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Transient Behavior of EMP Induced Currents on a Sphere with a Trailing Wire Antenna Part 1. Formulation of the Integral Equations

F. M. Tesche
The Dikewood Corporation

Abstract

The formulation for both a Pocklington and a Hallén frequency domain integral equation describing the currents flowing on a thin-wire attached to a conducting sphere is presented. This model is capable of studying the EMP response of a long trailing wire attached to an aircraft.

transient radiation, calculations, trailing wire antennas

I. Introduction

This note begins an investigation into the transient behavior of currents flowing on a thin-wire which protrudes from a perfectly conducting sphere. This particular idealized model may be used to study a number of different structures and how they interact with an EMP. One problem of interest is attempting to understand how an aircraft with a long trailing wire antenna will behave when subject to an EMP environment. By computing the isolated capacitance of the aircraft and then replacing the aircraft by a sphere with the same capacitance, an estimate can be made as to how the short circuit current at the input of the antenna will behave.

Another use of this model is to consider the sphere and wire to represent an EMP sensor and the structural boom which holds the sensor in place. As before, the transient behavior of the input current is of interest, as well as the perturbation of the electromagnetic fields in the region of the instrument.

Recently, there has been considerable interest as to the effects of EMP on satellites. Whereas this particular model does not address the case of system generated EMP and its effects, it does provide a means for the computing of the conventional EMP interaction with a satellite. Various antennas may be represented by radially directed wires and the induced currents and charges computed as a function of the EMP.

The analysis technique employed for this problem involves the formulation and solution to an integral equation for the wire current. By a proper choice of the boundary condition at the sphere surface which the dyadic Green's function for the problem satisfies, the range of the integral equation is limited to be over only the antenna wire. The effect of the sphere is then included in the kernel of the integral equation.

Specifically, two frequency domain integral equations (Pocklington and Hallén) are derived for the currents flowing on a single radially directed wire attached to a conducting sphere. Both the scattering and the driven antenna problems are considered, as it is necessary to compute not only the short circuit current, but also the input admittance of the long wire to characterize its operation. Explicit relations for the fields driving the integral equations for both of these cases are derived.

In this note, only the formulation of the problem is discussed. It is anticipated that the numerical techniques for summing the relatively slowly converging series for the kernel and the results of the calculations (in both frequency and time domains) which are being presently performed, will be reported in a future note.

II. Formulation of the Pocklington Integral Equation

To analyze the time dependent behavior of currents on a wire attached to a perfectly conducting sphere, a frequency domain integral equation for the wire currents may be solved at many different frequencies and then transformed to the time domain using the Fast Fourier Transform. Throughout this procedure, the presence of the sphere is accounted for by using the spherical Green's function instead of the free space Green's function.[12]

Consider the structure shown in Figure 1. The scattering electric field, $\bar{E}^{sca}$ produced by the current density on the wire and by the sphere surface currents (both of which are excited either by an incident plane wave or by a driving source at the junction of the wire and sphere) may be written as:

$$\bar{E}^{sca}(\bar{r}_o) = j\omega\mu \int_{\substack{\text{sphere} \\ \& \text{ wire}}} \bar{J}(\bar{r}_s) \cdot \Gamma_{fs}(\bar{r}_s, \bar{r}_o)\, ds, \tag{1}$$

where Γ_{fs} is the free space Green's tensor as described by Tai.[10] If, however, one uses the Green's tensor which satisfies

$$\hat{n}_s \times \underset{\sim}{\Gamma}(\bar{r}_s, \bar{r}_o) = 0 \tag{2}$$

for r_s on the sphere surface, it is possible to show that the integral in Eq. (1) extends only over the wire. Thus, it is necessary to determine only the wire currents to solve the problem.

For the case under consideration, only radially directed currents flow on the thin wire. Moreover, the component of the electric field needed to form the integral equation for the wire current is only the radial component. Noting that for the perfectly conducting wire, $E_r^{inc} + E_r^{sca} = 0$ on the wire surface, Eq. (1) yields the following integral equation for the wire current density:

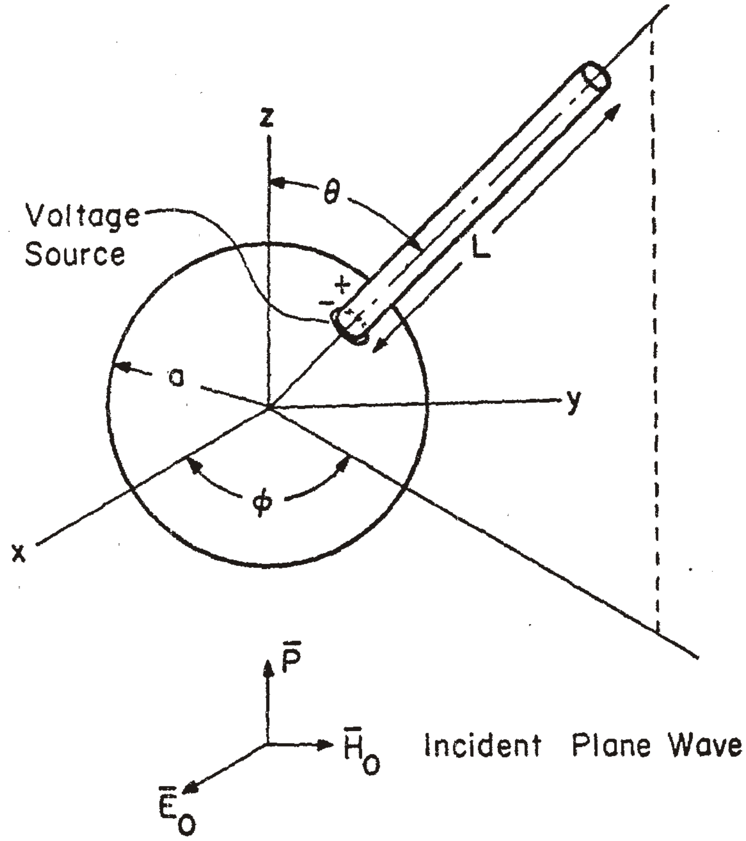

Figure 1. Perfectly conducting sphere with a thin-wire attached, excited by an incident plane wave or by a voltage source.

