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Circular Loop Antenna

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ABSTRACT

A new mathematical formulation is given for the calculation of currents and voltages in a thin wire circular loop receiving antenna. The resulting expressions facilitate a discussion of the convergence properties and validity range of the familiar results to this problem.

circular loop antennas, receiving antennas, thin wires

CONTENTS

CIRCULAR LOOP ANTENNA

1. INTRODUCTION

A mathematical treatment of the single-turn, circular loop antenna has existed in the literature for at least 34 years[1]. Numerous discussions and modifications of this work of Hallén, have appeared in the intervening years (e.g., References 2, 3, and 4). A recent review of the theoretical literature on this subject, as well as a comparison of theory and experiment, can be found in the book of King and Harrison[5].

The theoretical work of this paper is not intended to significantly expand the range of validity of the work referred to above, nor to improve the already good agreement with experiment. This work approaches the same problem using a somewhat different mathematical framework, and yields a different form for the final result. The results, when evaluated numerically, yield results which agree well with previous numerical results[4], but are in a form which provides a better analytic "feel" for the behavior of the results. By inspection of the analytical form, it is possible to see the effects of the approximations which are normally made, as well as more clearly defining the range of validity of the final results. A second, and as important, reason for presenting a new approach to an old problem is that it appears to provide a better basis upon which to build the solution to a related, but less investigated problem -- the multi-turn circular loop.

2. THE PROBLEM

The question to be addressed is the following: consider a conducting wire with a circular cross section, radius a; this wire is wound into the shape of a single circular loop, radius b (inside diameter of the loop is 2(b-a)); at the joining point of the loop is a generator capable of maintaining a potential difference across this point; external to the wire is a source of electromagnetic fields; what is the current on the wire due to the two sources?

Assumptions which are normally made, and which will be made here, are as follows; 1) $a \ll b$; 2) $ka \ll 1$ where k is the wave number of the incoming radiation; this restricts consideration to wavelengths much greater than the wire diameter; 3) conductivity of the wire is infinite. Assumptions 1) and 2) (the thin wire approximation) allow one to consider only the currents flowing parallel to the wire axis.

The integral equation which describes the situation of interest has a straightforward derivation. Assuming the antenna to be immersed in a nonconductive medium*, with permittivity ε and permeability μ, the potentials of the fields radiated from the antenna are related to the currents and charges on the antenna via Maxwell's equations (mks units);

$$(\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2}) \underset{\sim}{A}_R(\underset{\sim}{r},t) = - \mu \underset{\sim}{J}(\underset{\sim}{r},t) \tag{1}$$

$$(\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2}) \phi_R(\underset{\sim}{r},t) = - \frac{1}{\varepsilon} \rho(\underset{\sim}{r},t), \tag{2}$$

where $\underset{\sim}{A}_R$ and ϕ_R are the vector and scalar potentials for the radiated fields, $\underset{\sim}{J}$ and ρ are the current density and charge density on the antenna.

Consider the fields, and thus the potentials, to be harmonic with a time dependence $e^{-i\omega t}$. (This is of course no limitation -- an arbitrary time dependence can be built of linear combinations of terms of this form). Equations 1) and 2) thus take the form

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*For a discussion of both bare wire and insulated antennas in conductive media, see the Appendix.

$$(\nabla^2 + k^2)\underset{\sim}{A}_R(\underset{\sim}{r},\omega) = -\mu\ \underset{\sim}{J}(\underset{\sim}{r},\omega) \qquad (3)$$

$$(\nabla^2 + k^2)\phi_R(\underset{\sim}{r},\omega) = -\frac{1}{\varepsilon}\rho(\underset{\sim}{r},\omega) \qquad (4)$$

where $k^2 = \mu\varepsilon\omega^2$. (From this point onward, explicit reference to frequency dependence will be suppressed, and the notation $f(\underset{\sim}{r})$ will imply $f(\underset{\sim}{r},\omega)$ unless otherwise stated). These equations can be shown to be satisfied by potentials of the form

$$\underset{\sim}{A}_R(\underset{\sim}{r}) = \frac{\mu}{4\pi}\int d\underset{\sim}{r}'\ \underset{\sim}{J}(\underset{\sim}{r}')\ \frac{e^{ik|\underset{\sim}{r}-\underset{\sim}{r}'|}}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\ , \qquad (5)$$

$$\phi_R(\underset{\sim}{r}) = \frac{1}{4\pi\varepsilon}\int d\underset{\sim}{r}'\ \rho(\underset{\sim}{r}')\ \frac{e^{ik|\underset{\sim}{r}-\underset{\sim}{r}'|}}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\ . \qquad (6)$$

The fields due to the external source will be related to the fields due to the antenna sources by the boundary condition requiring the tangential component of the electric field to vanish on the surface of a perfect conductor. The radiated electric field can be derived from the potentials via

$$\underset{\sim}{E}_R(\underset{\sim}{r},t) = -\underset{\sim}{\nabla}\ \phi_R(\underset{\sim}{r},t) - \frac{\partial \underset{\sim}{A}_R(\underset{\sim}{r},t)}{\partial t} \qquad (7)$$

$$\underset{\sim}{E}_R(\underset{\sim}{r}) = -\underset{\sim}{\nabla}\phi_R(\underset{\sim}{r}) + i\omega\ \underset{\sim}{A}_R(\underset{\sim}{r})$$

