INTERACTION NOTE WL-EMP-IN-125

NOTE 125

SEPTEMBER 1972

# THE DIFFERENTIAL EQUATIONS OF A TRANSMISSION LINE EXCITED BY AN IMPRESSED FIELD WITH AXIAL MAGNETIC COMPONENT

SIDNEY FRANKEL
Sidney Frankel and Associates
Menlo Park, California 49025

transmission lines, differential equations

## ABSTRACT

When a TEM transmission line is excited by an external field having a magnetic component parallel to the line axis, the quasi-static concepts on which the simplicity of conventional line theory rests may, under certain conditions, become invalid.

While not rigorously argued, except for the case of a single, lossless, round conductor above a lossless ground plane, we believe, as a result of this study, that the conventional concepts for multiconductor lines remain valid, provided the transverse component of propagation wavelength is much greater than the maximum dimension of the line in the transverse direction of propagation.

Voltage measured between conductors in a transverse plane generally will not be independent of the position of the voltmeter connecting wires, even though confined to the transverse plane, but conductor currents should be as predicted by conventional theory. Under these conditions the axial component has no effect on the TEM response of the line.

An attempt to explore the effect of small, but non-vanishing radius of an isolated conductor in such a field yields the suggestion that the relative error in assuming quasi-static behavior of the line will be of the order of the ratio of that radius to the transverse propagation wavelength, and out of phase with it.

## CONTENTS

## 1. INTRODUCTION

The study of the TEM response of transmission lines to external fields has a long history, dating from early studies of inductive interference between power and telephone circuits, lightning discharge effects, and telephone crosstalk[1,2]. More recent studies have been concerned with disturbances, in control- and communication systems of airborne and ground installations, produced by high-energy radiations of short duration arriving from arbitrary directions. Such studies have concentrated on the effects of impressed fields having, at most, transverse electric and magnetic components of intensity, and an axial component of electric intensity[3-6].

When the magnetic intensity is transverse and the electric field is axial, two alternate methods of dealing with the problem yield equivalent results for solid-conductor lines[3,5,6]. However, in the case of a braided-shield cable, where the coupling to the interior of the cable depends on the total current and charge on the sheath exterior, the vector-potential method[3] yields the total exterior current, while the generalized transmission-line method[5] yields only the normal TEM current of the cable exterior. Nevertheless, in the case of a single thin wire above a perfectly conducting ground plane, Harrison has shown that only the transmission-line component of current is present[7]. Furthermore, the nature of the proof implies that the conclusion is valid for any number of conductors of any cross-section (much less than a wavelength ) above a perfectly conducting ground plane.

Thus if the exterior current of a braided-shield cable above ground is computed by the elementary transmission-line method, the subsequent analysis of the interior response remains valid for the stated excitation conditions.

The only question that remains for an arbitrary wave incident on a line above ground is the effect of an axial magnetic field. Intuitively one is inclined to dismiss this component on the grounds that it cannot contribute a forcing function to the voltage differential equation of the line. Nevertheless, the line differential equations cannot be written in the usual way without imposing certain restrictions. For a single conductor, -or for the exterior of a braided-shield cable- above perfect ground, these restrictions are ordinarily easily satisfied. In fact, insofar as the line mechanical structure is concerned, it is sufficient that the maximum transverse dimension of the system be much less than a wavelength, thus yielding quasi-static behavior in any transverse plane in the vicinity of the line. An attempt to explore this criterion quantitatively is made in section 2.3 by studying the deviation from quasi-static conditions for an isolated round wire excited by an appropriate wave, as the radius of the wire is increased.

## 2. TECHNICAL DISCUSSION

We consider, first, the case of a surface wave travelling in the transverse plane of the line, and then generalize to a surface wave at arbitrary incidence.

### 2.1 Surface Wave Transverse to Line

Consider a system of lossless conductors of arbitrary invariant cross-section above a lossless ground plane (fig. 1). With a coordinate system specified as shown, assume a plane surface wave travelling across the line in the positive-z direction, with vertical electric field intensity, $E_y^e\bar{j}$, and horizontal magnetic intensity, $H_x^e\bar{i}$, parallel to the line axis. Assume,

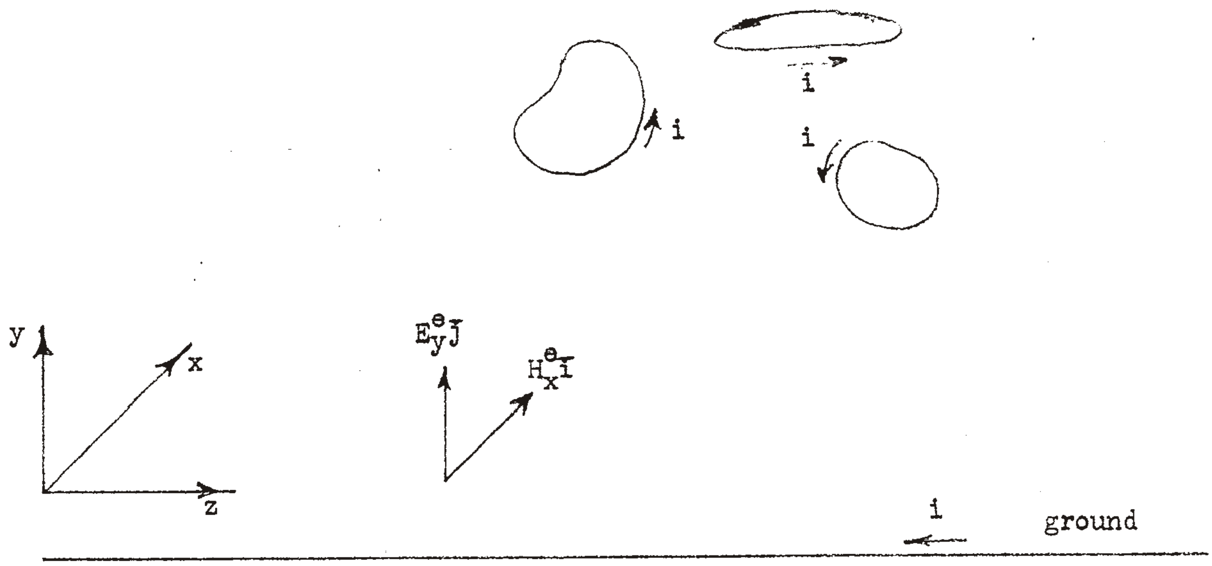



Fig.1. System of conductors above ground

further, that the conductors above ground (N in number) are so grouped and of such size that the maximum dimension of the system in the transverse z direction is much less than the wavelength of the travelling wave. Then the impressed fields appear as quasi-static fields to the conductor system, with the following implications:

1. In the (perfect) conductor interiors the magnetic and electric intensities are zero. Exterior to the conductors the magnetic intensity is everywhere the same, namely $H_x^e \bar{i}$, and the surface current densities on all conductors are also everywhere the same, namely,

$$i = H_x^e \tag{1}$$

   Since the current around the periphery of each conductor is independent of position, there is no accumulation of electric charge at any point of any conductor due to $H_x^e$.