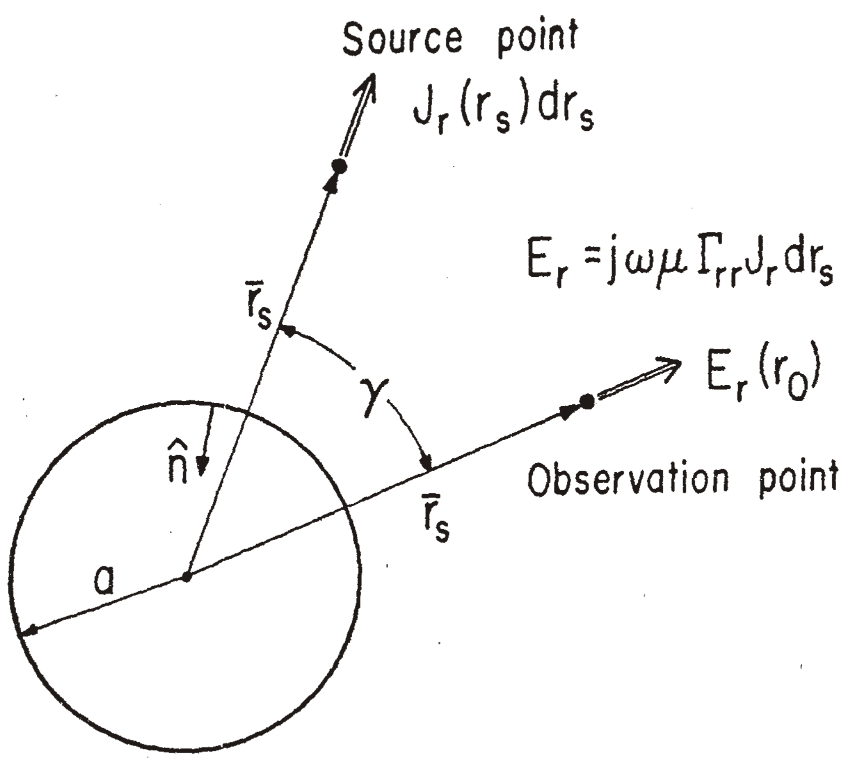

Figure 2. General locations of the source and observation points in the presence of the sphere.

$$- E_r^{inc}(\bar{r}_o) = j\omega\mu \int_{wire} J_r(\bar{r}_s) \cdot \Gamma_{rr}(\bar{r}_s, \bar{r}_o)\, ds \,. \tag{3}$$

As in most thin-wire problems, only the total current flowing in the wire is computed.

The (rr) component of the spherical Green's tensor is derived by Jones[5] and has the form

$$\Gamma_{rr}(\bar{r}_o, \bar{r}_s) = \sum_{n=1}^{\infty} \frac{n(n+1)(2n+1)}{r_o r_s} \Big[j_n(kr_<) h_n^{(2)}(kr_>) + T_n h_n^{(2)}(kr_o) h_n^{(2)}(kr_s) \Big] P_n(\cos\gamma) \tag{4}$$

where $r_{(\lessgtr)}$ represents the $\begin{pmatrix}\text{smaller}\\\text{larger}\end{pmatrix}$ of r_o and r_s, and j_n and $h_n^{(2)}$ represent the spherical Bessel and Hankel functions respectively. The angle γ is that between the two vectors $\bar{r}_o$ and $\bar{r}_s$ as shown in Figure 2. The factor T_n takes into account the presence of the sphere and has the value

$$T_n = -\frac{\frac{d}{da}\, a\, j_n(ka)}{\frac{d}{da}\, a\, h_n(ka)} \tag{5}$$

for a perfectly conducting sphere of radius a. For a sphere of finite conducting material, this term may be modified as outlined in Jones.[5]

It should be pointed out that the construction of $\tilde{\Gamma}$ by Jones does not explicitly include the $\bar{L}$ functions in the eigen-function expansion. These vector wave functions, along with the $\bar{M}$ and $\bar{N}$ functions, have been discussed by Stratton[9] and may be employed to represent an arbitrary vector field. For ψ being a solution to the scalar Helmholtz equation

$$\nabla^2\psi + k^2\psi = 0$$

the wave functions $\bar{L}$, $\bar{M}$ and $\bar{N}$ have the following form

$$\bar{L} = \nabla\psi$$
$$\bar{M} = \nabla \times \bar{a}\psi$$
$$\bar{N} = \frac{1}{k}\nabla \times \bar{M}$$

with $\bar{a}$ being a constant vector. From these relations, it is possible to verify that

$$\nabla \times \bar{L} = 0 \quad ; \qquad \nabla \cdot \bar{L} = -k^2\psi$$
$$\nabla \cdot \bar{M} = 0$$
$$\nabla \cdot \bar{N} = 0$$

Thus, if the electric field $\bar{E}$ is represented by an expansion of the form

$$\bar{E} = \sum_n a_n\bar{L}_n + b_n\bar{M}_n + c_n\bar{N}_n$$

where the coefficients a_n have to be determined, the presence of free charge is accounted for since

$$\nabla \cdot \bar{E} = \rho/\epsilon_o = \sum a_n\nabla \cdot \bar{L} .$$

If the point of observation lies outside of the region having charge, the $\bar{L}$ functions are not needed. Tai,(11) in a recent correction to his text(10) discusses this point. In the present analysis, however, this problem is of no concern, as the source and observation points are always separated at least by the antenna wire radius due to the thin wire approximation which is employed as a computational aid.

Another point which is important to understand is how the currents and charges on the sphere affect the incident field along the antenna. Consider the scattering problem as shown in Figure 3a, where it is assumed that the wire end at r = a is not touching the sphere. Since there is no current flowing out of the sphere, the net charge on the sphere must be

