IN 106

SCATTERING OF ELECTROMAGNETIC RADIATION BY APERTURES;

I. NORMAL INCIDENCE ON THE SLOTTED PLANE

by:

L. F. Libelo
U.S. Naval Ordnance Laboratory

and
J. Bombardt
USA/MERDC, Fort Belvoir

ABSTRACT: We report the results of the first of a series of investigations into the diffraction of electromagnetic radiation by apertures in conducting screens. We present here calculations of the fields everywhere for linearly polarized plane electromagnetic radiation normally incident on a slotted conducting plane. Although this problem has been solved by others earlier, it serves as a useful application in that it provides a test of our method of solution. Comparison of our work with the standard solution shows indeed that our technique yields results in excellent agreement with those in the literature. Our results were obtained by a technique of approximation in which infinite series representations of the fields are systematically truncated by utilizing the device of applying the usual boundary conditions but only at selected points of the boundaries. In this work the mathematical boundary used is one constructed in a manner convenient to the geometry of the diffracting system in precisely the same manner that had been shown earlier to be highly useful in static problems.

PUBLISHED 2 APRIL 1970

PHYSICS RESEARCH DEPARTMENT
U. S. NAVAL ORDNANCE LABORATORY
WHITE OAK, MARYLAND

NOLTR 70-58 2 April 1970

SCATTERING OF ELECTROMAGNETIC RADIATION BY APERTURES;
I. NORMAL INCIDENCE ON THE SLOTTED PLANE

The research reported herein was performed jointly at the Naval Ordnance Laboratory, Physics Research Department, and at the USA/MERDC, Fort Belvoir. This work was carried out as part of the Independent Research Program at NOL (Task Number MAT-03L 000/ZR011 01 01). This document is for information only.

The authors acknowledge the efforts of Mr. D. Lozier, presently with the National Bureau of Standards in Washington, D.C., for computer programming and valuable discussions concerning the numerical aspects of the work reported herein.

GEORGE G. BALL

D. F. Bleil

D. F. BLEIL
By direction

CONTENTS

ILLUSTRATIONS

I. GENERAL FORMULATION OF THE PROBLEM

We consider a plane, monochromatic, linearly polarized electromagnetic wave traveling in the direction of decreasing Y. This plane wave is assumed normally incident on an infinite perfectly conducting plane containing an infinite slot of width $2\rho_o$. We assume the regions above and below the slotted plane, which is taken to occupy the XZ plane, to be free space. The geometry of the problem for the case where the incident wave is polarized parallel to the slot is shown in Fig. 1.

Maxwell's equations in free space are:

$$\vec{\nabla} \times \vec{E} = 0 \tag{1}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \tag{4}$$

where we have used the constitutive equations:

$$\vec{D} = \epsilon_o \vec{E} \quad , \quad \vec{B} = \mu_o \vec{H} \tag{5}$$

Assuming time variation of the form $e^{-i\omega t}$ the fields in the incident wave are taken as

$$\vec{B}_i = -\vec{e}_x B_o \, e^{-i(ky+\omega t)} \tag{6a}$$

$$\vec{E}_i = \vec{e}_z E_o \, e^{-i(ky+\omega t)} \tag{6b}$$

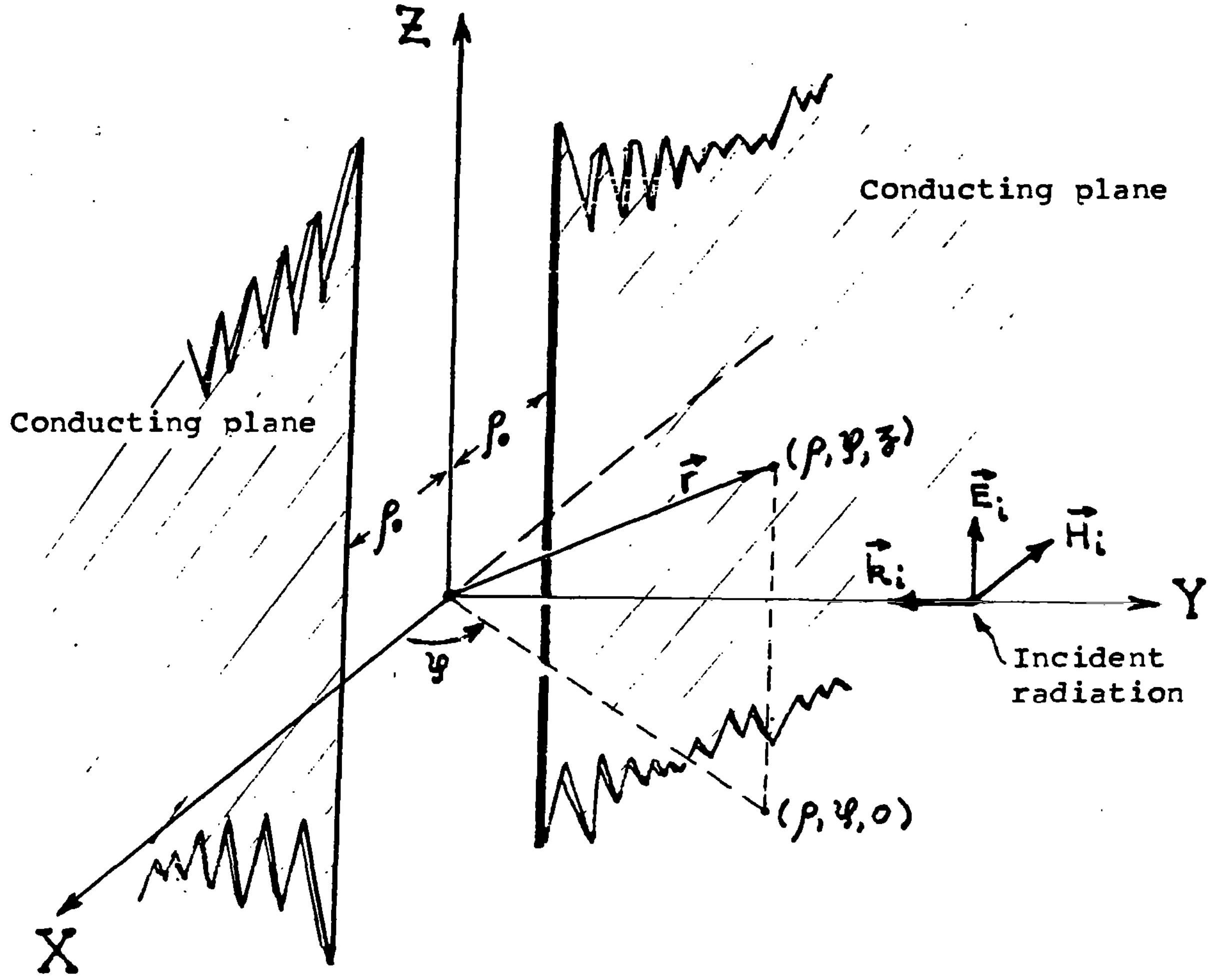

Figure 1. The Slotted Infinite Conducting Plane Showing the Direction of Incidence

where the propagation constant is

$$k = \omega / c \tag{7}$$

and also

$$E_o = Z_o H_o = c B_o$$

where $Z_o = \sqrt{\mu_o / \varepsilon_o}$ is the impedance of free space.

We represent the fields in free space by means of a vector potential only, so that we have

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{8}$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} \tag{9}$$

where the vector potential satisfies

$$\vec{\nabla} \cdot \vec{A} = 0 \tag{10}$$

In the usual manner we obtain via the Maxwell equations the wave equation for $\vec{A}$

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = 0 \tag{11}$$

On the conducting plane the surface current density is in the direction of the electric field in the incident wave. Then the vector potential has only a single component and we write

$$\vec{A} = \vec{e}_z A_z e^{-i\omega t} \tag{11a}$$

Substituting this into eq. (11) yields

$$\nabla^2 A_z + k^2 A_z = 0 \tag{12}$$

which is just the scalar Helmholtz equation. Since the Cartesian z-component of the vector potential is identically the same as the corresponding z-component in circular cylindrical coordinates we can work in cylindrical coordinates.

Our problem now is to completely determine A_z everywhere as this will lead to solutions of the fields everywhere, since we have from eqs. (8), (9) and (11)

$$E_z(\rho, \varphi) = i\omega A_z(\rho, \varphi) \tag{13}$$

$$B_\rho(\rho, \varphi) = \frac{1}{\rho}\frac{\partial}{\partial \varphi} A_z(\rho, \varphi) \tag{14}$$

$$B_\varphi(\rho, \varphi) = -\frac{\partial}{\partial \rho} A_z(\rho, \varphi) \tag{15}$$

In solving this problem we employ a device used earlier by Kaden[1] to solve static field problems involving circular apertures in conducting plane screens. This device, in our calculation, consists merely of symmetrically introducing an imaginary "slot cylinder", or Kaden cylinder, of radius ρ_o with axis along the center line of the slot so as to provide a cylindrical boundary that divides all space into three regions, namely,

(i) region (1) : $\rho > \rho_o \quad , \quad 0 < \varphi < \pi$

(ii) region (2) : $\rho < \rho_o \quad , \quad 0 < \varphi < 2\pi$

(iii) region (3) : $\rho > \rho_o \quad , \quad \pi < \varphi < 2\pi$

What we must do then is determine the vector potential in each of these regions (see Fig. 2) such that the boundary conditions are satisfied on the conducting plane, on the upper half of the Kaden cylinder, and on the bottom half of the Kaden cylinder as well. Simultaneously the radiation condition must be satisfied for $\rho \to \infty$ in regions (2) and (3) (i.e. above and below the conducting plane but outside the Kaden cylinder). In the conventional manner we then write three vector potential functions

