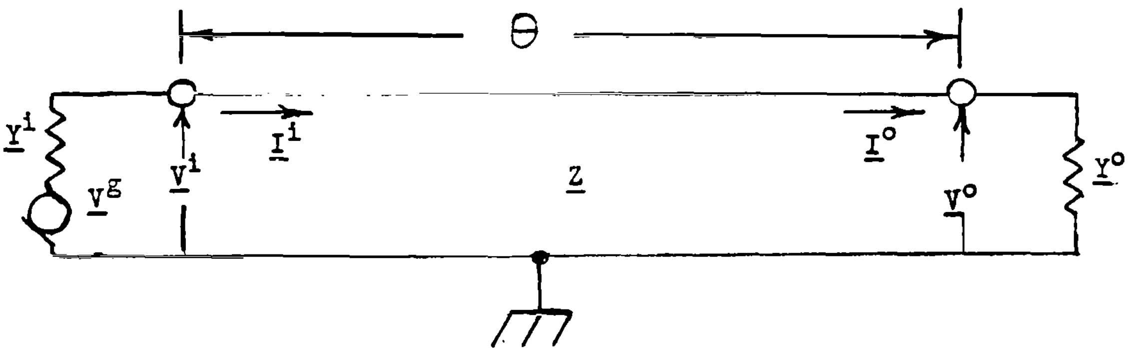


Fig.1. Parameters and Terminations of an N-line.

$$I = \frac{V_g}{Z_g + Z_L} \quad .$$

In other words, the network with two accessible terminals appears externally like an emf, V_g, in series with an impedance, Z_g. The voltage, V_g, is the "open-circuit" source voltage.

A generalization of this theorem for an $(N + 1)$-port is derived in Appendix A. See Fig. 2, which shows an $(N + 1)$-terminal generator (N terminals above ground) connected to a driving point admittance, Y^L, representing the input to an N-line.

Suppose the line disconnected from the source, so that the N terminals of the source above ground are open-circuited. Measure the open-circuit potentials, V_k^g, $k = 1, \ldots, N$, of these terminals with respect to ground. Next, set all internal emf's within the termination equal to zero and measure the $\frac{1}{2} N(N + 1)$ admittances between all pairs of the $(N + 1)$ terminals. Call the resulting admittance matrix $\underline{Y}^i = (\underline{Z}^i)^{-1}$. Then it is shown in Appendix A that the generator termination is externally equivalent to a passive termination, $\underline{Y}^i$, which has an emf, V_k^g, in series with its k^{th} terminal, $k = 1, \ldots, N$, (Fig. 3). This is the meaning to be attached to the source of impedance, $\underline{Y}^i$, and voltage, $\underline{V}^g$, of Fig. 1.

2.1.1 Further Discussion of the Terminal Networks

In Ref. 1, cable terminations were described in terms of $(N + 1)$ nodes, corresponding to the N terminals of the cable plus the shield. However, the termination may have additional internal nodes which are not connected to any conductors of the N-line. (To simplify the mathematical description we will assume that every conductor of the N-line is connected to the terminal networks. In case some line, say the k^{th}, is not so connected, this fact merely specifies that the connected admittances are $Y_{kj}^L = 0$, $j = 1, \ldots, N$).

Suppose the number of additional internal nodes is P, so that the total number of nodes is $(N + P)$. This additional complexity may be handled in one of several different ways, depending on the requirements of the problem.

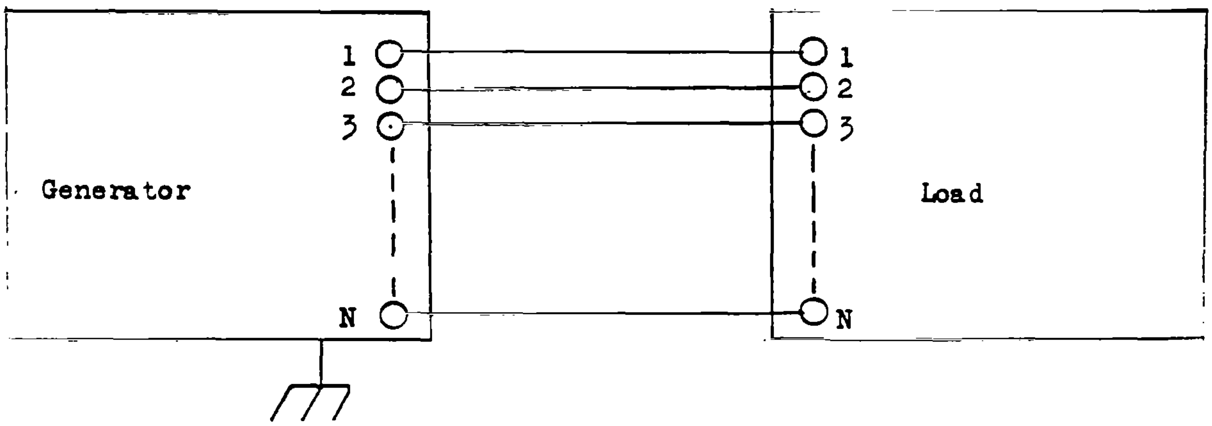


Fig.2. N-port Generator Connected to N-port Load

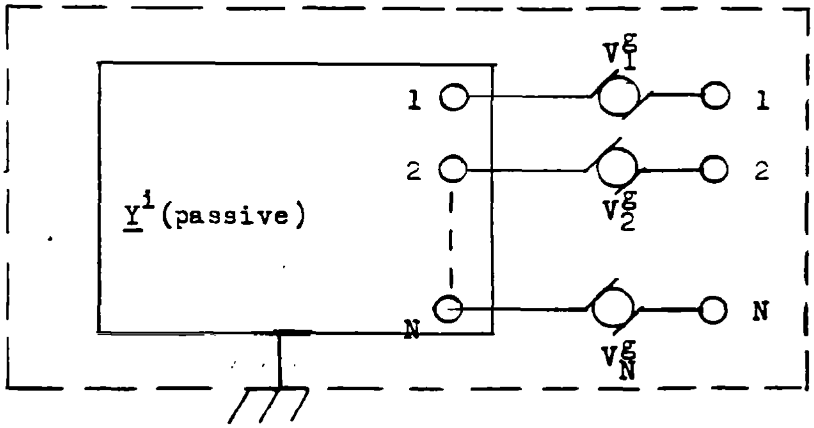


Fig. 3. Generalized Thévenin Generator.

Seen from the N-line the termination (assumed linear, passive, reciprocal) is, as we have seen, specified by all the admittance measurements that can be made between any combination of the (N + 1) terminals (ground included), the number of such measurements being $\frac{1}{2}$ N(N + 1), corresponding to the number of independent Y^i_{jk}, Y^o_{jk} (j, k = 1,..., N) in the termination matrices.

If these Y^i_{jk}, Y^o_{jk} are to be determined analytically from specified termination networks, the method to use depends on the complexity of the termination and the generality of the required solution. If a single configuration with specific numbers is to be used, actual measurements might prove simplest, unless a simulating computer program happens to be already available. If only behavior at the line terminals is of interest, a formulation yielding the ${}_{(N+1)}C_2$ elements of the terminal admittance matrices is adequate. On the other hand, if quantities internal to the terminations are required, such as the current in some branch, or the potential between one of the P internal nodes and another node, then the formulation has to consist of a mixed combination of network equations and transmission line equations. An example of such a formulation is the problem dealt with in Ref. 3. (See, especially, Appendix D of the cited reference.)

3. <u>Line Analysis: Thévenin Source and Load</u>

For simplicity, and without loss of generality insofar as line behavior is concerned, we assume a generalized Thévenin source specified, for an N-line, by N open-circuit terminal potentials and $\frac{1}{2}$ N(N + 1) available admittance parameters. We further assume the load to have the same general description (with generally different parameters) as the Thévenin source with all emf's set equal to zero. Such a load will be termed a Thévenin load.

Referring now to Fig. 1 for notation we have [Equations (14), Ref. 1],

$$\left.\begin{aligned} \underline{V}^i &= a\,\underline{Z}\,\underline{I}^i + b\,\underline{Z}\,\underline{I}^o \\ \underline{V}^o &= -b\,\underline{Z}\,\underline{I}^i - a\,\underline{Z}\,\underline{I}^o \end{aligned}\right\} \qquad (1)$$

where

$$\left.\begin{aligned} a &= -j \cot \theta \\ b &= j \csc \theta \quad . \end{aligned}\right\} \tag{2}$$

Terminal conditions are

$$\left.\begin{aligned} &\text{at the load end:} \quad \underline{I}^{o} = \underline{Y}^{o}\, \underline{V}^{o} \\ &\text{at the source end:} \quad \underline{V}^{g} - \underline{Z}^{i}\, \underline{I}^{i} = \underline{V}^{i} \end{aligned}\right\} \tag{3}$$

where $\underline{Z}^{i} = (\underline{Y}^{i})^{-1}$. Use Equations (3) to eliminate $\underline{V}^{i}$ and $\underline{I}^{o}$ in Equations (1).

$$\left.\begin{aligned} \underline{V}^{g} - \underline{Z}^{i}\, \underline{I}^{i} &= a\, \underline{Z}\, \underline{I}^{i} + b\, \underline{Z}\, \underline{Y}^{o}\, V^{o} \\ \underline{V}^{o} &= -b\, \underline{Z}\, \underline{I}^{i} - a\, \underline{Z}\, \underline{Y}^{o}\, \underline{V}^{o} \quad . \end{aligned}\right\} \tag{4}$$

Write

$$\underline{P}^{o} = \underline{Z}\, \underline{Y}^{o}$$

$$\underline{\mathcal{I}} = \text{unit matrix}$$

and

$$\left.\begin{aligned} \underline{M} &= a\, \underline{\mathcal{I}} + \underline{P}^{o} \\ \underline{N} &= \underline{\mathcal{I}} + a\, \underline{P}^{o} \quad . \end{aligned}\right\} \tag{5}$$

Then after re-arranging, collecting terms, and solving, we have from the second of Equations (4)

$$\underline{V}^{o} = -b\ \underline{N}^{-1}\ \underline{Z}\ \underline{I}^{i} \quad . \tag{6}$$

Substituting Equation (6) in the first of Equations (4)

$$\underline{V}^{g} - \underline{Z}^{i}\ \underline{I}^{i} = a\ \underline{Z}\ \underline{I}^{i} + b\ \underline{P}^{o}\ [-b\ \underline{N}^{-1}\ \underline{Z}\ \underline{I}^{i}] \quad .$$

Re-arranging and collecting terms,

$$\{[a\ \underline{\mathcal{I}} - b^{2}\ \underline{P}^{o}\ \underline{N}^{-1}]\ \underline{Z} + \underline{Z}^{i}\}\ \underline{I}^{i} = \underline{V}^{g} \quad . \tag{7}$$

The coefficient of Z in Equation (7) is reduced as follows:

$$\begin{aligned}
a\ \underline{\mathcal{I}} - b^{2}\ \underline{P}^{o}\ \underline{N}^{-1} &= \underline{P}^{o}\ \underline{N}^{-1}\ [a\ \underline{N}\ (\underline{P}^{o})^{-1} - b^{2}\ \underline{\mathcal{I}}] \\
&= \underline{P}^{o}\ \underline{N}^{-1}\ [a(\underline{\mathcal{I}} + a\ \underline{P}^{o})(\underline{P}^{o})^{-1} - b^{2}\ \underline{\mathcal{I}}] \\
&= \underline{P}^{o}\ \underline{N}^{-1}\ [a(\underline{P}^{o})^{-1} + (a^{2} - b^{2})\ \underline{\mathcal{I}}] \\
&= \underline{P}^{o}\ \underline{N}^{-1}\ [a(\underline{P}^{o})^{-1} + \underline{\mathcal{I}}] \\
&= \underline{P}^{o}\ \underline{N}^{-1}\ (\underline{P}^{o})^{-1}\ \underline{M} \\
&= \underline{N}^{-1}\ \underline{M} \qquad (8)
\end{aligned}$$

since

$$\underline{P}^{o}\ \underline{N}^{-1}\ (\underline{P}^{o})^{-1} = [\underline{P}^{o}\ \underline{N}\ (\underline{P}^{o})^{-1}]^{-1} = \underline{N}^{-1} \quad .$$

