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On the Singularity Expansion Method for the Case of First Order Poles

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Abstract

This note presents the singularity expansion terms for first order poles, deriving them from an appropriate, but somewhat arbitrary, integral equation. After presenting the equations for the natural frequencies, modes and coupling vectors, the coupling coefficient is considered. Various forms of coupling coefficients are derived from the integral equation and coupling coefficients are also generalized to coupling operators. All of these coupling coefficients and operators satisfy the pole residue requirement but have different forms for complex frequencies not equal to the appropriate natural frequencies. A convenient notation for the integral operations in the formulas is introduced to simplify the form of the equations and make summaries somewhat more compact for presentation.

singularity expansion method (SEM), first order poles, electromagnetic coupling

I. Introduction

The singularity expansion method (SEM) has been discussed in several notes in various of its theoretical aspects and the electromagnetic responses of a few specific geometries have been calculated using this technique.[1-7] The basic idea involved in this technique is to expand the solution to an electromagnetic interaction problem (antenna problem, propagation problem, or any linear problem (not necessarily electromagnetic)) in terms of the singularities of the response in the complex frequency plane. Such singularities can take various forms such as poles, branch points (and associated branch cuts), essential singularities, and singularities at infinity. For restricted classes of objects, such as finite size objects in free space, these s-plane singularities are limited to poles and possible singularities at infinity. However, in certain cases (at least) the coupling coefficient form(s) can be chosen such that the singularities at infinity are not present or are in effect contained in the coupling coefficients.

This note considers only the case of first order poles as the s-plane singularities. Other forms of singularities are ignored. There may be higher order poles in special cases but such are neglected here. This note is then somewhat tutorial in that it sets down the format of the singularity expansion in a simplified form. Various subtleties are neglected in this note in the interest of simplicity of presentation.

The general problem is illustrated in figure 1. We have some finite size object described by $\vec{r} \in V$ or S although one might also include infinite size objects if other than poles are included in the response. Figure 1 shows a plane wave incident on the object. However, other types of electromagnetic problems are also appropriate for singularity expansion such as antennas driven at some source region (gap) for which some source field takes the place of the incident wave in an interaction (or scattering) problem. In general singularity expansion would seem applicable to any physical system described by linear equations (including scalar, vector, tensor quantities) such as in acoustics, mechanics, circuits, etc.

In constructing a singularity expansion of the solution of some electromagnetic boundary value problem one needs to first formulate the problem in some set of equations which admit of a unique solution. There are many ways to go about this. One of the more general ways to formulate the problem is to set up an integral equation in the form

$$\int_V \tilde{\vec{\vec{K}}}(\vec{r},\vec{r}\,';s) \cdot \tilde{\vec{J}}(\vec{r}\,',s)\,dV' = \tilde{\vec{F}}(\vec{r},s) \qquad \vec{r} \in V \text{ or } S \tag{1.1}$$

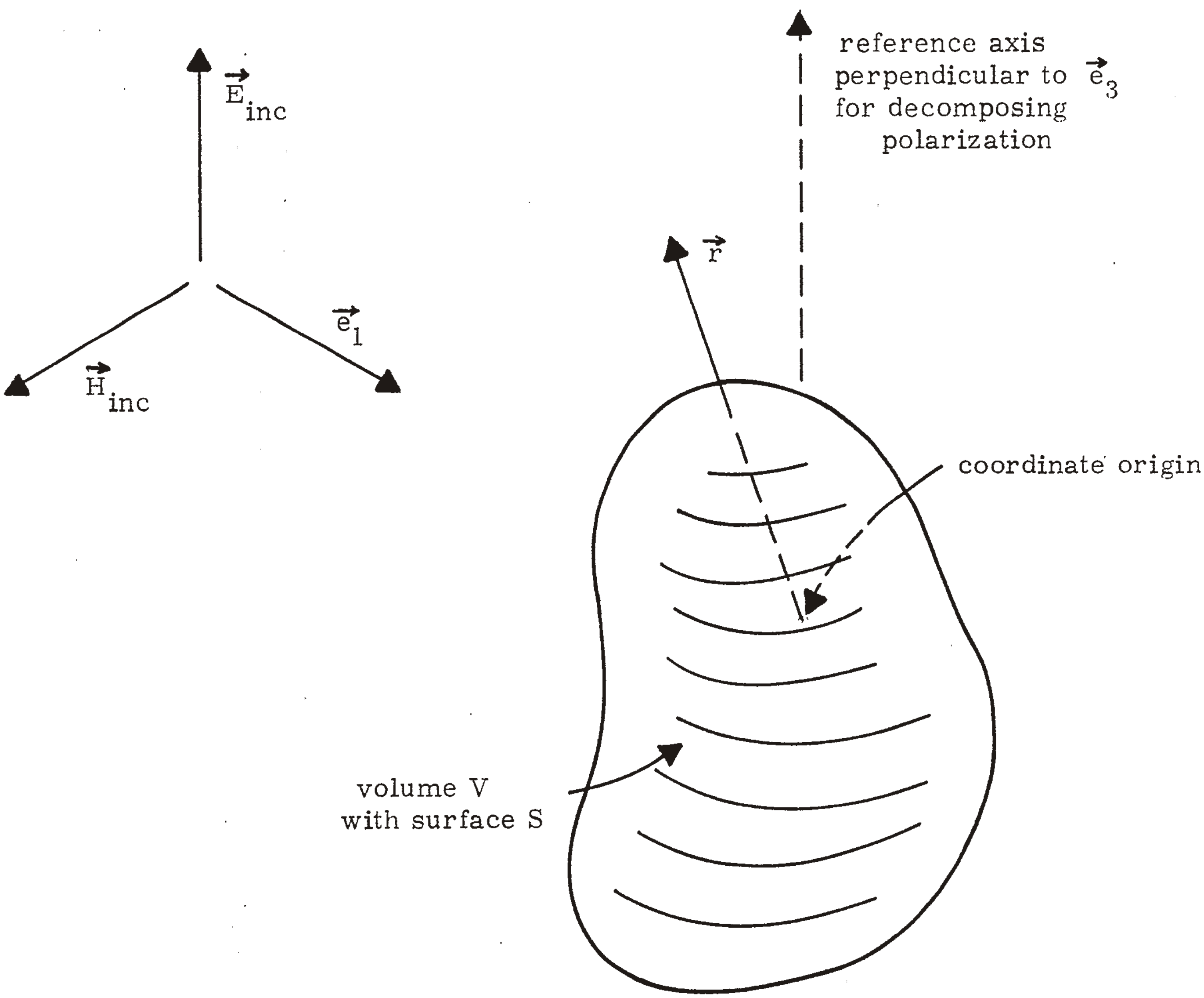

Figure 1. Interaction of an Incident Electromagnetic Wave with an Object

where s is the Laplace transform variable (with respect to t) and a tilde ~ over a quantity indicates the Laplace transform (two sided). In this equation $\tilde{\vec{J}}$ is the current density response to an incident (or forcing) function $\tilde{\vec{I}}$ which might be an incident electric or magnetic field or some other derived electromagnetic quantity which is known (a "given" of the problem). The kernel $\tilde{\vec{\vec{K}}}$ is a function of two coordinate sets and s and relates the response to the forcing function. Here $\vec{r}\,'$ is taken as the object coordinates over which the integral operator (the kernel with the integral) operates where $\vec{r}\,' \in$ V or S. Note that in some cases the integral equation reduces to one over a surface S with dS' replacing dV' and a surface current density $\vec{J}_s$ replacing $\vec{J}$. Without loss of generality we can consider the case of surface integral equations in the general form of equation 1.1.

The form of the singularity expansion of the current density response for an assumed incident delta function uniform plane wave (and some other forms of incident waves as well) is[1]

$$\tilde{\vec{U}}_p^{(\vec{J})}(\vec{r}\,',s) = \sum_\alpha \tilde{\eta}_\alpha(\vec{e}_1,s)\vec{\nu}_\alpha^{(\vec{J})}(\vec{r}\,')(s-s_\alpha)^{-n_\alpha} + \tilde{\vec{W}}_p^{(\vec{J})}(\vec{e}_1,\vec{r}\,',s) \quad (1.2)$$

where $\tilde{\vec{W}}_p$ contains the singularities for $s \to \infty$, any essential singularities, and any branch cuts. The subscript p refers to one of two orthogonal plane waves arriving with propagation direction $\vec{e}_1$. For other than plane waves the notation can be changed appropriately. For our present purposes $\tilde{\vec{W}}_p$ will generally be ignored and n_α will be assumed to be 1.

