INTERIOR FIELDS OF A SLOTTED CYLINDER RADIATED WITH AN ELECTROMAGNETIC PULSE

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FOREWORD

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This technical report has been reviewed and is approved.

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ABSTRACT

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The problem geometry is an infinite conducting cylinder stationed in free space. The cylinder has a slit or aperture of width 2α. An electromagnetic pulse (polarized with the electric field parallel to the axis of the cylinder) is normally incident upon the cylinder. The pulse can be expressed as a spectrum of plane waves. Formulas for the field interior to the cylinder are computed for a single frequency incident plane wave. These formulas are dependent upon the value of the electric field in the aperture. The electric field in the aperture is found to be approximately defined by a single constant V. The interior field of the cylinder, in response to a pulse, is then an integral over the frequency spectrum and V becomes a function of frequency, V (ω). Graphs of V (ω) are given for apertures of 10 and 60 degrees and frequencies from 1 to 500 MHz. For certain frequencies the cylinder will be resonant. At these frequencies the formulas for the interior fields are greatly simplified; furthermore, these formulas are independent of the aperture opening.

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SECTION I

INTRODUCTION

The problem is to compute the electromagnetic fields interior to a slotted cylinder when an electromagnetic pulse is normally incident upon the slot. This is shown in Figure 1. The infinite cylinder is of radius R and perfectly conducting. The aperture is open for $-\alpha \leq \phi \leq \alpha$. The electromagnetic pulse is traveling from $x = +\infty$ to $x = -\infty$ with the $\overline{E}$ field parallel to the z-axis.

The pulse as a function of time will have a Fourier transform:

$$E_z(\vec{r},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_o(\vec{r},\omega)\ e^{i\omega t}\ d\omega \tag{1}$$

$$E_o(\vec{r},\omega) = \int_{-\infty}^{\infty} E_z(\vec{r},t)\ e^{-i\omega t}\ dt \tag{2}$$

where it is assumed that the transform of the incoming pulse can be expressed as

$$E_o(\vec{r},\omega) = E_o(\omega)\ E_z^{inc}\ (\vec{r},\omega) \tag{3}$$

In other words the electromagnetic pulse can be considered to be made up of a continuous spectrum of plane waves, each wave having a frequency ω and an amplitude $E_o(\omega)$.

Each of these component plane waves when intercepting the cylinder will have some of the energy reflected or scattered away from the cylinder and some of the energy transmitted through the aperture. Let $E_z^{int}\ (\vec{r},\omega)$ be the transform field interior to the cylinder in response to $E_z^{inc}\ (\vec{r},\omega)$, a plane wave of frequency ω and unit amplitude. Then the field interior to the slotted cylinder will be a summation of the fields of different frequencies, each frequency having an amplitude of $E_o(\omega)\ E_z^{int}\ (\vec{r},\omega)$ or

$$E_z(\vec{r},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_o(\omega)\ E_z^{int}\ (\vec{r},\omega)\ e^{i\omega t}\ d\omega \tag{4}$$