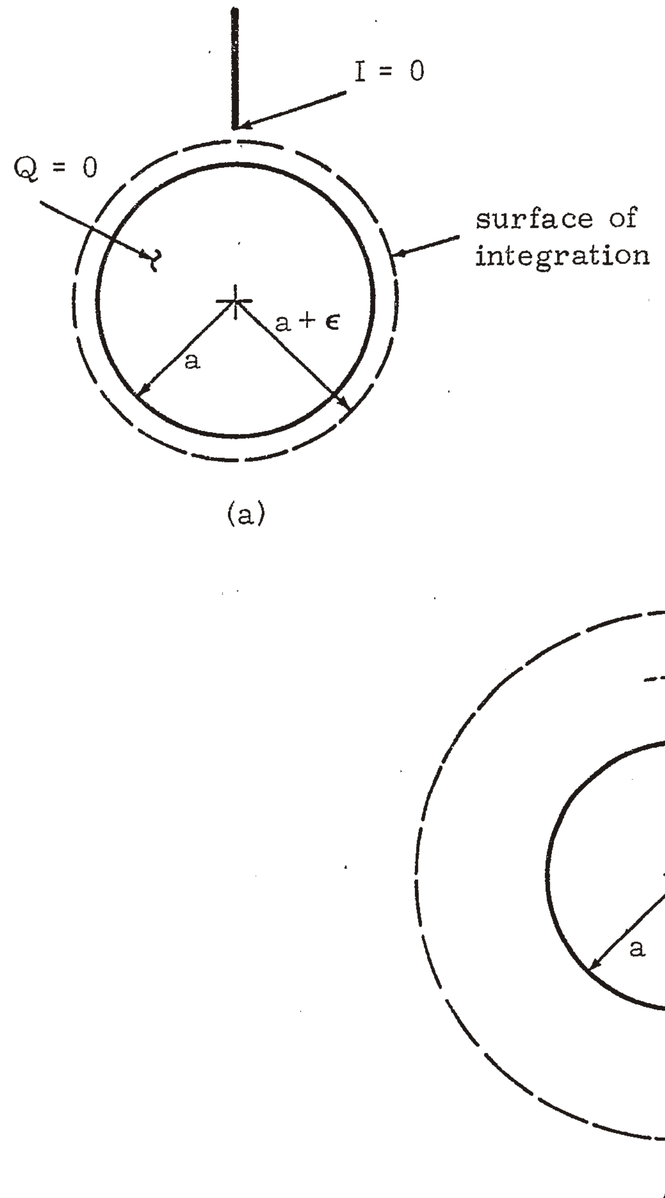

Figure 3. Sphere, current element and associated charge.

zero. Indeed, integrating the radial E field of Eq. (3), which is due to the charges (and currents) induced on the sphere, over the sphere surface gives $\int E_r a^2 \sin\theta \, d\theta \, d\phi = 0$, since the summation starts from n = 1. There is no term in the series which contributes to a net charge.

An interesting question is to ask what happens when the wire actually touches the sphere. Physically, the current can then flow at the wire end and leave a net charge on the sphere, having a value related to the input current by $Q = I_o/j\omega$. If one attempts to compute this quantity by integrating the radial electric field of Eq. (3), it is seen that zero still results because the summation has not been changed from the previous case. This is in conflict with the equation of continuity. The difficulty with the latter approach is that all of the charge within the Gaussian surface is included in the integral of $\bar{E} \cdot d\bar{S}$ over the surface, not just the charge on the sphere. Consider the case shown in Figure 3c. If the current element $I\,d\ell$ is such that $r_s > r_o$, then clearly $\int E_r dS = 0$ since there is no net charge on the sphere. If $r_s < r_o$ then again $\int E_r dS = 0$ since there is still no free charge within the surface of integration. If $r_s = a$, the same still holds. The total charge enclosed is still zero, even though there is now a net charge on the sphere. One can think of the dipole moment $I\,d\ell$ in this case consisting of two oscillating charges, one on the sphere and the other of opposite polarity, a distance $d\ell$ away from the surface of the sphere. The Gaussian surface encloses both of these charges. Hence, although it is possible to speak of the total charge on the sphere, it is not necessary to compute it exactly in order to determine the effect of the sphere on the electric field tangential to the radial antenna wire. This follows from the fact that Γ_{rr} in Eq. (4), as observed at some observation point r_o and with a current element at $r_s = a$, accounts for the total E_r produced by the current element and the sphere currents acting together, not separately.

The integral equation Eq. (3) for the current on the antenna looks similar to the integro-differential equation of the Pocklington type for a single isolated scatterer or antenna.[7] This type of equation is written as

$$-E_r^{inc}(r_o) = \frac{1}{j\omega\epsilon}\left(\frac{d^2}{dr_o^2} + k^2\right)\int_{wire} J_r(\bar{r}_s)\,\frac{e^{-jk|\bar{r}_o-\bar{r}_s|}}{4\pi|\bar{r}_o-\bar{r}_s|}\,ds\ , \qquad (6)$$

and is known to have a triplet-type singularity in the kernel as $r_o \rightarrow r_s$ if the indicated derivatives are carried out explicitly. This necessitates the use of the finite difference technique in order to determine a solution.[2,6] By analogy it is expected that the infinite sum in Eq. (4) will contain a similar triplet singularity as $r_o \rightarrow r_s$, so it is desirable to rewrite Eq. (3) as an integro-differential equation as was done in the isolated dipole case.

If $g_n(kr)$ represents any spherical Bessel function, then by definition[1]

$$\frac{d^2 g_n}{dr^2} + \frac{2}{r}\frac{dg_n}{dr} + \left(k^2 - \frac{n(n+1)}{r^2}\right) g_n = 0\ . \qquad (7)$$

Looking at the function $f_n(kr) = krg_n(kr)$ and taking the appropriate derivatives, it is found from Eq. (7) that

$$\left[\left(\frac{d^2}{dr^2} + k^2\right) - \frac{n(n+1)}{r^2}\right] f_n(kr) = 0 \qquad (8)$$

or,

$$\frac{n(n+1)g_n(kr)}{r} = \left(\frac{d^2}{dr^2} + k^2\right) r\, g_n(kr)\ . \qquad (9)$$

Substituting this back into Eqs. (3) and (4), an equation of the Pocklington form results:

$$- E_r^{inc}(\bar{r}_o) = j\omega\mu \left(\frac{d^2}{dr_o^2} + k^2 \right) \int_{wire} J_r(\bar{r}_s) K(\bar{r}_s, \bar{r}_o)\, ds \qquad (10)$$

where the kernel $K(\bar{r}_s, \bar{r}_o)$ is given by

$$K(\bar{r}_s, \bar{r}_o) = \frac{j}{4\pi k} \sum_{n=1}^{\infty} \frac{r_o}{r_s} (2n+1) \Big[j_n(kr_<) h_n^{(2)}(kr_>)$$

$$+ T_n h_n^{(2)}(kr_o) h_n^{(2)}(kr_s) \Big] P_n(\cos \gamma) \; . \qquad (11)$$

In solving Eq. (10), it is necessary to evaluate the kernel K in an efficient manner. If the calculation of K requires more time than to solve the equivalent set of coupled integral equations using the simpler free space Green's tensor, this method will not be a useful one. From the isolated thin-wire problem, it is known that the kernel in Eq. (6) has a singularity of the form $|r_o - r_s|^{-1}$. If there is a similar singularity in the kernel of Eq. (11), it is expected that the series would converge very slowly at points near $r_o \approx r_s$ and, in fact, diverge when $r_o = r_s$. Thus, it would be advantageous to put Eq. (11) in closed form for rapid numerical computation.

In looking at Eq. (11), it is seen that there are two terms in the summation. The first term involving j_n and $h_n^{(2)}$ represents the direct contribution of the source on the observed electric field, while the second term, involving the factor T_n, represents the effect of the sphere on the observed field. Thus, the singularity in the kernel will occur in the first term as $\bar{r}_o \rightarrow \bar{r}_s$.