The current density response for plane wave incidence is constructed as

$$\begin{aligned} \tilde{\vec{J}}(\vec{r}\,',s) &= \tilde{\vec{J}}_2(\vec{r}\,',s) + \tilde{\vec{J}}_3(\vec{r}\,',s) \\ \tilde{\vec{J}}_p(\vec{r}\,',s) &= E_o \Sigma \tilde{f}_p(s)\tilde{\vec{U}}_p^{(\vec{J})}(\vec{r}\,',s) \end{aligned} \quad (1.3)$$

where Σ is a normalizing constant with dimensions of conductivity (Sm^{-1}). The singularity expansion is carried further by separating the singularities associated with the object in $\tilde{\vec{U}}_p$ from those associated with the incident waveform in $\tilde{f}_p$ as

$$\begin{aligned} \tilde{\vec{V}}_p^{(\vec{J})}(\vec{r}\,',s) &= \tilde{f}_p(s)\tilde{\vec{U}}_p^{(\vec{J})}(\vec{r}\,',s) \\ &= \tilde{\vec{V}}_{p_o}^{(\vec{J})}(\vec{r}\,',s) + \tilde{\vec{V}}_{p_w}^{(\vec{J})}(\vec{r}\,',s) \end{aligned} \quad (1.4)$$

where the object part for first order object poles is

$$\tilde{\vec{V}}^{(\vec{J})}_{p_o}(\vec{r}\,',s) = \sum_{\alpha} \tilde{f}_p(s_\alpha)\tilde{\eta}_\alpha(\vec{e}_1,s)\vec{\nu}^{(\vec{J})}_\alpha(\vec{r}\,')(s - s_\alpha)^{-1} \qquad (1.5)$$

and the waveform part is

$$\tilde{\vec{V}}^{(\vec{J})}_{p_w}(\vec{r}\,',s) = \sum_{\alpha} \tilde{\eta}_\alpha(\vec{e}_1,s)\vec{\nu}^{(\vec{J})}_\alpha(\vec{r}\,')\frac{\tilde{f}_p(s) - \tilde{f}_p(s_\alpha)}{s - s_\alpha} \qquad (1.6)$$

where the object and waveform singularities have been assumed to be separate in the s-plane.

This brings us to the point of some definitions of the terms used in the singularity expansion.

s_α — Natural frequency
This is a complex frequency for which the problem has a nontrivial solution with no forcing function. Each s_α depends only on the object and applies all over the object.

$\vec{\nu}_\alpha$ — Natural mode
This is the nontrivial solution at s_α (for $\vec{J}$). A superscript indicates which EM quantity it applies to—often a current or current density in which case it may be suppressed. This can be a vector or scalar (such as for charge) or anything else as appropriate. It depends only on the object parameters including where one samples it on the object.

$\tilde{\eta}_\alpha$ — Coupling coefficient
This gives the strength of the natural oscillation in terms of the incident wave parameters. It depends on the object parameters but not on where one samples the result on the object. It can be frequency dependent as long as it satisfies the residue requirement at s_α. More general forms of the coupling coefficient are also possible.

$\vec{\mu}_\alpha$ — Coupling vector
This is a term which is used in obtaining the coupling coefficient (evaluated at the natural frequency). For symmetric integral equation kernels the coupling vector is the same as the natural mode (for $\vec{J}$) within a constant multiplier.

$\vec{\tilde{U}}_p$ Delta function response of object (normalized) A superscript indicates which quantity is being considered.

f_p Incident waveform

$\vec{V}_p$ Response including incident waveform

$\vec{V}_{p_o}$ Object part of the response

$\vec{V}_{p_w}$ Incident waveform part of the response

Rewriting equation 1.1 for the response due to an incident delta function plane wave of fixed polarization gives

$$\int_V \tilde{\vec{\vec{K}}}(\vec{r},\vec{r}\,';s) \cdot \tilde{\vec{U}}_p^{(\vec{J})}(\vec{r}\,',s)\,dV' = \tilde{\vec{I}}_p(\vec{r},s) \;, \quad \vec{r} \in V \text{ or } S \qquad (1.7)$$

where $\tilde{\vec{I}}_p$ is the forcing function for an appropriately normalized incident delta function plane wave. It is not necessarily the plane wave itself but some modification of it depending on the form of integral equation used.

Appendix A defines an implied integral notation to write the various integral operators. In this form and dropping all additional arguments, subscripts, superscripts, etc. the problem of the singularity expansion with first order poles in the delta function response reduces to

$\langle \tilde{\vec{\vec{K}}} ; \tilde{\vec{U}} \rangle = \tilde{\vec{I}}$	Integral equation	(1.8)
$\tilde{\vec{U}} = \sum_\alpha \tilde{\eta}_\alpha \vec{\nu}_\alpha (s - s_\alpha)^{-1}$	Form of solution	

where the domain of integration is V. Again this ignores many complexities but one purpose of this note is to simplify and abbreviate the singularity expansion for the reader. The results of reference 1 are presented here in continuous operator form instead of the numerical vector and matrix form in that reference. The remainder of this note deals with finding the terms in the second of equations 1.8 from the terms of the first of these equations, as well as generalizing the form of the coupling coefficient somewhat.

II. Singularity Expansion with the Coupling Coefficients Evaluated at the Natural Frequencies

To find the terms in the second of equations 1.8 first find the solutions of the homogeneous equation

$$\int_V \overleftrightarrow{K}(\vec{r},\vec{r}\,';s_\alpha) \cdot \vec{\nu}_\alpha^{(\vec{J})}(\vec{r}\,')dV' = \vec{0} \tag{2.1}$$

This gives the natural frequencies s_α and the natural modes $\vec{\nu}_\alpha$. Next with $s = s_\alpha$ find for each α the coupling vectors $\vec{\mu}_\alpha$ from

$$\int_V \vec{\mu}_\alpha(\vec{r}) \cdot \overleftrightarrow{K}(\vec{r},\vec{r}\,';s_\alpha)dV = \vec{0} \tag{2.2}$$

Note that if $\overleftrightarrow{K}$ is symmetric then $\vec{\mu}_\alpha(\vec{r})$ is a complex constant times $\vec{\nu}_\alpha(\vec{r})$.

Next make a Taylor series expansion of the kernel near s_α as

$$\begin{aligned} \overleftrightarrow{K}(\vec{r},\vec{r}\,';s) &= \sum_{\ell=0}^{\infty}(s - s_\alpha)^\ell \overleftrightarrow{K}_{\alpha_\ell}(\vec{r},\vec{r}\,') \\ \overleftrightarrow{K}_{\alpha_\ell}(\vec{r},\vec{r}\,') &= \frac{1}{\ell!}\left[\frac{d^\ell}{ds^\ell}\overleftrightarrow{K}(\vec{r},\vec{r}\,';s)\right]_{s=s_\alpha} \end{aligned} \tag{2.3}$$

where the kernel used in equations 2.1 and 2.2 is just

$$\overleftrightarrow{K}(\vec{r},\vec{r}\,';s_\alpha) = \overleftrightarrow{K}_{\alpha_o}(\vec{r},\vec{r}\,') \tag{2.4}$$