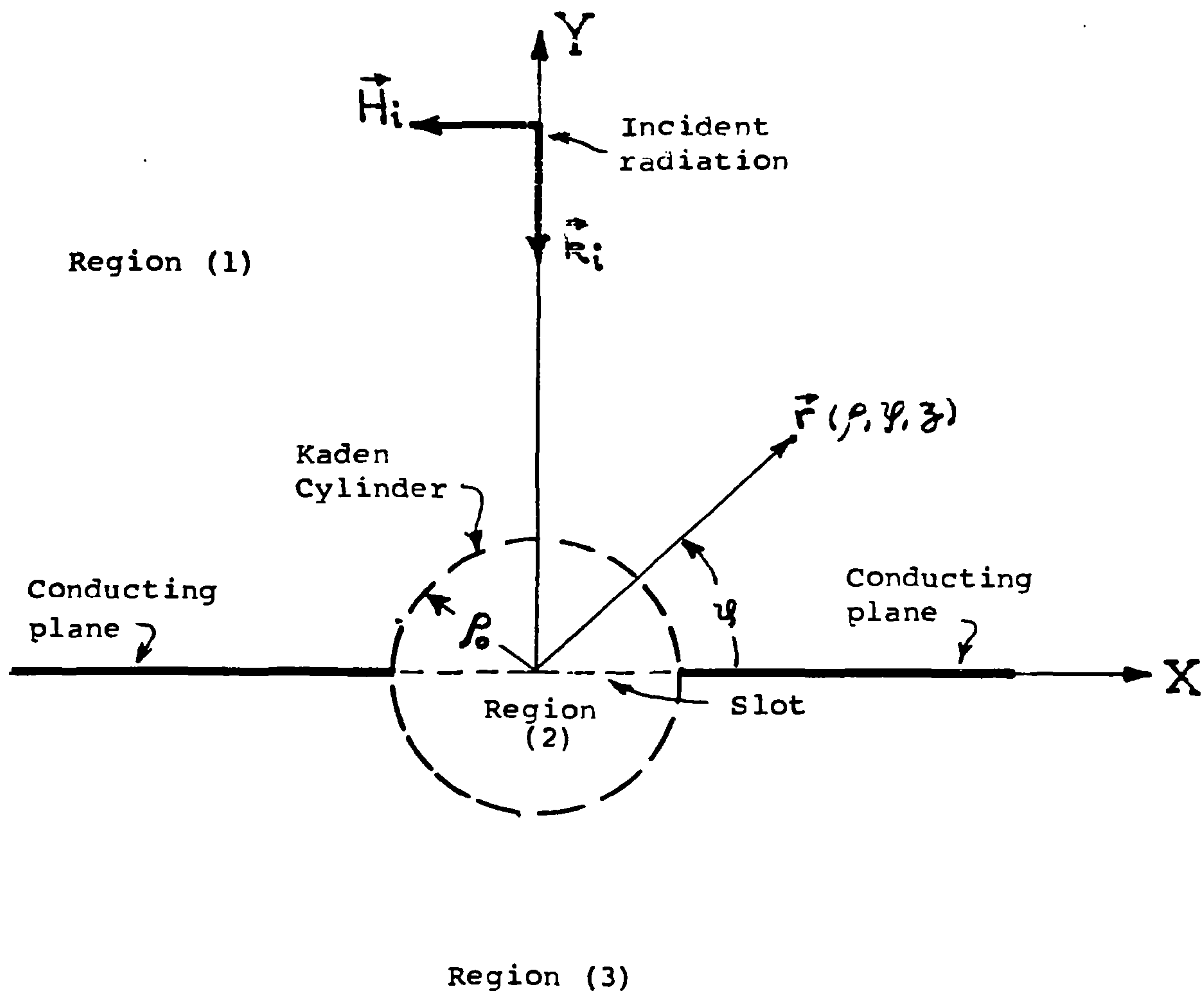

Figure 2. Cross-Section of Three Scattering Regions Used In Formulating The Diffraction Problem.

$$A_z^{(1)} = A_z^{(1)}(\rho, \varphi) \quad ; \qquad \rho > \rho_0 \ , \ 0 < \varphi < \pi \tag{16}$$

$$A_z^{(2)} = A_z^{(2)}(\rho, \varphi) \ ; \qquad \rho < \rho_0 \ , \ 0 < \varphi < 2\pi \tag{17}$$

$$A_z^{(3)} = A_z^{(3)}(\rho, \varphi) \ ; \qquad \rho > \rho_0 \ , \ \pi < \varphi < 2\pi \tag{18}$$

II. BOUNDARY CONDITIONS

Clearly at the boundary of the Kaden cylinder all field components must be continuous. On the conducting plane in region (1)

$$E_z^{(1)}(\rho, 0) = i\omega A_z^{(1)}(\rho, 0) \tag{19a}$$

$$E_z^{(1)}(\rho, \pi) = i\omega A_z^{(1)}(\rho, \pi) \tag{19b}$$

$$B_\varphi^{(1)}(\rho, 0) = \frac{1}{\rho}\left[\frac{\partial A_z^{(1)}(\rho, \varphi)}{\partial \varphi}\right]_{\varphi = 0} \tag{20a}$$

$$B_\varphi^{(1)}(\rho,\pi) = \frac{1}{\rho}\left[\frac{\partial A_z^{(1)}(\rho,\varphi)}{\partial \varphi}\right]_{\varphi=\pi} \tag{20b}$$

and in region (3)

$$E_z^{(3)}(\rho,\pi) = i\omega A_z^{(3)}(\rho,\pi) \tag{21a}$$

$$E_z^{(3)}(\rho,2\pi) = i\omega A_z^{(3)}(\rho,2\pi) \tag{21b}$$

$$B_\varphi^{(3)}(\rho,\pi) = \frac{1}{\rho}\left[\frac{\partial A_z^{(3)}(\rho,\varphi)}{\partial \varphi}\right]_{\varphi=\pi} \tag{22a}$$

$$B_\varphi^{(3)}(\rho,2\pi) = \frac{1}{\rho}\left[\frac{\partial A_z^{(3)}(\rho,\varphi)}{\partial \varphi}\right]_{\varphi=2\pi} \tag{22b}$$

III. SOLUTION ABOVE THE PLANE AND OUTSIDE THE SLOT CYLINDER

We now formulate the general solution for $A_z^{(1)}(\rho,\varphi)$. In the region (1) the total resultant vector potential at any point will consist of linear superposition of the potential due to the incident wave, which we shall call $A_z^{(i)}$, and a potential due to the reflected wave, which we call $A_z^{(r)}$. Thus we have

$$A_z^{(1)} = A_z^{(i)} + A_z^{(r)} \tag{23}$$

The incident wave potential is readily obtained since

$$\vec{E}_i = i\omega A_z^{(i)} \vec{e}_z e^{-i\omega t} = E_o \vec{e}_z e^{-iky-i\omega t}$$

and hence

$$A_z^{(i)} = \frac{E_o}{i\omega} e^{-iky} = \frac{E_o}{i\omega} e^{-ik\rho\sin\varphi} \tag{24}$$

We expand the exponential in equation (24) in a standard Fourier-Bessel series[2] to obtain for the incident wave.

$$A_z^{(i)} = \frac{E_0}{i\omega}\left[J_0(k\rho) + 2\sum_{n=2,4,\ldots}^{\infty} J_n(k\rho)\cos n\varphi - 2i\sum_{n=1,3,\ldots}^{\infty} J_n(k\rho)\sin n\varphi\right] \tag{25}$$

To obtain the reflected wave vector potential we must solve the scalar Helmholtz equation,

$$\nabla^2 A_z^{(r)} + k^2 A_z^{(r)} = 0$$

which separates in circular cylindrical coordinates under the assumption

$$A_z^{(r)} = a_1(\rho)\, a_2(\varphi)$$

to yield the pair of ordinary differential equations.

$$\frac{1}{a_1(\rho)}\rho\frac{d}{d\rho}\left\{\rho\frac{d a_1(\rho)}{d\rho} + k^2\rho^2 a_1(\rho)\right\} = n^2 \tag{26}$$

$$\frac{1}{a_2(\varphi)}\frac{d^2 a_2(\varphi)}{d\varphi^2} = -n^2 \tag{27}$$

where of course, n^2 is the separation constant. The solutions of equations (26) and (27) are well known, and we then have

$$A_z^{(r)} = [B_{10}J_0(k\rho) + B_{20}N_0(k\rho)](C_1\varphi + C_2) + \sum_{n=1}^{\infty}[B_{1n}J_n(k\rho) + B_{2n}N_n(k\rho)][C_{1n}\cos n\varphi + C_{2n}\sin n\varphi] \tag{28}$$

Combining equations (25) and (28) as indicated in equation (23) we will have the general solution for the vector potential in region (1). This general solution must be periodic in the following sense

$$A_z^{(1)}(\rho,\varphi) = A_z^{(1)}(\rho,\varphi + 2\pi)$$

For this to hold we must have $C_2 \equiv 0$. We can then write

$$A_z^{(1)} = \frac{E_0}{i\omega}\left[J_0(k\rho) + \sum_{n=2,4,\ldots}^{\infty} J_n(k\rho)\cos n\varphi - 2i\sum_{n=1,3,\ldots}^{\infty} J_n(k\rho)\sin n\varphi\right] + \sum_{n=0}^{\infty}[B_{1n}J_n(k\rho) + B_{2n}N_n(k\rho)]C_{1n}\cos n\varphi + \sum_{n=1}^{\infty}[B_{1n}J_n(k\rho) + B_{2n}N_n(k\rho)]C_{2n}\sin n\varphi \tag{29}$$

when we have defined $C_{10} \equiv C_1$ We have maintained the constants

C_{1n} and B_{1n} uncombined merely for convenience in applying the boundary conditions. Applying the boundary conditions in equations (19) we find we must have

$$C_{1n} \equiv 0 \qquad \text{for } n = 1,3,5,\ldots \tag{30}$$

and also

$$\frac{E_0}{i\omega}\left[J_0(k\rho) + 2\sum_{n=2,4,\ldots}^{\infty} J_n(k\rho)\right] + \sum_{n=0,2,4,\ldots}^{\infty} C_{1n}\left[B_{1n}J_n(k\rho) + B_{2n}N_n(k\rho)\right] = 0 \tag{31}$$

From equations (31) we find that we must have

$$B_{2n} \equiv 0 \qquad \text{for } n = 0,2,4,\ldots \tag{32}$$

Taking this into account we find

$$C_{10}\, B_{10} = -\,E_{zo}/i\omega \tag{33}$$

$$C_{1n}\, B_{1n} = -\,2E_{zo}/i\omega \qquad \text{for } n = 2,4,6,\ \ldots \tag{34}$$

Using equations (30), (32), (33) and (34) equation (28) becomes

$$A_z^{(1)} = -\frac{2E_0}{\omega}\sum_{n=1,3,\ldots}^{\infty} J_n(k\rho)\sin n\varphi + \sum_{n=1}^{\infty}\left[B_{1n}J_n(k\rho) + B_{2n}N_n(k\rho)\right]\times$$
$$\times\, C_{2n}\sin n\varphi \equiv A_z^{(i)} + A_z^{(r)} \tag{35}$$

where each mode of the reflected wave vector potential must consist of only an outgoing wave. For the n-th mode this requires that at large distances from the slot in the conducting plane the following relation must hold

$$\lim_{k\rho\to\infty}\left[B_{1n}J_n(k\rho) + B_{2n}N_n(k\rho)\right] \longrightarrow \frac{e^{ik\rho}}{\sqrt{2\pi k\rho}} \tag{36}$$

Since the Hankel function of the first kind exhibits this asymptotic property,

$$H_n^{(1)}(k\rho) \equiv J_n(k\rho) + i\,N_n(k\rho) \xrightarrow[k\rho\to\infty]{} \frac{e^{ik\rho}}{\sqrt{2\pi k\rho}} \tag{37}$$

we can conclude that

$$B_{2n} = i\,B_{1n} \qquad \text{for } n = 2,\ 4,\ 6,\ \ldots \tag{39}$$

Defining the composite constants

$$A_n \equiv C_{2n} B_1^n \tag{40}$$

we can now write our vector potential in region (1) as

$$A_z^{(1)}(\rho,\varphi) = -\frac{2E_0}{\omega}\sum_{n=1,3,\dots}^{\infty} J_n(k\rho)\sin n\varphi + \sum_{n=1,3,\dots}^{\infty} A_n H_n^{(1)}(k\rho)\sin n\varphi \tag{41}$$

IV. SOLUTION BELOW THE SLOTTED PLANE AND OUTSIDE THE KADEN CYLINDER

In region 3 we have

$$\vec{E}^{(3)}(\vec{r},t) = i\omega A_z^{(3)}\vec{e}_z e^{-i\omega t} \tag{42}$$