The addition theorem for spherical Hankel function[1] will permit the summation of the first part of the kernel K. It may be shown that

$$h_o^{(2)}\left(\left|r_o - r_s\right|\right) = \sum_{n=0}^{\infty} (2n+1)\, j_n(r_<)\, h_n^{(2)}(r_>)\, P_n(\cos\gamma) \tag{12}$$

and that

$$h_o^{(2)}(R) = j\,\frac{e^{-jR}}{R} \quad . \tag{13}$$

Thus, upon noting that Eq. (12) is essentially the first part of Eq. (11), aside from the n = 0 term, it is possible to express the kernel K as:

$$K(\bar{r}_s, \bar{r}_o) = \frac{j}{4\pi k}\,\frac{r_o}{r_s}\left[\frac{j\,e^{-jk\left|\bar{r}_o - \bar{r}_s\right|}}{k\left|\bar{r}_o - \bar{r}_s\right|} - j_o(kr_<)\,h_o^{(2)}(kr_>) + \sum_{n=1}^{\infty} T_n (2n+1)\, h_n^{(2)}(kr_s)\, h_n^{(2)}(kr_o)\, P_n(\cos\gamma)\right] \quad . \tag{14}$$

Hence, the first portion of the kernel, which is singular at $\bar{r}_o = \bar{r}_s$, may be summed in closed form, leaving only the reflection contribution to be summed numerically. This kernel as given by Eq. (14) and the relation in Eq. (10) describe the Pocklington integro-differential equation for the currents flowing on the wire in the presence of the sphere.

It should be pointed out that, for the case of a single wire on the sphere as shown in Figure 1, the angle γ between the source point ($\bar{r}_s$) and the observation point ($\bar{r}_o$) is zero, thereby causing the $P_n(\cos\gamma)$ term to be unity for all values of n. For the more general case where there are other wires on the sphere, this term needs to be included.

The solution of Eq. (10) is often facilitated by assuming that the surface current J on the antenna wire can be replaced by the same amount of total current $I = 2\pi bJ$ which flows along the axis of the wire. The factor b is the radius of the wire. With this thin-wire approximation, the factor $\left|\bar{r}_o - \bar{r}_s\right|$ in Eq. (14) is never singular. Care must be exercised in using this approximation, however, as is discussed in Ref. (6).

III. Hallén's Integral Equation

Since the integro-differential equation (10) is of the same form as Pocklington's equation for a straight wire, it is expected that a Hallén type integral equation can be derived for the present problem.

By letting $\pi(r_o)$ be represented by the integral

$$\pi(r_o) = \int J(r_s) K(\bar{r}_s, \bar{r}_o)\, ds \tag{15}$$

a differential equation for π is obtained directly from Eq. (10) and has the form

$$\left(\frac{d^2}{dr_o^2} + k^2\right) \pi(r_o) = -\frac{1}{j\omega\mu} E_r^{inc}(\bar{r}_o) \ . \tag{16}$$

It is interesting to note in passing that this function $\pi(r_o)$ is actually proportional to the radially directed vector potential A_r which produces fields which are TM with respect to $\hat{r}$. By comparing Eq. (6.28) of Harrington[3] to the present equation, it can be shown that

$$A_r(r_o) = -k^2 \pi(r_o) \ .$$

The solution of Eq. (16) consists of two parts, a homogeneous solution and a complimentary solution. Obtaining the homogeneous solution is straightforward. Determining the particular solution is accomplished by the use of the method of variation of parameters.[4] If $U_1(r_o)$ and $U_2(r_o)$ are both solutions to the homogeneous equation, then a particular solution is given by

$$U_p(r_o) = \int_a^{r_o} -\frac{1}{j\omega\mu} E_r^{inc}(\xi) \frac{\left[U_1(\xi)U_2(r_o) - U_2(\xi)U_1(r_o)\right]}{W\left[U_1(\xi), U_2(\xi)\right]}\, d\xi \ , \tag{17}$$

where W represents the Wronskian.

Noting that $U_1 = \cos(kr_o)$ and $U_2 = \sin(kr_o)$, after some algebra the resulting solution to the differential equation in Eq. (16) becomes

$$\pi(r_o) = C_1 \cos(kr_o) + C_2 \sin(kr_o) - \frac{1}{(j\omega\mu)k}\int_a^{r_o} E_r^{inc}(\xi)\sin k(r_o - \xi)\, d\xi \ . \tag{18}$$

Using the definition of $\pi(r_o)$ from Eq. (15), the following Hallén type integral equation is obtained for a wire antenna on the sphere.

$$\int_{ant.} J_r(\bar{r}_s) K(\bar{r}_s, \bar{r}_o)\, ds = C_1 \cos(kr_o) + C_2 \sin(kr_o) - \frac{1}{(j\omega\mu)k}\int_a^{r_o} E_r^{inc}(\xi)\sin k(r_o - \xi)\, d\xi \ . \tag{19}$$

The unknown constants in this equation must be evaluated by the application of suitable constraints on the solution of the integral equation. It is known that $I = 0$ at the end of the antenna due to the termination of the metal there. This gives one constraint. The other is found from the requirement that for a perfectly conducting sphere,

$$\left. \frac{\partial \pi(r_o)}{\partial r_o} \right|_{r_o = a} = 0 \tag{20}$$

This requirement is necessary since the tangential electric field on the sphere is given in terms of the radially directed vector potential A_r by

$$E_\theta(a) = \frac{1}{j\omega\epsilon}\, \frac{1}{a} \left. \frac{\partial A(r_o)}{\partial r_o} \right|_{r_o = a} = 0 \ . \tag{21}$$

Since this expression must be zero and since $A(r_o)$ is proportional to $\pi(r_o)$, it is readily seen that Eq. (20) is required. Similarly, it can be shown by taking derivatives explicitly, that

$$\frac{\partial}{\partial r_o} \int_{ant} J(r_s) K(\bar{r}_s, \bar{r}_o)\, ds \bigg|_{r_o=a} = 0 \ . \tag{22}$$

Thus, upon taking the derivative of Eq. (19) at $r_o = a$, the following expression is obtained

$$0 = -kC_1 \sin(ka) + kC_2 \cos(ka) - \frac{1}{j\omega\mu k} \frac{\partial}{\partial r_o} \int_a^{r_o} E_r^{inc}(\xi) \sin k(r_o - \xi)\, d\xi \bigg|_{r_o=a} \tag{23}$$

Taking the derivative of the last term yields:

$$\int_a^{r_o} E_r^{inc}(\xi)\, k \cos k(r_o - \xi)\, d\xi + E_r^{inc}(r_o) \sin k(r_o - r_o) \bigg|_{r_o=a} = 0 \ . \tag{24}$$

Hence, the second constraint is given by the first two terms of Eq. (23) and can be expressed as:

$$C_1 \sin(ka) = C_2 \cos(ka) \ . \tag{25}$$

This relation, along with Eq. (19) and the condition that $I(r) = 0$ at the end of the wire, provide an alternate integral equation for the current flowing on the wire in the presence of the spherical obstacle.