Similarly expand $\overleftrightarrow{U}_p$ near s_α as

$$\overleftrightarrow{U}_p^{(\vec{J})}(\vec{r}\,',s) = \tilde{\eta}_\alpha(\vec{e}_1,s_\alpha)\vec{\nu}_\alpha^{(\vec{J})}(\vec{r}\,')(s - s_\alpha)^{-1} + \overleftrightarrow{U}_\alpha'(\vec{r}\,',s) \tag{2.5}$$

where $\overleftrightarrow{U}_\alpha'$ is analytic at s_α. Note that p is readily absorbed into the α index set. Then expand the term associated with the delta function source in a Taylor series as

$$\tilde{\vec{I}}_p(\vec{r},s) = \sum_{\ell=0}^{\infty}(s - s_\alpha)^\ell \vec{I}_{\alpha_\ell}(\vec{r})$$

$$\vec{I}_{\alpha_\ell}(\vec{r}) = \frac{1}{\ell!}\left[\frac{d^\ell}{ds^\ell}\tilde{\vec{I}}_p(\vec{r},s)\right]_{s=s_\alpha} \tag{2.6}$$

$$\tilde{\vec{I}}_p(\vec{r},s_\alpha) = \vec{I}_{\alpha_0}(\vec{r})$$

Now multiply the expansions for $\tilde{\vec{\vec{K}}}$ and $\tilde{\vec{U}}_p$ and equate to the expansion for $\tilde{\vec{I}}_p$. The term proportional to $(s - s_\alpha)^{-1}$ gives the result of equation 2.1. The constant term $(s - s_\alpha)^0$ gives

$$\int_V \vec{\vec{K}}_{\alpha_0}(\vec{r},\vec{r}\,') \cdot \tilde{\vec{U}}_\alpha'(\vec{r}\,',s_\alpha)dV' + \tilde{\eta}_\alpha(\vec{e}_1,s_\alpha)\int_V \vec{\vec{K}}_{\alpha_1}(\vec{r},\vec{r}\,') \cdot \vec{\nu}_\alpha^{(\vec{J})}(\vec{r}\,')dV' = \tilde{\vec{I}}_{\alpha_0}(\vec{r}) \tag{2.7}$$

Multiply on the left by $\vec{\mu}_\alpha(\vec{r})$ in a dot product sense and integrate over $\vec{r}$. Note that

$$\int_V\int_V \vec{\mu}_\alpha(\vec{r}) \cdot \vec{\vec{K}}_{\alpha_0}(\vec{r},\vec{r}\,') \cdot \tilde{\vec{U}}_\alpha'(\vec{r}\,',s_\alpha)dV'dV = \int_V \vec{0} \cdot \tilde{\vec{U}}_\alpha'(\vec{r}\,',s_\alpha)dV' = 0 \tag{2.8}$$

giving

$$\tilde{\eta}_\alpha(\vec{e}_1,s_\alpha) = \frac{\int_V \vec{\mu}_\alpha(\vec{r}) \cdot \vec{I}_{p_{\alpha_0}}(\vec{r})dV}{\int_V\int_V \vec{\mu}_\alpha(\vec{r}) \cdot \vec{\vec{K}}_{\alpha_1}(\vec{r},\vec{r}\,') \cdot \vec{\nu}_\alpha^{(\vec{J})}(\vec{r}\,')dV'dV} \tag{2.9}$$

This is the general form to which any coupling coefficients for first order poles must reduce in order to preserve the required pole residues. This is only a requirement at s_α and for other $s \neq s_\alpha$ the form of $\tilde{\eta}_\alpha$ can be somewhat different.

Note that in some cases the natural modes for a given s_α from equation 2.1 might be degenerate. In that case the index set α can have an index to indicate which of the degenerate modes is being considered. An orthogonal set of the degenerate modes can readily be obtained if one knows the order of the degeneracy. For symmetric kernels one can let $\vec{\mu}_\alpha = \vec{\nu}_\alpha$ to get the coupling coefficients for the separate degenerate modes, although almost any $\vec{\mu}_\alpha$ will still work.

Now let us summarize these results leading to the coupling coefficients evaluated at the natural frequencies. Drop the subscripts α, the superscripts, and coordinates, and put the results in the implied integral (or symmetric product) form. Note that these results are to be applied for each α, including the different s_α, p, and degenerate modes.

$$\left\langle \tilde{\vec{K}}_o \, ; \vec{\nu} \right\rangle = \vec{0}$$

$$\left\langle \vec{\mu} \, ; \tilde{\vec{K}}_o \right\rangle = \vec{0}$$

Natural frequencies, natural modes, and coupling vectors

$$\tilde{\vec{K}} = \sum_{\ell=o}^{\infty} (s - s_\alpha)^{\ell} \tilde{\vec{K}}_\ell$$

$$\tilde{\vec{U}} = \tilde{\eta}\vec{\nu}(s - s_\alpha)^{-1} + \tilde{\vec{U}}'$$

$$\tilde{\vec{I}} = \sum_{\ell=o}^{\infty} (s - s_\alpha)^{\ell} \tilde{\vec{I}}_\ell \tag{2.10}$$

Expansion near s_α

$$\tilde{\eta}(s_\alpha) = \frac{\left\langle \vec{\mu} \, ; \tilde{\vec{I}}_o \right\rangle}{\left\langle \vec{\mu} \, ; \tilde{\vec{K}}_1 \, ; \vec{\nu} \right\rangle}$$

Coupling coefficient at s_α

III. More General Forms of the Coupling Coefficients

In section II the value of the coupling coefficient was found at s_α to satisfy the residue requirement at each first order pole. The coupling coefficients can have various forms for $s \neq s_\alpha$ and still satisfy the residue requirement at s_α. To consider these forms in a unified manner consider two integral operators, $\overset{\rightrightarrows}{S}$ and $\overset{\rightrightarrows}{T}$, inverse to each other as

$$\int_V \overset{\rightrightarrows}{S}(\vec{r},\vec{r}\,'';s) \cdot \overset{\rightrightarrows}{T}(\vec{r}\,'',\vec{r}\,';s)\,dV'' = \overset{\rightrightarrows}{I}\delta(\vec{r} - \vec{r}\,')$$

$$\langle \overset{\rightrightarrows}{S} ; \overset{\rightrightarrows}{T} \rangle = \overset{\rightrightarrows}{I}\delta(\vec{r} - \vec{r}\,') \tag{3.1}$$

where $\overset{\rightrightarrows}{I}$ is the identity matrix, $\overset{\rightrightarrows}{I}\delta(\vec{r} - \vec{r}\,')$ is the identity operator, and $\delta(\vec{r} - \vec{r}\,')$ is the delta function for three spatial dimensions.

Convert the integral equation for the delta function response from equations 1.8 by a similarity transformation to a new integral equation as

$$\langle \overset{\rightrightarrows}{K} ; \tilde{\tilde{U}} \rangle = \tilde{\tilde{I}}$$

$$\langle \overset{\rightrightarrows}{S} ; \overset{\rightrightarrows}{K} ; \overset{\rightrightarrows}{T} ; \overset{\rightrightarrows}{S} ; \tilde{\tilde{U}} \rangle = \langle \overset{\rightrightarrows}{S} ; \tilde{\tilde{I}} \rangle \tag{3.2}$$

$$\langle\langle \overset{\rightrightarrows}{S} ; \overset{\rightrightarrows}{K} ; \overset{\rightrightarrows}{T} \rangle ; \langle \overset{\rightrightarrows}{S} ; \tilde{\tilde{U}} \rangle\rangle = \langle \overset{\rightrightarrows}{S} ; \tilde{\tilde{I}} \rangle$$

As this indicates we have a new transformed integral equation with

kernel $\langle \overset{\rightrightarrows}{S} ; \overset{\rightrightarrows}{K} ; \overset{\rightrightarrows}{T} \rangle$, response $\langle \overset{\rightrightarrows}{S} ; \tilde{\tilde{U}} \rangle$, and

forcing function $\langle \overset{\rightrightarrows}{S} ; \tilde{\tilde{I}} \rangle$.

One can expand the solution of this integral equation in much the same manner as in section II, concentrating on the pole residues. Write the solution of the integral equation (at least for the first order pole contributions) then as

$$\left\langle \overset{\rightarrow}{\vec{S}} ; \overset{\rightarrow}{\tilde{U}} \right\rangle = \sum_{\alpha} \tilde{\eta}'_{\alpha} \vec{\nu}'_{\alpha} (s - s_{\alpha})^{-1} \tag{3.3}$$

where the modified natural modes and coupling vectors are

$$\begin{aligned} \vec{\nu}'_{\alpha} &= \left\langle \overset{\rightarrow}{\vec{S}}(s_{\alpha}) ; \vec{\nu}_{\alpha} \right\rangle \\ \vec{\mu}'_{\alpha} &= \left\langle \vec{\mu}_{\alpha} ; \overset{\rightarrow}{\vec{T}}(s_{\alpha}) \right\rangle \end{aligned} \tag{3.4}$$

as can be readily seen by operating these on the transformed kernel with $s = s_{\alpha}$ and noting the combinations of $\overset{\rightarrow}{\vec{S}}$ and $\overset{\rightarrow}{\vec{T}}$ that collapse allowing one to invoke equations 2.1 and 2.2.