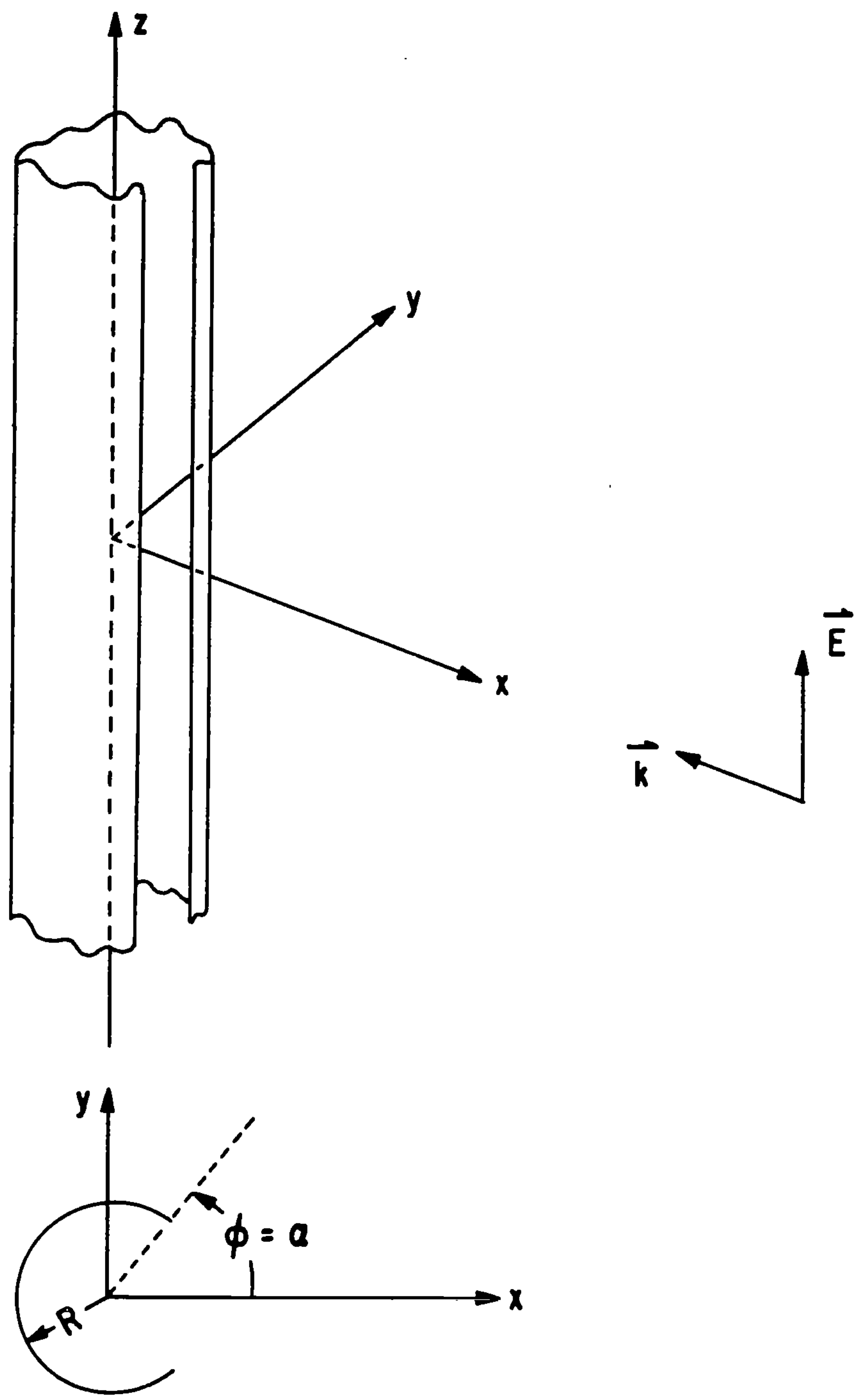

Figure 1. Problem Geometry

The strength and shape of the incident field in the time domain will determine $E_o(\omega)$ and for the purposes of this report will be assumed to be known. The problem then reduces to determining the function $E_z^{int}(\vec{r},\omega)$.

SECTION II

CALCULATION OF $E_z^{int}(\vec{r},\omega)$

$E_z^{inc}(\vec{r},\omega)$ is the Fourier transform of the field interior to a slotted cylinder when a plane wave of frequency ω and unit amplitude is incident upon the cylinder. As such, both the interior and exterior fields are a boundary value problem solution to Maxwell's equations. Because of the ext $(i\omega t)$ appearing in the Fourier transform and utilizing the constitutive equations for free space, Maxwell's equations reduce to

$$\nabla \times \vec{E} = -i\omega\mu_o \vec{H} \qquad \nabla \times \vec{H} = i\omega\varepsilon_o \vec{E} \tag{5}$$

In this case the incident plane wave traveling to the -x direction can be expressed as

$$E_z = e^{ikx} \qquad H_y = \frac{1}{\eta} e^{ikx} \tag{6}$$

where k is the wave number and equal to ω/c (c being the speed of light in the medium) and η the intrinsic impedance of the medium $\left(\sqrt{\mu/\varepsilon}\right)$.

Since for this problem there are no variations with respect to z, and since the incident magnetic field has no z-component, all of the fields of this problem will be of the transverse magnetic to z type. The fields will then be derivable from a scalar potential ψ according to (Ref. 1, p. 518):

$$E_z = i\omega\mu\,\psi \qquad H_\rho = \frac{1}{\rho}\frac{\delta\psi}{\delta\phi} \tag{7}$$

$$H_\phi = -\frac{\delta\psi}{\delta\rho} \qquad H_z = E_\rho = E_\phi = 0$$

where the cylindrical coordinate system is used (ρ,ϕ,z) and the wave function ψ satisfies the Helmholtz equation; in this case the wave functions will have the form

$$\psi = Z_n(k\rho)\cos n\phi \tag{8}$$

where Z_n denotes any Bessel function.

Using the wave transformation

$$e^{ikx} = \sum_{n=0}^{\infty} \varepsilon_n i^n J_n(k\rho) \cos n\phi \tag{9}$$

(where ε_n is Neumann's number and is equal to 1 for $n = 0$ and equal to 2 for all other n), the incident field wave function will be

$$\psi^{inc} = \frac{-1}{i\omega\mu_o} \sum_{n=0}^{\infty} \varepsilon_n i^n J_n(k\rho) \cos n\phi \tag{10}$$

Since the wave functions must have the form of equation (8) the wave function describing the total interior fields will be chosen as

$$\psi^{int} = \frac{-1}{i\omega\mu_o} \sum_{n=0}^{\infty} A_n J_n(k\rho) \cos n\phi \tag{11}$$

where A_n are unknown constants and the Bessel functions of the first kind are chosen because they are the only ones which are finite at $\rho = 0$. In the same manner the wave functions describing the scattered external fields will be

$$\psi^{sct} = \frac{-1}{i\omega\mu_o} \sum_{n=0}^{\infty} B_n H_n^{(2)}(k\rho) \cos n\phi \tag{12}$$

where the B_n are unknown constants and the Hankel functions of the second kind are chosen to represent an outgoing wave because of the exp $(i\omega t)$. The total exterior fields will be described by the sum of the external wave functions:

$$\psi^{ext} = \psi^{inc} + \psi^{sct} \tag{13}$$

The $\vec{E}$ field must be zero at the conducting surface and continuous in the aperture. Because of equation (7) this means that the external and internal wave functions are equal at $\rho = R$. This relation and the orthogonality of the cosine functions can be used to express the unknown constants B_n in terms of A_n:

$$B_n = \left(A_n - \varepsilon_n i^n\right) \frac{J_n(kR)}{H_n^{(2)}(kR)} \tag{14}$$

The remaining boundary condition to find the A_n's is that the magnetic field be continuous in the aperture: $-\alpha \leq \phi \leq \alpha$. Since this is not a complete coordinate surface, the orthogonality of the cosine functions cannot be used. Instead the following method will be used to find the A_n's. At $\rho = R$ the $\vec{E}$ field will be (from equations (7) and (11))

$$E_z^{int}(\rho=R,\phi) = \sum_{n=o}^{\infty} A_n J_n(kR) \cos n\phi \qquad (15)$$

This series can be recognized as a Fourier cosine series in ϕ. It will be remembered that an even function of period 2π may be expanded in a Fourier cosine series as (Ref. 2, p. 411)

$$f(\phi) = \sum_{n=o}^{\infty} a_n \cos n\phi \qquad (16)$$

$$a_n = \frac{\varepsilon_n}{\pi} \int_o^{\pi} f(\phi) \cos n\phi \, d\phi \qquad (17)$$