Either equation is capable of yielding correct answers and the choice of which equation is used is often a matter of personal taste.

In the derivation of the Pocklington or Hallén equation, it was assumed that both the current $\bar{J}$ and the incident electric field $\bar{E}_{inc}$ were in the radial direction only and therefore related simply by the Green's tensor component Γ_{rr}. In the limiting case of a very thin cylindrical dipole mounted on a relatively large sphere this is a good approximation but it is never exact, due to the finite thickness of the wire. Therefore, if an exact solution is desired, both the $\hat{\theta}$ and $\hat{r}$ components of $\bar{E}$ and $\bar{J}$ must be considered and suitably related through the four components of the Green's tensor, Γ_{rr}, $\Gamma_{r\theta}$, $\Gamma_{\theta r}$ and $\Gamma_{\theta\theta}$.

Through the use of appropriate approximations, it is possible to simplify the above problem to a certain extent. The tangential longitudinal current flowing at a point $\bar{r}_s$ on the antenna wire is given by

$$J_{tan}(\bar{r}_s) = J_r(\bar{r}_s)\cos(\theta) - J_\theta(\bar{r}_s)\sin(\theta)$$

where θ is the angle between the tangent $\hat{z}$ to the antenna wire surface and the radially directed vector $\bar{r}_s$ at the source point is given by the relation:

$$\cos(\theta) = b/r_s ,$$

b being the wire radius. See Fig. 4 for a graphical description of this problem.

Since the antenna wire surface is not a constant co-ordinate surface, the angle θ is a function of position along the wire. Assuming that $b/r_s \ll 1$ for all r_s, it is then possible to approximate the current as being completely radially directed. The largest possible value for the term b/r_s is when r_s is equal to the sphere radius. This requirement may therefore be written as $b \ll a$.

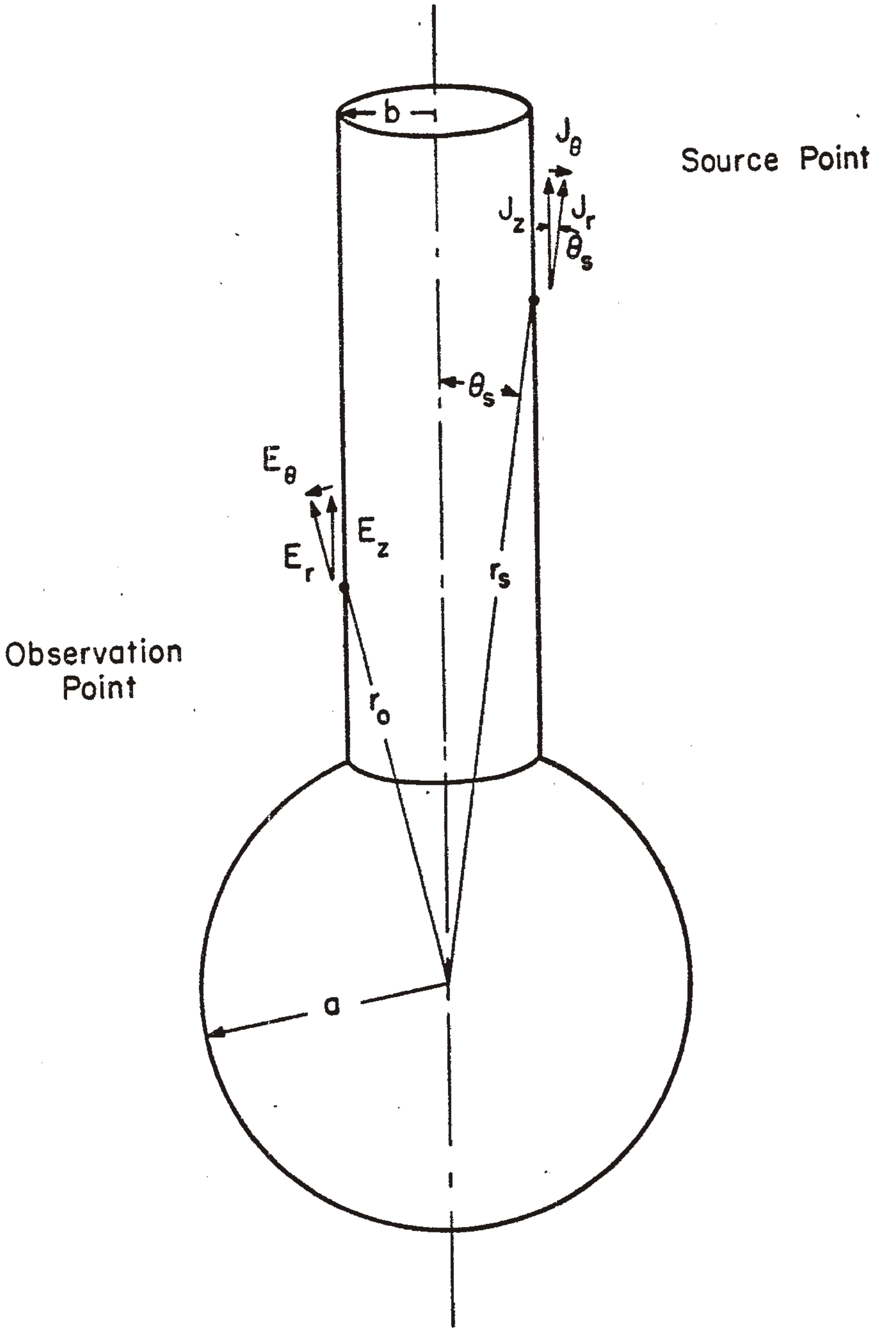

Figure 4. Pictorial representation of the currents on the wire antenna, the components of the electric field and the pertinent antenna dimensions.

III. Antenna Loading and Finitely Conducting Sphere

In the two previous sections, the Pocklington and Hallén integral equations have been derived assuming infinite conductivity for the sphere and the trailing antenna wire. In many practical cases, it may be desirable to include the effects of the finite conductivity of the metals from which the antenna and sphere are constructed. Also, being able to consider the effects of placing a load somewhere on the length of the antenna wires would be useful.