The solution for the modified coupling coefficient at s_{α} carries over directly from equation 2.9 to give

$$\tilde{\eta}'_{\alpha}(s_{\alpha}) = \frac{\left\langle \vec{\mu}'_{\alpha} ; \overset{\rightarrow}{\vec{S}}(s_{\alpha}) ; \vec{I}_{o} \right\rangle}{\left\langle \vec{\mu}'_{\alpha} ; \frac{\partial}{\partial s} \left\langle \overset{\rightarrow}{\vec{S}} ; \overset{\rightarrow}{\vec{K}} ; \overset{\rightarrow}{\vec{T}} \right\rangle \Big|_{s=s_{\alpha}} ; \vec{\nu}'_{\alpha} \right\rangle} \tag{3.5}$$

Expand the derivative of the kernel at $s = s_{\alpha}$ as

$$\begin{aligned} \frac{\partial}{\partial s} \left\langle \overset{\rightarrow}{\vec{S}} ; \overset{\rightarrow}{\vec{K}} ; \overset{\rightarrow}{\vec{T}} \right\rangle &= \left\langle \overset{\rightarrow}{\vec{S}} ; \overset{\rightarrow}{\vec{K}}_{1} ; \overset{\rightarrow}{\vec{T}} \right\rangle \\ &+ \left\langle \frac{\partial}{\partial s} \overset{\rightarrow}{\vec{S}} ; \overset{\rightarrow}{\vec{K}}_{o} ; \overset{\rightarrow}{\vec{T}} \right\rangle + \left\langle \overset{\rightarrow}{\vec{S}} ; \overset{\rightarrow}{\vec{K}}_{o} ; \frac{\partial}{\partial s} \overset{\rightarrow}{\vec{T}} \right\rangle \end{aligned} \tag{3.6}$$

Using equations 3.4 with 3.1 for $\overset{\rightarrow}{\vec{S}}$ and $\overset{\rightarrow}{\vec{T}}$ together with equations 2.1 and 2.2 then we have at $s = s_{\alpha}$

$$\tilde{\eta}'_{\alpha}(s_{\alpha}) = \frac{\left\langle \vec{\mu} ; \vec{I}_{o} \right\rangle}{\left\langle \vec{\mu} ; \overset{\rightarrow}{\vec{K}}_{1} ; \vec{\nu} \right\rangle} = \tilde{\eta}(s_{\alpha}) \tag{3.7}$$

Operating on equation 3.3 by $\overset{\rightarrow}{\vec{T}}$ gives

$$\tilde{\vec{U}} = \sum_\alpha \tilde{\eta}_\alpha(s_\alpha)\left\langle \vec{\vec{T}}(s)\,;\,\vec{\vec{S}}(s_\alpha)\,;\,\vec{\nu}_\alpha \right\rangle (s - s_\alpha)^{-1} \tag{3.8}$$

The more general coupling coefficient resulting from this equation is then an operator of the form

$$\vec{\vec{\eta}}_\alpha(\vec{r},\vec{r}\,';\vec{e}_1,s) = \tilde{\eta}_\alpha(s_\alpha)\left\langle \vec{\vec{T}}(s)\,;\,\vec{\vec{S}}(s_\alpha)\right\rangle \tag{3.9}$$

which we term a class 1 coupling operator and which for $s = s_\alpha$ reduces to

$$\vec{\vec{\eta}}_\alpha(\vec{r},\vec{r}\,';\vec{e}_1,s_\alpha) = \tilde{\eta}_\alpha(s_\alpha)\vec{\vec{I}}\delta(\vec{r} - \vec{r}\,') \tag{3.10}$$

which is consistent with the requirement of equations 2.10 for the coupling coefficient at a first order pole. This form of coupling coefficient has the disadvantage in its general form of effectively altering the natural mode for frequencies away from s_α, although there may be some special applications for which this is useful.

If $\tilde{\vec{\vec{S}}}$ is used as a matrix instead of a matrix operator, and similarly $\tilde{\vec{\vec{T}}}$ with the requirement

$$\vec{\vec{T}}(s) \cdot \vec{\vec{S}}(s) = \vec{\vec{I}} \tag{3.11}$$

then the integral equation and its pole terms reduce to

$$\left\langle \vec{\vec{S}} \cdot \vec{\vec{K}} \cdot \vec{\vec{T}}\,;\,\vec{\vec{S}} \cdot \tilde{\vec{U}} \right\rangle = \vec{\vec{S}} \cdot \vec{\vec{I}}$$

$$\vec{\nu}\,'_\alpha = \vec{\vec{S}}(s_\alpha) \cdot \vec{\nu}_\alpha$$

$$\vec{\mu}\,'_\alpha = \vec{\mu}_\alpha \cdot \vec{\vec{T}}(s_\alpha)$$

$$\tag{3.12}$$

$$\tilde{\eta}'(s_\alpha) = \tilde{\eta}(s_\alpha) = \frac{\langle \vec{\mu} ; \vec{I}_o \rangle}{\langle \vec{\mu} ; \vec{K}_1 ; \vec{\nu} \rangle}$$

$$\tilde{\vec{\vec{U}}} = \sum_\alpha \tilde{\eta}_\alpha(s_\alpha)\tilde{\vec{\vec{T}}}(s) \cdot \tilde{\vec{\vec{S}}}(s_\alpha) \cdot \vec{\nu}_\alpha(s - s_\alpha)^{-1}$$

If $\tilde{\vec{\vec{T}}}$ is further reduced to a scalar coefficient then we have

$$\tilde{T}(s)\tilde{S}(s) = 1$$

$$\langle \tilde{\vec{\vec{K}}} ; \tilde{S}\tilde{\vec{\vec{U}}} \rangle = \tilde{S}\vec{I}$$

$$\tilde{\eta}(s_\alpha) = \frac{\langle \vec{\mu} ; \vec{I}_o \rangle}{\langle \vec{\mu} ; \vec{K}_1 ; \vec{\nu} \rangle}$$

$$\tilde{\vec{\vec{U}}} = \sum_\alpha \frac{\tilde{S}(s)}{\tilde{S}(s_\alpha)} \tilde{\eta}_\alpha(s_\alpha)\vec{\nu}_\alpha(s - s_\alpha)^{-1} \tag{3.13}$$

$$= \tilde{S}(s) \sum_\alpha \tilde{c}_\alpha \vec{\nu}_\alpha(s - s_\alpha)^{-1}$$

$$c_\alpha \equiv \tilde{T}(s_\alpha)\tilde{\eta}(s_\alpha) = \tilde{T}(s_\alpha)\frac{\langle \vec{\mu} ; \vec{I}_o \rangle}{\langle \vec{\mu} ; \vec{K}_1 ; \vec{\nu} \rangle}$$

where c_α is also referred to as a coupling coefficient. This is also written as

$$\tilde{\vec{U}} = \sum_{\alpha} \tilde{\eta}_{\alpha}(s)\vec{\nu}_{\alpha}(s - s_{\alpha})^{-1}$$

$$\tilde{\eta}_{\alpha}(s) = \frac{\tilde{S}(s)}{\tilde{S}(s_{\alpha})}\tilde{\eta}_{\alpha}(s_{\alpha}) \qquad (3.14)$$

$$= \frac{\tilde{S}(s)}{\tilde{S}(s_{\alpha})} \frac{\langle \vec{\mu} ; \tilde{\vec{I}}_{o} \rangle}{\langle \vec{\mu} ; \tilde{\vec{K}}_{1} ; \vec{\nu} \rangle}$$

There are various forms that $\tilde{S}$ can take as discussed in a previous note.[1] Let it have the form

$$\tilde{S}(s) = \frac{1}{\tilde{T}(s)} = e^{-st'}$$

$$\tilde{\eta}_{\alpha}(s) = e^{(s_{\alpha}-s)t'}\tilde{\eta}_{\alpha}(s_{\alpha}) \qquad (3.15)$$

giving

$$\boxed{\tilde{\eta}_{\alpha}(s) = e^{(s_{\alpha}-s)t'} \frac{\langle \vec{\mu} ; \tilde{\vec{I}}_{o} \rangle}{\langle \vec{\mu} ; \tilde{\vec{K}}_{1} ; \vec{\nu} \rangle}} \qquad \text{Class 1 coupling coefficient for turn-on time } t' \qquad (3.16)$$

where t' is some chosen turn-on time for the solution, each term being zero in time domain for $t < t'$. Some choices for our coupling coefficients in class 1 are given by choosing t' as

$t' = t_o$	Type 1:	time when incident wave first reaches object
$t' = t_i$	Type 2:	time when incident wave first reaches position of interest on object (varies with $\vec{r}'$)
$t' = t_r$	Type 3:	time when resultant fields can first reach position of interest on object (normally $t_r \geq t_i$)

(3.17)

All three of these turn-on times satisfy the requirement of having the solution terms individually non zero for times at or before they are needed. If one were to take $t' > t_r$ then one would clearly need the additional entire function $\vec{W}$ as in equation 1.2 to give a solution for $t_r < t < t'$.