Comparing equations (16) and (15), and since E_z is zero except in the aperture:

$$A_n = \frac{\varepsilon_n}{\pi J_n(kR)} \int_o^{\alpha} E_z^{int}(\rho=R,\phi) \cos n\phi \, d\phi \qquad (18)$$

The solution to the problem is therefore known when the value of the electric field is known in the aperture. One method of approximating a solution is to assume some series of functions for the value of E_z in the aperture; at the same time a set of unknown constants is used to adjust the amplitude of these functions. For example, assume

$$\begin{aligned} E_z(\rho=R,\phi) &= \sum_{q=1}^{m} V_q f_q(\phi) \qquad -\alpha \leq \phi \leq \alpha \\ &= 0 \qquad \text{elsewhere} \end{aligned} \qquad (19)$$

where the V_q are unknown. Using equation (19) in (18) will yield for $A_n J_n(kR)$ functions of the form:

$$A_n J_n(kR) = \sum_{q=1}^{M} V_q F_q(n,\alpha) \qquad (20)$$

The values of the V_q's can then be found by matching the H_ϕ field in the slot at m different points.

Specifically, the value of H_ϕ is found from equations (7), (10), (11), (12), and (13). Then the function ERROR is defined as

$$\begin{aligned} \text{ERROR} &= \left(H_\phi^{\text{int}} - H_\phi^{\text{ext}}\right)_{\rho=R} \\ &= \frac{2}{\pi R\omega\mu} \sum_{n=o}^{\infty} \left(A_n - \varepsilon_n i^n\right) \frac{\cos n\phi}{H_n^{(2)}(kR)} \end{aligned} \tag{21}$$

The function ERROR should be zero for those values of ϕ in the aperture since the magnetic field is continuous there. In particular, we designate ϕ_m as some m different values of ϕ in the aperture, and using equation (20) for A_n in (21) will result in

$$\sum_{n=o}^{\infty} \frac{\Sigma V_q F_q(n,\alpha)}{J_n(kR)H_n^{(2)}(kR)} \cos \phi_m = \sum_{n=o}^{\infty} \frac{\varepsilon_n i^n}{H_n^{(2)}(kR)} \cos n\phi_m \tag{22}$$

Equation (22) can be rewritten as

$$\sum_{q=1}^{m} V_q C_{mq} = S_m \tag{23}$$

where

$$C_{mq} = \sum_{n=o}^{\infty} \frac{F_q(n,\alpha)}{J_n(kR)H_n^{(2)}(kR)} \cos n\phi \tag{24}$$

$$S_m = \sum_{n=o}^{\infty} \frac{\varepsilon_n i^n}{H_n^{(2)}(kR)} \cos n\phi \tag{25}$$

Equation (23) for m different values of ϕ will yield m equations for the m different values of V_q.

At this point one may begin to consider the functions $f_q(\phi)$ which are used in equation (13) to produce $F_q(n,\alpha)$. If these functions exactly produce E_z in the aperture, the problem will be exactly solved. On the other hand the difference between these functions and E_z in the aperture will produce an error in the solutions. A measure of this error can be made by calculating the function ERROR (given in equation (21)) for the values of ϕ different from ϕ_m. It should be noted that theoretically an exact solution could be found by letting m approach infinity and having the set of functions be a complete set over the aperture; in this case all boundary conditions of the problem would be satisfied.

Morse and Feshbach have examined the problem in this manner (Ref. 3, p. 1387). In their work they assumed that the value of E_z in the aperture would be the same (except for a multiplying constant) as for a plane wave incident upon a plane covering the y-z axis with a narrow slit of width 2α. The problem of a plane wave incident upon a conducting plane with a slit has been solved exactly when there are no reflections from the $x < 0$ area (Ref. 4, p. 895). For a slit narrow with respect to the wavelength, the value of E_z in the slit is approximately

$$E_z \sim V\sqrt{1-\left(\frac{\phi}{\alpha}\right)^2} \tag{26}$$

where V is the magnitude of the field in the center of the slit.

Morse and Feshbach used equation (26) for (19) which then resulted for equation (20):

$$A_n = \frac{V}{J_n(kR)} F(n) \tag{27}$$

$$F(n) = \frac{\alpha}{4} \tag{28}$$

$$F(n) = \frac{1}{n} J_1(n\alpha) \qquad n > o \tag{29}$$

Using these expressions the resulting magnetic field was matched at $\phi_m = 0$ to produce a single equation for V (where q is now equal to 1):

$$V = \frac{S_o}{C_o} \tag{30}$$

$$C_o = \sum_{n=o}^{\infty} D(n)\ F(n) + i \sum_{n=o}^{\infty} D_2(n)\ F(n) \tag{31}$$

$$\mathrm{Re}\left(S_o\right) = \sum_{\substack{n=o \\ n=\text{even}}}^{\infty} (-1)^{\frac{n}{2}} \varepsilon_n D_3(n) + 2 \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} (-1)^{\frac{n+1}{2}} D_4(n) \tag{32}$$

$$\mathrm{Im}\left(S_o\right) = \sum_{\substack{n=o \\ n=\text{even}}}^{\infty} (-1)^{\frac{n}{2}} \varepsilon_n D_4(n) - 2 \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} (-1)^{\frac{n+1}{2}} D_3(n) \tag{33}$$

where $D_1(n)$, $D_2(n)$, $D_3(n)$, and $D_4(n)$ are representations for the real and imaginary Bessel function combinations, i.e.:

$$D_1(n) = \left(J_n^{\,2}(kR) + Y_n^{\,2}(kR)\right)^{-1} \tag{34}$$

$$D_2(n) = \left(\frac{J_n^{\,3}(kR)}{Y_n\ (kR)} + J_n(kR)\ Y_n(kR)\right)^{-1} \tag{35}$$

$$D_3(n) = J_n(kR)\ D_1(n) \tag{36}$$

$$D_4(n) = \left(\frac{J_n^{\,2}(kR)}{Y_n\ (kR)} + Y_n(kR)\right)^{-1} \tag{37}$$

Morse and Feshbach did not produce either numerical results or graphs of their work. This results in two unanswered questions:

(1) What is the effect of the reflections of the wave from the interior of the cylinder; i.e., is the shape of the aperture distribution still given by equation (26)?