From considerations of the current density on the conducting plane it has been shown by others [3,4] that we must have, by reason of symmetry

$$\vec{n}\times\vec{E}^{(r)}(\vec{r}\,') = \vec{n}\times\vec{E}^{(3)}(\vec{r}) \tag{43}$$

where $\vec{n}\equiv\vec{e}_y$ is the outward directed unit vector to the conducting plane in region (1), and $\vec{r}\,'$ is the mirror image point of $\vec{r}$ (see Fig.3) we then have

$$i\omega e^{-i\omega t}(\vec{e}_y\times\vec{e}_z)A_z^{(r)}(\vec{r}\,') = i\omega e^{-i\omega t}(\vec{e}_y\times\vec{e}_z)A_z^{(3)}(\vec{r})$$

which simplifies to

$$A_z^{(3)}(\vec{r}) = A_z^{(r)}(\vec{r}\,') \tag{44}$$

or more explicitly we have

$$A_z^{(3)}(\rho,\varphi) = A_z^{(r)}(\rho,-\varphi) \tag{45}$$

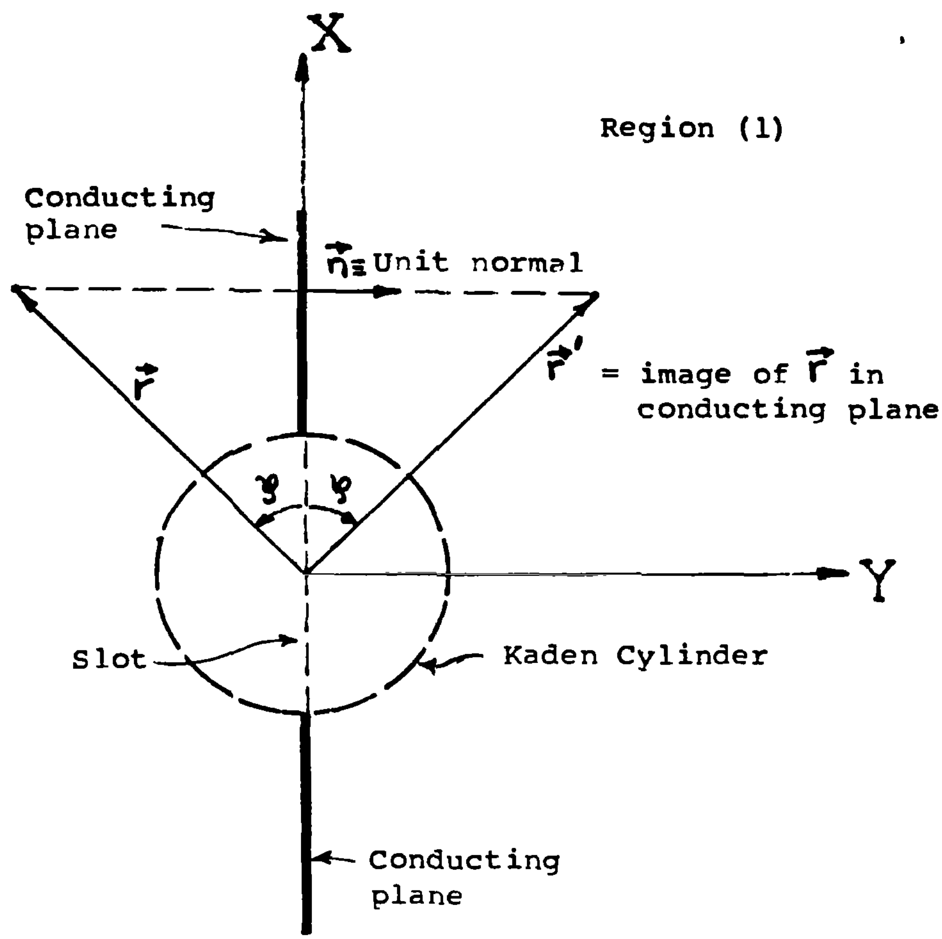

Figure 3. Cross-Section of Slotted Conducting Plane Showing Image Points in this Plane

By equation (41) we have more explicitly

$$A_z^{(3)}(\rho,\varphi) = -\sum_{n=1,3,..}^{\infty} A_n H_n^{(1)}(k\rho) \sin n\varphi \tag{46}$$

Note that this result satisfies the boundary conditions stated in equations (21) and furthermore is properly periodic in φ i.e.

$$A_z^{(3)}(\rho,\varphi) = A_z^{(3)}(\rho,\varphi+2\pi)$$

Note also that this solution behaves properly at large distances from the slot i.e. each mode is merely an outgoing wave as required by the radiation condition.

Before continuing on to the solution in region (2) we state here that by introducing the two functions

$$g_n(\varphi) \equiv \begin{cases} \sin n\varphi & 0<\varphi<\pi \\ -\sin n\varphi & \pi<\varphi<2\pi \end{cases} \tag{47}$$

and

$$h_n(\varphi) \equiv \begin{cases} \sin n\varphi & 0<\varphi<\pi \\ 0 & \pi<\varphi<2\pi \end{cases} \tag{48}$$

we can combine the information in equations (41) and (46) into one equation which gives the vector potential everywhere outside of the "slot cylinder"; This expression is

$$\begin{aligned} A_z^{(1,3)}(\rho,\varphi) &= -\frac{2E_0}{\omega}\sum_{n=1,3,\ldots}^{\infty} J_n(k\rho)\, h_n(\varphi) + \sum_{n=1,3,\ldots}^{\infty} A_n H_n^{(1)}(k\rho)\, g_n(\varphi) \\ &= -\frac{E_0}{\omega}\sum_{n=1,3,..}^{\infty} J_n(k\rho)\sin n\varphi - \frac{E_0}{\omega}\sum_{n=1,3,..}^{\infty}\ \sum_{m=0,2,\ldots}^{\infty} C_m^n J_n(k\rho)\cos m\varphi\ + \\ &\quad + \sum_{n=1,3,\ldots}^{\infty}\ \sum_{m=0,2,\ldots}^{\infty} A_n C_m^n H_n^{(1)}(k\rho)\cos m\varphi \end{aligned} \tag{49}$$

where

$$C_m^n \equiv \begin{cases} \frac{2}{n\pi} & \text{for } m=0 \; ; \; n=1,3,5,\ldots \\ \frac{2}{\pi}\left[\frac{1}{(n-m)} + \frac{1}{(n+m)}\right] & \text{for } \begin{array}{l} m=2,4,6,\ldots \\ n=1,3,5,\ldots \end{array} \end{cases} \tag{50}$$

This relation is easily derived from Fourier series representations of g_n.

V. SOLUTION INSIDE THE KADEN CYLINDER

Inside the Kaden cylinder, which constitutes region (2), the fields must remain finite as one approaches the axis, $\rho = 0$. We can consider the potential in this region as a linear superposition of a contribution from the incident wave and one scattered in from the conducting plane. For the contribution from the incident wave we see from equation (49) and the boundary conditions at the cylinder that this quantity should be

$$-\frac{E_o}{\omega}\sum_{n=1,3,\ldots}^{\infty} J_n(k\rho)\sin n\varphi$$

after taking into account symmetry and the non-singular nature of the vector potential at $\rho=0$ we see that the contribution scattered in by the conducting plane should be of the form

$$\sum_{m=0,2,4,\ldots}^{\infty} B_m J_m(k\rho)\cos m\varphi$$

Thus in region 2 the general solution for the potential is

$$A_z^{(2)} = -\frac{E_o}{\omega}\sum_{n=1,3,\ldots}^{\infty} J_n(k\rho)\sin n\varphi + \sum_{m=0,2,\ldots}^{\infty} B_m J_m(k\rho)\cos m\varphi \tag{51}$$

VI. FORMAL DETERMINATION OF THE FIELDS

Expressions for the vector potentials are given in equations (49) and (51). These expressions still contain the undetermined constant coefficients i.e. the A_n's and the B_m's. The conventional technique for finding these coefficients is to match the fields across the Kaden cylinder boundary where all components must be continuous. We thus have the following conditions at the slot cylinder:

$$E_z^{(1,3)}(\rho_0, \varphi) = E_z^{(2)}(\rho_0, \varphi) \tag{52}$$

$$B_\rho^{(1,3)}(\rho_0, \varphi) = B_\rho^{(2)}(\rho_0, \varphi) \tag{53}$$

$$B_\varphi^{(1,3)}(\rho_0, \varphi) = B_\varphi^{(2)}(\rho_0, \varphi) \tag{54}$$

from equations (52) and (53) we have respectively

$$A_z^{(1,3)}(\rho_0, \varphi) = A_z^{(2)}(\rho_0, \varphi) \tag{55a}$$

$$\left.\frac{\partial A_z^{(1,3)}}{\partial \varphi}\right|_{\rho=\rho_0} = \left.\frac{\partial A_z^{(2)}}{\partial \varphi}\right|_{\rho=\rho_0} \tag{55b}$$

We can write these in terms of the explicit series representations as the single expression.

$$-\frac{E_0}{\omega}\sum_{n=1,3,\ldots}^{\infty} J_n(k\rho_0)\begin{Bmatrix}\sin n\varphi \\ n\cos n\varphi\end{Bmatrix} - \frac{E_0}{\omega}\sum_{m=0,2,\ldots}^{\infty}\sum_{n=1,3,\ldots}^{\infty} C_m^n J_n(k\rho_0)\begin{Bmatrix}\cos m\varphi \\ -m\sin m\varphi\end{Bmatrix}$$

$$+\sum_{m=0,2,\ldots}^{\infty}\sum_{n=1,3,\ldots}^{\infty} A_n C_m^n H_n^{(1)}(k\rho_0)\begin{Bmatrix}\cos m\varphi \\ -m\sin m\varphi\end{Bmatrix} = \sum_{m=0,2,\ldots}^{\infty} B_m J_m(k\rho_0) \times \tag{56}$$

$$\times\begin{Bmatrix}\cos m\varphi \\ -m\sin m\varphi\end{Bmatrix} - \frac{E_0}{\omega}\sum_{n=1,3,\ldots}^{\infty} J_n(k\rho_0)\begin{Bmatrix}\sin n\varphi \\ n\cos n\varphi\end{Bmatrix}.$$

This relation can be rewritten in the more useful form

$$\sum_{m=0,2,\ldots}^{\infty} \left[-\frac{E_o}{\omega} \sum_{n=1,3,\ldots}^{\infty} C_m^n J_n(k\rho_o) + \sum_{n=1,3,\ldots}^{\infty} A_n C_m^n H_n^{(1)}(k\rho_o) \right] \left\{ \begin{matrix} \cos m\varphi \\ -m \sin m\varphi \end{matrix} \right\} =$$

$$= \sum_{m=0,2,\ldots}^{\infty} B_m J_m(k\rho_o) \left\{ \begin{matrix} \cos m\varphi \\ -m \sin m\varphi \end{matrix} \right\} \tag{57}$$

From this we see that only the single following condition is obtained from both boundary conditions of eqs. (52) and (53) :

$$-\frac{E_o}{\omega} \sum_{n=1,3,\ldots}^{\infty} C_m^n J_n(k\rho_o) + \sum_{n=1,3,\ldots}^{\infty} A_n C_m^n H_n^{(1)}(k\rho_o) = B_m J_m(k\rho_o)$$

for m=o, 2, 4, ...