Furthermore, for purposes of comparison with the broken cable shield analysis [1] it is worth noting that the product $\underline{N}^{-1}\,\underline{M}$ is commutative. For we have

$$\underline{N}\,\underline{M} = (\underline{\mathscr{I}} + a\,\underline{P}^o)(a\,\underline{\mathscr{I}} + \underline{P}^o)$$

$$= a\,\underline{\mathscr{I}} + (1 + a^2)\,\underline{P}^o + a\,(\underline{P}^o)^2$$

while

$$\underline{M}\,\underline{N} = (a\,\underline{\mathscr{I}} + \underline{P}^o)(\underline{\mathscr{I}} + a\,\underline{P}^o)$$

$$= a\,\underline{\mathscr{I}} + (1 + a^2)\,\underline{P}^o + a(\underline{P}^o)^2 = \underline{N}\,\underline{M} \quad . \tag{8a}$$

Multiplying both sides of the equation on the left and on the right by $\underline{M}^{-1}$ then yields

$$\underline{M}^{-1}\,(\underline{M}\,\underline{N})\,\underline{M}^{-1} = \underline{M}^{-1}\,(\underline{N}\,\underline{M})\,\underline{M}^{-1}$$

whence

$$\underline{N}\,\underline{M}^{-1} = \underline{M}^{-1}\,\underline{N} \quad . \tag{9}$$

Using (8) in the coefficient of $\underline{I}^i$ in (7) gives

$$\begin{aligned}\{[a\,\underline{\mathscr{I}} - b^2\,\underline{P}^o\,\underline{N}^{-1}]\,\underline{Z} + \underline{Z}^i\} &= \underline{N}^{-1}\,\underline{M}\,\underline{Z} + \underline{Z}^i = \underline{M}\,\underline{N}^{-1}\,\underline{Z} + \underline{Z}^i \\ &= (\underline{M} + \underline{Z}^i\,\underline{Y}\,\underline{N})\,\underline{N}^{-1}\,\underline{Z} \\ &= (\underline{M} + \underline{Q}^i\,\underline{N})\,\underline{N}^{-1}\,\underline{Z}\end{aligned} \tag{10}$$

where we have written

$$\underline{Q}^i = \underline{Z}^i\,\underline{Y} = (\underline{P}^i)^{-1} = (\underline{Z}\,\underline{Y}^i)^{-1} \quad . \tag{11}$$

Substituting (10) in (7) and solving for $\underline{I}^i$

$$\underline{I}^i = \underline{Y}\ \underline{N}\ (\underline{M} + \underline{Q}^i\ \underline{N})^{-1}\ \underline{V}^g \quad . \tag{12}$$

Next, solving for $\underline{V}^o$ by substituting (12) in (6)

$$\begin{aligned} \underline{V}^o &= -b\ \underline{N}^{-1}\ \underline{Z}\ \underline{Y}\ \underline{N}\ (\underline{M} + \underline{Q}^i\ \underline{N})^{-1}\ \underline{V}^g \\ &= -b\ (\underline{M} + \underline{Q}^i\ \underline{N})^{-1}\ \underline{V}^g \quad . \end{aligned} \tag{13}$$

Next, since

$$\underline{I}^o = \underline{Y}^o\ \underline{V}^o$$

Equation (13) yields

$$\underline{I}^o = -b\ \underline{Y}^o\ (\underline{M} + \underline{Q}^i\ \underline{N})^{-1}\ \underline{V}^g \quad . \tag{14}$$

Finally, using (12) and the second of Equations (3),

$$\begin{aligned} \underline{V}^i &= \underline{V}^g - \underline{Z}^i\ [\underline{Y}\ \underline{N}\ (\underline{M} + \underline{Q}^i\ \underline{N})^{-1}]\ \underline{V}^g \\ &= [\mathcal{I} - \underline{Q}^i\ \underline{N}\ (\underline{M} + \underline{Q}^i\ \underline{N})^{-1}]\ \underline{V}^g \\ &= [(\underline{M} + \underline{Q}^i\ \underline{N}) - \underline{Q}^i\ \underline{N}]\ (\underline{M} + \underline{Q}^i\ \underline{N})^{-1}\ \underline{V}^g \\ &= \underline{M}\ (\underline{M} + \underline{Q}^i\ \underline{N})^{-1}\ \underline{V}^g \quad . \end{aligned} \tag{15}$$

Collecting the results [Equations (13) - (15)] for convenience, we have

$$\left.\begin{aligned} \underline{V}^i &= \underline{M}\ (\underline{M} + \underline{Q}^i\ \underline{N})^{-1}\ \underline{V}^g \\ \underline{I}^i &= \underline{Y}\ \underline{N}\ (\underline{M} + \underline{Q}^i\ \underline{N})^{-1}\ \underline{V}^g \\ \underline{V}^o &= -b\ (\underline{M} + \underline{Q}^i\ \underline{N})^{-1}\ \underline{V}^g \\ \underline{I}^o &= -b\ \underline{Y}^o\ (\underline{M} + \underline{Q}^i\ \underline{N})^{-1}\ \underline{V}^g \end{aligned}\right\} \tag{16}$$

where

$$\left.\begin{aligned}
\underline{M} &= a\,\underline{\mathscr{L}} + \underline{P}^{o} \\
N &= \underline{\mathscr{L}} + a\,\underline{P}^{o} \\
\underline{P}^{o} &= \underline{Z}\,\underline{Y}^{o} \\
\underline{P}^{o} &= \underline{Z}\,\underline{Y}^{i} = (\underline{Q}^{i})^{-1} \\
\underline{Q}^{i} &= \underline{Z}^{i}\,\underline{Y}
\end{aligned}\right\} \qquad (17)$$

3.1 Special Cases: Termination Matrices Proportional to Line Matrix

Let

$$\left.\begin{aligned}
\underline{Y}^{i} &= k_i\,\underline{Y} \\
\underline{Y}^{o} &= k_o\,\underline{Y}
\end{aligned}\right\} \qquad (18)$$

then

$$\left.\begin{aligned}
\underline{P}^{i} &= k_i\,\underline{\mathscr{L}} \;;\quad \underline{Q}^{i} = k_i^{-1}\,\underline{\mathscr{L}} \\
\underline{P}^{o} &= k_o\,\underline{\mathscr{L}} \\
\underline{M} &= (a + k_o)\,\underline{\mathscr{L}} \\
\underline{N} &= (1 + a\,k_o)\,\underline{\mathscr{L}} \\
\underline{M} + \underline{Q}^{i}\,\underline{N} &= \gamma\,(k_i, k_o\,;\,a)\,\underline{\mathscr{L}}
\end{aligned}\right\} \qquad (19)$$

where

$$\gamma = \frac{1 + a\,(k_i + k_o) + k_i\,k_o}{k_i} \quad . \tag{20}$$

Then Equations (16) easily reduce to

$$\left.\begin{aligned}
\underline{V}^i &= \frac{a + k_o}{\gamma}\,\underline{V}^g \\
\underline{I}^i &= \frac{1 + a\,k_o}{\gamma}\,\underline{Y}\,\underline{V}^g \\
\underline{V}^o &= -\,\frac{b}{\gamma}\,\underline{V}^g \\
\underline{I}^o &= -\,\frac{b\,k_o}{\gamma}\,\underline{Y}\,\underline{V}^g \quad .
\end{aligned}\right\} \tag{21}$$

The effect of the source admittance is contained entirely within the scalar quantity, γ. Some special cases follow.

3.1.1 $k_i = \infty$: Constant-Voltage Source

For $k_i = \infty$ we have, from (20),

$$\gamma\,(\infty,\,k_o;\,a) \rightarrow \frac{a\,k_i + k_i\,k_o}{k_i} = a + k_o \quad .$$

Equations (21) become

$$\left.\begin{aligned}
\underline{V}^i &= \underline{V}^g \\
\underline{I}^i &= \frac{1 + a\,k_o}{a + k_o}\,\underline{Y}\,\underline{V}^g \\
\underline{V}^o &= -\,\frac{b}{a + k_o}\,\underline{V}^g \\
\underline{I}^o &= -\,\frac{b\,k_o}{a + k_o}\,\underline{Y}\,\underline{V}^g \quad .
\end{aligned}\right\} \tag{22}$$

3.1.2 $k_i = 1$: Matched Source

For $k_i = 1$ we have, from (20),

$$\gamma(1, k_o; a) = 1 + a(1 + k_o) + k_o = (1 + a)(1 + k_o) \quad .$$

Equations (21) become

$$\left.\begin{aligned}
\underline{V}^i &= \frac{a + k_o}{(1 + a)(1 + k_o)} \; \underline{V}^g \\
\underline{I}^i &= \frac{1 + a\, k_o}{(1 + a)(1 + k_o)} \; \underline{Y}\, \underline{V}^g \\
\underline{V}^o &= - \frac{b}{(1 + a)(1 + k_o)} \; \underline{V}^g = (1 + k_o)^{-1}\, e^{-j\theta}\, \underline{V}^g \\
\underline{I}^o &= \frac{k_o\, e^{-j\theta}}{1 + k_o} \; \underline{Y}\, \underline{V}^g \quad .
\end{aligned}\right\} \tag{23}$$

3.1.3 $k_i \to 0$: Constant-Current Source

For $k_i \to 0$ we have, from (20),

$$\gamma(k_i \to 0, k_o; a) \to \frac{1 + a\, k_o}{k_i} \quad .$$

Equations (21) become

$$\left.\begin{aligned}
\underline{V}^i &\to \frac{a + k_o}{1 + a\, k_o} \; k_i\, \underline{V}^g \\
\underline{I}^i &\to k_i\, \underline{Y}\, \underline{V}^g \\
\underline{V}^o &\to - \frac{b}{1 + a\, k_o} \; k_i\, \underline{V}^g \\
\underline{I}^o &\to - \frac{b\, k_o}{1 + a\, k_o} \; k_i\, \underline{Y}\, \underline{V}^g \quad .
\end{aligned}\right\} \tag{24}$$

Now as $k_i \to 0$, assume that $\underline{V}^g \to \underline{\infty}$ (at least one element of $\underline{V}^g \to \infty$) in such a way that $k_i \underline{V}^g$ remains, in the limit, finite and different from $\underline{0}$. Then by the second of Equations (24), this finite value is given by

$$k_i \underline{V}^g = \underline{Z}\, \underline{I}^i \quad .$$

Substituting this value in the remaining Equations (24) we get

$$\left.\begin{aligned} \underline{V}^i &= \frac{a + k_o}{1 + a\, k_o}\, \underline{Z}\, \underline{I}^i \\ \underline{V}^o &= - \frac{b}{1 + a\, k_o}\, \underline{Z}\, \underline{I}^i \\ \underline{I}^o &= - \frac{b\, k_o}{1 + a\, k_o}\, \underline{I}^i \quad . \end{aligned}\right\} \qquad (25)$$

3.2 Source Matrix Alone Proportional to Line Matrix

In this section, although we permit the load matrix to be completely general, we write its value in such a way that it becomes relatively easy to use it for simple deviations from the cases discussed in the previous section. Thus, as before, take

$$\underline{P}^i = k^i \mathcal{L}$$

but write, without loss of generality

$$\underline{Y}^o = k_o\, \underline{Y} + \underline{\Delta Y}^o$$

and

$$\underline{P}^o = \underline{Z}\, \underline{Y}^o = \underline{Z}\,(k_o\, \underline{Y} + \underline{\Delta Y}^o)$$

$$= k_o \mathcal{L} + \underline{Z}\, \underline{\Delta Y}^o \quad .$$

Therefore

$$\underline{Q}^i = k_i^{-1} \underline{\mathcal{I}}$$

$$\underline{M} = (a + k_o) \underline{\mathcal{I}} + \underline{Z}\, \underline{\Delta Y}^o$$

$$\underline{N} = (1 + a\, k_o) \underline{\mathcal{I}} + a\, \underline{Z}\, \underline{\Delta Y}^o$$

$$\begin{aligned} \underline{M} + \underline{Q}^i\, \underline{N} &= \frac{1}{k_i} (k_i\, \underline{M} + \underline{N}) \\ &= \frac{1}{k_i} \left\{ k_i \left[(a + k_o) \underline{\mathcal{I}} + \underline{Z}\, \underline{\Delta Y}^o \right] + \left[(1 + a\, k_o) \underline{\mathcal{I}} + a\, \underline{Z}\, \underline{\Delta Y}^o \right] \right\} \\ &= \gamma\, \underline{\mathcal{I}} + \frac{a + k_i}{k_i} \underline{Z}\, \underline{\Delta Y}^o \quad . \end{aligned}$$