(2) Does the aperture distribution change when the opening of the aperture, α, is large as compared with the radius of the cylinder R?

To answer these questions the author of this report considered a series of functions with a series of unknown constants for the aperture distribution given in equation (19). In this case with several points across the aperture being matched, the selection of the functions $f_q(\phi)$ are somewhat arbitrary with the exception of three considerations. One of the considerations is the following:

the constants C_{mq} and S_m are infinite series as given in equations (24) and (25). For computational purposes we wish these series to converge as fast as possible. But the denominator of the series for C_{mq} contains the terms $J_n(kR)$ $H_n(kR)$. Using the asumptotic forms for these Bessel functions where n >> kR, we have (Ref. 5, p. 365)

$$J_n(kR) \longrightarrow \frac{1}{\sqrt{2\pi n}}\left(\frac{ekR}{2n}\right)^n \tag{38}$$

$$\begin{aligned} H_n^{(2)}(kR) &\longrightarrow -i\ Y_n(kR) \\ &\longrightarrow i\sqrt{\frac{2}{\pi n}}\left(\frac{ekR}{2n}\right)^{-n} \end{aligned} \tag{39}$$

The denominator term of this series then approaches for large n

$$J_n(kR)\ H_n^{(2)}(kR) \longrightarrow \frac{i}{\pi n} \tag{40}$$

This means that for the series to converge, the transform of $f_q(\phi)$, or $F_q(n,\alpha)$, must approach zero faster than 1/n for large n. In particular, it would be desirable to have the transforms $F_q(n,\alpha)$ approach zero at least as fast as $1/n^3$.

To aid in the selection of $f_q(\phi)$, one of the basic principles of Fourier series can be used; that is, if a function and its various derivatives all satisfy the Dirichlet conditions, and if the k-th derivative is the first which is not everywhere continuous, then the Fourier coefficients of the function approach zero as n to the minus (k + 1) power for sufficiently large n (Ref. 6, p. 407).

Finally there are two other considerations when chosing the functions $f_q(\phi)$ to represent E_z in the slot. These are that from the physical consideration of the problem the functions $f_q(\phi)$ should be symmetrical about ϕ equal to zero and should be zero at ϕ equal to $\pm\ \alpha$. The series of functions corresponding to equation (19) used in this report was then

$$E_z(\rho=R,\phi) = \sum_{q=1}^{m} V_q\left(\left(\frac{\phi}{\alpha}\right)^2-1\right)\cos\left(\left(2_q-1\right)\frac{\pi}{2\alpha}\ \phi\right) \tag{41}$$

It can be seen that the above set of functions is symmetrical about ϕ equal to zero, and both equation (41) and its first derivative are zero at ϕ equal to $\pm\alpha$. Using equation (41) in (18) will produce

$$F_q(n,\alpha) = (-1)^{q-1}\left(\frac{\varepsilon_n}{\pi}\right)\left(W_1(n)\ \sin n_\alpha - W_2(n)\ \cos n_\alpha\right) \tag{42}$$

$$W\ (n) = \frac{4tn}{Q^2\alpha} \tag{43}$$

$$W\ (n) = \frac{3t^3+6tn^2}{Q^3\alpha^2} \tag{44}$$

$$t = (2_q-1)\ \frac{\pi}{2\alpha} \tag{45}$$

$$Q = t^2 - n^2 \tag{46}$$

and when t is equal to n, the equation (42) reduces to

$$F_q(n,\alpha) = -\frac{2\alpha}{3\pi} - \frac{2}{\pi^3(29-1)^2} \tag{47}$$

The unknown V_q's were then determined by use of equation (23) where the ϕ_m were equally distributed across the aperture, i.e.,

$$\phi_m = (p-1)\ \frac{\alpha}{m} \qquad\qquad 1 \leq p \leq m \tag{48}$$

The distribution of the electric field was then found for various aperture openings and frequencies.

The results of this investigation were that the distribution used by Morse and Feshbach is approximately correct for wide aperture openings and high frequencies. For example, the V_q's shown in Table I are for a frequency of 500 MHz, an aperture opening of 60 degrees, a cylinder radius of 1 meter, and for m equal to 10. A comparison of the Morse and Feshbach distribution and the distribution of equation (41), for the V_q's in Table I, is shown in Figure 2. In this case the V_q's listed in the table were normalized to 1 and the unknown

Table I

VALUES OF V_q (500 MHz, 60°, m = 10)

Q	Re (V)	Im (V)
1	0.461446	2.529478
2	-1.226267	-0.880974
3	0.693587	0.549577
4	-0.491884	-0.374646
5	0.360717	0.268753
6	-0.263716	-0.193839
7	0.186795	0.136116
8	-0.123022	-0.-89152
9	0.068577	0.049536
10	-0.021123	-0.015235