The basic feature of these coupling coefficient forms comes from expanding the transformed integral equation (equations 3.2) by expanding

the transformed kernel $\langle \tilde{\vec{S}} ; \tilde{\vec{K}} ; \tilde{\vec{T}} \rangle$,

the transformed response $\langle \tilde{\vec{S}} ; \tilde{\vec{U}} \rangle$, and

the transformed excitation $\langle \tilde{\vec{S}} ; \tilde{\vec{I}} \rangle$

around $s = s_\alpha$. Call the coupling coefficients (operators, etc.) obtained this way class 1 coupling coefficients. While this class is defined by expanding $\tilde{\vec{U}}$ and

$$\langle \tilde{\vec{S}} , \tilde{\vec{I}} \rangle ,$$

define another class (class 2) by expanding $\tilde{\vec{U}}$ while keeping

$$\langle \tilde{\vec{S}} , \tilde{\vec{I}} \rangle$$

unexpanded. This will still give the correct coupling coefficients at the natural frequencies but have a different form for other frequencies. This is equivalent to expanding the inverse operator (inverse of

$$\langle \tilde{\vec{S}} ; \tilde{\vec{K}} ; \tilde{\vec{T}} \rangle)$$

in terms of poles $(s - s_\alpha)^{-1}$ (plus any other singularities) and applying each term to the transformed forcing function

$$\langle \tilde{\vec{S}} ; \tilde{\vec{I}} \rangle$$

(unexpanded).

Following the same steps, equations 3.1 through 3.4, we have

$$\tilde{\eta}'_\alpha(s) = \frac{\langle \vec{\mu}'_\alpha ; \vec{\vec{S}}(s) ; \vec{\tilde{I}}(s) \rangle}{\langle \vec{\mu}'_\alpha ; \frac{\partial}{\partial s} \langle \vec{\vec{S}} ; \vec{\vec{K}} ; \vec{\vec{T}} \rangle \Big|_{s=s_\alpha} ; \vec{\nu}'_\alpha \rangle}$$

$$= \frac{\langle \vec{\mu} ; \vec{\vec{T}}(s_\alpha) ; \vec{\vec{S}}(s) ; \vec{\tilde{I}}(s) \rangle}{\langle \vec{\mu} ; \vec{\vec{K}}_1 ; \vec{\nu} \rangle} \quad (3.18)$$

Next operating on equation 3.3 by $\vec{\vec{T}}$ gives the new form for the delta function current density response as

$$\vec{\tilde{U}} = \sum_\alpha \tilde{\eta}'_\alpha(s) \langle \vec{\vec{T}}(s) ; \vec{\vec{S}}(s_\alpha) ; \vec{\nu} \rangle (s - s_\alpha)^{-1}$$

$$= \sum_\alpha \frac{\langle \vec{\mu} ; \vec{\vec{T}}(s_\alpha) ; \vec{\vec{S}}(s) ; \vec{\tilde{I}}(s) \rangle}{\langle \vec{\mu} ; \vec{\vec{K}}_1 ; \vec{\nu} \rangle} \langle \vec{\vec{T}}(s) ; \vec{\vec{S}}(s_\alpha) ; \vec{\nu} \rangle (s - s_\alpha)^{-1} \quad (3.19)$$

For $s = s_\alpha$ the corresponding individual term in the expansion reduces to the form in equations 2.10. This defines a more general coupling operator as

$$\vec{\vec{\eta}}_\alpha(\vec{r},\vec{r}';\vec{e}_1,s) = \frac{\langle \vec{\mu} ; \vec{\vec{T}}(s_\alpha) ; \vec{\vec{S}}(s) ; \vec{\tilde{I}}(s) \rangle}{\langle \vec{\mu} ; \vec{\vec{K}}_1 ; \vec{\nu} \rangle} \langle \vec{\vec{T}}(s) ; \vec{\vec{S}}(s_\alpha) \rangle \quad (3.20)$$

This is similar to the coupling operator in equations 3.7 and 3.9 except for the additional two operators with the excitation function and the dependence of the excitation function on s. For $s = s_\alpha$ this class 2 coupling operator reduces to exactly the same form as the class 1 coupling operator in equation 3.10.

If $\vec{\vec{S}}$ and $\vec{\vec{T}}$ are used as matrices instead of matrix operators we have

$$\vec{\vec{T}}(s) \cdot \vec{\vec{S}}(s) = \vec{\vec{I}}$$

$$\left\langle \vec{\vec{S}} \cdot \vec{\vec{K}} \cdot \vec{\vec{T}} ; \vec{\vec{S}} \cdot \tilde{\vec{U}} \right\rangle = \vec{\vec{S}} \cdot \tilde{\vec{I}}$$

$$\vec{\nu}'_\alpha = \vec{\vec{S}}(s_\alpha) \cdot \vec{\nu}_\alpha$$

$$\vec{\mu}'_\alpha = \vec{\mu}_\alpha \cdot \vec{\vec{T}}(s_\alpha) \tag{3.21}$$

$$\tilde{\eta}'_\alpha(s) = \frac{\left\langle \vec{\mu} \cdot \vec{\vec{T}}(s_\alpha) ; \vec{\vec{S}}(s) \cdot \tilde{\vec{I}}(s) \right\rangle}{\left\langle \vec{\mu} ; \vec{K}_1 ; \vec{\nu} \right\rangle}$$

$$\tilde{\vec{U}} = \sum_\alpha \tilde{\eta}'_\alpha(s) \vec{\vec{T}}(s) \cdot \vec{\vec{S}}(s_\alpha) \cdot \vec{\nu}_\alpha (s - s_\alpha)^{-1}$$

If $\vec{\vec{T}}$ is further reduced to a scalar coefficient then we have

$$\tilde{T}(s)\tilde{S}(s) = 1$$

$$\left\langle \vec{\vec{K}} ; \tilde{S}\tilde{\vec{U}} \right\rangle = \tilde{S}\tilde{\vec{I}}$$

$$\tilde{\eta}'_\alpha(s) = \frac{\tilde{S}(s)}{\tilde{S}(s_\alpha)} \frac{\left\langle \vec{\mu} ; \tilde{\vec{I}}(s) \right\rangle}{\left\langle \vec{\mu} ; \vec{K}_1 ; \vec{\nu} \right\rangle}$$

$$\tilde{\vec{U}} = \sum_\alpha \tilde{\eta}'_\alpha(s) \frac{\tilde{S}(s_\alpha)}{\tilde{S}(s)} \vec{\nu}_\alpha \tag{3.22}$$

$$= \sum_\alpha \frac{\left\langle \vec{\mu}_\alpha ; \tilde{\vec{I}}(s) \right\rangle}{\left\langle \vec{\mu}_\alpha ; \vec{K}_{\alpha_1} ; \vec{\nu}_\alpha \right\rangle} \vec{\nu}_\alpha (s - s_\alpha)^{-1}$$

Note how the scalars $\tilde{T}$ and $\tilde{S}$ have cancelled out of the final result. This gives a coupling coefficient for class 2 as

$$\tilde{\eta}_\alpha(\vec{e}_1,s) = \frac{\langle \vec{\mu} \,;\, \tilde{\vec{I}}(s) \rangle}{\langle \vec{\mu} \,;\, \vec{\vec{K}}_1 \,;\, \vec{\nu} \rangle}$$

Class 2 coupling coefficient for case of transformation operators reduced to scalars (3.23)

In reference 1 this is referred to as a type 4 coupling coefficient. Other more general forms of class 2 coupling operators as in equations 3.20 and 3.21 can also be considered in which case the transformation operators

$$\tilde{\vec{\vec{T}}} \text{ and } \tilde{\vec{\vec{S}}}$$

do not drop out so simply.