Substituting these results in Equations (16) yields

$$\left. \begin{aligned} \underline{V}^i &= \frac{a + k_o}{\gamma} \left[\underline{\mathcal{I}} + (a + k_o)^{-1} \underline{Z}\, \underline{\Delta Y}^o \right] \left[\underline{\mathcal{I}} + \frac{a + k_i}{\gamma\, k_i} \underline{Z}\, \underline{\Delta Y}^o \right]^{-1} \underline{V}^g \\ \underline{I}^i &= \frac{1 + a\, k_o}{\gamma} \underline{Y} \left[\underline{\mathcal{I}} + \frac{a + k_i}{\gamma\, k_i} \underline{Z}\, \underline{\Delta Y}^o \right]^{-1} \left[\underline{\mathcal{I}} + \frac{a}{1 + a\, k_o} \underline{Z}\, \underline{\Delta Y} \right] \underline{V}^g \\ \underline{V}^o &= - \frac{b}{\gamma} \left[\underline{\mathcal{I}} + \frac{a + k_i}{\gamma\, k_i} \underline{Z}\, \underline{\Delta Y}^o \right]^{-1} \underline{V}^g \\ \underline{I}^o &= - \frac{b\, k_o}{\gamma} \underline{Y} \left[\underline{\mathcal{I}} + k_o^{-1} \underline{Z}\, \underline{\Delta Y}^o \right] \left[\underline{\mathcal{I}} + \frac{a + k_i}{\gamma\, k_i} \underline{Z}\, \underline{\Delta Y}^o \right]^{-1} \underline{V}^g \quad . \end{aligned} \right\} \qquad (26)$$

Let

$$\beta = \frac{a + k_i}{\gamma\, k_i} = \frac{a + k_i}{1 + a(k_i + k_o) + k_i\, k_o} \quad . \tag{27}$$

We are particularly interested in further manipulation of the third and fourth of Equations (26). Substituting Equation (27) we have

$$\left.\begin{aligned} \underline{V}^o &= -\frac{b}{\gamma}\left(\mathcal{I} + \beta\, \underline{Z}\, \underline{\Delta Y}^o\right)^{-1} \underline{V}^g \\ \underline{I}^o &= -\frac{b\, k_o}{\gamma}\, \underline{Y}\left(\mathcal{I} + k_o^{-1}\, \underline{Z}\, \underline{\Delta Y}^o\right)\left(\mathcal{I} + \beta\, \underline{Z}\, \underline{\Delta Y}^o\right)^{-1} \underline{V}^g \end{aligned}\right\} \tag{28}$$

or, more compactly,

$$\left.\begin{aligned} \underline{V}^o &= -\frac{b}{\gamma}\, \underline{A}^{-1}\, \underline{V}^g \\ \underline{I}^o &= -\frac{b\, k_o}{\gamma}\, \underline{Y}\, \underline{B}\, \underline{A}^{-1}\, \underline{V}^g \end{aligned}\right\} \tag{29}$$

where

$$\left.\begin{aligned} \underline{A} &= \mathcal{I} + \beta\, \underline{Z}\, \underline{\Delta Y}^o \\ \underline{B} &= \mathcal{I} + k_o^{-1}\, \underline{Z}\, \underline{\Delta Y}^o \quad . \end{aligned}\right\} \tag{30}$$

3.2.1 Effect of Varying a Single Load Admittance, Y^L_{kk}, Starting With Proportional Terminations

The remainder of this section deals with a simple case of $\underline{\Delta Y}^o$ in which only the admittance between the k^{th} output terminal and ground has an incremental component different from zero. That is,

$$\Delta Y^L_{kk} \neq 0$$

$$\Delta Y^L_{ij} = 0, \ i \text{ or } j \neq k \quad .$$

Therefore

$$\Delta Y^o_{kk} = \Delta Y^L_{kk} \neq 0$$

while

$$\Delta Y^o_{ij} = 0, \ i \text{ or } j \neq k \quad .$$

Thus,

$$\underline{\Delta Y}^o = \begin{bmatrix} 0 & & & \\ & \ddots & & 0 \\ & & \Delta Y^L_{kk} & \\ & 0 & & \ddots \\ & & & & 0 \end{bmatrix}$$

$$\underline{Z}\ \underline{\Delta Y}^o = \begin{bmatrix} Z_{11}, & \cdots, & Z_{1N} \\ \cdots & \cdots & \cdots \\ Z_{N1}, & \cdots, & Z_{NN} \end{bmatrix} \begin{bmatrix} 0. & & & \\ & \ddots & & 0 \\ & & Y^L_{kk} & \\ & 0 & & \ddots \\ & & & & 0 \end{bmatrix}$$

$$= \begin{array}{c} \qquad\qquad\qquad (k) \\ \left[\begin{array}{ccccccc} 0, & 0, & \ldots & 0, & Z_{1k}\,\Delta Y^{L}_{kk}, & 0, \ldots, & 0 \\ 0, & 0, & \ldots & 0, & Z_{2k}\,\Delta Y^{L}_{kk}, & 0, \ldots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & \ldots & 0, & Z_{Nk}\,\Delta Y^{L}_{kk}, & 0, \ldots, & 0 \end{array}\right] \end{array} \tag{30a}$$

where the sumbol (k) above the column of non-zero quantities indicates that it is the k^{th} column. Then

$$\underline{A} = \underline{\mathcal{I}} + \beta\, \underline{Z}\, \underline{\Delta Y^{o}} = \begin{array}{cc} & \qquad\qquad\qquad\qquad (k) \\ \begin{array}{c} \\ \\ \\ (k) \\ \\ \\ \end{array} & \left[\begin{array}{ccccccccc} 1, & 0, & 0, & \ldots, & 0, & \beta\, Z_{1k}\,\Delta Y^{L}_{kk}, & 0, & \ldots, & 0 \\ 0, & 1, & 0, & \ldots, & 0, & \beta\, Z_{2k}\,\Delta Y^{L}_{kk}, & 0, & \ldots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ 0, & 0, & 0, & \ldots, & 0, & 1 + \beta\, Z_{kk}\,\Delta Y^{L}_{kk}, & 0, & \ldots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ 0, & 0, & 0, & \ldots, & 0, & \beta\, Z_{Nk}\,\Delta Y^{L}_{kk}, & 0, & \ldots, & 1 \end{array}\right] \end{array} \tag{31}$$

where, again, the symbol (k) above and to the left of the matrix indicates the k^{th} column and the k^{th} row respectively.

We require next the inverse of (31). Use the standard relation [4]

$$\underline{A}^{-1} = D_A^{-1}\, \underline{\hat{A}} \tag{32}$$

where D_A is the determinant of $\underline{A}$ and $\underline{\hat{A}}$ is the adjoint of $\underline{A}$. If a_{ij} is the element of the i^{th} row, j^{th} column of $\underline{A}$ and B_{ij} is the cofactor of a_{ij} in $\underline{A}$, then the element of the i^{th} row, j^{th} column of $\underline{\hat{A}}$ is

$$\hat{a}_{ij} = B_{ji}, \; i, j = 1, \ldots, N \tag{33}$$

that is, $\underline{\hat{A}}$ can be obtained by replacing each a_{ij} in $\underline{A}$ by its cofactor B_{ij} and then transposing the resulting matrix.

The elements of D_A are the same as the elements of $\underline{A}$ given in Equation (31). The value of D_A can be obtained by summing the products of the elements of any row or column by their associated cofactors:

$$D_A = \sum_{i=1}^{N} a_{ij} B_{ij} = \sum_{j=1}^{N} a_{ij} B_{ij}, \ i, j = 1, \ldots, N \quad . \tag{34}$$

In particular the summation may be performed in the k^{th} column

$$D_A = \sum_{i=1}^{N} a_{ik} B_{ik} \tag{35}$$

where

$$\left. \begin{aligned} a_{ik} &= \beta Z_{ik} \Delta Y^L_{kk} , \quad i \neq k \\ a_{kk} &= 1 + \beta Z_{kk} \Delta Y^L_{kk} \quad . \end{aligned} \right\} \tag{36}$$

To obtain a cofactor, B_{ik}, in the k^{th} column of D_A, the i^{th} row and k^{th} column are removed; B_{ik} is the resulting (N - 1) st order determinant, except for sign.* Removing the k^{th} column leaves the k^{th} row as a row of zeros. Any determinant containing that row will be zero. The only non-zero determinant is the one obtained by striking out the k^{th} row, and it is clear that this is a diagonal determinant with all diagonal elements equal to one. Thus

$$\left. \begin{aligned} B_{ik} &= 0, \ i \neq k \\ B_{kk} &= 1 \end{aligned} \right\} \tag{37}$$

*The appropriate sign is $(-1)^{i+k}$

and, consequently

$$D_A = a_{kk} = 1 + \beta \ Z_{kk} \ \Delta Y^L_{kk} \quad . \tag{38}$$

Analysis of Cofactors of D_A

Consider the cofactors, B_{1j}, of the elements of the first row of D_A. We have

$$B_{1j} = (-1)^{1+j} \ D_{1j}, \quad j = 1, \ldots, N \tag{39}$$

where D_{1j} is the determinant obtained by striking out the first row, j^{th} column of D_A. But this leaves the first column as a column of zeros [see Equations (31)] unless $j = 1$. With $j = 1$ the resulting determinant has the same form as D_A except that it is of a lower order. The same reasoning as used for evaluating D_A then yields

$$D_{11} = a_{kk}$$

so that

$$\left.\begin{aligned} B_{11} &= a_{kk} \\ B_{1j} &= 0, \ j \neq 1 \quad . \end{aligned}\right\} \tag{40}$$

Considering, next, the cofactors, B_{2j}, we have

$$B_{2j} = (-1)^{2+j} \ D_{2j} \tag{41}$$

where D_{2j} is the determinant obtained by striking out the second row, j^{th} column of D_A. The resulting second column is a column of zeros. Reasoning in the same way as before we get

$$\left.\begin{aligned} B_{22} &= a_{kk} \\ B_{2j} &= 0, \quad j \neq 2 \quad . \end{aligned}\right\} \tag{42}$$

Similar arguments apply for all B_{ij}, $i < k$, $j = 1, \ldots , N$. That is, we have

$$\left.\begin{aligned} B_{ii} &= a_{kk} \,, \quad i < k \\ B_{ij} &= 0 \,, \quad\; j \neq i \\ & \qquad\quad\; j = 1, \ldots, N \quad . \end{aligned}\right\} \tag{43}$$

For $i > k$ the reasoning is again the same as above, so that in (43), the restriction $i < k$ becomes $i \neq k$.

Finally we have to evaluate the cofactors B_{kj}, $j = 1, \ldots, N$. For $j = 1$ we have

$$B_{k1} = \left[(-1)^{k+1}\right] \begin{array}{c|cccccccc|} & (2) & (3) & & & (k) & (k+1) & & \\ & 0, & 0, & \dots, & 0, & a_{1k}, & 0, & \dots, & 0 \\ & 1, & 0, & \dots, & 0, & a_{2k}, & 0, & \dots, & 0 \\ & 0, & 1, & \dots, & 0, & a_{3k}, & 0, & \dots, & 0 \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ (k-1) & 0, & 0, & \dots, & 1, & a_{k-1,k}, & 0, & \dots, & 0 \\ (k+1) & 0, & 0, & \dots, & 0, & 1, & 0, & \dots, & 0 \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & 0, & 0, & \dots, & 0, & a_{Nk}, & 0, & \dots, & 1 \end{array} \quad .$$

Expanding in terms of the first row yields only one term, namely the product of a_{1k} and its cofactor. The minor of a_{1k} is the unit diagonal determinant, and since a_{1k} is the (k - 1) st element of the determinant of B_{k1}, we have, altogether

$$B_{k1} = \left[(-1)^{k+1}\right]\left[(-1)^{(k-1)+1} a_{1k}\ (1)\right] = -a_{1k} \quad . \tag{44}$$

For B_{k2} the same process applies, except that expansion is done in the second row to yield

$$B_{k2} = -a_{2k}$$

and, in general,

$$B_{kj} = -a_{jk}, \quad j \neq k \tag{44a}$$

while it is evident by inspection of Equation (31) that

$$B_{kk} = 1 \quad . \tag{44b}$$

To summarize,

$$\hat{\underline{A}} = \left[B_{ji} \right]$$

where,

$$\left.\begin{array}{l} \text{for } i \neq k, \\ \quad B_{ij} = 0,\ j \neq i \\ \quad B_{ii} = a_{kk} \\ \text{for} \quad i = k \\ \quad B_{kj} = -a_{jk},\ j \neq k \\ \quad B_{kk} = 1 \\ \quad a_{jk} = \beta\ Z_{jk}\ \Delta Y^{L}_{kk},\ j \neq k \\ \quad a_{kk} = 1 + \beta\ Z_{kk}\ \Delta Y^{L}_{kk} \quad . \end{array}\right\} \qquad (45)$$

Thus

$$
\underline{\hat{A}} = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ \\ (k) \\ \\ (N) \end{array}
\begin{array}{c}
\begin{array}{ccccccccccc} (1) & (2) & (3) & & & & (k) & & & & (N) \end{array} \\
\left[\begin{array}{ccccccccccc}
a_{kk}, & 0, & 0, & . & . & . & ,-a_{1k}, & . & . & . & ,0 \\
0, & a_{kk}, & 0, & . & . & . & ,-a_{2k}, & . & . & . & ,0 \\
0, & 0, & a_{kk}, & . & . & . & ,-a_{3k}, & . & . & . & ,0 \\
. & . & . & & . & & . & & . & & . \\
0, & 0, & 0, & . & . & . & ,1, & . & . & . & ,0 \\
. & . & . & & . & & . & & . & & . \\
0, & 0, & 0, & . & . & . & ,-a_{Nk}, & . & . & . & ,a_{kk}
\end{array}\right]
\end{array}
\tag{46}
$$

while

$$
\underline{A}^{-1} = a_{kk}^{-1}\,\underline{\hat{A}} \tag{47}
$$

by (32) and (38).