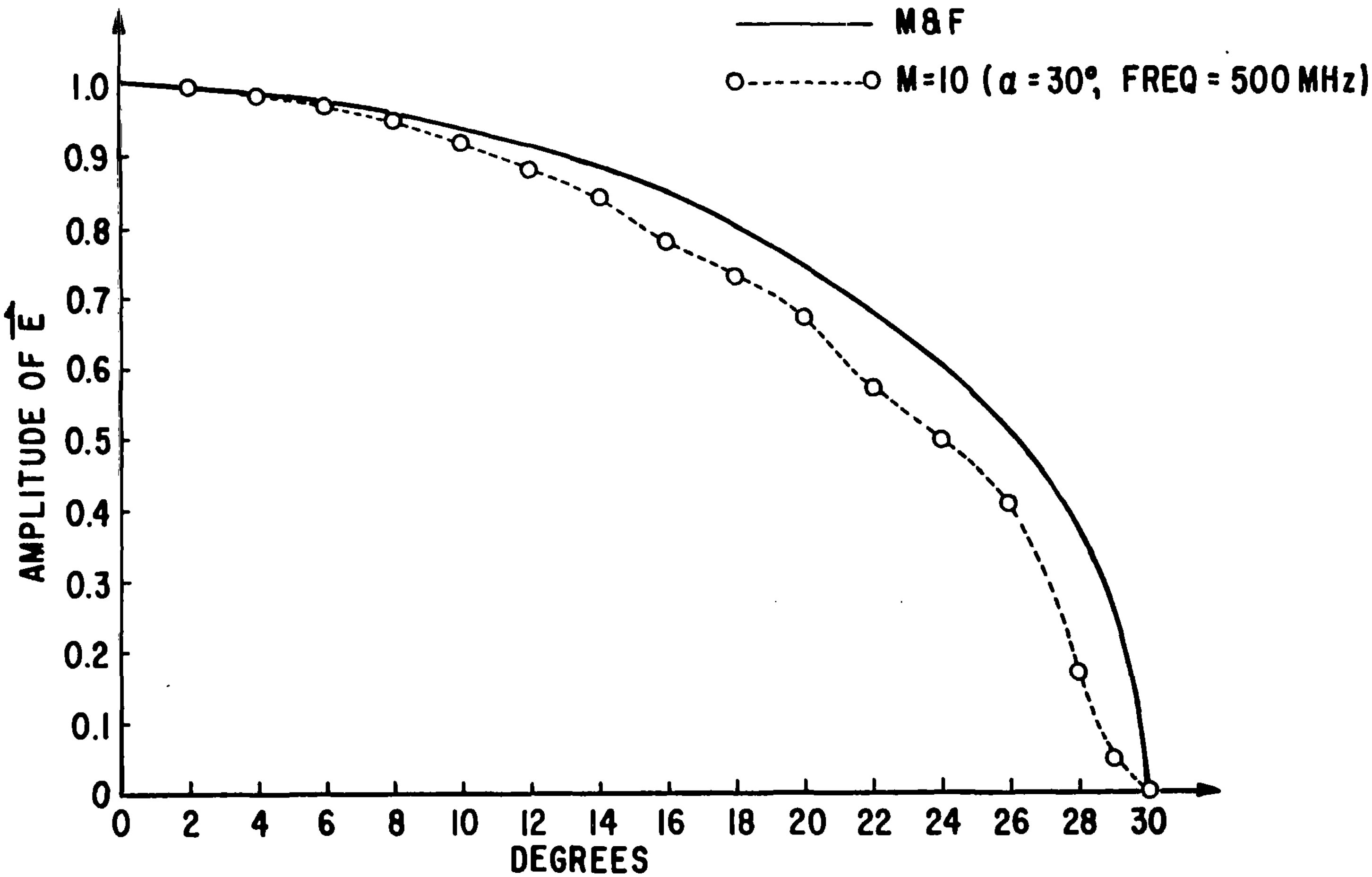

Figure 2. A Comparison of Aperture Field Strengths

V of equation (19) was set to 1. It can be seen that the shape of the aperture distribution is approximately the same. Therefore, a good approximation to the aperture field distribution is the Morse and Feshbach approximation (equation (26)) where only one unknown need be solved for (i.e., equation (30)).

One small difficulty in using the Morse and Feshbach approximation might be mentioned. This is the summing of the series for the imaginary part of C_o. As shown above (equation (40)) $D_2(n)$ for large n approaches minus $n\pi$. For large n the Bessel function $J_1(n\alpha)$ has an asymptotic form (Ref. 5, p. 364):

$$J_1(n\alpha) \longrightarrow \sqrt{\frac{2}{\pi n\alpha}} \cos\left(n\alpha - \frac{3\pi}{4}\right) \tag{49}$$

Therefore, for large n, the imaginary part of C_o approaches

$$I_m\left(C_o\right) \longrightarrow -\sqrt{\frac{2\pi}{\alpha}} \sum_{n=K}^{\infty} \frac{1}{\sqrt{n}} \cos\left(n\alpha - \frac{3\pi}{4}\right) \tag{50}$$

where K is some number such that the approximation in equation (40) is valid. This is a slowly converging series.

SECTION III

COMPUTATIONS AND ACCURACY

All computations used in this report were made by the Burrough's B-5500 digital computer. This computer has a Bessel function call-up as described by the Burrough's Mathematical Report Series MRS-139 (1 June 64). This call-up uses either the defining series or a polynominal approximation to compute $J_o(z)$, $J_1(x)$, $Y_o(z)$, and $Y_1(z)$. Then the following recurrence formula is used to compute all other orders:

$$\left(\frac{2n}{z}\right) Z_n(z) = Z_{n-1}(z) + Z_{n+1}(z) \qquad (51)$$

The accuracy of zero and first order computations is better than 4×10^{-6}. Since for n larger than z the $Y_n(z)$ function is a numerically increasing function, the rounding errors when using equation (51) are not large and the order of accuracy is the same as for the first and second orders. On the other hand the $J_n(z)$ function is a numerically decreasing function for n larger than z, and the round-off error in using (51) is severe. Experience has shown that for z smaller than n, 20 iterations of equation (51) will produce values of $J_n(z)$, whose values are always smaller than or equal to 1, of the order of 1000.