In Ref. (5), it is shown that the boundary condition on $\underset{\sim}{\Gamma}$ is given by Eq. (2) if the sphere is a perfect conductor. If the sphere is composed of some sort of penetrable material, the boundary condition on $\underset{\sim}{\Gamma}$ must be such that the tangential components of $\bar{E}$ and the tangential components of $\bar{H}$ are continuous across the spherical surface. Under these conditions, the term T_n in Eq. (5) takes on the following form for a sphere made of material with constants $\mu_1, \epsilon_1, \sigma \neq \infty$:

$$T_n = \frac{\dfrac{\mu_o k_1}{\mu_1 k} \dfrac{d}{da}\left(a\, j_n(ka)\right) k_1 a\, j_n(k_1 a) - ka\, j_n(ka) \dfrac{d}{da}\left(a\, j_n(k_1 a)\right)}{\dfrac{d}{da}\left(a\, j_n(k_1)\right) ka\, h_n^{(2)}(ka) - \dfrac{\mu_o k_1}{\mu_1 k} \dfrac{d}{da}\left(a h_n^{(2)}(ka)\right) k_1 a\, j_n(k_1 a)}$$

where $k_1 = \omega\sqrt{\mu_1(\epsilon_1 + j\omega\sigma)}$. The use of this parameter in evaluating the kernel K in Eq. (11) correctly accounts for the finite conductivity of the sphere for both the Pocklington and Hallén formulation.

If the Hallén equation is employed in the case of the finitely conducting sphere, it is important to realize that the boundary condition given by Eq. (25) is no longer correct, since the derivative of π in Eq. (20) is no longer zero. This derivative, evaluated at the spherical surface, must be known and should replace the left-hand side of Eq. (23). Unfortunately, this derivative depends upon the unknown current on the antenna. By taking the derivative of Eq. (19) and utilizing the results of Eq. (24) the following relation is determined:

$$\int_{\text{ant.}} J_r(r_s) \frac{\partial}{\partial r_o} K(r_s, r_o)\bigg|_{r_o=a} ds + kC_1 \sin(ka) = kC_2 \cos(ka) . \tag{26}$$

This relates the unknown current to the two unknown constants and provides the extra relationship needed for the solution of Hallén's equation. Notice that this relation reduces to Eq. (25) for the perfectly conducting sphere.

To include the effects of loading and finite conductivity in the antenna wire, it is noted that under these circumstances

$$E_r^{tot}(r_o) = E_r^{inc}(r_o) + E_r^{sca}(r_o) \neq 0 . \tag{27}$$

For the case of the finite conductivity of the antenna wires, it is convenient to represent the relationship between the tangential electric and magnetic fields on the wire surfaces by a surface impedance approximation. The impedance per unit length of a cylindrical wire in which a total current I flows is known to be given by [8]

$$Z_s \equiv \frac{E_s}{I} = -\frac{\gamma J_o(\gamma b)}{2\pi b\sigma J_o'(\gamma b)} . \tag{28}$$

Here E_s is the electric field tangent to the conductor surface, $\gamma^2 = -j\omega\mu\sigma$ σ = wire conductivity, b is the wire radius, and J_o is the cylindrical Bessel function. With this constraint Eq. (27) becomes

$$Z_s I(\bar{r}_o) = E_r^{inc}(\bar{r}_o) + E_r^{ref}(\bar{r}_o) \tag{29}$$

The Pocklington equation of (10) is therefore slightly modified, yielding the following inhomogeneous integro-differential equation,

$$- E_r^{inc}(\bar{r}_o) = - Z_s I(\bar{r}_o) + j\omega\mu \left(\frac{d^2}{dr_o^2} + k^2\right) \int_{ant.} I(\bar{r}_s) K(\bar{r}_s, \bar{r}_o)\, dr_s \ , \qquad (30)$$

where the integral over the current distribution has been approximated by a simple integral over the length of the antenna by use of the thin-wire approximation.

The relation for the impedance of the wire as given in Eq. (28) can be approximated in many practical cases. For very high frequencies such that $|\gamma b| \equiv |\sqrt{-j\omega\mu\sigma}\ b| \gg 1$, the asymptotic form of the Bessel functions may be employed and the impedance expressed as

$$Z_s \approx \frac{(1 + j)}{2\pi b \sigma \delta} \qquad (31)$$

where

$$\delta = \sqrt{\frac{1}{\pi f \mu \sigma}}$$

represents the effective skin depth as defined for a plane surface.

The impedance term Z_s of Eq. (30) which arises from the finite conductivity of the wire can also be thought of as arising from impedance loading of the antenna wires. In this way both the effects of conductivity and loading may be included by a solution of the modified integral equation.

A similar modification of the Hallén equation can be made so that wire conductivity and loading effects may be considered. In this case, the effects of the conductivity will occur within the integral of the right-hand side of Eq. (19). The result is given by:

$$\int_{ant.} I(\bar{r}_s) K(\bar{r}_s, \bar{r}_o)\, dr_s = C_1 \cos(kr_o) + C_2 \sin(kr_o) - \frac{1}{(j\omega\mu)k} \int_a^{r_o} (E_r^{inc}(\xi) - Z_s I(\xi)) \sin k(a-\xi)\, d\xi \qquad (32)$$

where again the integral over the source current density has been approximated by a line integral.

IV. Determination of the Incident Field: Scattering Problem

The source term for both the Pocklington and Hallén equations depend on the tangential component of the electric field incident on the wire due to a driving source on the wire itself (antenna problem) or due to an incident plane wave (scattering problem). Note that, in evaluating this tangential field, the presence of the conducting sphere must be accounted for. In the scattering problem, the field incident on the wire consists of two parts, as shown in Figure 5. One is a direct contribution of the incident field and the other is reflected off of the spherical obstacle.

The scattering of an incident field by a conducting sphere is presented in detail by Harrington.[3] For the geometry as shown in Figure 6 where the incident plane wave of magnitude E_o is $\hat{x}$ polarized and propagates in the $\hat{\zeta}$ direction, the radial component of the electric field observed at the point $\bar{r}_o = (r_o, \theta, \phi)$ is given by

$$E_r(\bar{r}_o) = \frac{1}{j\omega\epsilon}\left(\frac{d^2}{dr_o^2} + k^2\right) A_r(\bar{r}_o) \quad . \tag{33}$$