To continue the process of evaluating Equations (29) we write next, using (30a) in (30)

$$
\underline{B} = \underline{\mathscr{I}} + k_o^{-1}\,\underline{Z}\,\underline{\Delta Y}^o
$$

$$
= \begin{array}{c} \\ \\ \\ \\ \\ (k) \\ \\ \\ \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} & & & & & & & (k) & & & & \end{array} \\
\left[\begin{array}{cccccccccccc}
1, & 0, & 0, & . & . & . & 0, & b_{1k}, & 0, & . & . & . & ,0 \\
0, & 1, & 0, & . & . & . & 0, & b_{2k}, & 0, & . & . & . & ,0 \\
0, & 0, & 1, & . & . & . & 0, & b_{3k}, & 0, & . & . & . & ,0 \\
. & . & . & & . & & & . & & . & & & . \\
0, & 0, & 0, & . & . & . & 0, & b_{kk}, & 0, & . & . & . & ,0 \\
. & . & . & & . & & & . & & . & & & . \\
0, & 0, & 0, & . & . & . & 0, & b_{Nk}, & 0, & . & . & . & ,1
\end{array}\right]
\end{array}
\tag{48}
$$

where [cf. Equation (30a)]

$$
\left.
\begin{aligned}
b_{jk} &= k_o^{-1}\, Z_{jk}\, \Delta Y_{kk}^{L},\ j \neq k \\
b_{kk} &= 1 + k_o^{-1}\, Z_{kk}\, \Delta Y_{kk}^{L} \quad .
\end{aligned}
\right\}
\tag{49}
$$

Thus

$$
\underline{B}\ \underline{A}^{-1} = a_{kk}^{-1}\ \underline{B}\ \hat{\underline{A}}
$$

$$
= a_{kk}^{-1}
\begin{array}{r|ccccccccc|}
 & & & & & & (k) & & & \\
 & a_{kk}, & 0, & 0, & \ldots, & 0, & b_{1k} - a_{1k}, & 0, & \ldots, & 0 \\
 & 0, & a_{kk}, & 0, & \ldots, & 0, & b_{2k} - a_{2k}, & 0, & \ldots, & 0 \\
 & 0, & 0, & a_{kk}, & \ldots, & 0, & b_{3k} - a_{3k}, & 0, & \ldots, & 0 \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
(k) & 0, & 0, & 0, & \ldots, & 0, & b_{kk}, & 0, & \ldots, & 0 \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & 0, & 0, & 0, & \ldots, & 0, & b_{nk} - a_{Nk}, & 0, & \ldots, & a_{kk}
\end{array} \quad .
\tag{50}
$$

By (36) and (49)

$$
\left.
\begin{aligned}
b_{ik} - a_{ik} &= (k_o^{-1} - \beta)\ Z_{ik}\, \Delta Y_{kk}^{L},\ i \neq k \\
&= c_{ik}, \text{ say} \quad .
\end{aligned}
\right\}
\tag{51}
$$

Write $b_{kk} = c_{kk}$ for uniformity of notation. Then the first of Equations (29) becomes

$$
\underline{V}^{o} = -\frac{b}{\gamma a_{kk}}
\begin{array}{c}
\\ \\ \\ \\ \\ (k) \\ \\ \\
\end{array}
\begin{array}{c}
\begin{array}{cccccc} & & & (k) & & \end{array} \\
\begin{bmatrix}
a_{kk}, & 0, & \dots, & -a_{1k}, & \dots, & 0 \\
0, & a_{kk}, & \dots, & -a_{2k}, & \dots, & 0 \\
0, & 0, & \dots, & -a_{3k}, & \dots, & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0, & 0, & \dots, & 1, & \dots, & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0, & 0, & \dots, & -a_{Nk}, & \dots, & a_{kk}
\end{bmatrix}
\end{array}
\begin{bmatrix}
V_1^g \\ V_2^g \\ V_3^g \\ \cdot \\ V_k^g \\ \cdot \\ V_N^g
\end{bmatrix} . \tag{52}
$$

For the second of Equations (29) we first find

$$
\underline{Y}\,\underline{B}\,\underline{\hat{A}} =
\begin{bmatrix}
Y_{11}, & \dots & Y_{1N} \\
\cdot & \cdot\ \cdot & \cdot \\
Y_{N1}, & \dots & Y_{NN}
\end{bmatrix}
\begin{bmatrix}
a_{kk}, & 0, & \dots, & 0, & c_{1k}, & 0, & \dots, & 0 \\
0, & a_{kk}, & \dots, & 0, & c_{2k}, & 0, & \dots, & 0 \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\
0, & 0, & \dots, & 0, & c_{kk}, & 0, & \dots, & 0 \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\
0, & 0, & \dots, & 0, & c_{Nk}, & 0, & \dots, & a_{kk}
\end{bmatrix}
$$

$$
= \begin{array}{c}
\begin{array}{ccccccc} & & & & (k) & & \end{array} \\
\begin{bmatrix}
Y_{11}\,a_{kk}, & Y_{12}\,a_{kk}, & Y_{13}\,a_{kk}, & \dots, & \sum_1^N Y_{1i}\,c_{Nk}, & \dots, & Y_{1N}\,a_{kk} \\
Y_{21}\,a_{kk}, & Y_{22}\,a_{kk}, & Y_{23}\,a_{kk}, & \dots, & \sum_1^N Y_{2i}\,c_{ik}, & \dots, & Y_{2N}\,a_{kk} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
Y_{k1}\,a_{kk}, & Y_{k2}\,a_{kk}, & Y_{k3}\,a_{kk}, & \dots, & \sum_1^N Y_{ki}\,c_{ik}, & \dots, & Y_{kN}\,a_{kk} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
Y_{N1}\,a_{kk}, & Y_{N2}\,a_{kk}, & Y_{N3}\,a_{kk}, & \dots, & \sum_1^N Y_{Ni}\,c_{ik}, & \dots, & Y_{NN}\,a_{kk}
\end{bmatrix}
\end{array}
$$

$$
= a_{kk} \begin{array}{c} \hphantom{Y_{11}, Y_{12}, \ldots,}\ (k) \\ \begin{bmatrix} Y_{11}, & Y_{12}, & \ldots, & a_{kk}^{-1} \sum_{1}^{N} Y_{1i}\ c_{ik}, & \ldots, & Y_{1N} \\ Y_{21}, & Y_{22}, & \ldots, & a_{kk}^{-1} \sum_{1}^{N} Y_{2i}\ c_{ik}, & \ldots, & Y_{2N} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ Y_{N1}, & Y_{N2}, & \ldots, & a_{kk}^{-1} \sum_{1}^{N} Y_{N1},\ c_{ik}, & \ldots, & Y_{NN} \end{bmatrix} \end{array} = a_{kk}\ \underline{Y}^{1}, \text{ say,}
$$

and
$$
\underline{Y}\ \underline{B}\ \underline{A}^{-1} = \underline{Y}\ \underline{B}\ \hat{\underline{A}}\ a_{kk}^{-1} = \underline{Y}^{1} \quad .
$$

Then the second of Equations (29) becomes

$$
\underline{I}^{o} = -\frac{b\ k_o}{Y} \begin{array}{c} \hphantom{Y_{11}, Y_{12}, \ldots,}\ (k) \\ \begin{bmatrix} Y_{11}, & Y_{12}, & \ldots, & a_{kk}^{-1} \sum_{1}^{N} Y_{1i}\ c_{ik}, & \ldots, & Y_{1N} \\ Y_{21}, & Y_{22}, & \ldots, & a_{kk}^{-1} \sum_{1}^{N} Y_{2i}\ c_{ik}, & \ldots, & Y_{2N} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ Y_{N1}, & Y_{N2}, & \ldots, & a_{kk}^{-1} \sum_{1}^{N} Y_{Ni}\ c_{ik}, & \ldots, & Y_{NN} \end{bmatrix} \end{array} \begin{bmatrix} V_1^g \\ V_2^g \\ \vdots \\ V_N^g \end{bmatrix} \quad . \tag{53}
$$

The c_{ik} being defined by (51) for $i \neq k$, and $c_{kk} = b_{kk}$.

To simplify the problem still further, assume all applied emf's are equal:

$$
V_i^g = V_g,\ i = 1, \ldots, N \quad . \tag{54}
$$

In that case, Equation (52) becomes

$$
\underline{V}^{o} = -\frac{b\ V_g}{Y\ a_{kk}} \begin{bmatrix} a_{kk} - a_{1k} \\ \vdots \\ a_{kk} - a_{Nk} \end{bmatrix} \quad .
$$

That is,

$$V_i^o = -\frac{b\ V_g}{\gamma\ a_{kk}}\ (a_{kk} - a_{ik})$$

$$= -\frac{b\ V_g}{\gamma\ a_{kk}}\ \left[1 + \beta\,(Z_{kk} - Z_{ik})\ \Delta Y_{kk}^L\right] \quad . \tag{55}$$

From (53) we have, in general,

$$I_j^o = -\frac{b\ k_o\ V_g}{\gamma}\left\{\sum_{i=1}^{N} Y_{ji} + a_{kk}^{-1}\sum_{i=1}^{N} Y_{ji}\ c_{ik}\right\}\ ,\ j = 1, \ldots, N$$

$$= -\frac{b\ k_o\ V_g}{\gamma\ a_{kk}}\left\{\sum_{i=1}^{N} (a_{kk} + c_{ik})\ Y_{ji} - a_{kk}\ Y_{jk}\right\} \quad . \tag{56}$$

From (45) and (51) we have, for $i \neq k$, the coefficient of Y_{ji} in (56):

$$a_{kk} + c_{ik} = 1 + \left[\beta\ Z_{kk} + (k_o^{-1} - \beta)\ Z_{ik}\right] \Delta Y_{kk}^L\ ,\ i \neq k \quad . \tag{57}$$

For $i = k$, the whole coefficient of Y_{jk} in (56) is

$$(a_{kk} + c_{kk}) - a_{kk} = b_{kk} = 1 + k_o^{-1}\ Z_{kk}\ \Delta Y_{kk}^L \quad .$$

But this is exactly the same value as would be obtained by setting $i = k$ in the right member of (57). Therefore, this expansion may be substituted for <u>all</u> i in (56) to yield

$$I_j^o = - \frac{b\, k_o\, V_g}{Y\, a_{kk}} \sum_{i=1}^{N} \left\{ 1 + \left[\beta\, Z_{kk} + (k_o^{-1} - \beta)\, Z_{ik} \right] \Delta Y_{kk}^L \right\} Y_{ji}$$

$$= - \frac{b\, k_o\, V_g}{Y\, a_{kk}} \left\{ \sum_{i=1}^{N} Y_{ji} + \beta\, Z_{kk}\, \Delta Y_{kk}^L \sum_{i=1}^{N} Y_{ji} \right.$$

$$\left. + (k_o^{-1} - \beta)\, \Delta\, Y_{kk}^L \sum_{i=1}^{N} Z_{ik}\, Y_{ji} \right\} \quad . \tag{58}$$

<u>Define</u>

$$Y_j^e = \sum_{i=1}^{N} Y_{ji} \tag{59}$$

= common-mode, or even-mode characteristic admittance of the j^{th} conductor .