For this problem the following procedure was used. The $Y_o(z)$ and $Y_1(z)$ functions were called up from the Bessel call-up. Then the value of $Y_n(z)$ was computed by equation (51) until the value of $Y_n(z)$ exceeded 10 to the power plus 35; let the value of n for this value of $Y_n(z)$ be N. Then initial values of $J_n(z)$ were computed for orders N and N-1 by using the asymptotic values given in equation (26). The recurrence formula (51) was then used to compute the values of $J_n(z)$ from N down to zero, where now $J_n(z)$ is a numerically increasing function. All values of $J_n(z)$ were then normalized by the formula (Ref. 5, p. 361)

$$1 = J_o(x) + 2\left(J_2(z) + J_4(z) + J_6(z) + \cdots\right) \qquad (52)$$

Experience has shown that from N-5 to zero orders the $J_n(z)$ functions are accurate to at least the sixth significant figure. At the N-5 order, the values of $Y_n(z)$ are larger than 10 to the plus 30 power and the values of the $J_n(z)$ function are smaller than 10 to the minus 25 power. From the N-5 orders down to the zero order these computed values of $J_n(z)$ and $Y_n(z)$ are used in computing $D_1(n)$, $D_2(n)$, $D_3(n)$, and $D_4(n)$. For those orders above N-5, the values of these D-functions (with the exception of $D_2(n)$) are all smaller than 10 to the minus 30 power and these functions are then set to zero.

For the values of $D_2(n)$ where n is larger than N-5 the Debye asymptotic formulas were used (Ref. 5, p. 366). These formulas were then reduced to

$$J_n(z)\ Y_n(z) \doteq \frac{-1}{n\pi}\left(1 - \left(\frac{z}{n}\right)^2\right)^{-1/2} \tag{53}$$

For n larger than N-5 the worst error in using this formula was found to be in the fifth significant figure.

The largest value of N used in the different calculations was for N equal to 59 when the frequency was 500 MHz and R was equal to 1.

In the results presented in this report all the infinite sums were carried out to 1000 terms, or until the terms were smaller than 10 to the minus 30 power, with the exception of the imaginary part of C_o. This sum was computed to 1000 terms using the approximations given in equation (53) and also using Hankel's asymptotic expansion for $J_1(n\alpha)$ for $n\alpha$ larger than 5. (Ref. 5, p. 364)

$$J_1(n\alpha) = \sqrt{\frac{2}{\pi n A}}\left(P(n\alpha)\ \cos x - Q(n\alpha)\ \sin x\right) \tag{54}$$

$$P(n\alpha) = 1 + \frac{15}{128(n\alpha)^2} - \frac{(15)(21)(45)}{(24)(64)(64)(n\alpha)^4} \tag{55}$$

$$Q(n\alpha) = \frac{3}{8n\alpha} - \frac{(15)(21)}{(48)(64)(n\alpha)^4} \tag{56}$$

$$x = n\alpha - \frac{3\pi}{4} \tag{57}$$

This approximation is correct to the fifth significant place. From 1,000 to 1,000,000 terms the factor $D_2(n)$ was approximated by $-n\pi$ and the sum S' was formed:

$$S' = -\sqrt{\frac{2\pi}{\alpha}} \sum_{n=1,000}^{n=1,000,000} \frac{1}{\sqrt{n}} \left(P(n\alpha) \cos x - Q(n\alpha) \sin x\right) \tag{58}$$

so that the imaginary part of C_o is equal to

$$\mathrm{Im}\left(C_o\right) = \sum_{N=0}^{n=1,000} D_2(n)\ F(n) + S' \tag{59}$$

SECTION IV

RESULTS

A Burroughs B-5500 computer was used to calculate the values of V from equations (30) through (33). For these calculations the cylinder radius was scaled to a value of 1 meter and frequencies from 1 MHz to 500 MHz were computed. These results are presented in Figures 3 and 4 for aperture values of 10 and 60 degrees. For other cylinder radii these results will be valid by scaling the frequency according to

$$\omega R = c \tag{60}$$

With reference to equation (4), for any given single frequency and a unit incident field the interior electric field will be

$$E_z^{int}(\rho,\phi) = V \sum_{n=o}^{\infty} F(n) \frac{J_n(k\rho)}{J_n(kR)} \cos n\phi \tag{61}$$

where F(n) is given by equations (28) and (29). For a spectrum of frequencies, V becomes a continuous function of ω given in Figures 3 and 4. The electric field interior to the cylinder as a function of time will then be

$$E_z^{int}(\vec{r},t) = \frac{1}{2\pi} \sum_{n=o}^{\infty} F(n) \cos n\phi \int_{-\infty}^{\infty} E_o(\omega)\, V(\omega) \frac{J_n(k\rho)}{J_n(kR)} e^{i\omega t}\, d\omega \tag{62}$$

One fact should be noted about equations (61) and (62) and the graphs of $V(\omega)$. From the graphs of $V(\omega)$ it can be seen that $V(\omega)$ is zero at certain frequencies. At these frequencies the functions $D_2(n)$, given in equation (35), are infinite. (This is the only one of the four D functions which are infinite.) The reason that $D_2(n)$ is infinite is that $J_n(kR)$ is an oscillatory function for $kR > n$. For these particular frequencies the values of A_n and V can be rewritten from equations (27) and (30) as follows. Designating