2. The impressed electric field, $E_y^e$, induces charges on the conductors in accordance with quasi-static concepts, since variations of the field are felt simultaneously over the cross-section of the system.

As a result of these considerations the total charge on any conductor (say the k<u>th</u>) results only from the impressed quasi-static electric intensity, $E_y^e$, and the potentials, $V_i$ (i = 1,..., N) on the various conductors associated with TEM waves, if any, on the line, Furthermore, the conductor traces are equipotentials in any transverse plane. Thus, the law of current continuity applied to the line axial currents and total linear charge densities yields the conventional result[5]

$$\frac{dI}{dx} + \underline{\eta}\,\underline{V} = \underline{H}^e(x) = j\omega\,\underline{C}^e\,E_y^e \tag{2}$$

where[5,8] $\underline{I}$ and $\underline{V}$ are the line current and voltage column vectors respectively, $\underline{C}^e$ is the impressed electric field coupling column vector, and

$$\underline{\eta} = j\omega\,\underline{C} \tag{3}$$

where $\underline{C}$ is the N x N matrix of Maxwell's coefficients of capacitance for the system. The vector, $\underline{H}^e(x)$, is defined by the third member of equation (2). It may be represented as a continuous distribution of current sources along the line.

To write the voltage equation we proceed along conventional lines as indicated in figure 2. Curve, C, is an integration path for the electric intensity, $\overline{E}$. Segments 1 and 3 are corresponding transverse segments at x and x + Δx respectively. Segments 2 and 4 are at the surfaces of the k<u>th</u> conductor and ground respectively. We have, by Faraday's law,

$$\int_C \overline{E}\cdot\overline{ds} = -\,j\omega\phi_k\cdot\Delta x \tag{4}$$

where $\phi_k\cdot\Delta x$ is the normal magnetic flux through any surface bounded by C. Since the conductors are lossless, $\overline{E} \equiv 0$ along segments 2 and 4, and

$$-\,j\omega\phi_k\cdot\Delta x = \int_1 \overline{E}(x;\,y,z)\cdot\overline{ds} + \int_3 \overline{E}\,(x+\Delta x;\,y,z)\cdot\overline{ds}$$

$$= \int_1 [\overline{E}_t(x;\,y,z) - \overline{E}_t(x+\Delta x;\,y,z)]\cdot\overline{ds}$$

where $\overline{E}_t$ is the transverse component of $\overline{E}$. For $\Delta x \to 0$,

$$\frac{\partial}{\partial x}\int_1 \overline{E}_t\cdot\overline{ds} = j\omega\phi_k \tag{5}$$

In terms of the electrodynamic potentials,

$$\overline{E}_t = -\,\overline{\nabla}_t V - j\omega\mu_o\overline{A}_t \tag{6}$$

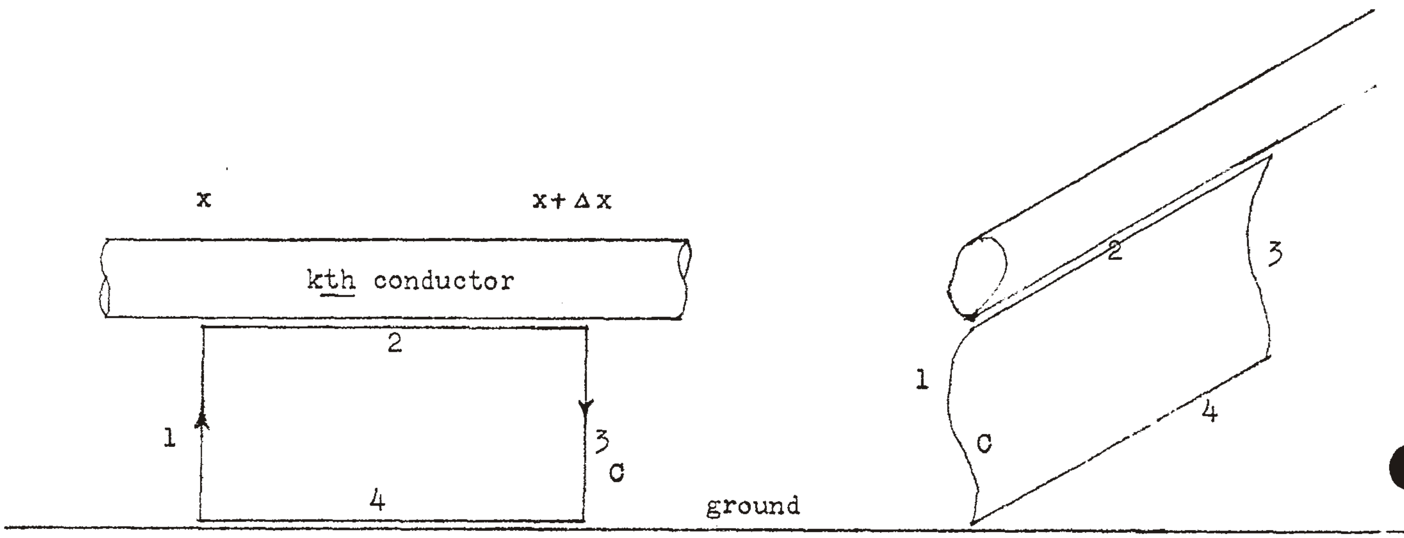



Fig.2. Integration path for voltage equation.

where V and $\overline{A}$ are the scalar and vector potentials of the total field, respectively. Thus equation (5) becomes

$$\frac{\partial}{\partial x} \int_1 \overline{\nabla}_t V \cdot \overline{ds} + j\omega\mu_o \frac{\partial}{\partial x} \int_1 \overline{A}_t \cdot \overline{ds} + j\omega\phi_k = 0 \qquad (7)$$

The first term in the left member is $(dV_k/dx)$. Since $H_x^e$ is independent of x, so is the transverse current density, i (equation (1)). Consequently, $\overline{A}_t$ is independent of x, whence the second term in the left member is zero. The final term is a summation of the effects of axial currents on all conductors above ground, so that equation (7) becomes the conventional result in the absence of transverse magnetic intensity:

$$\frac{dV_k}{dx} + j\omega \sum_{i=1}^{N} L_{ik} I_i = 0, \; k = 1,\ldots, N \qquad (8)$$

or, in matrix notation,

$$\frac{d\underline{V}}{dx} + \underline{\zeta}\, \underline{I} = 0 \qquad (9)$$

where we have written

$$\underline{\zeta} = j\omega \underline{L} \qquad (10)$$

and $\underline{L}$ is the N x N matrix of line inductance coefficients[8].