Also note [5, Chapter 2] that since

$$\underline{Z}\, \underline{Y} = \underline{Y}\, \underline{Z} = \underline{\mathcal{I}} \ ,$$

$$\sum_{i=1}^{N} Z_{ik}\, Y_{ji} = \delta_k^j \tag{60}$$

where

$$\delta_k^j = \text{Kronecker's delta}$$
$$= 0, \; k \neq j$$
$$= 1, \; k = j \; .$$

Substitution of (59) and (60) in (58) yields

$$I_j^o = -\frac{b\, k_o\, V_g}{\gamma\, a_{kk}} \left\{ \left[1 + \beta\, Z_{kk}\, \Delta Y_{kk}^L \right] Y_j^e + (k_o^{-1} - \beta)\, \delta_k^j\, \Delta Y_{kk}^L \right\} \; . \qquad (61)$$

The <u>bulk output current</u> is

$$I_T^o = \sum_{j=1}^{N} I_j^o$$
$$= -\frac{b\, k_o\, V_g}{\gamma\, a_{kk}} \left\{ \left[1 + \beta\, Z_{kk}\, \Delta Y_{kk}^L \right] \sum_{j=1}^{N} Y_j^e + (k_o^{-1} - \beta)\, \Delta Y_{kk}^L \right\} \; .$$

<u>Define</u>

Y_o^e = common-mode, or even-mode characteristic admittance of the whole line

$$= \sum_{j=1}^{N} Y_j^e$$

and the last equation becomes

$$I_T^o = -\frac{b\, k_o\, V_g}{\gamma\, a_{kk}} \left\{ Y_o^e + \left[\beta\, Y_o^e\, Z_{kk} + (k_o^{-1} - \beta) \right] \Delta Y_{kk}^L \right\} \; . \qquad (62)$$

A quantity of particular interest in this study is the ratio of output voltage on any conductor to the bulk output current:

From (55) and (62) we get

$$\sigma_i = \frac{V_i^o}{I_T^o} = \frac{1 + \beta\ (Z_{kk} - Z_{ik})\ \Delta Y_{kk}^L}{k_o\ Y_o^e + [\beta\ k_o\ (Y_o^e\ Z_{kk} - 1) + 1]\ \Delta Y_{kk}^L}\ ,\ i = 1, \ldots, N\ . \quad (63)$$

A particular question to be investigated is: For what values of the available parameters do the σ_i ($i = 1, \ldots, N$) remain relatively independent of variations in ΔY_{kk}^L? This question is pursued in the following sub-section.

3.2.1.1 Ranges of Parameters for Which the Ratios of Output Voltages to Bulk Currents are Relatively Insensitive to Variations in Y_{kk}^L.

In (63) write

$$\left.\begin{aligned} b &= \beta\ (Z_{kk} - Z_{ik}) \\ c &= k_o\ Y_o^e \\ d &= \beta\ k_o\ (Y_o^e\ Z_{kk} - 1) + 1 \\ x &= \Delta Y_{kk}^L\ . \end{aligned}\right\} \quad (64)$$

Then (63) may be written

$$\sigma_i = \frac{1 + bx}{c + dx} \quad (65)$$

and we inquire what combinations of the quantities b, c, and d make σ_i relatively insensitive to x.

Assume all source and load admittances are pure resistances, so that k_i and k_o are real. Assume also that $\Delta Y_{kk}^{L} = x$ is real.

A natural attack on the problem is to expand the right member of (65) in a MacLaurin's series. One easily obtains as a result of such an expansion

$$\sigma_i(x) = \frac{1}{c} + \frac{bc - d}{c^2}\, x \left[1 - \left(\frac{dx}{c}\right) + \left(\frac{dx}{c}\right)^2 \ . \ . \ . \right] \quad . \tag{66}$$

It is immediately evident that σ_i is independent of x if

$$bc - d = 0 \quad . \tag{67}$$

However, we will now prove that this result is unrealizeable. Substitution of the appropriate quantities from Equations (64) in (67) yields

$$\beta\, k_o\, Y_o^e\, (Z_{kk} - Z_{ik}) - \beta\, k_o\, (Y_o^e\, Z_{kk} - 1) - 1 = 0$$

whence

$$\beta\, k_o = \frac{1}{1 - Y_o^e\, Z_{ik}} = A, \text{ say} \tag{68}$$

where, in virtue of the middle member of (68), A is real. But by Equation (27)

$$\beta\, k_o = k_o \frac{k_i + a}{(1 + k_i\, k_o) + (k_i + k_o)a} \quad . \tag{69}$$

In (69), all quantities are real except

$$a = -j \cot \theta$$

But since $\beta\, k_o$ is real in virtue of (68), we must have, in (69),

$$\frac{k_i}{1} = \frac{1 + k_i\, k_o}{k_i + k_o}$$

whence

$$k_i^2 = 1 \text{ and } k_i = 1 \quad . \tag{70}$$

Equation (70) is a necessary condition that σ_i be completely independent of x. Substitution of (70) in (69) then yields

$$\beta\, k_o = \frac{k_o}{1 + k_o} \tag{71}$$

so that this quantity is also independent of a, or cot θ. (This is, of course, evident by Thévenin's theorem when $k_i = 1$.) Equation (71) in (68) then yields

$$k_o = \frac{1}{\frac{1}{A} - 1} = - \frac{1}{Y_o^e\, Z_{ik}} \tag{72}$$

which is impossible, since both Y_o^e and Z_{ik} are greater than zero. Thus, σ_i cannot be made completely independent of x under the specified assumptions.

Next, inspection of Equation (66) suggests that the effects of varying x might be minimized by making c large. The second of Equations (64) suggests that this implies making k_o large.

For $k_o \to \infty$, we get

$$\beta \to \frac{a + k_i}{a\, k_o + k_i\, k_o} = \frac{1}{k_o}$$

$$b \to \frac{1}{k_o} (Z_{kk} - Z_{ik})$$

$$c = k_o\, Y_o^e$$

$$d \to Y_o^e\, Z_{kk}$$

$$bc - d \to Y_o^e\, Z_{ik}$$

$$\frac{d}{c} \to \frac{Z_{kk}}{k_o} \quad .$$

Therefore, Equation (65) becomes

$$\sigma_i(u) \to \frac{1}{k_o\, Y_o^e} \; \frac{1 + (1 - K_{ik})\, u}{1 + u} \tag{73}$$

where

$$\left.\begin{aligned} u &= \frac{Z_{kk}}{k_o}\, x = \frac{Z_{kk}}{k_o}\, \Delta Y_{kk}^L \\ K_{ik} &= \text{a voltage coupling coefficient} \\ &= Z_{ik}/Z_{kk} \quad . \end{aligned}\right\} \tag{74}$$

Further inspection of Equation (66) suggests that letting $d \to 0$ could minimize variation with respect to x. However, analysis quickly shows that this leads to the requirement

$$k_o = -\frac{1}{Y_o^e Z_{kk}}$$

which is impossible.

Thus the only clue to minimal variation is Equation (73). This result is discussed in the next section.

3.2.2 Discussion of Results

Equation (73) is independent of source impedance and line length. However, the result is predicated on the assumption that k_o is "large." More specifically, it is assumed large enough so that, in Equation (27), $\beta \to 1/k_o$. This, in turn, implies that

$$k_o \gg \left| \frac{1 + a\, k_i}{a + k_i} \right| \quad ;$$

that is

$$k_o \gg \left[\frac{1 + k_i^2 \cot^2 \theta}{k_i^2 + \cot^2 \theta} \right]^{\frac{1}{2}} \quad .$$

Write

$$F = \frac{1 + k_i^2 \cot^2 \theta}{k_i^2 + \cot^2 \theta} \tag{75}$$

and say that a satisfactory approximation is achieved if

$$k_o \geq 10 \sqrt{F} \quad . \tag{76}$$

The quantity, $\sqrt{F}$, may be recognized as the conventional generator-impedance magnitude transformation for a 1-line, with k_i or $1/k_i$ representing the VSWR of the source. Thus $\sqrt{F}$ fluctuates between the values k_i and $1/k_i$ as θ ranges over a complete cycle (2π radius). For Equation (73) to hold within an approximation limited by (76), we must have

$$\frac{k_o}{10} \geq \text{the larger of } k_i \text{ or } \frac{1}{k_i} \quad .$$

One has to keep in mind that this problem has been investigated for an artificially simple situation in order to derive the beginnings of some insight into the line behavior. A possible next step might be a statistical approach, assuming some reasonable distribution of source and load admittances.

4. Miscellaneous Additional Results

Section 2.1 and Appendix A discuss the derivation of a Thévenin generator matrix. During the course of this phase of the study it was thought that generalizations of other network theorems and other N-line formulations would prove useful; accordingly these results were obtained. Although this report contains no applications of those theorems, they are set down here for the record for possible future use. We will discuss (a) a generalization of Norton's theorem, which is the network dual of Thévenin's theorem (b) a generalization of the compensation theorem (compensating-current form) and (c) the response of an N-line to current sources. These are discussed in turn in the succeeding sections.

4.1 Norton's Theorem

The conventional (scalar) form of Norton's theorem is the circuit dual of Thévenin's theorem. It may be stated as follows [cf. Ref. 6]:

The voltage across any admittance, Y_R, connected to two terminals of a network, is the same as if Y_R were connected to a constant-current generator, whose generated current is equal to the current which flows through the two terminals when these terminals are short-circuited, the terminals being shunted with an admittance equal to the admittance of the network looking back from the terminals in question when all generated currents are set equal to zero.

The derivation of the corresponding generalized theorem for an N-port source is straightforward, and, in fact, is the dual of the derivation of the N-port Thévenin generator (see Appendix A).

Fig. 4(a) shows an N-port source with all terminals grounded. The corresponding short-circuit currents flowing toward ground are the elements of the current vector

$$\underline{I}^s = \begin{bmatrix} I^s \\ \\ I_2^s \\ \cdot \\ \cdot \\ I_N^s \end{bmatrix} \quad . \tag{77}$$

Add another set of currents, $-\underline{I}^s$, (with respect to ground) as shown in Fig. 4(b), so that the net current flowing to ground in each of the N circuits is zero. Under this condition any admittance may be inserted between any current node and ground, or between any pair of nodes. Thus we can consider the grounds to be replaced by a minimally general load $\underline{Y}^L$, [1], of $\frac{1}{2}$ N (N + 1) independent components, as in Fig. 5.