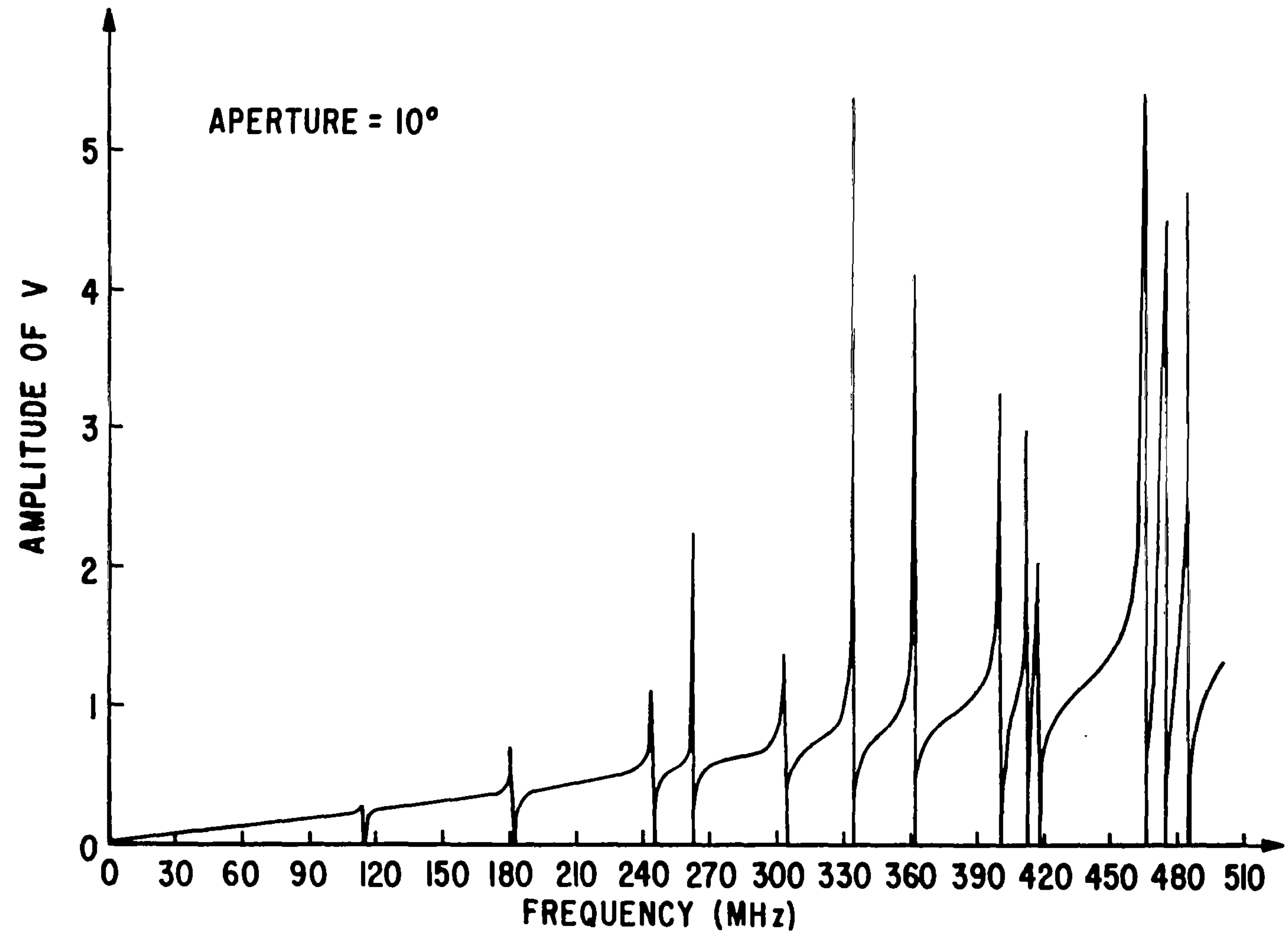

Figure 3. Amplitude of V (0-500 MHz, 10° Aperture)