(58)

Further information can be obtained from the remaining boundary condition, equation (54). This condition requires

$$\left. \frac{\partial A_z^{(1,3)}}{\partial \rho} \right|_{\rho=\rho_o} = \left. \frac{\partial A_z^{(2)}}{\partial \rho} \right|_{\rho=\rho_o}$$

or equivalently,

$$\sum_{m=0,2,\ldots}^{\infty} \left[-\frac{E_o}{\omega} \sum_{n=1,3,\ldots}^{\infty} C_m^n \left. \frac{\partial J_n(k\rho)}{\partial \rho} \right|_{\rho=\rho_o} + \sum_{n=1,3,}^{\infty} A_n C_m^n \left. \frac{\partial H_n^{(1)}(k\rho)}{\partial \rho} \right|_{\rho=\rho_o} \right] \cos m\varphi$$

$$= \sum_{m=0,2,\ldots}^{\infty} B_m \left. \frac{\partial J_m(k\rho)}{\partial \rho} \right|_{\rho=\rho_o} \cos m\varphi \tag{59}$$

From this we obtain a second condition relating the A_n's and B_m's

$$-\frac{E_o}{\omega} \sum_{n=1,3,\ldots}^{\infty} C_m^n J_n'(k\rho_o) + \sum_{n=1,3,\ldots}^{\infty} A_n C_m^n H_n^{(1)\prime}(k\rho_o) = B_m J_m'(k\rho_o) \tag{60}$$

where we use the notation of a prime to denote differentiation of a function with respect to the argument of that function.

We now have in equations (58) and (60) two sets of equations which, in principle, completely define the coefficients A_n and B_m. In practice, however, it is in general an intractable task to explicitly evaluate these coefficients in closed form. One situation in which such a solution can actually be effected is the limiting case where

$$k\rho_o << 1 \tag{61}$$

We shall consider this special case next.

VII. THE LONG WAVELENGTH APPROXIMATION

In the long wavelength approximation we are in effect assuming the wavelength of the incident radiation is very large compared to the width of the slot in the conducting plane. This is essentially equivalent to having static fields near the slot.

To determine the A_n and B_m as they are given by equations (58) and (60) for the limiting situation of equation (61) we require the small argument forms of the pertinent Bessel and Hankel functions. These are[5]

$$J_\nu(k\rho_o) \approx (k\rho_o/2)^\nu / \Gamma(\nu+1) \qquad \nu = 0, 1, 2, \ldots \tag{62a}$$

$$H_\nu^{(1)}(k\rho_o) \approx (-i/\pi)\,\Gamma(\nu)\,(k\rho_o/2)^{-\nu} \qquad \nu = 1, 3, 5, \ldots \tag{62b}$$

From the recursion relation for the ordinary Bessel function:

$$J_\nu'(k\rho_o) = -J_{\nu+1}(k\rho_o) + (\nu/k\rho_o)J_\nu(k\rho_o) \tag{62c}$$

we obtain also

$$J_o'(k\rho_o) \approx -(k\rho_o/2) \tag{63a}$$

$$J_\nu'(k\rho_o) \approx \left(\frac{\nu}{k\rho_o}\right)\frac{(k\rho_o/2)^\nu}{\Gamma(\nu+1)} \qquad \nu = 1, 2, 3, \ldots \tag{63b}$$

Similarly from the recursion function

$$H_{\nu}^{(1)\prime}(k\rho_o) = -H_{\nu+1}^{(1)}(k\rho_o) + (\nu/k\rho_o)H_{\nu}^{(1)}(k\rho_o)$$

we find

$$H_{\nu}^{(1)\prime}(k\rho_o) \approx \frac{i\nu}{\pi k\rho_o}\Gamma(\nu)(k\rho_o/2)^{-\nu} \qquad \nu = 1,3,5,\ldots \tag{64}$$

Substituting these Asymptotic forms into equation (58) and (60) we obtain

$$-\frac{E_o}{\omega}\sum_{n=1,3,\ldots}^{\infty} C_m^n \frac{(k\rho_o/2)^n}{\Gamma(n+1)} + \sum_{n=1,3,\ldots}^{\infty} A_n C_m^n\left[\frac{-i}{\pi k\rho_o}\Gamma(n)(k\rho_o/2)^{-n}\right]$$
$$= B_m(k\rho_o/2)^m/\Gamma(m+1) \tag{65}$$

and
$$-\frac{E_o}{\omega}\sum_{n=1,3,\ldots}^{\infty} C_m^n\left[\frac{n}{k\rho_o}\frac{(k\rho_o/2)^n}{\Gamma(n+1)}\right] + \sum_{n=1,3,\ldots}^{\infty} A_n C_m^n\left[\frac{in}{\pi k\rho_o}\Gamma(n)(k\rho_o/2)^{-n}\right]$$
$$= \frac{B_m(k\rho_o/2)^m}{\Gamma(m+1)}\left(\frac{m}{k\rho_o}\right) \tag{66}$$

We can eliminate the B_m's from these and solve the remaining set of equations for all the A_n's. Since these A_n's are actually only obtained for $m \neq o$ we must show that they still hold when $m=o$. We do this next. The asymptotic forms of equations (58) and (56) for $m=o$ are

$$-\frac{E_o}{\omega}\sum_{n=1,3,\ldots}^{\infty} C_o^n (k\rho_o/2)\frac{(k\rho_o/2)^n}{\Gamma(n+1)} - \frac{i}{\pi}\sum_{n=1,3,\ldots}^{\infty} A_n\left(\frac{k\rho_o}{2}\right) C_o^n\,\Gamma(n)(k\rho_o/2)^{-n} \tag{67}$$
$$= B_o\,(k\rho_o/2)$$

$$-\frac{E_o}{\omega}\sum_{n=1,3,\ldots}^{\infty} C_o^n (n/k\rho_o)\frac{(k\rho_o/2)^n}{\Gamma(n+1)} + \frac{i}{\pi}\sum_{n=1,3,\ldots}^{\infty} A_n\left(\frac{n}{k\rho_o}\right) C_o^n\,\Gamma(n)(k\rho_o/2)^{-n} \tag{68}$$
$$= -B_o\,(k\rho_o/2)$$

Now eliminate the B_m for $m \neq o$ from equations (65) and (66) to obtain the set of simultaneous equations for the A_n's:

$$-\frac{E_o}{\omega}\sum_{n=1,3,\ldots}^{\infty}(m-n)\frac{C_m^n(k\rho_o/2)^n}{\Gamma(n+1)} = \frac{i}{\pi}\sum_{n=1,3,\ldots}^{\infty}(m+n)A_n C_m^n\,\Gamma(n)(k\rho_o/2)^{-n} \tag{69}$$

Similarly eliminate B_o from equation (67) and (68) to obtain

$$-\frac{E_o}{\omega}\sum_{n=1,3}^{\infty}\left[\left(\frac{n}{k\rho_o}\right)+\frac{k\rho_o}{2}\right]C_o^n\frac{(k\rho_o/2)^n}{\Gamma(n+1)} = -\frac{i}{\pi}\sum_{n=1,3,}^{\infty}\left(\frac{n}{k\rho_o}-\frac{k\rho_o}{2}\right)C_o^n A_n\Gamma(n)\left(\frac{k\rho_o}{2}\right)^{-n} \tag{70}$$

For even the smallest value of n, namely unity, we have in the long wave-length approximation

$$\frac{n}{k\rho_o} \gg \frac{k\rho_o}{2}$$

Thus equation (70) can be written in this limit.

$$\frac{E_o}{\omega}\sum_{n=1,3,\ldots}^{\infty} n C_o^n \frac{(k\rho_o/2)^n}{\Gamma(n+1)} = \frac{i}{\pi}\sum_{n=1,3,\ldots}^{\infty} n A_n C_o^n \Gamma(n)(k\rho_o/2)^{-n} \tag{71}$$

which is just what we would obtain from equation (69) with m=o. We can then consider the A_n's determined from equation (69) for m = 0, 2, 4, ... Now let us find the A_n's explicitly. Consider the left hand side of eq. (69) first. We observe that using eq. (50)

$$(m-n)C_m^n = (m-n)\left(\frac{2}{\pi}\right)\left[\frac{1}{n-m}+\frac{1}{n+m}\right] = \frac{2}{\pi}\left[\frac{m-n}{m+n}-1\right]$$

then we have

$$\lim_{n\to\infty}\frac{2}{\pi\Gamma(n+1)}\left[\frac{m-n}{m+n}-1\right]\to 0 \qquad \text{for} \quad m=2,4,6,\ldots$$

Similarly for m=o we have

$$-nC_o^n = -n\left(\frac{2}{n\pi}\right) = -\frac{2}{\pi}$$

and

$$\lim_{n\to\infty}\frac{1}{\Gamma(n+1)}\left(-\frac{2}{\pi}\right)\to 0$$

Furthermore we know in the long wave-length approximation

$$k\rho_o \gg (k\rho_o)^3 \gg (k\rho_o)^5 \gg \ldots$$

We can thus write approximately

$$\frac{E_o}{\omega}\sum_{n=1,3,\ldots}^{\infty}(m-n)C_m^n\frac{(k\rho_o/2)^n}{\Gamma(n+1)} \approx \frac{E_o k\rho_o(m-1)}{2\omega}C_m^1 \quad \text{for } m=0,2,4,\ldots \tag{72}$$

Substituting this result into eqs. (69) and (71) we find after some simple algebra

$$1 = \sum_{n=1,3,\ldots}^{\infty} \frac{a_n}{(n+1)} + m \sum_{n=1,3,\ldots}^{\infty} \frac{a_n}{(n-m)} \tag{73}$$

where for convenience we have introduced the new label

$$a_n \equiv \frac{2i(n+1)}{\pi} \frac{\Gamma(n)\,\omega}{E_o k \rho_o} \left(\frac{2}{k\rho_o}\right)^n A_n \tag{74}$$

Now let us relabel the index as follows:

$$n \equiv 2N+1$$

and we obtain for equation (73)

$$1 = \frac{1}{2} \sum_{N=0}^{\infty} \frac{a_{2N+1}}{(N+1)} + \frac{m}{2} \sum_{N=0}^{\infty} \frac{a_{2N+1}}{\left[N + \left(\frac{1-m}{2}\right)\right]} \tag{75}$$

We can use the series definition of the beta-function[6]

$$B(\xi, \tfrac{1}{2}) = \frac{\Gamma(\xi)\Gamma(1/2)}{\Gamma(\xi + 1/2)} = \sum_{N=0}^{\infty} \frac{(2N)!}{2^{2N}(N!)^2} \frac{1}{(N+\xi)} \tag{76}$$

to prove that

$$a_{2N+1} = (2N)!/2^{2N}(N!)^2 \tag{77}$$

Substitute from eq. (77) into eq. (75), as a trial solution, to get on the right hand side

$$\frac{1}{2} \frac{\Gamma(1)\Gamma(1/2)}{\Gamma(1+1/2)} + \frac{m}{2} \frac{\Gamma(\frac{1-m}{2})\Gamma(1/2)}{\Gamma(\frac{1-m}{2} + \frac{1}{2})}$$

The first term is readily evaluated:

$$\frac{1}{2} \frac{\Gamma(1)\Gamma(1/2)}{\Gamma(3/2)} = 1$$

Clearly, the second term vanishes for m=o. For the remaining m values we see also the second term vanishes

$$\frac{m}{2} \frac{\Gamma(\frac{1-m}{2})\Gamma(1/2)}{\Gamma(1-\frac{m}{2})} = 0 \qquad \text{for } m=2,4,\ldots$$