Thus, for this particular case of excitation, the conclusion is that the axial magnetic intensity does not affect the TEM response of a line of sufficiently small dimensions.

2.2 Surface Wave at Arbitrary Incidence

Consult figure 3. $\overline{P}$, the poynting vector for the surface wave makes an angle, $\alpha$, with the positive x-axis. The direction cosines of $\overline{P}$ are

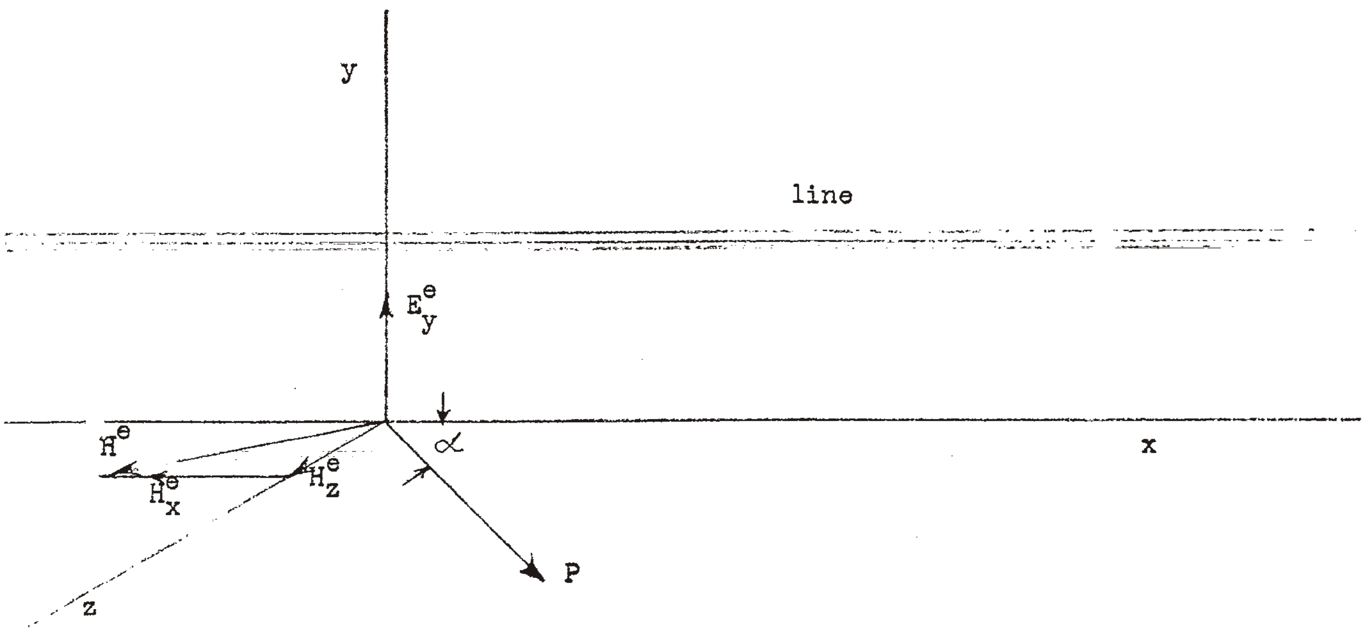



Fig.3. Field components of surface wave at oblique incidence to line.

$$\left.\begin{aligned} \lambda &= \cos\alpha \\ \mu &= \cos\beta = \cos\frac{\pi}{2} = 0 \\ \nu &= \cos\gamma = \sin\alpha \end{aligned}\right\} \quad (11)$$

As before, the electric intensity is $E_y^e \bar{j}$, while the magnetic intensity is

$$\bar{H}^e = H_x^e \bar{i} + H_z^e \bar{k}$$

$$= H^e \, ( - \nu \bar{i} + \lambda \bar{k}) \quad (12)$$

To show the relative phase of $H^e$ explicitly write

$$H^e = H_o^e \exp[-jk_o(\lambda x + \upsilon z)] \quad (13)$$

where

$$k_o = \frac{2\pi}{\lambda_o} \quad (14)$$

and $\lambda_o$ is the wavelength of the impressed field in the direction of propagation. Then, by equations (12) and (13)

$$\left.\begin{aligned} H_x^e &= - \upsilon \, H_o^e \exp[-jk_o(\lambda x + \upsilon z)] \\ H_z^e &= \lambda \, H_o^e \exp[-jk_o(\lambda x + \upsilon z)] \end{aligned}\right\} \quad (15)$$

The wavelength of propagation transverse to the line is given by

$$k_z = \frac{2\pi}{\lambda_z} \quad (16)$$

that is,

$$\lambda_z = \frac{\lambda_o}{\upsilon} = \lambda_o \csc\alpha \quad (17)$$

Thus, a sufficient condition on the wavelength of the impressed field is that $\lambda_o \csc\alpha$ be much greater than the z-dimension of the line. As

$\alpha \to 0$, $\lambda_0$ may have any value greater than zero, for any line cross-section, in accordance with previous concepts.

It follows immediately that the current equation (2) is valid under the lesser restriction that $\lambda_z$ be much greater than the line's greatest z-dimension.

However, in trying to reduce equation (7) to the form of equation (9), we are in difficulty with the second term of the left member of equation (7). For, since the wave has a propagation factor, $\exp(-jk_o\lambda x)$ in the x-direction ($\lambda \neq 0$), we have

$$\frac{\partial}{\partial x} \int_1 \overline{A}_t \cdot \overline{ds} = - jk_o\lambda \int_1 \overline{A}_t \cdot \overline{ds} \neq 0$$

generally.

The difficulty can be resolved if a path, (1), can be found, such that

$$\int_1 \overline{A}_t \cdot \overline{ds} = 0 \qquad (18)$$

Leaving general considerations aside, we can certainly satisfy this criterion in one elementary case of special interest. Figure 4 shows a conductor of circular cross-section above a ground plane, excited by an uniform, axial, quasi-static magnetic field, which we can imagine to be supported by currents travelling in the ground, and in a source plane far above the conductor-ground system, in planes transverse to the conductor axis. As a result of the impressed field, a uniform current circulates on the wire surface, with the same density as the ground current.