Now suppose the currents of the original source (marked "Source") all set equal to zero, and find the effects of the external (temporary) source. On the k^{th} line we have the node equation

$$I_k^s + I_k^g + I_k^L = 0$$

where I_k^g is the current flowing toward the quiescent generator and I_k^L is the current flowing toward the load. If the line voltage vector is

$$\underline{V} = \begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ V_N \end{bmatrix} \tag{78}$$

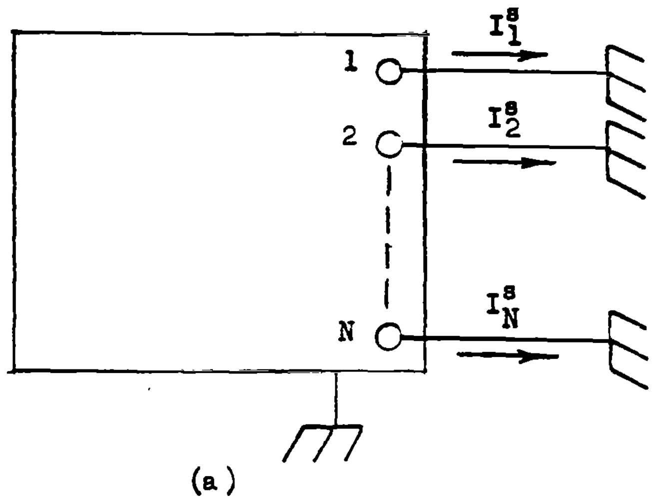
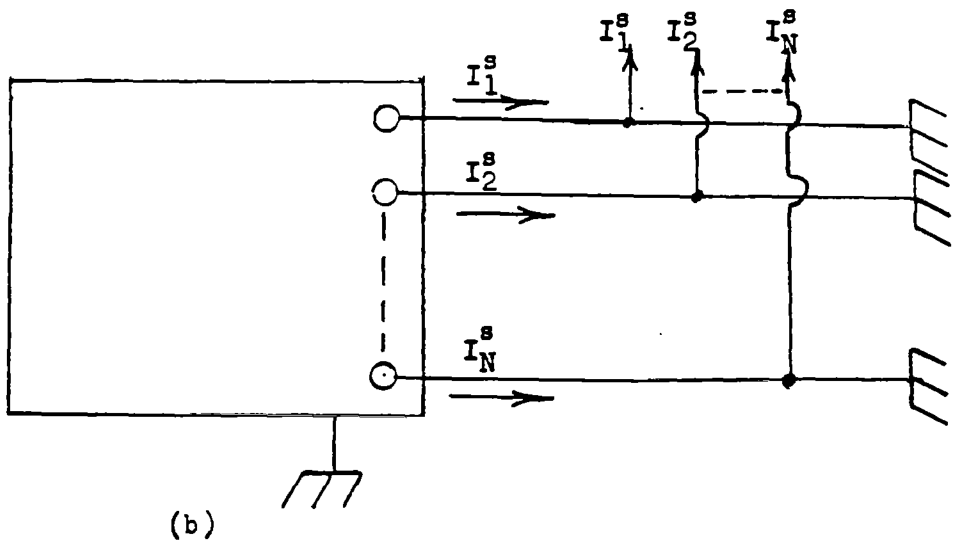


Fig.4. Short-circuited N-port Source (a) without added external vector current source (b) with added external source



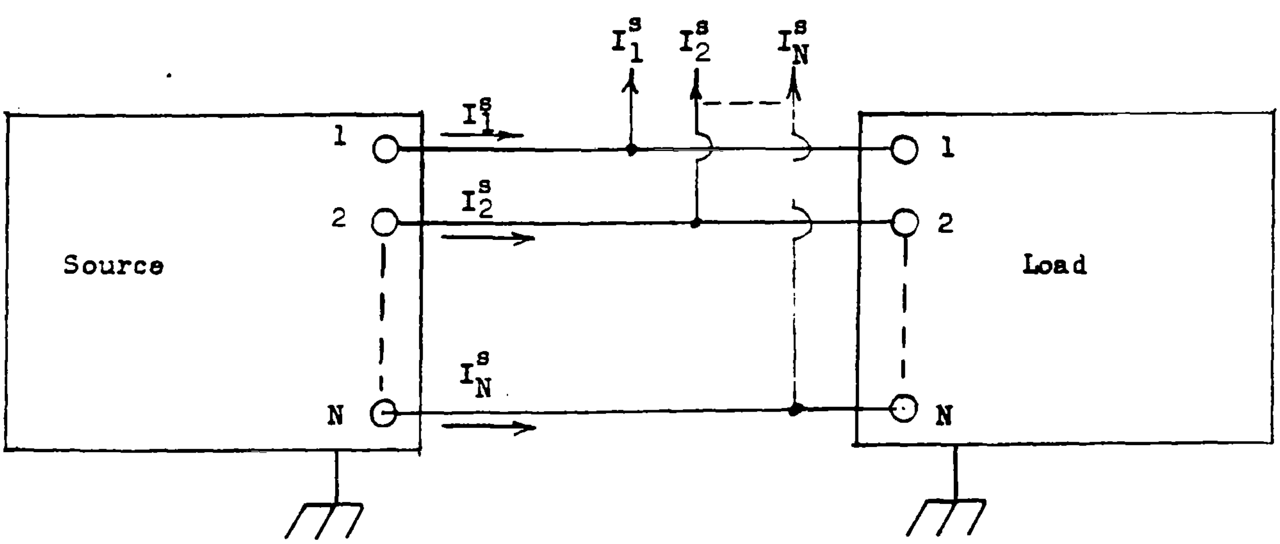


Fig.5. Source/Load System in which No Current Is Delivered to the Load.

then

$$\left.\begin{aligned} I_k^g &= \sum_{j=1}^{N} Y_{kj}^g V_j \\ I_k^L &= \sum_{j=1}^{N} Y_{kj}^L V_j \end{aligned}\right\} \tag{79}$$

where

$$\underline{Y}^g = \begin{bmatrix} Y_{11}^g, & . & . & . & , Y_{1N}^g \\ . & . & . & . & . \\ Y_{N1}^g, & . & . & . & , Y_{NN}^g \end{bmatrix} \tag{80}$$

is the admittance matrix of the quiescent source. Thus the current flowing from the external constant-current source is

$$\underline{I}^s = -(\underline{Y}^g + \underline{Y}^L)\,\underline{V}$$

whence the resulting voltage vector is

$$\underline{V} = -(\underline{Y}^g + \underline{Y}^L)^{-1}\,\underline{I}^s \quad .$$

Therefore the current in the load due to the external source above is

$$\begin{aligned} (\underline{I}^L)_{ext} &= \underline{Y}^L\,\underline{V} \\ &= -\underline{Y}^L\,(\underline{Y}^g + \underline{Y}^L)^{-1}\,\underline{I}^s \quad . \end{aligned}$$

The current in the load due to the original source alone must be the reverse of this; thus

$$\underline{I}^L = \underline{Y}^L\,(\underline{Y}^g + \underline{Y}^L)^{-1}\,\underline{I}^s \tag{81}$$

which is the same as the current that would flow from a constant-current source to the load, $\underline{Y}^L$, when it is paralleled by an admittance, $\underline{Y}^g$, which is the admittance looking into the source at its terminals when all internal sources are set equal to zero. This is the generalization of the scalar Norton theorem.

The result is also readily obtained directly from Thévenin's theorem. From Appendix A we have

$$\underline{V}^g = (\underline{Z}^g + \underline{Z}^L)\ \underline{I}^L$$

where $\underline{V}^g$ is the Thévenin emf vector of the source.

The output short-circuit current of the source ($\underline{Z}^L = \underline{0}$) is given by

$$\underline{V}^g = \underline{Z}^g\ \underline{I}^s \quad .$$

Combining the last two equations gives

$$(\underline{Z}^g + \underline{Z}^L)\ \underline{I}^L = \underline{Z}^g\ \underline{I}^s$$

$$\underline{I}^L = (\underline{Z}^g + \underline{Z}^L)^{-1}\ \underline{Z}^g\ \underline{I}^s$$

whence, using $\underline{Z}^g = (\underline{Y}^g)^{-1}$ and $\underline{Z}^L = (\underline{Y}^L)^{-1}$ and simplifying readily yields Equation (81).

4.2 Compensation Theorem

The compensation theorem is useful for finding the behavior of a network in terms of the behavior of another network from which it may be derived by making a small number of changes in the impedance (admittance) elements of the latter.

The theorem may be stated in terms of voltages compensating for impedance changes or currents compensating for admittance changes.

In scalar form (one impedance change) the voltage-compensation theorem may be stated as follows [2]:

If a network is modified by making a change ΔZ in the impedance of one of its branches, the current increment thereby produced in any conductor of the network is equal to the current that would be produced in that conductor by a compensating emf, acting in series with the modified branch, whose value is - $I \cdot \Delta Z$, where I is the original current which flowed in the modified branch.

In computing the effect of this compensating emf it is assumed that all other sources in the network are temporarily set equal to zero. Subsequently the total currents are computed by superimposing the original currents and the currents of the compensating emf.

Correspondingly the scalar case of the current-compensating form of the theorem may be stated as follows:

If a network is modified by making a change, ΔY is the admittance between a pair of its nodes, the voltage increment thereby produced between any pair of nodes of the network is equal to the voltage that would be produced between those nodes by a compensating constant-current generator connected to the first pair of nodes, acting in parallel with the modified node-pair, whose value is - $V \cdot \Delta Y$, where V is the original voltage across the node pair.

In computing the effect of this compensating current, it is assumed, as in the previous case, that all other sources are temporarily set equal to zero.

If these concepts are to be generalized to apply to the terminal matrices of an N-line, then, consistent with ideas already noted in this study, the more convenient terminal formulation is in the Y-matrix form. It follows that a generalized theorem is more useful in the current-compensation form. The theorem, derived in Appendix B, may be stated as follows:

If an N-port network, driven by a system of sources in another N-port network, is modified by making a change, $\Delta \underline{Y}$, in its driving-point admittance matrix, $\underline{Y}$, the voltage increment thereby produced between any pair of nodes, either in the driving, - or the driven network, is equal to the voltage that would be produced between those nodes by a compensating (vector) constant-current generator, $\underline{i}$, applied to the modified network between the N terminals and ground, given by

$$\underline{i} = - (\underline{\Delta Y}) \cdot \underline{V}$$

where $\underline{V}$ is the original voltage vector at the terminals of the N-port.

Again, this compensating effect is superposed on the original voltage obtained before the network was modified.

4.3 Response of An N-Line to Current Sources

In the discussion of the compensation theorem of the preceding section, no restriction was placed on the driving network other than that it be a (linear) N-port. Such an N-port could actually be an N-line with an N-port source at the far end. In order to determine the effect of the compensating current-source, $\underline{i}$, of the preceding section, it is therefore necessary to have a formulation for the response of an N-line to a current source. This formulation may be derived starting with an appropriate canonical formulation from Ref. 5. However, recall from section 4.1, that a Thévenin source is easily converted to a Norton source by the transformation

$$\underline{V}^g = \underline{Z}^i \, \underline{I}^g$$

where $\underline{Z}^i = (\underline{Y}^i)^{-1}$, and $\underline{I}^g$ is the strength of the current source.

Making this substitution in Equations (16) (Section 3),

$$\left.\begin{aligned}
\underline{V}^i &= \underline{M} \, (\underline{M} + \underline{N} \, \underline{Q}^i)^{-1} \, \underline{Z}^i \, \underline{I}^g \\
\underline{I}^i &= \underline{Y} \, \underline{N} \, (\underline{M} + \underline{Q}^i \underline{N})^{-1} \, \underline{Z}^i \, \underline{I}^g \\
\underline{V}^o &= -b \, (\underline{M} + \underline{Q}^i \, \underline{N})^{-1} \, \underline{Z}^i \, \underline{I}^g \\
\underline{I}^o &= -b \, \underline{Y}^o \, (\underline{M} + \underline{Q}^i \, \underline{N})^{-1} \, \underline{Z}^i \, \underline{I}^g \quad .
\end{aligned}\right\} \qquad (82)$$

5. Conclusions

The chief purpose of the phase of the study reported here was to discover whether there were conditions under which voltages appearing at output terminals of a driven cable tend to depend only on the bulk current entering the output terminals, and to be relatively insensitive to load variations, line length, frequency, and the number of conductors in the cable. The problem is generally impractical to analyze without machine assistance. However, we investigated an elementary special case in which we assumed source and load termination matrices to be proportional to the line admittance matrix, and all source emf's to be equal and then observed the effect of varying one diagonal element of the load admittance matrix. We found that the terminal voltages/bulk current ratio tended to be insensitive to this variation only if (1) the source were approximately matched to the line and (2) if the load admittance was much larger than the admittance of the line. In that case, a corollary of the first of these necessary conditions is that the ratio is also relatively insensitive to line length and frequency. All arguments and conclusions assume lossless lines. Existance of relatively large line losses may be expected to relieve the restrictions under which the conclusions hold.

No explicit variation of the voltage/bulk-current ratio with the number (N) of cable conductors was observed.

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Appendix A

Generalized Thévenin Generator

Refer to Fig. 2 of the main text which shows a generalized source matrix connected to a generalized load. If the load is disconnected from the source the following measurements can be made:

1. The Open-Circuit Potential Difference Between All Possible Pairs of Terminals

As usual, there are ${}_{(N+1)}C_2 = \frac{1}{2} N(N + 1)$ of these.* However, they are not all independent. Let V_k^g be the potential of the k^{th} terminal referred to ground, $(k = 1, \ldots, N)$, and let V_{jk}^g be the potential of the j^{th} terminal referred to the k^{th} terminal, $(j, k = 1, \ldots, N;\ j \neq k)$. Then

$$V_{jk}^g = - V_{kj}^g = V_j^g - V_k^g \ , \ j, k = 1, \ldots, N, \ j \neq k \quad .$$

These are ${}_NC_2 = \frac{1}{2} N(N - 1)$ of the V_{jk}^{g*}

which are thus linearly dependent on the V_k^g. The number of independent potentials is, as expected,

$${}_{(N+1)}C_2 - {}_NC_2 = \frac{1}{2} N(N + 1) - \frac{1}{2} N(N - 1) = N$$

and these are simply the V_k^g, $(k = 1, \ldots, N)$, or $\underline{V}^g$.