Since $\Gamma(1-\frac{m}{2}) = \Gamma(-\nu) = -\infty$ where $\nu = 0, 1, 2, \ldots$

We conclude then that equation (77) gives the solution for the coefficients a_{2N+1} of equation (75). From this result we then obtain

$$A_n = \frac{E_0 k\rho_0}{\omega} \frac{\pi}{i} \frac{(n-1)!}{2^n\left[\left(\frac{n-1}{2}\right)!\right]^2} \frac{1}{(n+1)\Gamma(n)} \left(\frac{k\rho_0}{2}\right)^n, \quad n = 1,3,5,\ldots \tag{78}$$

Using the result for A_n given in equation (78) we can explicitly state the vector potentials in regions (1) and (3):

$$A_z^{(1)} = -\frac{2E_0}{\omega}\sum_{n=1,3,\ldots}^{\infty} J_n(k\rho)\sin n\varphi + \frac{E_0 k\rho_0}{\omega}\frac{\pi}{i}\sum_{n=1,3,\ldots}^{\infty}\frac{(n-1)!}{2^n\left[\left(\frac{n-1}{2}\right)!\right]^2} \times \tag{79}$$

and

$$\times \frac{1}{(n+1)\Gamma(n)}\left(\frac{k\rho_0}{2}\right)^n H_n^{(1)}(k\rho)\sin n\varphi$$

$$A_z^{(3)} = -\frac{E_0 k\rho_0}{\omega}\left(\frac{\pi}{i}\right)\sum_{n=1,3,\ldots}^{\infty}\frac{(n-1)!}{2^n\left[\left(\frac{n-1}{2}\right)!\right]^2}\frac{1}{(n+1)\Gamma(n)}\left(\frac{k\rho_0}{2}\right)^n \times$$

$$\times H_n^{(1)}(k\rho)\sin n\varphi \tag{80}$$

and by equations (62) we rewrite these in a form more appropriate to the range $\rho-\rho_0 \ll 1$ where we shall be considering the boundary conditions on the slot cylinder.

$$A_z^{(1)} \approx -\frac{E_0}{\omega}k\rho\sin\varphi - \frac{E_0}{\omega}k\rho_0\sum_{n=1,3,\ldots}^{\infty}\frac{(n-1)!}{2^n\left[\left(\frac{n-1}{2}\right)!\right]^2}\frac{1}{(n+1)}\left(\frac{\rho_0}{\rho}\right)^n\sin n\varphi \tag{81}$$

$$A_z^{(3)} \approx \frac{E_0}{\omega}k\rho_0\sum_{n=1,3,\ldots}^{\infty}\frac{(n-1)!}{2^n\left[\left(\frac{n-1}{2}\right)!\right]^2}\frac{1}{(n+1)}\left(\frac{\rho_0}{\rho}\right)^n\sin n\varphi. \tag{82}$$

Using equations (79) and (80) for the vector potential just off the slot cylinder we can completely determine the potential within it by satisfying the boundary conditions as embodied by eqs. (58) and (60). We write these conditions in the appropriate limit as

$$-\frac{E_0}{\omega}C_m^1(k\rho_0/2) + \sum_{n=1,3,\ldots}^{\infty}A_n C_m^n\left[-\frac{i}{\pi}\Gamma(n)\left(\frac{k\rho_0}{2}\right)^{-n}\right] \simeq \tag{83}$$

$$\simeq \frac{B_m}{\Gamma(m+1)}\left(\frac{k\rho_0}{2}\right)^m$$

$$-\frac{E_0}{\omega} C^1_m\left(\frac{k\rho_0}{2}\right) + \sum_{n=1,3,\ldots}^{\infty} n A_n C^n_m \left[\frac{i}{\pi}\Gamma(n)\left(\frac{k\rho_0}{2}\right)^{-n}\right] \simeq$$

$$\simeq \frac{m B_m}{\Gamma(m+1)}\left(\frac{k\rho_0}{2}\right)^m \tag{84}$$

for m=0, 2, 4, ...

solving explicitly for the B_m with the help of equations (73) through (78) we obtain

$$B_m = -\frac{E_0 k\rho_0}{\omega}\,\frac{\Gamma(m+1)}{(1-m)}\,\frac{\Gamma\left(\frac{1+m}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma^2\left(\frac{m+2}{2}\right)}\left(\frac{2}{k\rho_0}\right)^m \tag{85}$$

Substituting this result back into equation (51) we obtain the vector potential within the Kaden cylinder. Since the inside the slot cylinder $\rho<\rho_0$ we have necessarily $k\rho \ll 1$.
Hence we find

$$A_z^{(2)} = -\frac{E_0 k\rho}{2\omega}\sin\varphi - \frac{E_0 k\rho_0}{2\pi\omega}\sum_{m=0,2,\ldots}^{\infty}\frac{\Gamma\left(\frac{1+m}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma^2\left(\frac{m+2}{2}\right)}\left(\frac{\rho}{\rho_0}\right)^m\frac{\cos m\varphi}{(1-m)} \tag{86}$$

Note that since $k/\omega = c$ = speed of light, we conclude that the vector potential we have found within the slot region as well as the corresponding electric and magnetic fields are frequency independent and thereby consistent with the long wavelength approximation. The magnetic fields can be calculated in this approximation and we obtain

$$B_\varphi^{(1)} = -\frac{\partial A_z^{(1)}}{\partial\rho} = \frac{E_0 k\sin\varphi}{\omega} - \frac{E_0 k}{\omega}\sum_{n=1,3,\ldots}^{\infty}\frac{n!}{2^n\left[\left(\frac{n-1}{2}\right)!\right]^2}\frac{1}{(n+1)}\left(\frac{\rho_0}{\rho}\right)^{n+1}\sin n\varphi \tag{87}$$

$$B_\rho^{(1)} = \frac{1}{\rho}\frac{\partial A_z^{(1)}}{\partial\varphi} = \frac{-E_0 k\cos\varphi}{\omega} - \frac{E_0 k}{\omega}\sum_{n=1,3,\ldots}^{\infty}\frac{n!}{2^n\left[\left(\frac{n-1}{2}\right)!\right]^2}\frac{1}{(n+1)}\left(\frac{\rho_0}{\rho}\right)^{n+1}\cos n\varphi \tag{88}$$

$$B_\varphi^{(3)} = -\frac{\partial A_z^{(3)}}{\partial\rho} = \frac{E_0 k}{\omega}\sum_{n=1,3,\ldots}^{\infty}\frac{n!}{2^n\left[\left(\frac{n-1}{2}\right)!\right]^2}\frac{1}{(n+1)}\left(\frac{\rho_0}{\rho}\right)^{n+1}\sin n\varphi \tag{89}$$

$$B_\rho^{(3)} = \frac{1}{\rho}\frac{\partial A_z^{(3)}}{\partial\varphi} = \frac{E_0 k}{\omega}\sum_{n=1,3,\ldots}^{\infty}\frac{n!}{2^n\left[\left(\frac{n-1}{2}\right)!\right]^2}\frac{1}{(n+1)}\left(\frac{\rho_0}{\rho}\right)^{n+1}\cos n\varphi \tag{90}$$

$$B_\varphi^{(2)} = -\frac{\partial A_z^{(2)}}{\partial \rho} = \frac{E_0 k}{2\omega} \sin\varphi - \frac{E_0 k}{2\pi\omega} \sum_{m=0,2,\ldots}^{\infty} \frac{m}{(m-1)} \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)} \left(\frac{\rho}{\rho_0}\right)^{m-1} \cos m\varphi \tag{91}$$

$$B_\rho^{(2)} = \frac{1}{\rho}\frac{\partial A_z^{(2)}}{\partial \varphi} = -\frac{E_0 k}{2\omega} \cos\varphi - \frac{E_0 k}{2\pi\omega} \sum_{m=0,2,\ldots}^{\infty} \frac{m}{(m-1)} \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)} \left(\frac{\rho}{\rho_0}\right)^{m-1} \sin m\varphi \tag{92}$$

These magnetic field expressions coincide as they should with the usual magnetostatic field solutions in each region obtained via the magnetic scalar potential[7] for the case of a uniform applied magnetic field of the form

$$\vec{B}_{\text{applied}}(\vec{r}) = \frac{E_0}{c}\,\vec{e}_x$$

We observe that our solution therefore reduces properly in the long wavelength limit stated in equation (61)

VIII. THE GENERAL SOLUTION BY NUMERICAL APPROXIMATION

Let us now return to the general problem of evaluating the coefficients in eqs. (58) and (60). This shall be effected by an approximation technique which we next discuss. If we use the identity

$$\sum_{n=1,3,5,\ldots}^{\infty} J_n(k\rho)\sin n\varphi \equiv \frac{1}{2}\sin(k\rho\sin\varphi) \tag{93}$$

we can rewrite the vector potentials of eqs. (49) and (51) as

$$A_z^{(1)}(\rho,\varphi) = -\frac{E_0}{\omega}\sin(k\rho\sin\varphi) + \sum_{n=1,3,\ldots}^{\infty} A_n H_n^{(1)}(k\rho)\sin n\varphi\,, \quad \rho > \rho_0,\ 0 < \varphi < \pi \tag{94}$$

$$A_z^{(2)}(\rho,\varphi) = -\frac{E_0}{2\omega}\sin(k\rho\sin\varphi) + \sum_{m=0,2,\ldots}^{\infty} B_m J_m(k\rho)\cos m\varphi\,, \quad \rho < \rho_0,\ 0 < \varphi < 2\pi \tag{95}$$

$$A_z^{(3)}(\rho,\varphi) = -\sum_{n=1,3,\dots}^{\infty} A_n H_n^{(1)}(k\rho)\sin n\varphi, \qquad \rho>\rho_0, \quad \pi<\varphi<2\pi \tag{96}$$

Consider the solutions for regions (2) and (3) first. At $\rho=\rho_0$ we can match the field expressions for each region at any particular value of φ, say φ_ℓ, where of course

$$\pi < \varphi_\ell < 2\pi$$

Thus at φ_ℓ we would have one boundary condition in the form

$$E_z^{(2)}(\rho_0,\varphi_\ell) = E_z^{(3)}(\rho_0,\varphi_\ell) \qquad \pi<\varphi_\ell<2\pi \tag{97}$$

i.e. evaluated at $\varphi=\varphi_\ell$ where more explicitly

$$E_z^{(2)}(\rho_0,\varphi_\ell) = i\omega\left\{\left(-\frac{E_0}{2\omega}\right)\sin(k\rho_0\sin\varphi_\ell) + \sum_{m=0,2,\dots}^{\infty} B_m J_m(k\rho_0)\cos m\varphi_\ell\right\}$$

and

$$E_z^{(3)}(\rho_0,\varphi_\ell) = i\omega\left\{-\sum_{n=1,3,\dots}^{\infty} A_n H_n^{(1)}(k\rho_0)\sin n\varphi_\ell\right\}$$

We can then rewrite eq. (97) as the single equation in the doubly infinite set of unknowns A_n, B_m;

$$\sum_{n=1,3,\dots}^{\infty} A_n H_n^{(1)}(k\rho_0)\sin n\varphi_\ell + \sum_{m=0,2,\dots}^{\infty} B_m J_m(k\rho_0)\cos m\varphi_\ell = \frac{E_0}{2\omega}\sin(k\rho_0\sin\varphi_\ell) \tag{98}$$