The dashed line represents a path along which $\overline{A}_t \cdot \overline{ds} = 0$ (a stronger condition than that specified by equation (18)). To see that this is so it is only necessary to show that $\overline{A}_t$ is everywhere normal to the path.

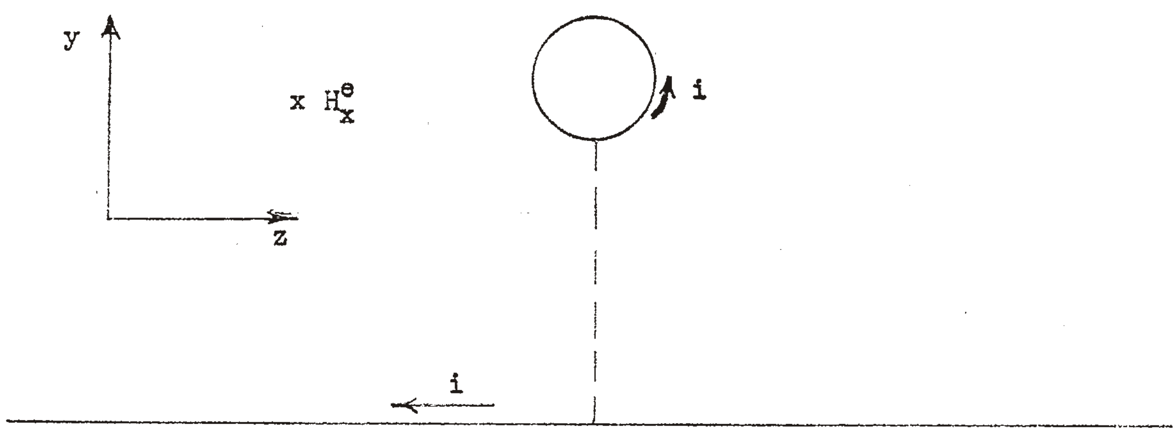



Fig.4. Circular cross-section wire excited by axial magnetic field.

We note first that the ground and source currents contribute only components of $\overline{A}_t$ normal to the path. On the circular conductor, current elements symmetrically disposed with regard to the path (and equidistant from any point on the path) yield horizontal components which add and vertical components which cancel. Thus the total $\overline{A}_t$ at any point of the path is normal to the path, whence $\overline{A}_t \cdot \overline{ds} = 0$ at every point.

The requirement that integration be limited to a specific path implies that the flux linking the conductor must be determined for a surface bounded by that path. In equation (7), since the first term in the left member is independent of the path, any variation in the second term due to change in path must be compensated by a corresponding change in the third, i.e., by a change in the normal flux. (See Appendix A).

Although we have not taken the time to explore this subject thoroughly, it appears offhand that there must be a singly-infinite family of paths satisfying the condition

$$\overline{A}_t \cdot \overline{ds} = 0 \qquad (19)$$

For equation (18) implies that

$$A_y dy + A_z dz = 0$$

or

$$\frac{dy}{dz} = -\frac{A_z}{A_y} \quad , \; A_y \neq 0 \qquad (20a)$$

or

$$\frac{dz}{dy} = -\frac{A_y}{A_z} \quad , \; A_z \neq 0 \qquad (20b)$$

Equation (20a), or, alternatively, equation (20b), is the first order differential equation of a one-parameter family of curves. The equation has a solution at any point $(y_o, z_o)$ for which a Taylor's expansion of the right member exists. For well-modelled physical functions this will generally be true unless both $A_y$ and $A_z$ are zero. However, at such points, equation (19) is automatically satisfied.

There is no guarantee, a priori, that **any** of the curves will intersect both a specified line conductor and ground. We have shown that at least one such curve exists in the case of a single round conductor above ground. In that case, also, the designated path yields the static value of inductive coupling parameter determined previously[13]. Further study appears warranted.

With the conclusion that the middle term of the left member of equation (7) may be dropped, the line equations, at least for the case of a single round wire (or the exterior of a braided cable) above ground, become[5]

$$\left.\begin{aligned} \frac{dV_c}{dx} + j\omega L I_c &= \mathcal{E}^e(x) = j\omega\, L^e\, H_z^e \\ \frac{dI_c}{dx} + j\omega\, C\, V_c &= \mathcal{H}^e(x) = j\omega\, C^e\, E_y^e \end{aligned}\right\} \qquad (21)$$

where $\mathcal{E}^e(x)$ may be represented as a continuous distribution of series voltage sources along the line.

2.3 Isolated Round Wire Excited by Plane Wave; Electric Vector Normal to Conductor Axis.

In this section we explore, briefly, the deviations from static charge and current distributions on an isolated round wire as a function of wire radius.

In figure 3 assume that no ground is present, and that the "line" consists of a single round conductor of radius, a, with axis coincident with the x-axis. In Appendix B we derive, by a simple extension of standard work[9,10], the approximate values of circumferential current density, $i_\phi$, and surface charge density, $\sigma$, as a function of polar angle, $\phi$, for small $k_z a$:

$$i_\phi \approx - \upsilon H_o^e \exp(-jk_x x) \{[1- \frac{1}{2} (k_z a)^2 \cos 2\phi] - j \frac{1}{2} (k_z a) \cos\phi\} \quad (22a)$$

$$\sigma \approx - 2 \varepsilon_o E_y^e \exp(-jk_x x) \{[\sin\phi - \frac{1}{8} (k_z a)^2 \sin 3\phi] - j \frac{1}{2} (k_z a) \sin 2\phi\} \quad (22b)$$

In the limit, as $k_z a \to 0$,

$$\left. \begin{aligned} i_\phi &\to - \upsilon H_o^e \exp(-jk_x x) \\ \sigma &\to - 2 \varepsilon_o E_y^e \exp(-jk_x x) \sin\phi \end{aligned} \right\} \quad (23)$$

In the limit, $i_\phi$ has the expected constant value at any transverse plane, while the charge distribution is identical with the familiar electrostatic distribution induced by a uniform field[12], except for the phase variation with x. To the extent that the actual charge distribution deviates from this behavior because of non-vanishing $k_z a$, the conductor surface cannot be an equipotential surface. Similarly, in equations (21), as the conductor diameter is increased, the meaning of $V_c$ can be expected to become increasingly vague in the presence of an axial component of magnetic intensity.