*Assuming that the distinction between the potential of the j^{th} terminal referred to the k^{th} and vice versa is trivial.

2. <u>The Various Admittances Between Pairs of Terminals When the emf's Internal to the Source Network Are Set Equal to Zero.*</u>

Again, there are $\frac{1}{2}N(N+1)$ of these, but this time they are all independent. In fact, whatever the physical nature of the network, it may (at least, at any one frequency) be replaced by a system of $\frac{1}{2}N(N+1)$ admittances joining all pairs of terminals.** Furthermore, these admittances are readily related to the measured terminal admittances of the network [1]. Designate this source admittance matrix seen at the (quiescent) source terminals by $\underline{Y}^i$.

Suppose these two sets of measurements to have been made. Restore the $\underline{V}^g$, so that the source is in its original condition, with the terminals still open-circuited. Then, if, in series with each terminal of the source we add a generator of emf, $-V_k^g$, the net emf appearing at each terminal is zero. If the load N-port is connected to this arrangement, no currents will flow in any of the connecting conductors to the load. The result is the same as though the added generators, $-V_k^g$, were respectively in series with cancelling generators, V_k^g. If the source generators (V_k^g) are once again turned off, the response of the system is to a source, $-\underline{V}^g$ acting on passive terminations $\underline{Z}^i$ and $\underline{Z}^o$ in series, where

$$\left.\begin{aligned} \underline{Z}^i &= (\underline{Y}^i)^{-1} \\ \underline{Z}^o &= (\underline{Y}^o)^{-1} \quad . \end{aligned}\right\}$$

The response of the load to the source generators, $\underline{V}^g$, alone, is the negative of the previous response; that is, it is the same as though $\underline{V}^g$ were in series with the terminals of the passive impedance matrices, $\underline{Z}^i$ and $\underline{Z}^o$.

*Or, as it is sometimes stated, when the generators in the source network are replaced by their internal impedances.

**Insofar as its external behavior is concerned.

Consider Fig. 1-A, which diagrams this result. **Clearly, we have**

$$\left.\begin{aligned} &\underline{I} = -\underline{Y}^i \underline{V}^i = \underline{Y}^o \underline{V}^o \\ \text{and}\quad & \\ &\underline{V}^i + \underline{V}^g = \underline{V}^o \quad . \end{aligned}\right\}$$

To eliminate $\underline{V}^o$ and $\underline{V}^i$ write

$$0 = \underline{Y}^i \underline{V}^i + \underline{Y}^o \underline{V}^o = \underline{Y}^i \underline{V}^i + \underline{Y}^o (\underline{V}^i + \underline{V}^g)$$

whence

$$\underline{V}^i = -(\underline{Y}^i + \underline{Y}^o)^{-1} \underline{Y}^o \underline{V}^g \quad .$$

Then

$$\begin{aligned} \underline{V}^o &= \underline{V}^i + \underline{V}^g \\ &= \left[\mathcal{I} - (\underline{Y}^i + \underline{Y}^o)^{-1} \underline{Y}^o\right] \underline{V}^g \\ &= (\underline{Y}^i + \underline{Y}^o)^{-1} \underline{Y}^i \underline{V}^g \end{aligned}$$

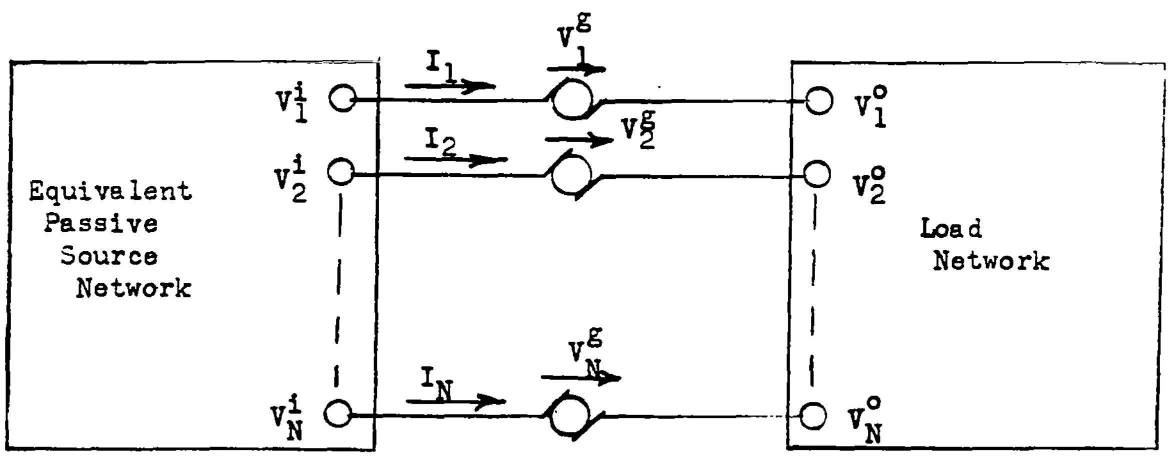


Fig. 1A. Generalized Thévenin Source and Load.

and

$$\underline{I} = \underline{Y}^{o}\,\underline{V}^{g}$$

$$= \underline{Y}^{o}\,(\underline{Y}^{i} + \underline{Y}^{o})^{-1}\,\underline{Y}^{i}\,\underline{V}^{g}$$

$$= (\underline{Z}^{o})^{-1}\,(\underline{Y}^{i} + \underline{Y}^{o})^{-1}\,(\underline{Z}^{i})^{-1}\,\underline{V}^{g}$$

$$= [\underline{Z}^{i}\,(\underline{Y}^{i} + \underline{Y}^{o})\,\underline{Z}^{o}]^{-1}\,\underline{V}^{g}$$

$$\underline{I} = (\underline{Z}^{o} + \underline{Z}^{i})^{-1}\,\underline{V}^{g}$$

$$= [(\underline{Y}^{o})^{-1} + (\underline{Y}^{i})^{-1}]^{-1}\quad \underline{V}^{g}\quad .$$

This justifies and interprets the symbolism of the source as given in Fig. 1 of the main text.

As an illustrative example consider the schematic of Fig. 2-A. This shows a source and load with N = 2. The source contains an emf of 1 volt. All impedances are resistors with values indicated adjacent to them.

Solving the network first in the standard manner we have the mesh equations

$$\left.\begin{aligned} 8i_1 - 4i_2 - i_3 &= 1 \\ -4i_1 + 15i_2 - 5i_3 &= 0 \\ -i_1 - 5i_2 + 7i_3 &= 0 \end{aligned}\right\}$$

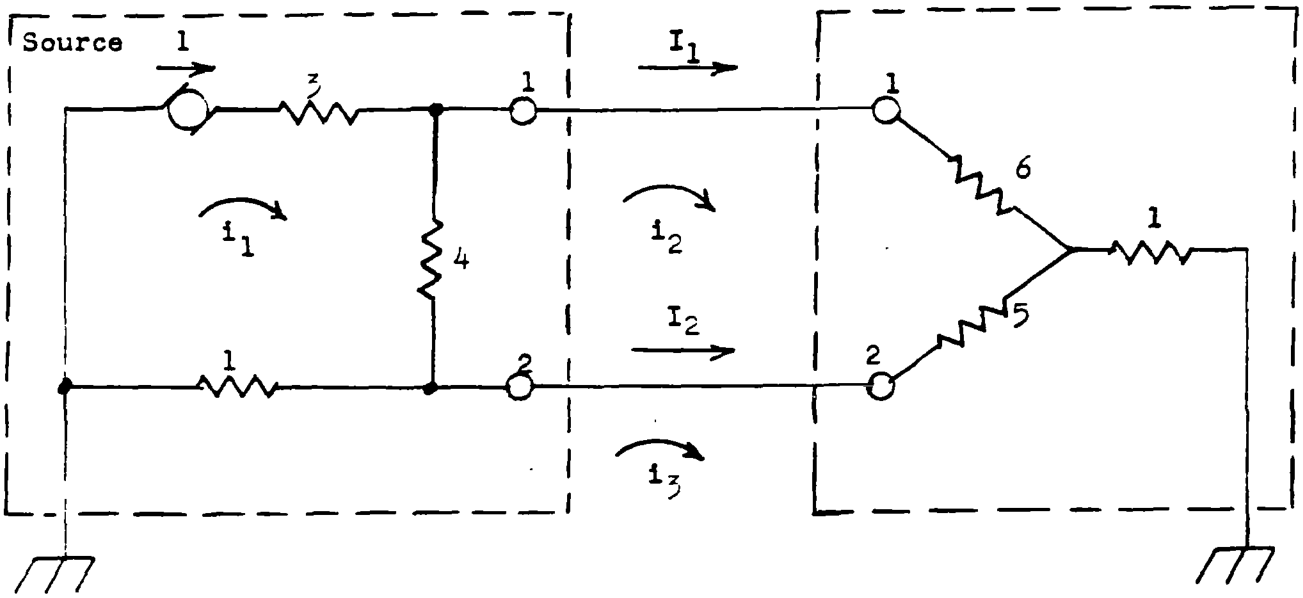


Fig.2A. Illustrative Example

with solutions

$$\left.\begin{aligned} i_1 &= 0.169 \\ i_2 &= 0.0698 \\ i_3 &= 0.0740 \quad . \end{aligned}\right\}$$

Then

$$\left.\begin{aligned} I_1 &= i_2 = 0.0698 \\ I_2 &= i_3 - i_2 = 0.00423 \end{aligned}\right\}$$

that is,

$$\underline{I} = \begin{bmatrix} 0.0698 \\ 0.00423 \end{bmatrix} \quad .$$

To apply Thévenin's theorem, disconnect the load. The source terminal voltages are, respectively,

$$V_1^g = \frac{4 + 1}{3 + 4 + 1} = \frac{5}{8}$$

$$V_2^g = \frac{1}{3 + 4 + 1} = \frac{1}{8}$$

that is,

$$\underline{V}^g = \frac{1}{8}\begin{bmatrix} 5 \\ 1 \end{bmatrix} \quad .$$

To compute the equivalent source impedance matrix and the load impedance matrix, it is convenient to first determine the admittance matrices. Fig. 3-A shows the separated quiescent source and load networks. In the source matrix (Fig. 3-A (a)) ground terminal No. 2, and apply a potential, V_1, to terminal No. 1. Then

$$Y^i_{11} = \frac{I_1}{V_1} \Bigg|_{V_2=0} .$$

This is just the admittance of the 3-ohm and the 4-ohm resistor in parallel:

$$Y^i_{11} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} \text{ mho.}$$

At the same time we get

$$Y^i_{12} = Y^i_{21} = \frac{I_2}{V_1} \Bigg|_{V_2=0} .$$

Noting that I_2 is the reverse of the current flowing in the 4-ohm resistor,

$$I_2 = -\frac{V_1}{4} \; ; \; Y^i_{12} = Y^i_{21} = -\frac{1}{4} \text{ mho.}$$

Next, grounding No. 1 terminal and applying a potential, V_2, to the No. 2 terminal we have

$$Y^i_{22} = \frac{I_2}{V_2} \Bigg|_{V_1=0}$$

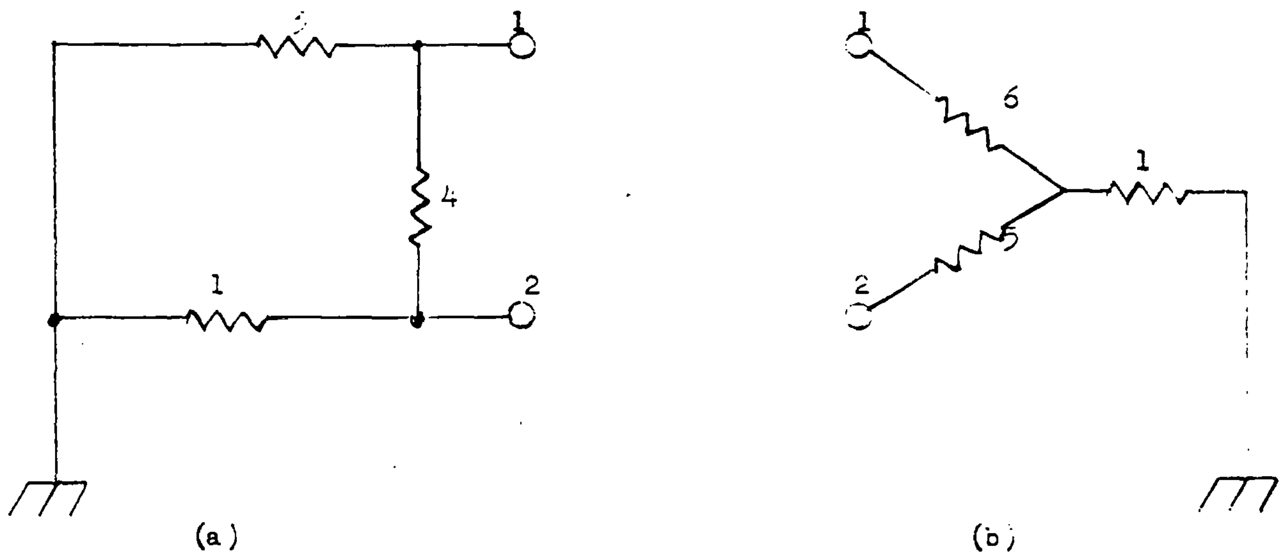


Fig. A. (a)Quiescent Source Network (b)Load Network with the 1-ohm and 4-ohm resistors in parallel to yield

$$Y_{22}^{i} = 1 + \frac{1}{4} = \frac{5}{4} \text{ mho.}$$

With this same setup it is, again, evident by inspection that $Y_{12}^{i} = -\frac{1}{4}$. In the same way one obtains for the load matrix,

$$\left.\begin{aligned} Y_{11}^{o} &= \frac{6}{41} \\ Y_{12}^{o} = Y_{21}^{o} &= -\frac{1}{41} \\ Y_{22}^{o} &= \frac{7}{41} \quad . \end{aligned}\right\}$$