Let's define some new notation for convenience

$$\eta \equiv k\rho_0 \tag{99}$$

$$a_n \equiv \frac{2c}{E_0\rho_0} A_n \tag{100}$$

$$b_m \equiv \frac{2c}{E_0\rho_0} B_m \tag{101}$$

We then can write eq. (98) as

$$\sum_{n=1,3,\dots}^{\infty} a_n H_n^{(1)}(\eta)\sin n\varphi_\ell + \sum_{m=0,2,\dots}^{\infty} b_m J_m(\eta)\cos m\varphi_\ell = \frac{\sin(\eta\sin\varphi_\ell)}{\eta} \tag{102}$$

Next we have the boundary condition

$$B_\rho^{(2)}(\rho_0,\varphi_\ell) = B_\rho^{(3)}(\rho_0,\varphi_\ell) \tag{103}$$

which can be readily shown to be

$$\sum_{n=1,3,\dots}^{\infty} a_n H_n^{(1)}(\eta)\, n \cos n\varphi_\ell - \sum_{m=0,2,\dots}^{\infty} b_m J_m(\eta)\, m \sin m\varphi_\ell = \tag{104}$$
$$= \cos\varphi_\ell \cos(\eta \sin\varphi_\ell)$$

which is exactly coincident with the derivative of eq. (102) with respect to φ_ℓ. Finally we have the boundary condition

$$B_\varphi^{(2)}(\rho_0, \varphi_\ell) = B_\varphi^{(3)}(\rho_0, \varphi_\ell) \tag{105}$$

which in the same manner yields the equation

$$\sum_{n=1,3,\dots}^{\infty} a_n H_n^{(1)\prime}(\eta) \sin n\varphi_\ell + \sum_{m=0,2,\dots}^{\infty} b_m J_m'(\eta) \cos m\varphi_\ell = \tag{106}$$
$$= \frac{\sin\varphi_\ell \cos(\eta \sin\varphi_\ell)}{\eta}$$

We note at this point that eqs. (102) and (106) are, for given φ_ℓ, independent equations in the undetermined constants a_n and b_m. By truncating these sets of constants and choosing an appropriate number of discrete values of φ we can obtain a finite set of simultaneous equations in a finite set of unknown a_n's and b_m's. These, when solved, will give us approximate values for the unknowns. This approximation technique has been shown earlier by one of us (L.F.L.) to yield good results[8] when applied to problems for which the exact answers are known. Thus to numerically evaluate the a_n and b_m let us arbitrarily terminate the series in equations (102) and (106) at $n = N$ and $m=M$. We shall then have the pair of equations

$$\sum_{n=1,3,\dots}^{N} a_n H_n^{(1)}(\eta) \sin n\varphi_\ell + \sum_{m=0,2,\dots}^{M} b_m J_m(\eta) \cos m\varphi_\ell = \frac{\sin(\eta \sin\varphi_\ell)}{\eta} \tag{107}$$

$$\sum_{n=1,3,\dots}^{N} a_n H_n^{(1)\prime}(\eta) \sin n\varphi_\ell + \sum_{m=0,2,\dots}^{M} b_m J_m'(\eta) \cos m\varphi_\ell = \frac{\sin\varphi_\ell \cos(\eta \sin\varphi_\ell)}{\eta} \tag{108}$$

Let us furthermore choose N and M such that

$$N-M = 1$$

Then we have N + 1 unknown coefficients, in two equations for a given value of φ_ℓ. If we repeat this procedure by choosing (N+1)/2 distinct values of φ we shall have a set of N+1 simultaneous equations in N+1 unknowns. This set of equations can be easily solved. Let us further introduce some notation so as to make the mathematical problem more compact. We define the index ν where

$$\nu \equiv \begin{cases} n & \nu = n = 1,3,5,\ldots \\ m & \nu = m = 0,2,4,\ldots \end{cases} \tag{110}$$

this results in ν running consecutively from $\nu = 0$ to $\nu = M+1 = N$. We also define the following set of quantities

$$c_{\ell\nu} \equiv \begin{cases} H_n^{(1)}(\eta) \sin n\varphi_\ell & \text{for} \quad \nu = n = 1,3,5\ldots \\ J_m(\eta) \cos m\varphi_\ell & \text{for} \quad \nu = m = 0,2,4,\ldots \end{cases} \tag{111}$$

$$d_{\ell\nu} \equiv \begin{cases} H_n^{(1)\prime}(\eta) \sin n\varphi_\ell & \text{for} \quad \nu = n = 1,3,5,\ldots \\ J_m'(\eta) \cos m\varphi_\ell & \text{for} \quad \nu = m = 0,2,4,\ldots \end{cases} \tag{112}$$

$$P_\ell \equiv \frac{\sin(\eta \sin\varphi_\ell)}{\eta} \tag{113}$$

$$Q_\ell \equiv \frac{\sin\varphi_\ell \cos(\eta \sin\varphi_\ell)}{\eta} \tag{114}$$

and relabel the coefficients in the following manner

$$X_\nu \equiv \begin{cases} a_n & \nu = n = 1,3,5,\ldots \\ b_m & \nu = m = 0,2,4,\ldots \end{cases} \tag{115}$$

In this more convenient notation equations (107) and (108) become respectively

$$\sum_{\nu=0}^{N} c_{\ell\nu} X_\nu = P_\ell \qquad \text{for} \quad \ell = 1,2,\ldots,\tfrac{1}{2}(N+1) \tag{116}$$

$$\sum_{\nu=0}^{N} d_{\ell\nu} X_\nu = Q_\ell \tag{117}$$

These constitute N+1 equations in N+1 unknows and by Cramers rule we can solve for the X_ν. Thus let's write

$$\Delta^{(N+1)} \equiv \begin{vmatrix} C_{10}; C_{11} & ; & \cdots & C_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ C_{(N+1)/2,0}; C_{(N+1)/2,1} & \cdots & & C_{(N+1)/2,N} \\ d_{10}; \; d_{11} & & \cdots & d_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ d_{(N+1)/2,0}; d_{(N+1)/2,1}; & \cdots & & d_{(N+1)/2,N} \end{vmatrix} \tag{118}$$

and

$$\Delta_\mu^{(N+1)} \equiv \begin{vmatrix} C_{10} ; & C_{11} ; & \cdots & C_{1,\mu-1} ; & P_1 ; & C_{1,\mu+1} ; & \cdots ; & C_{1N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ C_{\frac{N+1}{2},0} ; & C_{\frac{N+1}{2},1} ; & \cdots & C_{\frac{N+1}{2},\mu-1} ; & P_{\frac{N+1}{2}} ; & C_{\frac{N+1}{2},\mu+1} ; & \cdots & C_{\frac{N+1}{2},N} \\ d_{10} ; & d_{11} ; & \cdots & d_{1,\mu-1} ; & Q_1 ; & d_{1,\mu+1} ; & \cdots & d_{1,N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{\frac{N+1}{2},0} ; & d_{\frac{N+1}{2},1} ; & \cdots & d_{\frac{N+1}{2},\mu-1} ; & Q_{\frac{N+1}{2}} ; & d_{\frac{N+1}{2},\mu+1} ; & \cdots & d_{\frac{N+1}{2},N} \end{vmatrix} \tag{119}$$

In then follows, of course, that in our approximation

$$X_\nu \cong \frac{\Delta_\nu^{(N+1)}}{\Delta^{(N+1)}} \equiv X_\nu^{(N+1)} \qquad \text{for } \nu = 0, 1, 2, \ldots, N \tag{120}$$

when the super script "N+1" indicates the order of approximation. At this stage of the development it should be evident that we can, in principle at least, find X_ν and thereby the corresponding a_n or b_m to any order of approximation merely by choosing N sufficiently large. What we are doing actually is applying the boundary conditions along generators of the Kaden cylinder. For given k and ρ_o we expect that in the exact series solution the leading few coefficients A_n and B_m are the only ones that are significant. The remaining ones of course, will be quite small and hence negligible. The precise number of coefficients that will be significant will depend on the values of k and ρo.

Larger values of the product $k\rho_o$ will obviously require more coefficients than would be necessary for a smaller $k\rho_o$ value. We therefore expect that by choosing N sufficiently large for given $k\rho_o$ we can use our approximation scheme to determine the significant coefficients quite accurately. The corresponding higher order coefficients will be in considerable error. Since they are negligible anyway we should be able to numerically solve quite accurately for the fields. Results obtained via this approach for the slotted plane are presented below. In practice we have found the only limitation on the accuracy of our calculations to be the limitations of the available computer subroutines.

Let us interrupt our development to introduce and briefly discuss the transmission cross-section for our slot problem.

A. THE TRANSMISSION CROSS-SECTION PER UNIT LENGTH.

The transmission cross-section per unit length of the slot, σ_t is defined by

$$\sigma_t \equiv P_t \,/\, S_i \tag{121}$$

where

P_t ≡ total power per unit length transmitted through the slot.

S_i ≡ magnitude of the real part of the incident complex Poynting vector $\vec{\mathcal{S}}_i$ at the scatterer = $|\mathcal{R}e\,\frac{1}{2}\vec{E}_i \times \vec{H}_i^{\,*}| = |\mathcal{R}e\,\vec{\mathcal{S}}_i|$. Recalling from eq. (6) that the electric field in the incident plane wave is

$$\vec{E}_i = \vec{e}_z E_o \exp\{i\vec{k}_i\cdot\vec{r} - i\omega t\}$$

then

$$\vec{H}_i = \frac{1}{i\omega\mu_o}\vec{\nabla}\times\vec{E}_i$$

and

$$\vec{H}_i^{\,*} = -\frac{1}{i\omega\mu_o}\vec{\nabla}\times\vec{E}_i^{\,*}$$

which gives

$$\vec{H}_i^{\,*} = -\frac{E_o}{\omega\mu_o}\, e^{-i\vec{k}_i\cdot\vec{r} + i\omega t}\, \vec{e}_z \times \vec{k}_i \tag{122}$$

from this we get

$$S_i = |\mathcal{R}e\,\frac{1}{2}\vec{E}_i \times \vec{H}_i^{\,*}| = \frac{E_o^2}{2\mu_o c} \tag{123}$$

The total power, per unit length, transmitted through the slot is

$$\mathcal{P}_t = \int_{\pi}^{2\pi} d\varphi \lim_{\rho\to\infty} (\mathrm{Re}\ \tfrac{1}{2}\ \vec{E}^{(3)} \times \vec{H}^{(3)*}) \cdot \vec{\rho} \tag{124}$$

where

$$\vec{E}^{(3)} = E_z^{(3)}\,\vec{e}_z = -\,\vec{e}_z\, i\omega \sum_{n=1}^{\infty} A_n H_n^{(1)}(k\rho) \sin n\varphi$$

Then

$$\mathcal{P}_t = -\frac{1}{2}\int_{\pi}^{2\pi} d\varphi \left\{ \mathrm{Re} \lim_{\rho\to\infty} \rho\, E_z^{(3)} H_\varphi^{(3)*} \right\} \tag{125}$$

However we have from eq. (15)

$$H_\varphi^{(3)}(\rho,\varphi) = -\frac{1}{i\omega\mu_0}\frac{\partial E_z^{(3)}}{\partial\rho}$$

then we have in the transmitted region (3)

$$H_\varphi^{(3)*}(\rho,\varphi) = \frac{k}{\mu_0}\sum_{n=1}^{\infty} A_n^* H_n^{(2)\prime}(k\rho) \sin n\varphi$$