It appears quite difficult to estimate the effect of this deviation on the terminal response of a TEM line. The r.m.s. deviation of charge density over the conductor circumference is approximately

$$(\overline{\delta^2})^{1/2} = jK \frac{k_z a}{2\sqrt{2}}$$

where K is the factor outside the bracket in equation (22b). The r.m.s. value of the electrostatic limit is

$$(\overline{\sin^2\phi})^{1/2} = \frac{K}{\sqrt{2}}$$

The relative r.m.s. deviation is therefore

$$d = j\,\frac{1}{2}\,k_z a = j\pi\,\frac{a}{\lambda_z} = j\pi\,\frac{a}{\lambda_o}\,\sin\alpha \qquad (24)$$

Since this is out of phase with the electrostatic limit, the magnitude of d measures the r.m.s. "error" in the phase of the charge density. The r.m.s. error in its amplitude is measured by the square of the magnitude of d (Appendix C). As an example, for a cable sheath with outer diameter of 5 cm., we have, at a $\lambda_z$ of one meter,

$$|d| = 0.0785$$

$$|d|^2 = 6.16 \times 10^{-3}$$

3. CONCLUSION

The TEM terminal response of a multiconductor, isolated transmission line, or of a cable sheath above ground, to an external field impressed at arbitrary incidence, may be determined from the conventional analysis using only transverse components of the impressed field, provided that the impressed component of propagation wavelength transverse to the line is much greater than the maximum dimension of the line or cable in the same direction.

An attempt to explore the effect of small but non-vanishing radius of an isolated conductor in such a field yields the suggestion that the relative error in assuming the usual quasi-static behavior of the line will be of the order of the ratio of that radius to the transverse propagation wavelength, and out of phase with it.

ACKNOWLEDGEMENT

The work reported herein was done for Harry Diamond Laboratories under Contract No. DAAG39-73-C-0017.

## APPENDIX A

### Variation of Flux with Integration Path

To understand the nature of the variation of flux linkage with path, consult figure 5. This shows a length, $\Delta x$, of conductor above ground, for which two alternate paths of integration have been chosen. One of these is formed by the segments, $C_1$, at x and at $x + \Delta x$, and the lines joining their intersections with the conductor and with ground. The alternate path is formed similarly with $C_2$. Surfaces $S_1$ and $S_2$, respectively, are formed by axial line segments sweeping along $C_1$ and $C_2$ from conductor to ground. End-cap surfaces $S_3$ and $S_4$ are formed by portions of transverse planes at x and at $x + \Delta x$ intercepted by the closed path formed by $C_1$, $C_2$, the conductor, and ground.

The total normal flux into the volume enclosed by $S_1, \ldots, S_4$ must be zero:

$$\sum_{i=1}^{4} \mu_o \int_{S_i} \overline{H} \cdot \overline{n} \, dS_i = 0 \qquad \text{(A-1)}$$

Let

$\phi_1 \cdot \Delta x$ = normal flux <u>into</u> surface $S_1$

$$= \mu_o \int_{S_1} \overline{H} \cdot \overline{n} \, dS_1$$

$\phi_2 \cdot \Delta x$ = normal flux <u>out of</u> surface $S_2$

$$= - \mu_o \int_{S_2} \overline{H} \cdot \overline{n} \, dS_2$$

For the flux into $S_3$ we have, with the help of equation (15)

$$\phi_3 = \mu_o \int_{S_3} \overline{H} \cdot \overline{n} \, dS_3$$

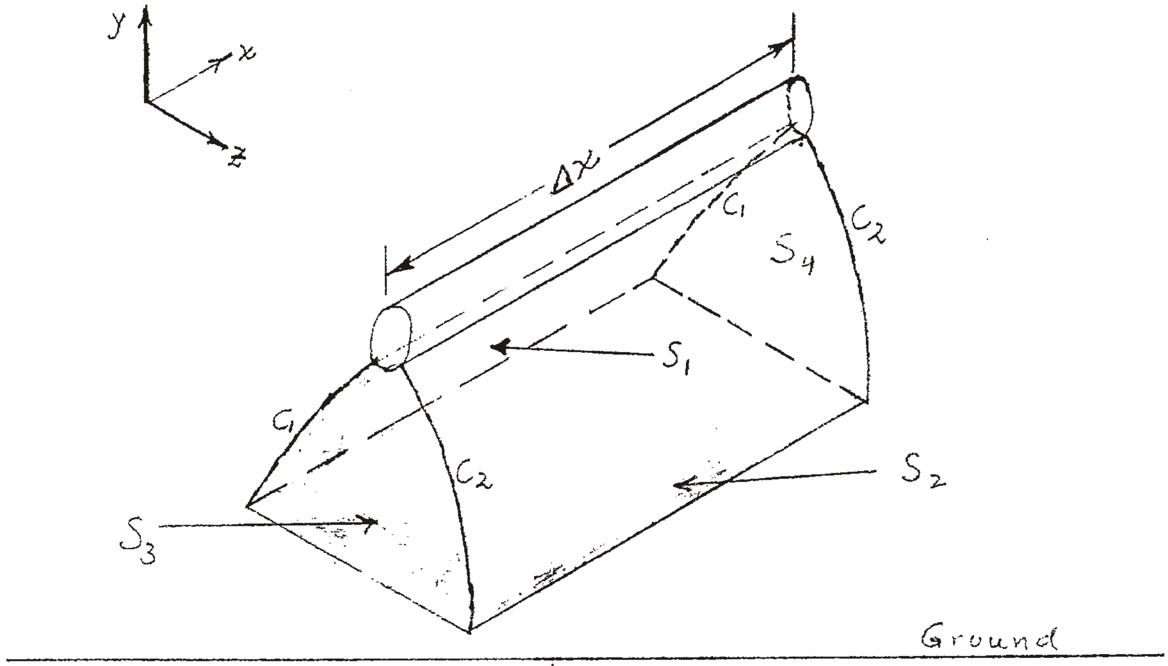



Fig.5. Diagram for determining variation of normal flux with integration path.

$$= \mu_o \int_{S_3} \{-\upsilon H_o^e \exp[-jk_o(\lambda x + \upsilon z)]\} \, dy \, dz$$

$$= - \mu_o \upsilon H_o^e \int_{S_3} \exp[-jk_o(\lambda x + \upsilon z)] \, dy \, dz$$

For the flux out of $S_4$ we have

$$\phi_4 = \mu_o \int_{S_4} \{-\upsilon H_o^e \exp[-jk_o(\lambda x + \lambda\Delta x + \upsilon z)]\} \, dy \, dz$$

$$= -\mu_o \upsilon H_o^e \exp(-jk_o\lambda \cdot \Delta x) \int_{S_3} \exp[-jk_o(\lambda x + \upsilon z)] \, dy \, dz$$

$$= \phi_3 \exp(-jk_o\lambda \cdot \Delta x)$$

Then equation (A-1) yields

$$(\phi_1 - \phi_2) \Delta x + \phi_3 (1 - \exp(-jk_o\lambda \cdot \Delta x) = 0$$