To compute the currents flowing between source and load we have the previous result

$$\underline{I} = \underline{Y}^{o} \left(\underline{Y}^{i} + \underline{Y}^{o}\right)^{-1} \underline{Y}^{i} \, \underline{V}^{g}$$

$$\underline{Y}^{i} + \underline{Y}^{o} = \begin{bmatrix} \frac{7}{12} + \frac{6}{41} & , & -\frac{1}{4} - \frac{1}{41} \\ -\frac{1}{4} - \frac{1}{41} & , & \frac{5}{4} + \frac{7}{41} \end{bmatrix}$$

$$= \begin{bmatrix} 0.730 & , & -0.274 \\ -0.274 & , & 1.421 \end{bmatrix} \quad .$$

To invert, we have the determinant

$$D^{s} = (0.730)(1.421) - (0.274)^2 = 0.963$$

$$(D^{s})^{-1} = 1.039 \quad .$$

Thus,

$$(\underline{Y}^{i} + \underline{Y}^{o})^{-1} = 1.039 \begin{bmatrix} 1.421 & , & 0.274 \\ 0.274 & , & 0.730 \end{bmatrix}$$

$$= \begin{bmatrix} 1.478 & , & 0.284 \\ 0.284 & , & 0.758 \end{bmatrix}$$

then

$$\underline{I} = \frac{1}{8} \begin{bmatrix} 0.147 & , & -0.0244 \\ -0.0244 & , & 0.171 \end{bmatrix} \begin{bmatrix} 1.478 & , & 0.284 \\ 0.284 & , & 0.758 \end{bmatrix} \begin{bmatrix} 0.583 & , & -0.250 \\ -0.250 & , & 1.250 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0701 \\ 0.00433 \end{bmatrix} \quad .$$

The discrepancy between this result and the conventionally obtained result

$$\underline{I} = \begin{bmatrix} 0.0698 \\ 0.00423 \end{bmatrix}$$

is undoubtedly attributable to slide rule error.

Appendix B

Generalized Compensation Theorem

(Compensating Current-Source Form)

The scalar case (single admittance change) is discussed first. Consider Fig. 1-B. At (a) are shown two nodes of a network between which the admittance to be modified is connected. At (b) an admittance, ΔY, and a current source of strength, i, are shown, If $i = V \cdot \Delta Y$, where V is the voltage drop across the terminals, no current flows out of the box, so the combination of i and ΔY leaves the rest of the network unaffected. If, now, another source, -i, is added, as at (c), the net physical modification to the system is the admittance change, ΔY, while the net dynamic change in the system is as though a current source,

$$-i = - V \cdot \Delta Y$$

had been added to the modified network.

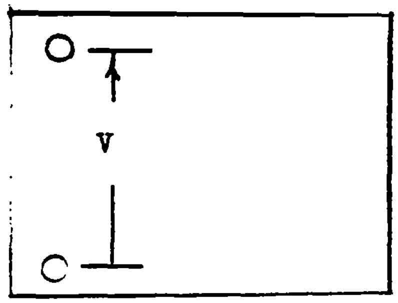
For our purposes we prefer to think of the system as consisting of a source network and a passive load network connected with a pair of wires as shown at Fig. 2 (main text), with N = 1 in that figure, rather than as the single network of Fig. 1-B. Obviously, the source network of Fig. 2 can be considered as being included in the box of Fig. 1-B.

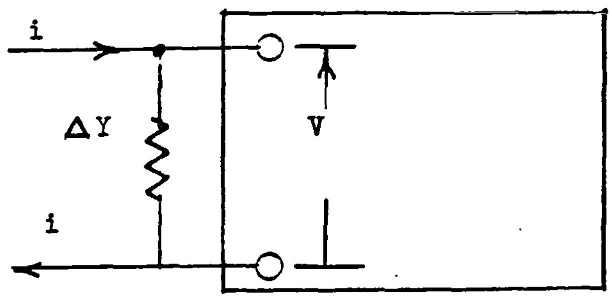
In Fig. 2 of the main text, assume the load to be in the minimal Thévenin form, so that each of its $\frac{1}{2} N(N + 1)$ admittances is accessible at some pair of its $(N + 1)$ terminals.

Consider changing an admittance, Y^L_{jk}, connected (internally) between the j^{th} and k^{th} terminals of the load. (The admittance Y^L_{kk} is connected between the k^{th} terminal and ground; see Fig. 2-B). The voltage across this admittance is

$$V = V_j - V_k \quad , \; j \neq k$$

$$= V_k \quad , \quad j = k$$





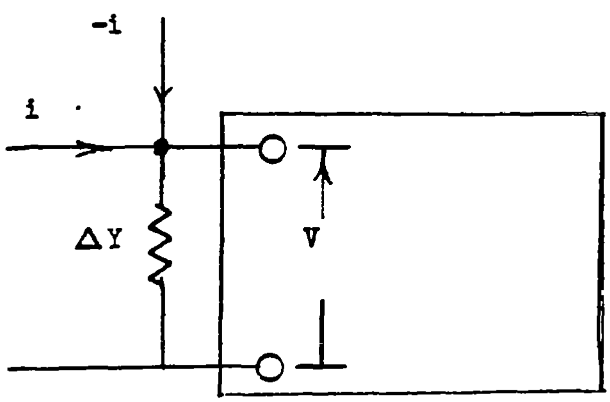


Fig.18. Compensation Theorem: Admittance Change

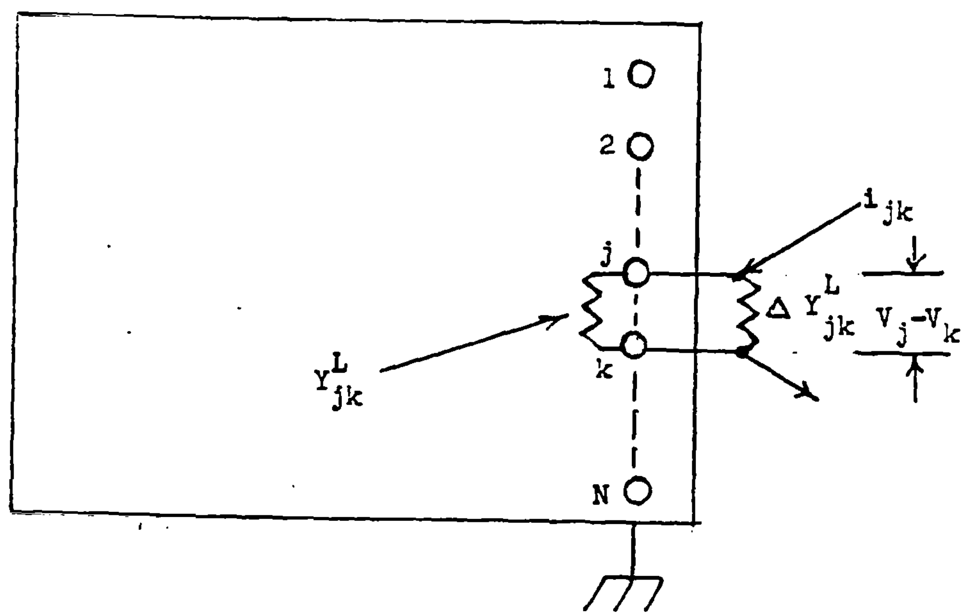


Fig.2B. Generalized Compensation Theorem:

Admittance Change

where V_j is the potential from the j^{th} terminal to ground.

The scalar compensation theorem just proved states that the effect of the modification, ΔY^L_{jk} is accounted for by adding a current source

$$i_{jk} = - \Delta Y^L_{jk} (V_j - V_k) \quad .$$

Now suppose that, in general, every admittance of the load network is modified. The method of derivation of the single port theorem is extended by adding, simultaneously, a compensating current source for each change, so that the net dynamic effect on the system is nil. Then, by adding the negatives of these sources at the same branches we conclude with a net physical modification consisting only of admittance changes, and a net dynamic modification represented by the second set of added current sources acting on the modified network. Their effects are then added to the original network response before modification to obtain the total modified response.

Admittances connected to the j^{th} terminal are Y^L_{jk}, $k = 1, \ldots, N$. The effects of changing all admittances connected to this terminal are summed as the current source

$$i_j = \sum_{k=1}^{N} i_{jk} = \sum_{\substack{k=1 \\ (j)}}^{N} \left[-\Delta Y^L_{jk} (V_j - V_k)\right] - \Delta Y^L_{jj} V_j \quad .$$

Note that $i_{jk} = -i_{kj}$. Continuing,

$$i_j = - \left[\sum_{k=1}^{N} \Delta Y^L_{jk} V_g - \sum_{\substack{k=1 \\ (j)}}^{N} \Delta Y^L_{jk} V_k \right]$$

$$= - \left[V_j \sum_{k=1}^{N} \Delta Y_{jk}^{L} - \sum_{\substack{k=1 \\ (j)}}^{N} \Delta Y_{jk}^{L} \cdot V_k \right] \quad . \tag{1-B}$$

Note that [1]

$$\left. \begin{aligned} Y_{jk}^{o} &= - Y_{jk}^{L} \; , \quad j \neq k \\ Y_{kk}^{o} &= \sum_{j=1}^{N} Y_{jk}^{L} \quad . \end{aligned} \right\}$$

For incremental load admittance changes these yield

$$\left. \begin{aligned} \Delta Y_{jk}^{o} &= - \Delta Y_{jk}^{L} \\ \Delta Y_{jj}^{o} &= \sum_{k=1}^{N} \Delta Y_{jk}^{L} \quad . \end{aligned} \right\}$$

Substituting these in Equation (1-B),

$$i_j = - \left[V_j \cdot \Delta Y_{jj}^{o} + \sum_{\substack{k=1 \\ (j)}}^{N} \Delta Y_{jk}^{o} \, V_k \right]$$

$$= - \sum_{k=1}^{N} \Delta Y_{jk}^{o} \, V_k \quad .$$

That is

$$\underline{i} = -\underline{\Delta Y}^{o} \cdot \underline{V}$$

where

$$\underline{i} = \begin{bmatrix} i_1 \\ \vdots \\ i_N \end{bmatrix}, \quad \underline{V} = \begin{bmatrix} V_1 \\ \vdots \\ V_N \end{bmatrix}$$

$$\underline{\Delta Y}^{o} = \begin{bmatrix} \Delta Y^{o}_{11}, & \cdot & \cdot & \cdot & , \Delta Y^{o}_{1N} \\ \cdot & & \cdot & \cdot & \cdot \\ \Delta Y^{o}_{N1}, & \cdot & \cdot & \cdot & , \Delta Y^{o}_{NN} \end{bmatrix} .$$