Classes 1 and 2 coupling coefficients are those which have been used in their simpler forms (scalar $\tilde{T}$ and $\tilde{S}$) in reports to date (references 1, 2, 4, 5, 6, 7) with some success in calculating responses of various objects. These are not the only classes of coupling coefficients that one could define. In defining different classes one merely expands certain terms in the transformed integral equation (equation 3.2) around s_α and leaves other terms unexpanded. In abbreviated form class 1 expands SKT, SU, and SI while class 2 only expands SKT and SU. The expansion of SKT, or at least K, is essential in finding the poles and of course SU or at least U must be expanded because this is the desired pole expansion that at least a major part of the singularity expansion is all about. This leaves other possibilities for defining classes of coupling coefficients or coupling operators. Expanding SKT one might expand SU and S (from SI), or SU and I (from SI) to obtain different forms. Similarly one might expand U (from SU) and any of S, I, and SI (from SI). In defining such classes of coupling operators one can combine some of the properties of class 1 and class 2 operators.

Another way to view the different classes of coupling operators (coefficients) is to write the general solution for the response from equation 3.20 as

$$\tilde{\vec{\vec{\eta}}}_\alpha(\vec{r},\vec{r}\,';\vec{e}_1,s) = \frac{\langle \vec{\mu} \,;\, \tilde{\vec{\vec{T}}} \,;\, \tilde{\vec{\vec{S}}} \,;\, \tilde{\vec{I}} \rangle}{\langle \vec{\mu} \,;\, \vec{\vec{K}}_1 \,;\, \vec{\nu} \rangle} \langle \tilde{\vec{\vec{T}}} \,;\, \tilde{\vec{\vec{S}}} \rangle$$

General form of coupling operator: T, S, I can have arguments s and/or s_α in any sequence (3.24)

Here let the excitation $\tilde{\vec{I}}$ and operators $\tilde{\vec{\vec{T}}}$ and $\tilde{\vec{\vec{S}}}$ have arguments s and/or s_α separately in various combinations. For example $\tilde{\vec{\vec{T}}}$ appears twice and might have argument s in one place and s_α in the other place. Note that for all such choices the coupling operator evaluated at $s = s_\alpha$ reduces to the form in equation 3.10 (with equation 3.7).

One then has many possible forms of coupling operators which satisfy the residue requirement at first order poles. Some are fairly simple and have been used already. Others are more complex but may have use for theoretical questions in the theory of the method. Various forms of coupling coefficients may make the additional entire function (see equation 1.2) zero. Clearly some forms of coupling coefficients make the final form of the expansion simpler (such as class 1 in equation 3.16), but these may not be the best forms for purposes of numerical accuracy as influenced by the rate of convergence of the singularity expansion series.

IV. Summary

We have now gone through the development of the singularity expansion for first order poles to obtain the natural frequencies, modes, and coupling vectors. The coupling coefficients come in several varieties and can also be generalized to coupling operators. Let us now summarize the results including the time domain forms.

The object delta function response is

$$\tilde{\vec{\vec{U}}}_p^{(\vec{J})}(\vec{r}',s) = \sum_\alpha \tilde{\eta}_\alpha(\vec{e}_1,s)\vec{\nu}_\alpha^{(\vec{J})}(\vec{r}')(s - s_\alpha)^{-1} + \tilde{\vec{\vec{W}}}_p(\vec{e}_1,\vec{r}',s) \tag{4.1}$$

where the additional function $\tilde{\vec{\vec{W}}}_p$ has been shown to be zero for certain objects with appropriate choice of coupling coefficients. The coupling coefficient is readily generalized to a coupling operator which operates on the natural mode. The natural frequencies, modes, and coupling vectors come from

$$\left\langle \vec{\vec{K}}_{\alpha_o} \; ; \; \vec{\nu}_\alpha \right\rangle = \vec{0} = \left\langle \vec{\mu}_\alpha \; ; \; \vec{\vec{K}}_{\alpha_o} \right\rangle \tag{4.2}$$

Dropping the additional function possibly associated with the delta function response we have the response to an incident waveform

$$\tilde{\vec{V}}_p^{(\vec{J})}(\vec{r}',s) = \tilde{f}_p(s)\tilde{\vec{U}}_p^{(\vec{J})}(\vec{r}',s) = \tilde{\vec{V}}_{p_o}^{(\vec{J})}(\vec{r}',s) + \tilde{\vec{V}}_{p_w}^{(\vec{J})}(\vec{r}',s)$$

$$\tilde{\vec{V}}_{p_o}^{(\vec{J})}(\vec{r}',s) = \sum_\alpha \tilde{f}_p(s_\alpha)\tilde{\eta}_\alpha(\vec{e}_1,s)\vec{\nu}_\alpha^{(\vec{J})}(\vec{r}')(s - s_\alpha)^{-1} \tag{4.3}$$

$$\tilde{\vec{V}}_{p_w}^{(\vec{J})}(\vec{r}',s) = \sum_\alpha \tilde{\eta}_\alpha(\vec{e}_1,s)\vec{\nu}_\alpha^{(\vec{J})}(\vec{r}')\frac{\tilde{f}_p(s) - \tilde{f}_p(s_\alpha)}{s - s_\alpha}$$

For an exponential waveform this is

$$\tilde{f}_p(s) = \frac{1}{s - s_w}\ , \qquad \tilde{f}_p(s_\alpha) = \frac{1}{s_\alpha - s_w}$$

$$\tilde{\vec{V}}^{(\vec{J})}_{p_w}(s) = \frac{1}{s - s_w}\sum_\alpha (s_w - s_\alpha)^{-1}\tilde{\eta}_\alpha(\vec{e}_1, s)\,\vec{\nu}^{(\vec{J})}_\alpha(\vec{r}\,') \tag{4.4}$$

For a step function waveform this reduces to

$$\tilde{f}_p(s) = \frac{1}{s}\ , \qquad \tilde{f}_p(s_\alpha) = \frac{1}{s_\alpha}$$

$$\tilde{\vec{V}}^{(\vec{J})}_{p_w} = \frac{1}{s}\sum_\alpha - \frac{1}{s_\alpha}\,\tilde{\eta}_\alpha(\vec{e}_1, s)\,\vec{\nu}^{(\vec{J})}_\alpha(\vec{r}\,') \tag{4.5}$$

There are various possible types of coupling coefficients. In the form of a more general coupling operator this is

$$\tilde{\vec{\vec{\eta}}}_\alpha(\vec{r}, \vec{r}\,'; \vec{e}_1, s) = \frac{\langle \vec{\mu}_\alpha\ ;\ \tilde{\vec{\vec{T}}}\ ;\ \tilde{\vec{\vec{S}}}\ ;\ \tilde{\vec{I}} \rangle}{\langle \vec{\mu}_\alpha\ ;\ \vec{K}_{\alpha_1}\ ;\ \vec{\nu}_\alpha \rangle}\langle \tilde{\vec{\vec{T}}}\ ;\ \tilde{\vec{\vec{S}}} \rangle \tag{4.6}$$

where $\tilde{\vec{\vec{T}}}$, $\tilde{\vec{\vec{S}}}$, and $\tilde{\vec{I}}$ can be regarded as functions of s or evaluated only at s_α in any combination desired. For coupling coefficients this has a somewhat less flexible form implied by making the term

$$\langle \tilde{\vec{\vec{T}}}\ ;\ \tilde{\vec{\vec{S}}} \rangle$$

reduce to a scalar times the identity operator for all s. For $s = s_\alpha$ all the coupling operator forms reduce to

$$\tilde{\vec{\vec{\eta}}}_\alpha(\vec{r}, \vec{r}\,'; \vec{e}_1, s_\alpha) = \tilde{\eta}_\alpha(\vec{e}_1, s_\alpha)\vec{\vec{I}}\delta(\vec{r} - \vec{r}\,')$$

$$\tilde{\eta}_\alpha(\vec{e}_1, s) = \frac{\langle \vec{\mu}_\alpha\ ;\ \vec{I}_{\alpha_o} \rangle}{\langle \vec{\mu}_\alpha\ ;\ \vec{K}_{\alpha_1}\ ;\ \vec{\nu}_\alpha \rangle} \tag{4.7}$$