For $\Delta x \to 0$,

$$(\phi_1 - \phi_2) \Delta x + \phi_3 (jk_o\lambda \cdot \Delta x) = 0$$

whence

$$\phi_2 - \phi_1 = jk_o\lambda\phi_3$$

$$= -j \frac{1}{2} \mu_o k_o H_o^e \sin 2\alpha \cdot \exp(-jk_o\lambda x) \int_{S_3} \exp[-jk_o\upsilon z] \, dy \, dz \qquad \text{(A-2)}$$

Evidently, the difference in flux through $S_1$ and $S_2$ is a consequence of (1) the area-weighted phase difference as measured by the integral in equation (A-2) (2) the magnitude of the transverse component, $\mu_o H_o^e \sin\alpha$, and (3) its phase shift along the line, as measured by the factor $k_o \cos\alpha$.

## APPENDIX B

### Effect of Conductor Diameter on Current and Charge Distribution Induced by a Travelling Wave with Axial Magnetic Component

The physical situation is again represented by figure 3, except that no ground is present, and the "line" consists of a single round conductor. In cross-section the situation is shown in figure 6, the conductor, of radius, a, being located at the origin of coordinates.

Analysis of the scattering effect of the conductor is well-documented[9,10] for $\alpha = \frac{\pi}{2}$, which is the extreme case of interest here. For the more general case, $- 0 \leqq \alpha \leqq \frac{\pi}{2}$, we start with the impressed field components (cf equations (15))

$$\left.\begin{aligned} H_x^e &= - \upsilon H_o^e \exp[-jk_o(\lambda x + \upsilon z)] \\ H_z^e &= \lambda H_o^e \exp[-jk_o(\lambda x + \upsilon z)] \\ E_y^e &= \eta_o H_o^e \exp[-jk_o(\lambda x + \upsilon z)] \end{aligned}\right\} \qquad \text{(B-1)}$$

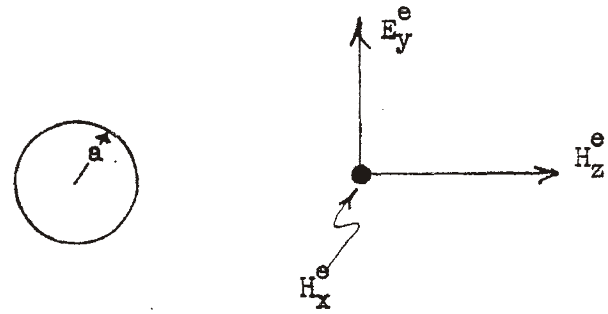
where $\eta_o$ is the wave impedance.

With obvious minor modifications the analysis proceeds as in the special case, $\alpha = \frac{\pi}{2}$. Begin by writing

$$H_x^e = - \upsilon H_o^e \exp(-jk_o\lambda x) \exp(-jk_o\upsilon z)$$

and using the well-known expansion

$$\begin{aligned} \exp(-jk_o\upsilon z) &= \exp(-jk_z z) = \exp(-jk_z\rho\cos\phi) \\ &= J_o(k_z\rho) + 2 \sum_{n=1}^{\infty} (-j)^n J_n (k_z\rho) \cos n\phi \end{aligned}$$




$$z = \rho \cos \phi$$

$$y = \rho \sin \phi$$

Fig.6. Circular conductor excited by travelling wave.

to get

$$H_x^e = - \upsilon H_o^e \exp(-jk_x x) \{J_o(k_z\rho) + 2 \sum_{n=1}^{\infty} (-j)^n J_n(k_z\rho) \cos n\phi\} \qquad \text{(B-2)}$$

This component of field induces circulating currents around the conductor periphery resulting in a scattered component $H_x^s$. The total x-component of magnetic intensity is therefore

$$H_x = H_x^e + H_x^s \qquad \text{(B-3)}$$

The scattered component must satisfy the cylindrical wave equation

$$\left.\begin{aligned} \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial x^2} + k_o^2\psi = 0 \\ k_o = \frac{\omega}{v} \end{aligned}\right\} \qquad \text{(B-4)}$$

To insure proper matching of fields continuously along x, the scattered component, like the impressed component, must have an appropriate propagation constant in the x-direction, thus:

$$\psi = T(\rho,\phi) \exp(-jk_x x)$$

$$\frac{\partial^2\psi}{\partial x^2} = - k_x^2 \psi$$

Noting that

$$k_o^2 - k_x^2 = k_\rho^2 = k_z^2$$

reduces equation (B-4) to

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + k_z^2 \psi = 0 \qquad \text{(B-5)}$$

Solutions of equation (B-5) representing outgoing scattered waves are of the form

$$\psi = H_n^{(2)} (k_z\rho) \exp(\pm jn\phi) \qquad \text{(B-6)}$$

where $H_n^{(2)}$ is an nth order Hankel function.

Thus equation (B-3) may be written

$$H_x = -\upsilon H_o^e \exp(-jk_x x) \{ [J_o(k_z\rho) + 2 \sum_{n=1}^{\infty} (-j)^n J_n(k_z\rho) \cos n\phi]$$

$$+ b_o H_o^{(2)}(k_z\rho) + \sum_{n=1}^{\infty} H_n^{(2)}(k_z\rho)[a_n \sin n\phi + b_n \cos n\phi]\} \qquad \text{(B-7)}$$

At the surface of the conductor the tangential component of electric field and the normal component of magnetic field are zero. From the appropriate field equation we have

$$j\omega\epsilon_o E_\phi = -jk_x H_\rho - \frac{\partial H_x}{\partial \rho}$$

yielding

$$\left.\frac{\partial H_x}{\partial \rho}\right|_{\rho = a} = 0$$

Using this in equation (B-7) and following standard procedures then yields

$$\left.\begin{aligned} a_n &= 0 \\ b_n &= -2(-j)^n \frac{J_n'(k_z a)}{H_n^{(2)\prime}(k_z a)} \qquad n = 1,2,3,\ldots. \end{aligned}\right\} \qquad \text{(B-8)}$$

where, as usual, primes denote derivatives with respect to the function arguments.