The quantity A_r is the radially directed magnetic vector potential which generates fields TM with respect to the $\hat{r}$ direction. This may be expressed in series form as shown by Harrington as

$$A_r(\bar{r}_o) = \frac{E_o}{\omega\mu} \cos\phi \sum_{n=0}^{\infty} kr_o\left[a_n j_n(kr_o) + b_n h_n^{(2)}(kr_o)\right] P_n^1(\cos\theta) \tag{34}$$

where

$$a_n = \frac{j^{-n}(2n+1)}{n(n+1)} \tag{35}$$

and

$$b_n = a_n T_n \tag{36}$$

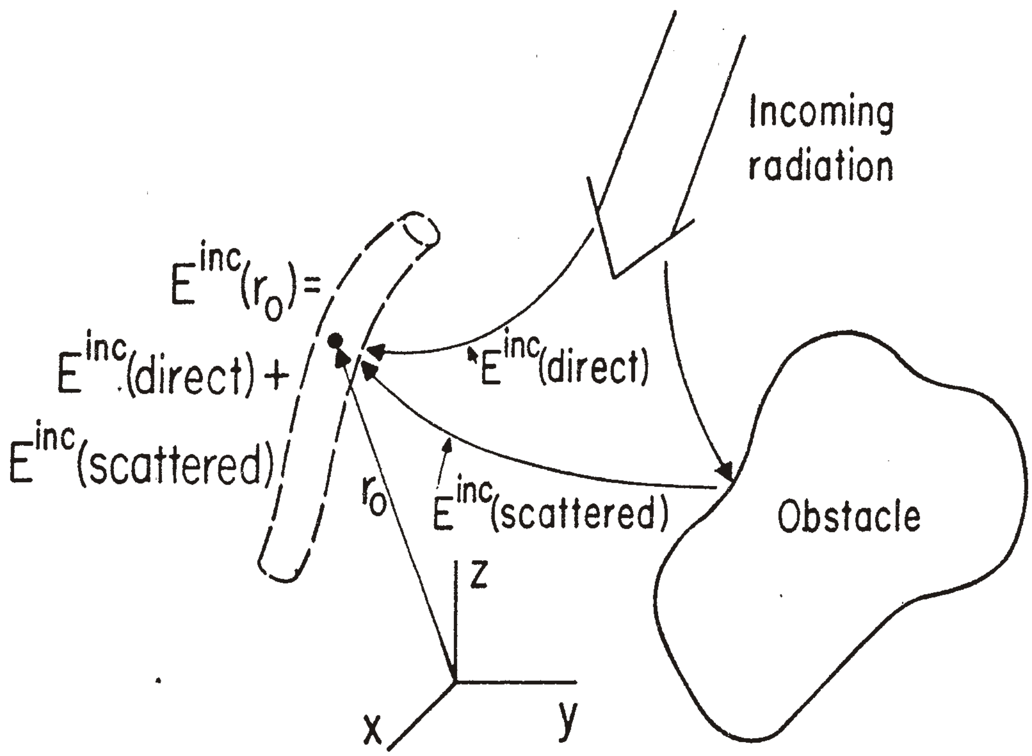

Figure 5. Representation of the incident field on a wire in the presence of an arbitrary scattering body.

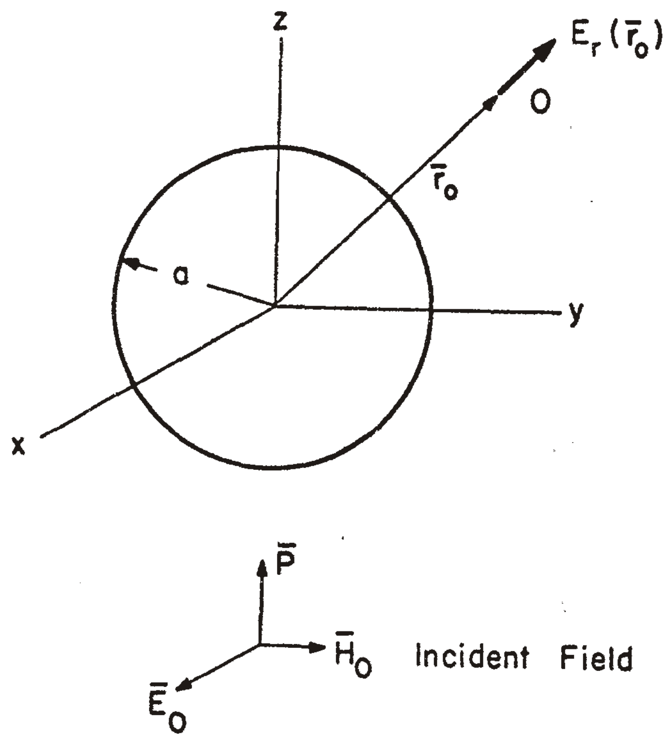

Figure 6. Incident plane wave striking a sphere and producing $E_r(\bar{r}_o)$ which is the source for the integral equation.

where T_n is defined by Eq. (5). Using the relation in Eq. (8), the expression for the field in Eq. (15) can be simplified to yield

$$E_r^{inc}(\bar{r}_o) = \frac{E_o \cos\phi}{jk} \sum \frac{j^{-n}(2n+1)}{r_o} P_n^1(\cos\theta) \left[j_n(kr_o) + T_n h_n^{(2)}(kr_o) \right] \quad (37)$$

This relation may be used to evaluate the forcing term in Eqs. (10) or (19). Since the field is not singular along the wire as is the kernel in Eq. (14), it is not necessary to separate out the singular term to sum it directly. The complete sum in Eq. (37) may be done numerically without difficulty.

V. Determination of the Incident Field: Antenna Problem

At this point it is desirable to consider the exact form of the electric field incident on the wire for the case of a driven antenna. This source which produces the incident field may be considered to be a small voltage source between the antenna and the sphere.

If the case of a single driven monopole on the sphere is to be considered, the antenna will be assumed to be fed by a co-axial line having the TEM excitation. This is shown in Fig. 7. By the equivalence principle, a mathematical surface can be drawn about the antenna wire and an equivalent source of $\hat{n} \times \bar{H} = \bar{J}$ placed on this surface, thereby allowing the removal of the antenna wire. The incident electric field tangent to the mathematical surface may then be calculated by considering the excited aperture in the sphere to be radiating without the wire present. Figure 8 shows the geometry for this problem.

The radially directed electric field for the source in this problem is easily determined from the Green's tensor $\underset{\sim}{\Gamma}$. From Jones[5], it may be shown that the radiated electric field produced by an impressed tangential electric field $\bar{E}_s$ on the surface of the sphere is given by

$$\bar{E}_{inc}(\bar{r}_o) = \int_{\substack{\text{sphere} \\ \text{surface}}} (\hat{n} \times \bar{E}(\bar{r}_s)) \cdot (\nabla_s \times \underset{\sim}{\Gamma}(\bar{r}_s, \bar{r}_o))\, ds \qquad (38)$$

where $\underset{\sim}{\Gamma}$ is the spherical Green's function.