Collecting terms we have

$$E_z^{(3)} H_\varphi^{(3)*} = \frac{-i\omega k}{\mu_0}\sum_{n,m=1}^{\infty} A_n A_m^* H_n^{(1)}(k\rho) H_m^{(2)\prime}(k\rho) \sin n\varphi \sin m\varphi$$

from which we obtain in the required limit

$$\lim_{\rho\to\infty} \rho\, E_z^{(3)} H_\varphi^{(3)*} = \frac{-i\omega k}{\mu_0}\,\frac{2}{\pi}\sum_{n,m=1}^{\infty} A_n A_m^* e^{i(m-n)\pi/2} \sin n\varphi \sin m\varphi \tag{126}$$

The transmission cross-section can then be written as usual

$$\sigma_t = \frac{\omega c}{E_0^2}\sum_{n=1}^{\infty} |A_n|^2 = \frac{ck}{E_0^2}\sum_{n=1}^{\infty} |A_n|^2 \tag{127}$$

For a normally incident plane wave geometric optics predicts the transmission cross-section per unit length for our slot to be

$$\sigma_t^{\text{optics}} = 2\rho_0 \tag{128}$$

This provides us with a convenient scaling factor to use in normalizing the transmission cross-section. We thus define the "transmission efficiency coefficient" for the slot as

$$T(\eta) \equiv \sigma_t(\eta)/\sigma_t^{\text{optics}} \tag{129}$$

using eqs. (99) and (100) we can write this

$$T(\eta) = \frac{1}{4}\eta \sum_{n=1,3,\dots}^{\infty} |a_n|^2 \tag{130}$$

It is informative to consider eq. (130) in the long-wave-length limit. This we do next.

B. TRANSMISSION CROSS-SECTION IN THE LONG WAVELENGTH APPROXIMATION.

For very long wavelengths $\eta<<1$ and we can write

$$\sigma_t \approx \frac{\omega c}{E_0^2}|A_1|^2 = \frac{\eta \rho_0}{2}|a_1|^2 \tag{131}$$

but

$$|a_1|^2 = \pi^2\eta^2/16$$

Thus in the Rayleigh approximation

$$\sigma_t \approx \rho_0 \frac{\pi^2\eta^3}{32} = \rho_0 \frac{\pi^2(2k\rho_0)^3}{256} \tag{132}$$

Now according to Babinet's principle the scattering cross-section σ, for a conducting strip of width $2\rho_0$, and of infinite length, with the incident magnetic field polarized parallel to the axis of the strip is equal to the scattering cross-section of an infinite slot of width $2\rho_0$ for the incident electric field polarized parallel to the slot axis. This scattering cross-section is simply twice the transmission cross section, i.e.

$$\sigma = 2\sigma_t \approx (2\rho_0)\,\pi^2(2\rho_0 k)^3/256 \tag{133}$$

King and Wu[9] give Bouwkamp's low-frequency result for this slot problem and the results in eq. (133) correspond to the leading term in the Bouwkamp solution.

IX. RESULTS AND DISCUSSION.

We show in Fig. 4 the results obtained for the transmission coefficient, T, using a seven term approximation and also a twelve term approximation in the range $0.8\leq\eta\leq2.0$. When these are compared with the corresponding results obtained by Skavlem[10] and also by Seshadri[11] we find that our results agree very closely with theirs. The results of Morse and Rubenstein[12] are extremely close to those of Seshadri and for convenience we only compare our results for T (η) with those of the latter. Interestingly, we note that the lower order approximation is in closer agreement with the results reported by others in the neighborhood of the first resonance. However, the higher order approximation indicates about 3% difference from the other results and this is not terribly significant, having as its source the numerical techniques used in the computer program. We shall see in the next publication, where we treat this problem as a special case, that a double precision computer calculation with the twelve term approximation rather than the single precision result given here brings our calculated behavior at the resonance into closer agreement with the seven term approximation. It should be pointed out before going on that this latter calculation was actually done in double precision. All further calculated data in this report consists of single precision computer results. Figure 4 also shows the results obtained for a fifteen term approximation at $\eta = 2.2$ and a nineteen term approximation for the range $2.4 \leq \eta \leq 5.6$. As we anticipated there is rather good agreement with the earlier work of others over a considerable range of η. We present in Table 1 the calculated values of the leading coefficients $|a_n|^2$, their sum and the corresponding values for the transmission coefficient $T(\eta)$. From this table we can clearly see how, for small η, $|a_1|^2$ is the predominant contributor to $T(\eta)$. Further as η increases $|a_3|^2$ begins to become significant even though $|a_1|^2$ is still the main contributor and in fact increases with η at first. As η_2 increases even further beyond the value 1.5 we note that $|a_1|^2$, although still the larger contributor, begins to decrease whereas $|a_3|^2$ is still growing. About $\eta = 4.2$ this behavior still persists. Now, however, $|a_1|^2$ and $|a_3|^2$ are comparable in value and, in addition, $|a_5|^2$ is now contributing to $T(\eta)$. For larger η, $|a_3|^2$ begins to exceed $|a_1|^2$ which is still decreasing while at the same time $|a_5|^2$ is increasing For still larger η say about $\eta = 5.0$, $|a_1|^2$ continues its decrease and $|a_3|^2$ starts dropping off, whereas $|a_5|^2$ is increasing and in addition $|a_7|^2$ starts contributing. All of this is consistent with our anticipation that for given η only the first few coefficients will be of practical significance in determining $T(\eta)$, and as η increases succeeding higher order coefficients will become significant. Figure 5 illustrates the dependence on η of the first three coefficients $|a_1|$, $|a_3|$, $|a_5|$ and merely displays graphically the behavior just discussed. We show in Fig. 6, the dependence, separately, of the real and imaginary parts of a_1 on η. This behavior is characteristic of each of the a_n except that the maxima are smaller and they occur at larger values of η, the further out the maxima the larger the index n.

It is worthwhile to point out that we merely chose the matching angle on the Kaden cylinder so as to more or less uniformly cover the cylinder. Obviously a different choice of this set of angles will yield different numerical results. However, by a larger order of approximation calculation these discrepancies can be made insignificant. Although we haven't attempted it, a variational calculation could probably yield the best set of angles to use.

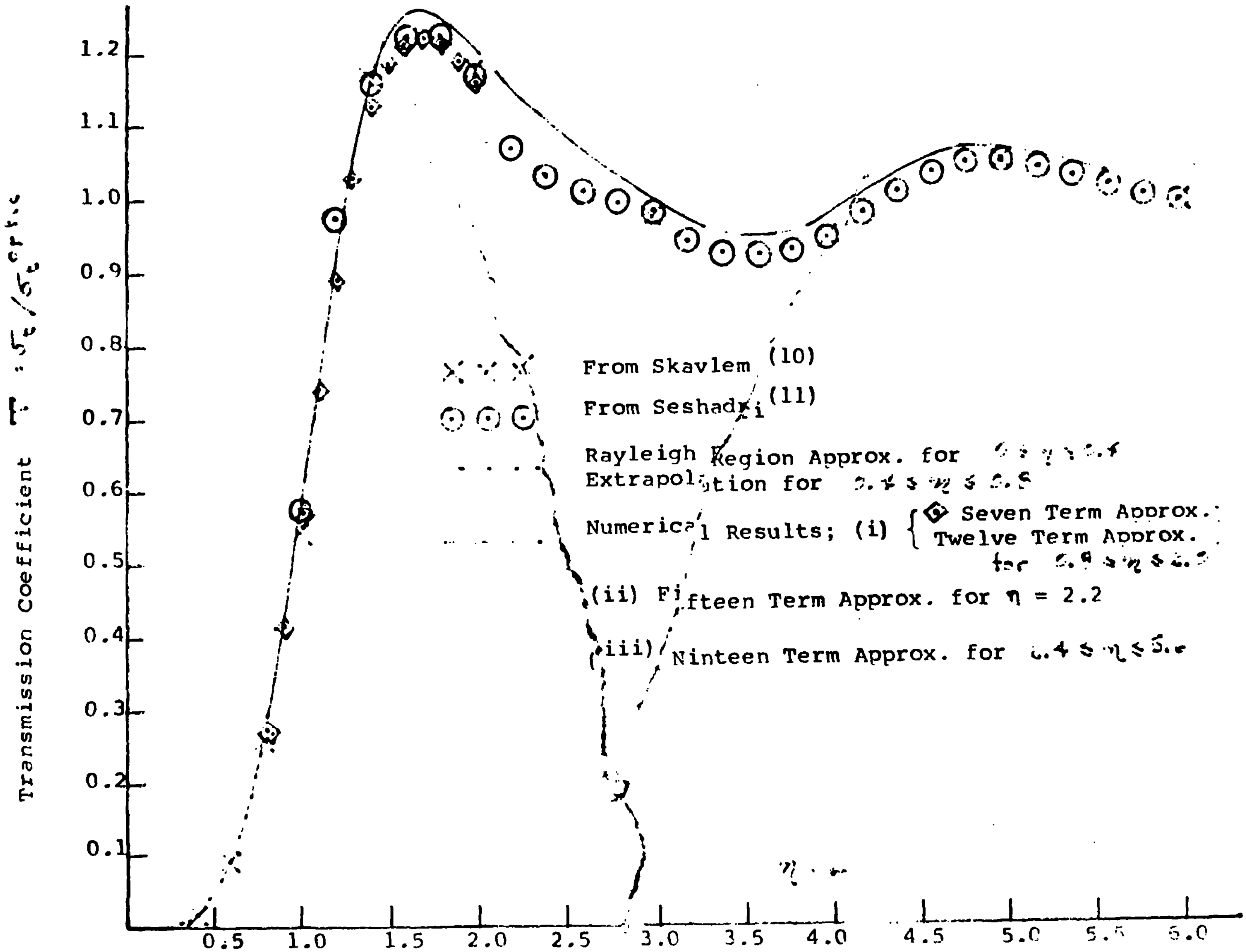

Figure 4. Transmission Coefficient As a Function of η.