The solution for the axial magnetic intensity is therefore

$$H_x = -\upsilon H_o^e \exp(-jk_x x) \{[J_o(k_z\rho) - \frac{J_o'(k_z a)}{H_o^{(2)\prime}(k_z a)} H_o^{(2)}(k_z\rho)]$$

$$+ 2 \sum_{n=1}^{\infty} (-j)^n [J_n(k_z\rho) - \frac{J_n'(k_z a)}{H_n^{(2)\prime}(k_z a)} H_n^{(2)}(k_z\rho)] \cos n\phi\} \qquad \text{(B-9)}$$

From appropriate components of Maxwell's equations we have

$$\left.\begin{aligned} \frac{\partial E_\phi}{\partial x} &= j\omega\mu_o H_\rho \\ \frac{\partial H_\rho}{\partial x} - \frac{\partial H_x}{\partial \rho} &= j\omega\varepsilon_o E_\phi \end{aligned}\right\}$$

Noting, once again, that

$$\frac{\partial E_\phi}{\partial x} = -jk_x E_\phi$$

etc., reduces these equations to

$$\left.\begin{aligned} -jk_x E_\phi &= j\omega\mu_o H_\rho \\ -jk_x H_\rho - \frac{\partial H_x}{\partial \rho} &= j\omega\varepsilon_o E_\phi \end{aligned}\right\}$$

Eliminating $E_\phi$,

$$-jk_x H_\rho - \frac{\partial H_x}{\partial \rho} = (j\omega\varepsilon_o)\left(-\frac{\omega\mu_o}{k_x}\right) H_\rho$$

$$= -j\,\frac{k_o^2}{k_x}\,H_\rho$$

whence

$$H_\rho = \frac{\lambda}{jk_o\upsilon^2}\,\frac{\partial H_x}{\partial \rho} \qquad \text{(B-10)}$$

Then from equation (B-9), noting that $k_z = k_o\upsilon$ ,

$$H_\rho = j\lambda H_o^e \exp(-jk_x x)\ \{[J_o'(k_z\rho) - \frac{J_o'(k_z a)}{H_o^{(2)\prime}(k_z a)} H_n^{(2)\prime}(k_z\rho)]$$

$$+ 2\sum_{n=1}^{\infty} (-j)^n\, [J_n'(k_z\rho) - \frac{J_n'(k_z a)}{H_n^{(2)\prime}(k_z a)} H_o^{(2)\prime}(k_z\rho)]\} \qquad \text{(B-11)}$$

Again, from Maxwell's equations,

$$\frac{\partial E_\rho}{\partial x} = - jk_x E_\rho = - j\omega\mu_o H_\phi$$

and

$$\frac{1}{\rho}\frac{\partial H_x}{\partial \phi} - \frac{\partial H_\phi}{\partial x} = \frac{1}{\rho}\frac{\partial H_x}{\partial \phi} + jk_x H_\phi = j\omega\varepsilon_o E_\rho$$

Eliminating $E_\rho$ and solving for $H_\phi$,

$$H_\phi = \frac{\lambda}{jk_o \upsilon^2 \rho}\frac{\partial H_x}{\partial \phi} \qquad \text{(B-12)}$$

Then equation (B-9) yields

$$H_\phi = \frac{2H_o^e \lambda}{jk_o \upsilon \rho} \exp(-jk_x x) \sum_{n=1}^{\infty} (-j)^n n [J_n(k_z \rho) - \frac{J_n'(k_z a)}{H_n^{(2)\prime}(k_z a)} H_n^{(2)}(k_z \rho)] \sin n\phi \qquad \text{(B-13)}$$

Finally, from results obtained previously,

$$\left.\begin{aligned} E_\phi &= - \frac{\omega\mu_o}{k_x} H_\rho = - \frac{\eta_o}{\lambda} H_\rho \\ E_\rho &= \frac{\omega\mu_o}{k_x} H_\phi = \frac{\eta_o}{\lambda} H_\phi \end{aligned}\right\} \qquad \text{(B-14)}$$

Components of surface current density on the conductor are given by

$$\left.\begin{aligned} i_x &= - H_\phi \Big|_{\rho=a} \\ i_\phi &= H_x \Big|_{\rho=a} \end{aligned}\right\} \qquad \text{(B-15)}$$

The surface charge density is

$$\sigma = \varepsilon_o E_\rho \Big|_{\rho=a} \qquad \text{(B-16)}$$

With the use of equation (B-9), the second of equations (B-15) yields

$$i_\phi = - \upsilon H_o^e \exp(-jk_x x)\left[\frac{J_o H_o^{(2)\prime} - J_o' H_o^{(2)}}{H_o^{(2)\prime}} + 2\sum_{n=1}^{\infty} (-j)^n \left[\frac{J_n H_n^{(2)\prime} - J_n' H_n^{(2)}}{H_n^{(2)\prime}}\right] \cos n\phi\right] \qquad \text{(B-17)}$$

where every Bessel function has the argument, $k_z a$.

By standard relations in Bessel function theory[11],

$$Z_n'(\xi) = \frac{1}{2}\left[Z_{n-1}(\xi) - Z_{n+1}(\xi)\right], \text{ n integer} \tag{B-18}$$

where $Z_n(\xi)$ is any Bessel function. In particular[11],

$$Z_o'(\xi) = -Z_1(\xi) \tag{B-18a}$$

Furthermore[11],

$$J_n H_{n-1}^{(2)} - J_{n-1} H_n^{(2)} = \frac{2}{j\pi\xi} \tag{B-19}$$

Thus, by equations (B-18), (B-18a), and (B-19),

$$J_o H_o^{(2)\prime} - J_o' H_o^{(2)} = J_1 H_o^{(2)} - J_o H_1^{(2)} = \frac{2}{j\pi\xi} \tag{B-20}$$

and

$$J_n H_n^{(2)\prime} - J_n' H_n^{(2)} = \frac{1}{2}\left\{\left[J_n H_{n-1}^{(2)} - J_{n-1} H_n^{(2)}\right] + \left[J_{n+1} H_n^{(2)} - J_n H_{n+1}^{(2)}\right]\right\} \tag{B-21}$$

Therefore, equation (B-17) becomes

$$i_\phi = -\frac{2\upsilon H_o^e \exp(-jk_x x)}{j\pi k_z a}\left[\frac{1}{H_o^{(2)\prime}(k_z a)} + 2\sum_{n=1}^{\infty}(-j)^n \frac{\cos n\phi}{H_n^{(2)\prime}(k_z a)}\right] \tag{B-22}$$

Next, using equations (B-14) and (B-13) in equation (B-16), and reducing in the same manner as the preceding,

$$\sigma = -\frac{4\varepsilon_o E_y^e \exp(-jk_x x)}{\pi(k_z a)^2}\sum_{n=1}^{\infty}(-j)^n \frac{n \sin n\phi}{H_n^{(2)\prime}(k_z a)} \tag{B-23}$$

where we have set $\eta_o H_o^e = E_y^e$.