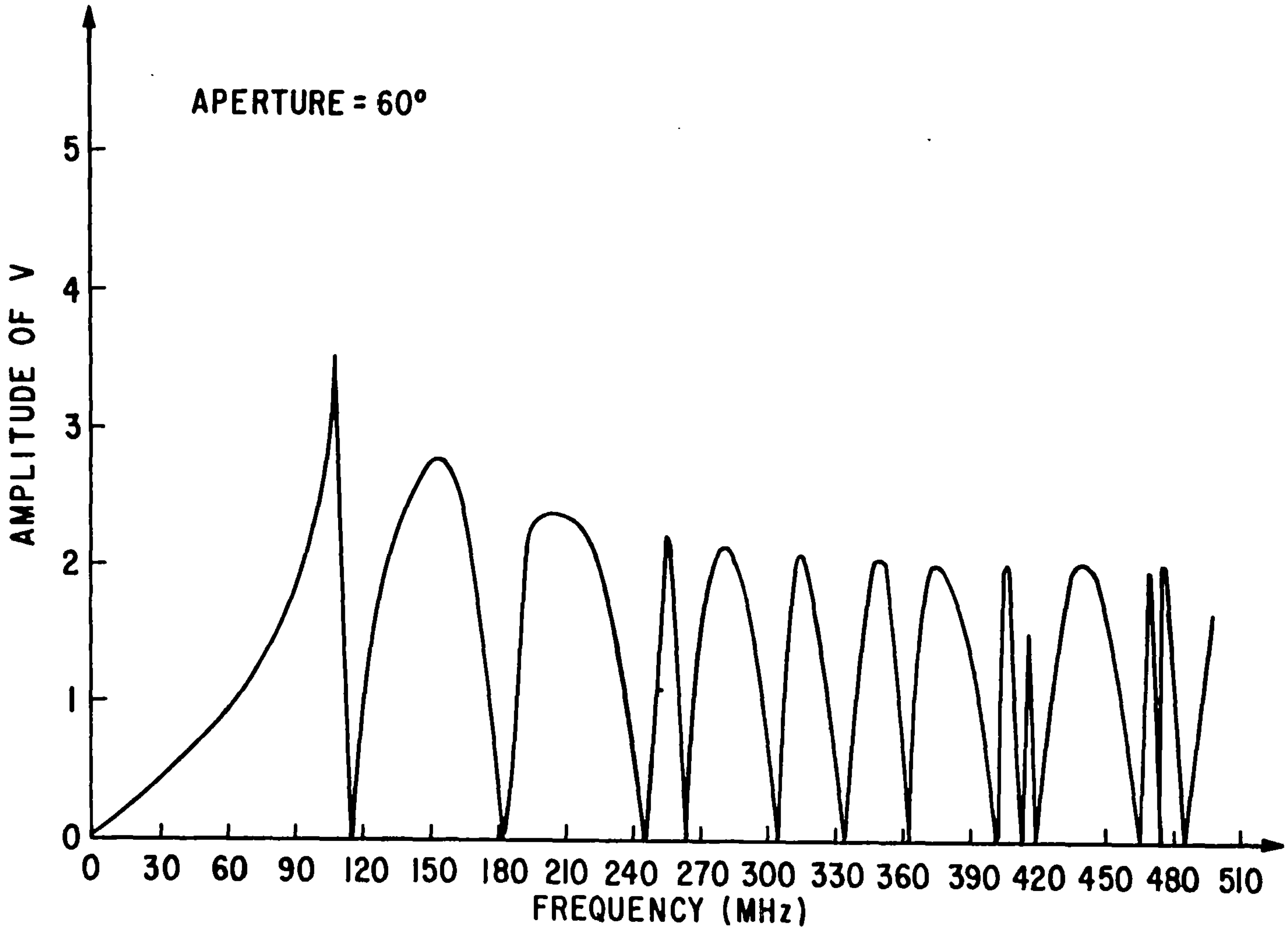

Figure 4. Amplitude of V (0-500 MHz, 60° Aperture)

$$J_n(kR) = J_p = 0 \tag{63}$$

for those values of kR and n = p for which the Bessel function is zero, C_o becomes from equation (31)

$$C_o = \frac{1}{J_p}\left[i\,\frac{F(p)}{Y_p(kR)}\right] \tag{64}$$

so that V from equation (30) becomes

$$V = J_p\left[-i\,S_o\,\frac{Y_p(kR)}{F(p)}\right] = 0 \tag{65}$$

The A_n in equation (30) is then zero for all n except for n = p, in which case

$$A_p = -i\,S_o\,Y_p(kR) \tag{66}$$

For these particular frequencies where resonances occur, the interior electric and magnetic fields will be

$$E_z^{int}\,(\rho,\phi) = E_o\left(\omega_p\right)\,A_p\,J_p(k\rho)\,\cos n\phi \tag{67}$$

$$H_\phi^{int} = -k\,E_o\left(\omega_p\right)\,A_p\,\mathring{J}_p(k\rho)\,\cos n\phi \tag{68}$$

$$H_\rho^{int} = \frac{p}{\rho}\,E_o\left(\omega_p\right)\,A_p\,J_p(k\rho)\,\sin n\phi \tag{69}$$

where ω_p and p are the frequencies and orders of the Bessel functions for which

$$J_p\left(\frac{\omega_p}{c}\,R\right) = 0 \tag{70}$$

It will be noted that at these resonant frequencies the field strength is independent of the size of the aperture opening.

Tables II and III record the following information:

(1) frequency for which J_p is zero for R = 1, f_p,

(2) the order of J_p, p,

(3) the argument of J_p for J_p equal to zero, ω_p/c,

(4) the absolute value of S_o for f_p,

(5) the value of $Y_p(\omega_p/c)$,

(6) the absolute value of A_p, and

(7) the phase of A_p.

Table II

CALCULATIONS FOR A_p (0-450 MHz)

Frequency (MHz) f_p	Order p	Argument ω_p/c	$\lvert S_o \rvert$	$Y_p(\omega_p/c)$	$\lvert A_p \rvert$	A_p (phase) (Degree)
114.743	0	2.40483	7.9891	0.50992	4.0738	38.27
182.824	1	3.83171	12.3839	0.41252	5.1086	122.97
245.038	2	5.13562	16.4238	0.36495	5.9938	-160.87
263.382	0	5.52009	17.6180	-0.33894	5.9714	41.48
304.419	3	6.38016	20.2935	0.33453	6.7888	- 88.66
334.738	1	7.01559	22.2730	-0.30236	6.7345	128.10
362.066	4	7.58834	24.0589	0.31262	7.5212	- 18.82
401.616	2	8.41724	26.6460	-0.27869	7.4261	-150.99
412.899	0	8.65373	27.3845	0.27101	7.4214	42.64
418.518	5	8.77148	27.7523	0.29571	8.2068	49.42

Table III

CALCULATIONS FOR A_p (450-565 MHz)

Frequency (MHz) f_p	Order p	Argument ω_p/c	$\lvert S_o \rvert$	$Y_p(\omega_p/c)$	$\lvert A_p \rvert$	A_p (phase) (Degree)
465.732	3	9.76102	30.8447	-0.26149	8.06566	- 73.58
474.086	6	9.93611	31.3922	0.28209	8.8554	- 63.50
485.411	1	10.1735	32.1344	0.25060	8.0529	130.16
527.936	4	11.0647	34.9224	-0.24813	8.6654	1.44
528.969	7	11.0864	34.9902	0.27076	9.4738	2.68
554.423	2	11.6198	36.6598	0.23570	8.6408	-146.64
562.615	0	11.7915	37.1972	-0.23225	8.6392	43.23

REFERENCES

1. Harrington, R. F.; Time-Harmonic Electromagnetic Fields, McGraw-Hill, New York, 1961.

2. Selby, S. M.; C. R. C. Standard Mathematical Tables, 14th Edition, The Chemical Rubber Co., Cleveland, Ohio, 1965.

3. Morse, P. M. and Feshbach, H.; Methods of Theoretical Physics, McGraw-Hill, New York, 1953.

4. Morse, P. M. and Rubenstein; "The Diffraction of Waves by Ribbons and by Slits," Phys. Rev., Vol. 54, No. 11, pp. 895-898, December 1938.

5. Abramowitz, M. and Stegun, A., Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards Applied Mathematics Series 55, U.S. Government Printing Office, Washington, D.C., 1965.

6. Sokolnikoff, I. S., Advanced Calculus, McGraw-Hill, New York, 1939.