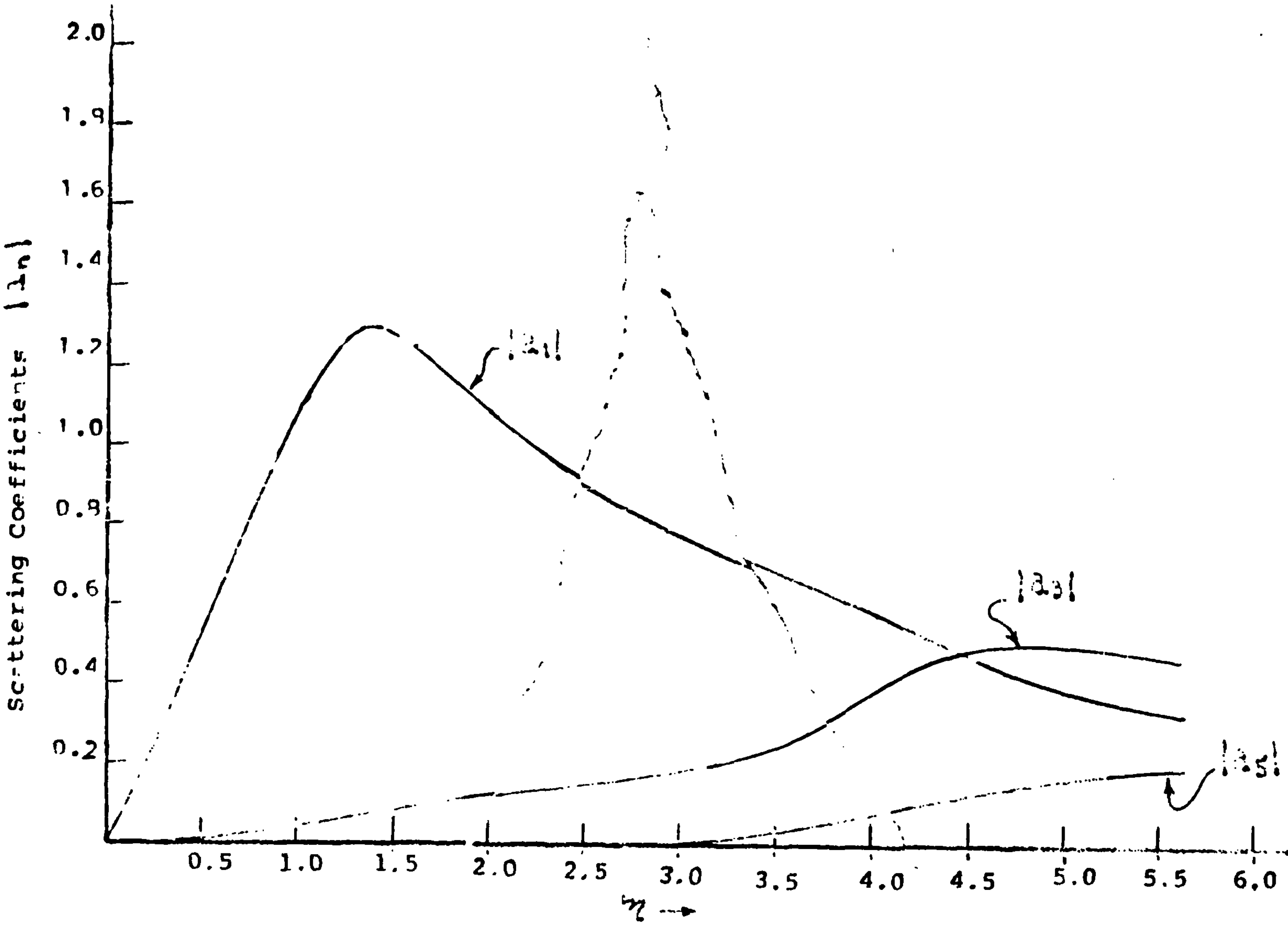

Figure 5. The Leading Scattering Coefficients As A Function of η

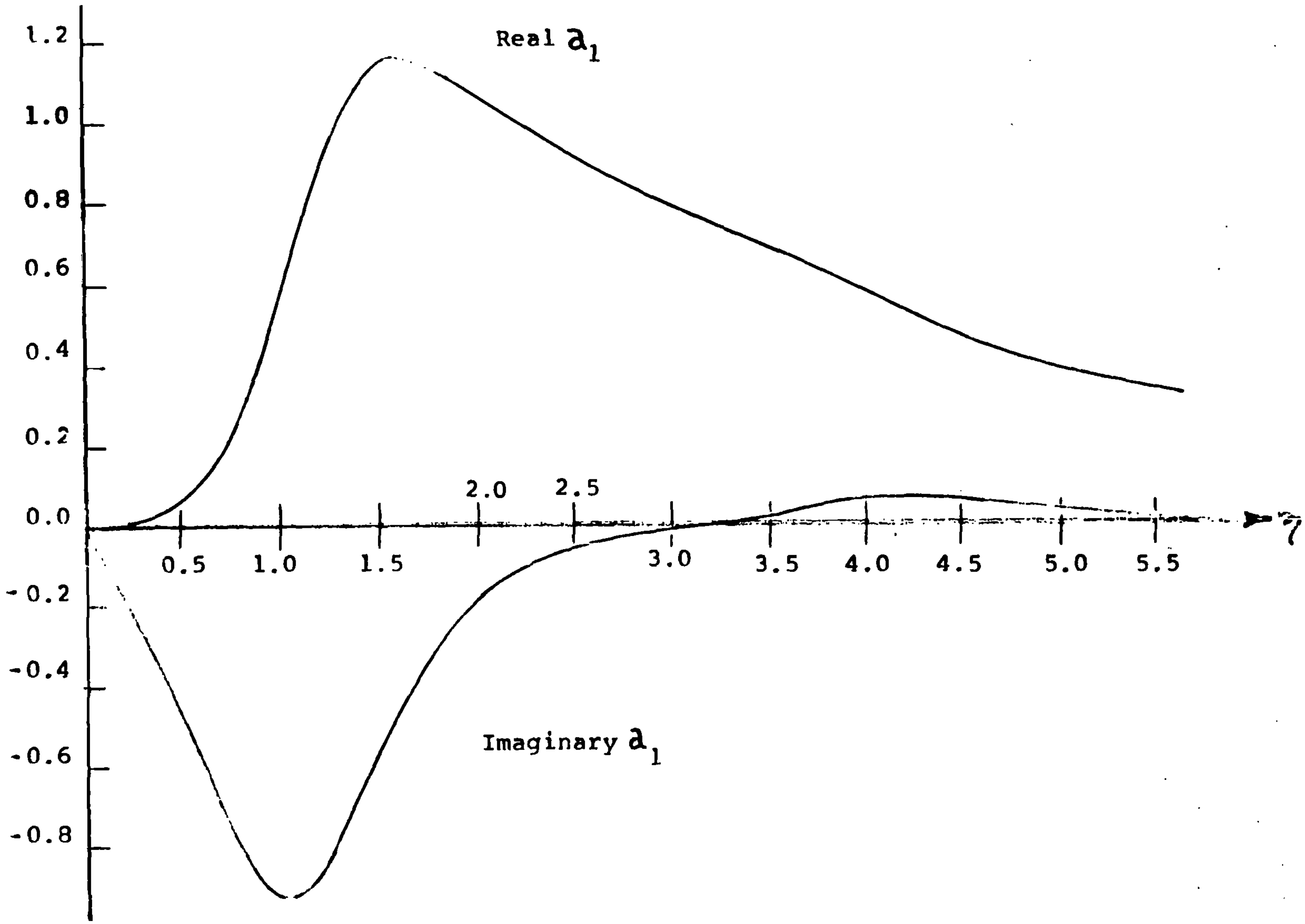

Figure 6. Dependence of Real & Imaginary Parts of a_1 on η

We have seen that the method of approximation used here gives good agreement with the results obtained by others for the relatively simple test problem investigated. In succeeding reports we shall present the results of applying the technique to more difficult scattering problems involving apertures.

η	$\lvert a_1 \rvert^2$	$\lvert a_3 \rvert^2$	$\sum_{n=1,3,5\ldots}^{\infty} \lvert a_n \rvert^2$	$T(\eta)$
0.80	0.7307		0.7307	0.2923
0.90	0.9673		0.9673	0.4353
1.00	1.2012	0.0012	1.2024	0.6012
1.10	1.4137	0.0020	1.4157	0.7786
1.20	1.5700	0.0031	1.5731	0.9439
1.30	1.6564	0.0045	1.6609	1.0796
1.40	1.6718	0.0060	1.6778	1.1745
1.50	1.6333	0.0076	1.6409	1.2307
1.60	1.5600	0.0092	1.5692	1.2554
1.70	1.4714	0.0109	1.4823	1.2600
1.80	1.3759	0.0125	1.3884	1.2496
1.90	1.2792	0.0141	1.2933	1.2286
2.00	1.1903	0.0158	1.2061	1.2061
2.20	1.0262	0.0189	1.0451	1.1496

Table I. The Leading Scattering Coefficients and the Transmission Coefficient

(i) Twelve Term Approximation for $0.8 \leq \eta \leq 2$

(ii) Fifteen Term Approximation at $\eta = 2.2$

(iii) Nineteen Term Approximation for $2.4 \leq \eta \leq 5.6$

Table I (continued)

η	$\lvert a_1\rvert^2$	$\lvert a_3\rvert^2$	$\lvert a_5\rvert^2$	$\lvert a_7\rvert^2$	$\sum_{n=1,3,5,\ldots}^{\infty} \lvert a_n\rvert^2$	$T(\eta)$
2.4	0.9048	0.0201			0.9249	1.1099
2.6	0.7973	0.0245	0.0001		0.8219	1.0685
2.8	0.7085	0.0289	0.0002		0.7376	1.0326
3.0	0.6322	0.0342	0.0003		0.6667	1.0001
3.2	0.5624	0.0428	0.0005		0.6057	0.9691
3.4	0.5047	0.0553	0.0009		0.5609	0.9535
3.6	0.4508	0.0746	0.0018		0.5272	0.9490
3.8	0.3953	0.1021	0.0034		0.5008	0.9515
4.0	0.3429	0.1388	0.0060		0.4877	0.9754
4.2	0.2913	0.1795	0.0098		0.4806	1.0093
4.4	0.2432	0.2125	0.0148	0.0002	0.4707	1.0355
4.6	0.2042	0.2364	0.0197	0.0003	0.4606	1.0594
4.8	0.1738	0.2473	0.0242	0.0005	0.4458	1.0699
5.0	0.1509	0.2480	0.0283	0.0006	0.4278	1.0695
5.2	0.1332	0.2423	0.0319	0.0008	0.4082	1.0613
5.4	0.1194	0.2332	0.0352	0.0011	0.3889	1.0500
5.6	0.1080	0.2224	0.0383	0.0014	0.3701	1.0363

REFERENCES

1. H. Kaden "Wirbelstrome and Schirmung in der Nachrichtentechnik," Technische Physik in Einzeldarstellungen <u>10</u>, Springer-Verlag (1959).

2. M. Abramowitz and I. Stegun, "Handbook of Mathematical Functions", page 361, U.S. Dept. of Commerce, N.B.S. Applied Mathematics Series 55, (1964)

3. M. Born and E. Wolf, page 558-9, Principles of Optics" 2nd Ed., Pergamon Press.

4. J. D. Jackson, page 289, "Classical Electrodynamics" J. Wiley, N. Y. (1963).

5. M. Abramowitz and I Stegun, page 360 OP CIT.

6. I. S. Gradshten and I. W. Rhyzik, Formula #3, Sect. 8.382, page 950, "<u>Table of Integrals, Series and Products</u>". Academic Press, N. Y. (1965)

7. W. R. Smythe, page 489 "Static & Dynamic Electricity" 3rd. Ed. McGraw Hill, N. Y. (1968).

8. J. M. Greenberg, A. C. Lind, R. T. Wang and L. F. Libelo "Electromagnetic Scattering" Eds. R. L. Rowell, R. S. Stein, Gordon & Breach, N. Y. (1967)

9. R. W. P. King and T. T. Wu, page 116 "The Scattering and Diffraction of Waves", Harvard Univ. Press, Cambridge, Mass.(1959).

10. S. Skavlem, Arch. Math. Naturvid. <u>51</u>, 61 (1951)

11. S. R. Seshadri, Scientific Report No. 20, Cruft Laboratory, Harvard Univ. (1958)

12. P. M. Morse and P. J. Rubenstein, Phys. Rev. <u>54</u>, 895 (1938).