We are particularly interested in the behavior of $i_\phi$ and $\sigma$ as $k_z a \to 0$.

To that end we can use the following approximations[11]:

As $\xi \to 0$,

$$H_n^{(2)}(\xi) \to -j\, N_n\,(\xi)$$

therefore

$$H_n^{(2)\prime}(\xi) \to -j\, N_n'\,(\xi)$$

where

$$N_n'\,(\xi) \to \frac{n!}{2\pi}\left(\frac{2}{\xi}\right)^{n+1}, \quad n \geq 1;\ 0! = 1$$

Using these results in equations (B-22) and (B-23) respectively yields

$$i_\phi = -\upsilon\, H_o^e \exp(-jk_x x)\left(1 + 4\sum_{n=1}^{\infty} (-j)^n \frac{(k_z a)^n}{2^n\, n!} \cos n\phi\right), \quad k_z a \ll 1 \qquad \text{(B-24)}$$

$$\sigma = -2\,\varepsilon_o\, E_y^e \exp(-jk_x x) \sum_{n=o}^{\infty} (-j)^n \frac{(k_z a)^n}{2^n n!} \sin\,(n+1)\,\phi, \quad k_z a \ll 1 \qquad \text{(B-25)}$$

Finally, approximating each of these to early terms,

$$\left.\begin{aligned} i_\phi &\approx -\upsilon\, H_o^e \exp(-jk_x x)\ \{[1 - \tfrac{1}{2}(k_z a)^2 \cos 2\phi] - j\,\tfrac{1}{2}(k_z a) \cos\phi\} \\ \sigma &\approx -2\,\varepsilon_o\, E_y^e \exp(-jk_x x)\ \{[\sin\phi - \tfrac{1}{8}(k_z a)^2 \sin 3\phi] - j\,\tfrac{1}{2}(k_z a) \sin 2\phi\} \end{aligned}\right\}$$

provided $k_z a \ll 1$.

## APPENDIX C

### Deviation from Quasi-static Distribution of Surface Charge Density

#### Amplitude

From equation (22b), the square of the magnitude of the surface charge is given by

$$|\sigma|^2 \approx K^2 \{[\sin\phi - \frac{1}{8}(k_z a)^2 \sin 3\phi]^2 + \frac{1}{4}(k_z a)^2 \sin^2 2\phi\}$$

where K is the factor outside the bracket in equation (22b). Expanding combining terms, and discarding a term in $(k_z a)^4$,

$$|\sigma|^2 \approx K^2[1 + \frac{1}{4}(k_z a)^2] \sin^2\phi$$

The static limiting case is

$$|\sigma_o| = K\,|\sin\phi|$$

The deviation from the static limit is

$$\delta_a = |\sigma| - |\sigma_o| = K\,|\sin\phi|\,\{[1 + \frac{1}{4}(k_z a)^2]^{1/2} - 1\}$$

The root mean square deviation averaged over $\phi$ is

$$[\overline{\delta_a^2}]^{1/2} = \frac{K}{\sqrt{2}}\{2 + \frac{1}{4}(k_z a)^2 - 2\,[1 + \frac{1}{4}(k_z a)^2]^{1/2}\}^{1/2}$$

$$\approx \frac{K}{\sqrt{2}}\,[\frac{1}{4}(k_z a)^2]$$

Since the r.m.s. value for the static case is $K/\sqrt{2}$, the relative r.m.s. deviation is

$$d_a = \frac{1}{4}(k_z a)^2 = \frac{1}{4}(2\pi\frac{a}{\lambda_z})^2 = \left(\frac{\pi a}{\lambda_z}\right)^2$$

## REFERENCES


1. Wright, C.A., and Puchstein, A.F., Telephone Communication, Chapter XVII. McGraw-Hill, New York, 1925.

2. Peek, F.W., Jr., Dielectric Phenomena in High Voltage Engineering, Chapter IX. McGraw-Hill, New York, 1929.

3. Harrison, C.W., Jr., "Generalized Theory of Impedance-Loaded Multiconductor Transmission Lines in an Incident Field", Sandia Laboratories Report No. SC-R-71 3303, July 1971, Interaction Note 82.

4. Licking, L.D., "The Response of an Arbitrarily Terminated Wire Near the Side of a Conducting Cylinder," Sandia Laboratories Report No. SC-TM-71 0319, July 1971, Interaction Note 94.

5. Frankel, S., "Response of a Multiconductor Transmission Line to Excitation by an Arbitrary Monochromatic Impressed Field Along the Line," Sandia Laboratories Report No. SC-CR-71 5076, April 1971. Also, AFWL Interaction Note No. 80, April 1971.

6. Taylor, C.D., R.S. Satterwhite, and C. W. Harrison, Jr., "The Response of a Terminated Two-Wire Transmission Line Excited by a Non-Uniform Electromagnetic Field," Sandia Laobratories Report SC-R-65-978A, November 1965, Interaction Note 66.

7. Harrison, C. W., Jr., "Bounds on the Load Currents of Exposed One- and Two-Conductor Transmission Lines Electromagnetically Coupled to a Rocket," IEEE EMC-14, No. 1, February 1972, pp. 4-9, SC-R-72-3209, Apr 1971, Interaction Note 71.

8. Frankel, S., "TEM Response of a Multiwire Transmission Line (Cable) to an Externally Impressed Electromagnetic Field: Recipe for Analysis," Frankel Associates Technical Memorandum FA-181, February 1972.

9. Kaden, H., Wirbelströme und Schirmung in der Nachrichtentechnik, pp. 118 ff. Springer-Verlag, Berlin, 1959.

10. Harrington, R.F., Time-Harmonic Electromagnetic Fields, Chapter 5, McGraw-Hill, New York, 1961.

11. Jahnke, E., and Emde, F., Tables of Functions with Formulae and Curves, Third Edition, B.G. Teubner, Leipzig. Also, Fourth Edition, Dover Publications, New York, 1945.

12. Smythe, W.R., Static and Dynamic Electricity, McGraw-Hill, New York, 1939, First Edition, Section 4.03, pp. 65 ff.

13. Frankel, S., "Field Coupling Parameters for a Single Round Wire Close to a Ground Plane or Two Large Round Wires in Free Space", Harry Diamond Laboratories Report No. HDL-TM-72-14, April 1972, Interaction Note 120.