There are special forms of coupling coefficients which have been used to obtain results for the currents and charges on various objects.

$$\tilde{\eta}_\alpha(\vec{e}_1,s) = e^{(s_\alpha - s)t'} \frac{\left\langle \vec{\mu}_\alpha \; ; \; \tilde{\vec{I}}(s_\alpha) \right\rangle}{\left\langle \vec{\mu}_\alpha \; ; \; \vec{\vec{K}}_{\alpha_1} \; ; \; \vec{\nu}_\alpha \right\rangle} \quad \text{Class 1}$$

$$\tilde{\eta}_\alpha(\vec{e}_1,s) = \frac{\left\langle \vec{\mu}_\alpha \; ; \; \tilde{\vec{I}}(s) \right\rangle}{\left\langle \vec{\mu}_\alpha \; ; \; \vec{\vec{K}}'_{\alpha_1} \; ; \; \vec{\nu}_\alpha \right\rangle} \quad \text{Class 2} \tag{4.8}$$

For class 1 one has t' as a turn on time for all the modes. It must be chosen such that for any given observer position it is less than the time at which the first signal can reach the observer, otherwise the additional function $\vec{\vec{W}}_p$ is certainly required.

The class 1 coupling coefficients give time domain responses (neglecting $\vec{\vec{W}}_p$) as

$$\begin{aligned} \vec{U}_p^{(\vec{J})}(\vec{r}',t) &= u(t-t')\sum_{\alpha_+} 2\mathrm{Re}\left[\tilde{\eta}_\alpha(\vec{e}_1,s_\alpha)\vec{\nu}_\alpha^{(\vec{J})}(\vec{r}')e^{s_\alpha t}\right] \\ &\quad + u(t-t')\sum_{\alpha_o} \tilde{\eta}_\alpha(\vec{e}_1,s_\alpha)\vec{\nu}_\alpha^{(\vec{J})}(\vec{r}')e^{s_\alpha t} \\ \vec{V}_{p_o}^{(\vec{J})}(\vec{r}',t) &= u(t-t')\sum_{\alpha_+} 2\mathrm{Re}\left[\tilde{f}_p(s_\alpha)\tilde{\eta}_\alpha(\vec{e}_1,s_\alpha)\vec{\nu}_\alpha^{(\vec{J})}(\vec{r}')e^{s_\alpha t}\right] \\ &\quad + u(t-t')\sum_{\alpha_o} \tilde{f}_p(s_\alpha)\tilde{\eta}_\alpha(\vec{e}_1,s_\alpha)\vec{\nu}_\alpha^{(\vec{J})}(\vec{r}')e^{s_\alpha t} \end{aligned} \tag{4.9}$$

For an exponential incident waveform the waveform part of the response is

$$\tilde{\vec{V}}^{(\vec{J})}_{p_w}(\vec{r}\,',s) = u(t-t')e^{s_w(t-t')}\left\{\sum_{\alpha_+} 2\mathrm{Re}\left[(s_w - s_\alpha)^{-1}\tilde{\eta}_\alpha(\vec{e}_1,s_\alpha)\vec{\nu}^{(\vec{J})}_\alpha(\vec{r}\,')\right]\right.$$

$$\left. + \sum_{\alpha_o}(s_w - s_\alpha)^{-1}\tilde{\eta}_\alpha(\vec{e}_1,s_\alpha)\vec{\nu}^{(\vec{J})}_\alpha(\vec{r}\,')\right\} \quad (4.10)$$

and for an incident step function waveform we have

$$\vec{V}^{(\vec{J})}_{p_w}(\vec{r}\,',s) = u(t-t')\tilde{\vec{U}}^{(\vec{J})}_p(\vec{r}\,',0) \quad (4.11)$$

so that the waveform part of the response is the static response times a unit step. Note that the summation over α_+ means the poles above the real s axis while the summation over α_o means the poles on the real s axis. The class 1 coupling coefficients then give rather simple forms for the resulting expansions. The object part of the response has only damped sinusoids turned on at t = t' and the object part of the response is fairly simple as well.

The class 2 coupling coefficients give time domain responses (neglecting $\tilde{\vec{W}}_p$) as

$$\vec{U}^{(\vec{J})}_p(\vec{r}\,',t) = \sum_{\alpha_+} 2\mathrm{Re}\left[\left[\left\langle\vec{\mu}_\alpha\,;\,\vec{\vec{K}}_{\alpha_1}\,;\,\vec{\nu}_\alpha\right\rangle\right]^{-1}\left[\left\langle\vec{\mu}_\alpha\,;\,\vec{I}(t)\right\rangle\right] * \left[u(t)e^{s_\alpha t}\right]\vec{\nu}^{(\vec{J})}_\alpha(\vec{r}\,')\right]$$

$$+ \sum_{\alpha_o}\left[\left\langle\vec{\mu}_\alpha\,;\,\vec{\vec{K}}_{\alpha_1}\,;\,\vec{\nu}_\alpha\right\rangle\right]^{-1}\left[\left\langle\vec{\mu}_\alpha\,;\,\vec{I}(t)\right\rangle\right] * \left[u(t)e^{s_\alpha t}\right]\vec{\nu}^{(\vec{J})}_\alpha(\vec{r}\,')$$

$$(4.12)$$

$$\vec{V}^{(\vec{J})}_{p_o}(\vec{r}\,',t) = \sum_{\alpha_+} 2\mathrm{Re}\left[\tilde{f}_p(s_\alpha)\left[\left\langle\vec{\mu}_\alpha\,;\,\vec{\vec{K}}_{\alpha_1}\,;\,\vec{\nu}_\alpha\right\rangle\right]^{-1}\left[\left\langle\vec{\mu}_\alpha\,;\,\vec{I}(t)\right\rangle\right] * \left[u(t)e^{s_\alpha t}\right]\vec{\nu}^{(\vec{J})}_\alpha(\vec{r}\,')\right]$$

$$+ \sum_{\alpha_o}\tilde{f}_p(s_\alpha)\left[\left\langle\vec{\mu}_\alpha\,;\,\vec{\vec{K}}_{\alpha_1}\,;\,\vec{\nu}_\alpha\right\rangle\right]^{-1}\left[\left\langle\vec{\mu}_\alpha\,;\,\vec{I}(t)\right\rangle\right] * \left[u(t)e^{s_\alpha t}\right]\vec{\nu}^{(\vec{J})}_\alpha(\vec{r}\,')$$

Note that an asterisk * indicates convolution of the two time functions. For an exponential incident waveform the waveform part of the response is

$$\vec{v}^{(\vec{J})}_{p_w}(\vec{r}',t) = \sum_{\alpha_+} 2\mathrm{Re}\left[(s_w - s_\alpha)^{-1}\left[\langle \vec{\mu}_\alpha ; \vec{\vec{K}}_{\alpha_1} ; \vec{\nu}_\alpha \rangle\right]^{-1}\left[\langle \vec{\mu}_\alpha ; \vec{I}(t)\rangle\right]\right.$$

$$\left. * \left[u(t)e^{s_w t}\right]\vec{\nu}^{(\vec{J})}_\alpha(\vec{r}')\right]$$

$$+ \sum_{\alpha_o} (s_w - s_\alpha)^{-1}\left[\langle \vec{\mu}_\alpha ; \vec{\vec{K}}_{\alpha_1} ; \vec{\nu}_\alpha \rangle\right]^{-1}\left[\langle \vec{\mu}_\alpha ; \vec{I}(t)\rangle\right]$$

$$* \left[u(t)e^{s_w t}\right]\vec{\nu}^{(\vec{J})}_\alpha(\vec{r}') \qquad (4.13)$$

For a step function incident waveform set $s_w = 0$. Note that if t_o is the time the incident wave first touches the body and t_1 is the time it just passes the body (not counting scattered fields) then the convolution basically affects the result only for $t_o \leq t \leq t_1$. For $t > t_1$ the convolution form or class 2 reduces in time domain to the simple damped sinusoids of class 1.