The electric field on the surface of the sphere is assumed to be related to the voltage across the co-axial line by the relation

$$E_\tau = \frac{V_o}{\tau \ln(c/b)} \approx E_\theta \qquad (39)$$

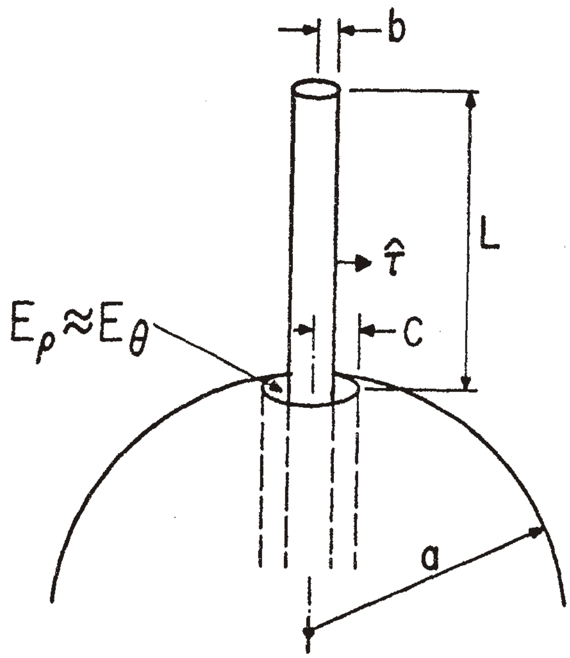

Figure 7. The assumed antenna feed is a co-axial transmission line which impresses an electric field E_θ over the annular ring at the antenna base.

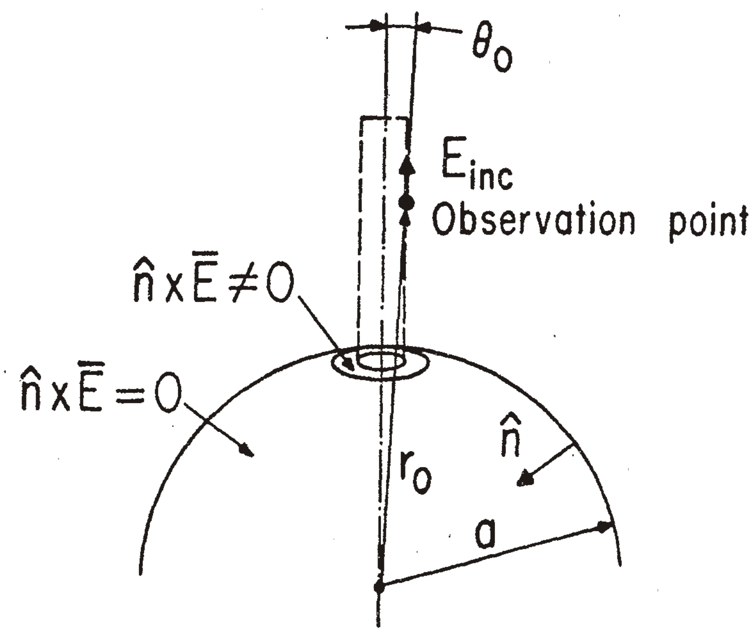

Figure 8. The geometry for calculating the incident electric field with the antenna wire removed.

where τ is the radius of the cylindrical co-ordinate system for the co-axial line and b and c are the inner and outer radii of the line. Noting $\tau = a \sin\theta$, the impressed electric field is given by

$$E_\theta \approx \frac{V_o}{a \ln(c/b) \sin\theta} \quad . \tag{40}$$

for θ within the co-axial region and zero elsewhere.

Since $\hat{n} \times \bar{E}$ is in the $-\hat{\phi}$ direction and we wish to find E^{inc} in the $\hat{r}$ direction, it is desired to extract the $\hat{\phi}\,\hat{r}^{th}$ component of $\nabla_s \times \underset{\sim}{\Gamma}(\bar{r}_s, \bar{r}_o)$, which has the following form:

$$(\nabla \times \underset{\sim}{\Gamma})_{\phi r} = -\frac{ik}{4\pi} \sum_{n=1}^{\infty} \frac{2n+1}{r_o} (j_n(kr_<) h_n^{(2)}(kr_>) + T_n h_n(kr_o) h_n^{(2)}(kr_s))$$

$$\cdot \frac{dP_n}{d\theta}(\cos\theta_o)\, P_n(\cos\theta_s) \tag{41}$$

where θ_o defines the observation point and θ_s defines the source point. Specializing this for $r_s \to a$ on the surface of a perfectly conducting sphere, and using the Wronskian to reduce the complexity of the Hankel functions, the following is obtained:

$$(\nabla \times \underset{\sim}{\Gamma})_{\phi r} = -\frac{k}{4\pi a r_o} \sum_{n=1}^{\infty} \frac{(2n+1)}{\frac{d}{da} a\, h_n(ka)} h_n^{(2)}(kr_o)\, P_n(\cos\theta_s) \frac{dP_n}{d\theta}(\cos\theta_o) \ . \tag{42}$$

Substituting this and Eq. (40) into (38) yields:

$$E_r^{inc}(r_o) = \frac{V_o k}{2a^2 \ln(c/b)} \int_{\theta_s = \sin^{-1} b/a}^{\theta = \sin^{-1} c/a} \sum_{n=1}^{\infty} \frac{2(n+1)}{\frac{d}{da}(a h_n(ka))} \frac{h_n(kr_o)}{kr_o}$$

$$\cdot\ P_n(\cos\theta_o) \frac{\frac{dP_n}{d\theta}(\cos\theta_s)}{\sin\theta_s} (a^2 \sin\theta_s\, d\theta_s) \tag{43}$$

where the integral over ϕ has already been carried out. Interchanging the order of summation and using the following relationship,

$$\int_{\theta_1}^{\theta_2} \frac{dP_n}{d\theta} \, d\theta = P_n(X_2) - P_n(X_1) \tag{44}$$

where

$$X_2 = \cos(\sin^{-1}(c/a))$$

and

$$X_1 = \cos(\sin^{-1}(b/a)) ,$$

the resulting equation for the incident radial electric field is

$$E_r^{inc}(\bar{r}_o) = \frac{V_o k}{2 \ln(c/b)} \sum_{n=1}^{\infty} \frac{2(n+1)}{\frac{d}{da} a h_n(ka)} \frac{h_n^{(2)}(kr_o)}{kr_o} P_n(\cos\theta_o) \left[P_n(X_2) - P_n(X_1) \right] . \tag{45}$$

This relation should be evaluated for observation points on the surface of the antenna wire and subsequently used in Pocklington's or Hallén's equation to determine the current on the driven antenna.

VI. Conclusions

The formulation for the two different frequency domain integral equations describing the currents flowing on a radially directed wire in the presence of a conducting sphere has been presented, as well as the relations for the incident tangential electric fields for the scattering and the antenna problems which act as forcing terms for the current.

In this note, only the frequency domain formulation has been considered. It is anticipated that the numerical methods used to solve the integral equations and convert the results to the time domain, as well as the computational results will be the subject of a future note.

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