The waveforms for charge density on an object are also important. The charge natural modes are related to the current natural modes by

$$\nu^{(\rho)}_\alpha(\vec{r}') = -a_\alpha \nabla' \cdot \vec{\nu}^{(\vec{J})}_\alpha(\vec{r}') \qquad (4.14)$$

where a_α is some convenient normalizing constant. By operating on the equations of this section by $-(1/s)\nabla' \cdot$ the current responses are each converted to charge responses. In time domain the $1/s$ becomes $\partial/\partial t$. Equation 4.14 gives an appropriate way of defining the resulting charge modes.

The charge response has the form

$$\tilde{U}^{(\rho)}_p(\vec{r}',s) = \sum_\alpha \tilde{\eta}_\alpha(\vec{e}_1,s)\,\nu^{(\rho)}_\alpha(\vec{r}')\left(\frac{sa_\alpha}{c}\right)^{-1}(s - s_\alpha)^{-1} + \tilde{W}^{(\rho)}_p(\vec{e}_1,\vec{r}',s)$$

$$\tilde{\rho}_p(\vec{r}',s) = E_o \frac{\Sigma}{c}\tilde{f}_p(s)\,\tilde{U}^{(\rho)}_p(\vec{r}',s) \qquad (4.15)$$

where Σ is a normalizing constant as used in equations 1.3. For an arbitrary incident waveform we have, dropping $\tilde{W}_p$,

$$\tilde{v}_p^{(\rho)}(\vec{r}\,',s) = \tilde{f}_p(s)\tilde{U}_p^{(\rho)}(\vec{r}\,',s) = \tilde{v}_{p_o}^{(\rho)}(\vec{r}\,',s) + \tilde{v}_{p_w}^{(\rho)}(\vec{r}\,',s)$$

$$\tilde{v}_{p_o}^{(\rho)}(\vec{r}\,',s) = \sum_\alpha \tilde{f}_p(s_\alpha)\tilde{\eta}_\alpha(\vec{e}_1,s)\nu_\alpha^{(\rho)}(\vec{r}\,')\left(\frac{s_\alpha a_\alpha}{c}\right)^{-1}(s-s_\alpha)^{-1} \qquad (4.16)$$

$$\tilde{v}_{p_w}^{(\rho)}(\vec{r}\,',s) = \sum_\alpha \tilde{\eta}_\alpha(\vec{e}_1,s)\nu_\alpha^{(\rho)}(\vec{r}\,')\left(\frac{s_\alpha a_\alpha}{c}\right)^{-1}\frac{\tilde{f}_p(s)-\tilde{f}_p(s_\alpha)}{s-s_\alpha}$$

Note then that the results for the current response carry over directly to the charge response by the addition of a factor $(sa_\alpha/c)^{-1}$ in the terms of the expansion.

For class 1 coupling coefficients the object response has the form in time domain

$$v_{p_o}^{(\rho)}(\vec{r}\,',t) = u(t-t')\sum_{\alpha_+} 2\mathrm{Re}\left[\tilde{f}_p(s_\alpha)\tilde{\eta}_\alpha(\vec{e}_1,s_\alpha)\frac{c}{s_\alpha a_\alpha}\nu_\alpha^{(\rho)}(\vec{r}\,')e^{s_\alpha t}\right]$$

$$+\, u(t-t')\sum_{\alpha_o}\tilde{f}_p(s_\alpha)\tilde{\eta}_\alpha(\vec{e}_1,c)\frac{c}{s_\alpha a_\alpha}\nu_\alpha^{(\rho)}(\vec{r}\,')e^{s_\alpha t} \qquad (4.17)$$

The waveform part for step excitation is

$$v_{p_w}^{(\rho)}(\vec{r}\,',t) = u(t-t')\tilde{U}_p^{(\rho)}(\vec{r}\,',0) \qquad (4.18)$$

The class 2 coupling coefficients can also be used for the charge response by substituting $(sa_\alpha/c)^{-1}$ times the charge mode for the current mode in equations 4.12 and 4.13. Again the class 2 forms are somewhat more complicated (but perhaps more convergent). For $t > t_1$ the class 2 time domain forms for the charge response reduce to the class 1 forms of equations 4.17 and 4.18.

Appendix A: A Notation for Abbreviating Integral Operations

For some purposes, such as in this note, where there are many integral operations and even repeated integral operations it is useful to have a more compact notation to use in the equations. This appendix briefly defines and discusses such a notation which might be thought of as an implied integral notation.

Consider integrals such as

$$\int_V \overrightarrow{\overrightarrow{A}}(\vec{r},\vec{r}\,'') \cdot \overrightarrow{\overrightarrow{B}}(\vec{r}\,'',\vec{r}\,')\,dV'' = \overrightarrow{\overrightarrow{C}}(\vec{r},\vec{r}\,')$$

$$\int_V \overrightarrow{\overrightarrow{A}}(\vec{r},\vec{r}\,') \cdot \vec{b}(\vec{r}\,')\,dV' = \vec{c}(\vec{r}) \tag{A1}$$

Write them respectively as

$$\left\langle \overrightarrow{\overrightarrow{A}} \;;\; \overrightarrow{\overrightarrow{B}} \right\rangle = \overrightarrow{\overrightarrow{C}}$$

$$\left\langle \overrightarrow{\overrightarrow{A}} \;;\; \vec{b} \right\rangle = \vec{c} \tag{A2}$$

This is basically a symmetric product as distinguished from an inner product. In an inner product one of the terms is conjugated in a complex variable sense before integrating. No conjugation is implied in the present notation but must be specifically exhibited such as by a bar ‾ above the quantity. In such a manner the present notation can also be used for inner products. In this form an integral over spatial coordinates is indicated by a comma between terms with $\langle\,\rangle$ around the expression. The spatial coordinates for integration are the last listed coordinates in the term before the comma and the first listed coordinates in the term after the comma such as indicated in equation A1. Note that the terms can have other variables such as time, complex frequency, etc. and symbols associated with these (such as for Laplace transform, convolution, etc.).

Vector, matrix, and dyadic multiplication symbols carry over and are listed above the comma. Equations A2 then indicate dot product. This notation is applicable to repeated integrals as well, such as

$$\int_V \int_V \vec{a}(\vec{r}) \times (\vec{\vec{B}}(\vec{r},\vec{r}'') \cdot \vec{\vec{C}}(\vec{r}'',\vec{r}'))\, dV dV''$$

$$\equiv \langle \vec{a} \overset{\times}{;} (\vec{\vec{B}} ; \vec{\vec{C}}) \rangle$$

$$= \langle \vec{a} \overset{\times}{;} \langle \vec{\vec{B}} ; \vec{\vec{C}} \rangle \rangle \qquad \text{(A3)}$$

where either form could be used. Another example (a rather common one) is

$$\int_V \int_V \vec{a}(\vec{r}) \cdot \vec{\vec{B}}(\vec{r},\vec{r}') \cdot \vec{c}(\vec{r}')\, dV dV'$$

$$\equiv \langle \vec{a} ; \vec{\vec{B}} ; \vec{c} \rangle$$

$$= \langle \vec{a} ; \langle \vec{\vec{B}} ; \vec{c} \rangle \rangle$$

$$= \langle \langle \vec{a} ; \vec{\vec{B}} \rangle ; \vec{c} \rangle \qquad \text{(A4)}$$

Convolution can also be readily adapted to this notation, as for example

$$\int_V \vec{a}(\vec{r},t) \overset{*}{\cdot} \vec{b}(\vec{r},t)\, dV = \langle \vec{a} \overset{*}{;} \vec{b} \rangle \qquad \text{(A5)}$$

where the asterisk indicates a convolution with respect to t in this case where the multiplication (which is part of convolution) is in the dot product sense.

In using this implied integral notation various of the variables can be listed with their appropriate terms, such as $\vec{I}(t)$, etc. if needed for clarity such as, for example, if t were to be assigned some particular value like t_o. Various degrees of compactness are then possible provided the variables not listed are clearly implied. In this regard the domain of integration implied in the symbol $\langle\,\rangle$ with commas whether it be a volume V, a surface S, or whatever should be clearly stated or implied. If there is more than one domain of integration then the one intended in each case should be clearly implied or an appropriate symbol such as a subscript added to distinguish the different domains of integration.

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