This expression can be simplified using the Lorentz condition

$$\nabla \cdot \vec{A}_R(\vec{r},t) + \mu\varepsilon \frac{\partial\phi_R(\vec{r},t)}{\partial t} = 0 \tag{8}$$

$$\nabla \cdot A_R(\vec{r}) - i\mu\varepsilon\omega\ \phi_R(\vec{r}) = 0$$

yielding

$$E_R(\vec{r}) = \frac{1}{\mu\varepsilon\omega} \nabla(\nabla\cdot A_{\vec{R}}(\vec{r})) + i\omega\vec{A}_R(\vec{r})$$

$$= \frac{i}{4\pi\varepsilon\omega} \int d^3\vec{r}' \left\{ \nabla_r\left(\nabla_r\cdot \vec{J}(\vec{r}')\ \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\right) + k^2\vec{J}(\vec{r}')\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \right\}$$

$$= \frac{i}{4\pi\varepsilon\omega} \int d^3\vec{r}'\ \vec{J}(r') \left\{ \cdot\nabla_r\left(\nabla_r\ \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\right) + k^2\ \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \right\} . \tag{9}$$

Specializing this equation to the thin wire geometry discussed earlier, allows the following simplifications: $\underset{\sim}{J}(\underset{\sim}{r}')$ has components only parallel to the wire axis; $\underset{\sim}{J}(\underset{\sim}{r}')$ as well as the Green's function are assumed not to vary across the wire (the point r=r' will be considered in detail later). Equation (9) can, with these assumptions, be reduced to a one dimensional relationship for the ϕ component of E at the wire surface,

$$E_{R_\phi}(\phi) = \frac{ib}{4\pi\epsilon\omega}\int_{-\pi}^{\pi} d\phi' \; I(\phi') \left\{ \frac{1}{b^2}\frac{\partial^2}{\partial\phi^2} + k^2\cos(\phi-\phi') \right\} \frac{e^{ik|\underset{\sim}{r}-\underset{\sim}{r}'|}}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \tag{10}$$

where ϕ is the angular distance about the center of the loop, and $I(\phi)$ is the current in the loop at angular position ϕ.

It is now expedient to expand $E_{R_\phi}(\phi)$ and $I(\phi')$ into Fourier series in the appropriate angular variable

$$E_{R_\phi}(\phi) = \sum_{n=-\infty}^{\infty} E_n e^{in\phi}$$

$$I(\phi') = \sum_{m=-\infty}^{\infty} I_m e^{im\phi'}, \tag{11}$$

such that

$$E_{R_n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} E_{R_\phi} (\phi) e^{-in\phi} d\phi ,$$

$$I_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} I(\phi') e^{-im\phi'} d\phi' \quad . \tag{12}$$

These Fourier coefficients obey the equation

$$E_{R_m} = \frac{i}{2} \sqrt{\frac{\mu}{\varepsilon}} \sum_{m=-\infty}^{\infty} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \; e^{-in\phi} \frac{1}{2\pi} \right.$$

$$\left. \int_{-\pi}^{\pi} d\phi' \; e^{im\phi'} \left[\frac{1}{kb} \frac{\partial}{\partial \phi^2} + kb \cos(\phi - \phi') \right] \frac{e^{ik|\underset{\sim}{r} - \underset{\sim}{r}'|}}{|\underset{\sim}{r} - \underset{\sim}{r}'|} \right\} I_m . \tag{13}$$

The matrix in the curly brackets will hereafter be referred to as α_{nm}.

If the Green's function is a function of the difference $\phi-\phi'$ only (as it will always be for the purposes of this paper) then the matrix reduces to a diagonal one, with diagonal elements α_n. If one represents the Green's function by a Fourier series in the differences $(\phi-\phi')$,

$$\frac{e^{ik|\underset{\sim}{r}-\underset{\sim}{r}'|}}{|\underset{\sim}{r}-\underset{\sim}{r}'|} = \sum_{n=-\infty}^{\infty} K_n \, e^{in(\phi-\phi')},$$

$$K_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} d(\phi-\phi') e^{in(\phi-\phi')} \frac{e^{ik|\underset{\sim}{r}-\underset{\sim}{r}'|}}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \tag{14}$$

then it is straightforward to express the α's in terms of the above Fourier coefficients

$$\alpha_n = \left(\frac{-n^2}{kb} K_n + \frac{kb}{2}(K_{n+1} + K_{n-1})\right) \tag{15}$$

The resulting equation, relating the radiated field to the current now reads

$$E_{R_n} = \frac{i}{2}\sqrt{\frac{\mu}{\varepsilon}}\,\alpha_n I_n = \frac{i}{2}\sqrt{\frac{\mu}{\varepsilon}}\left(\frac{-n^2}{kb} K_n + \frac{kb}{2}(K_{n+1}+K_{n-1})\right) I_n. \tag{16}$$

The boundary conditions on Maxwell's equations require a vanishing tangential electric field on the surface of the perfectly conducting wire. Thus the total of the incoming field, field due to a load on the antenna, and radiated field due to currents on the antenna, has to have a vanishing tangential component on the wire. Assuming the antenna gap to be infinitesimally small, the boundary condition reads

$$E_R(\phi) + E(\phi) + E_L(\phi) = E_R(\phi) + E(\phi) - \frac{I(\phi=0)Z_L}{b}\delta(\phi)=0 \tag{17}$$

where $E(\phi)$ is the incident electric field, $E_L(\phi)$ is the field due to the impressed potential difference at the load, Z_L is the load impedance (located at $\phi=0$), and $\delta(\phi)$ is the Dirac delta function. Fourier transforming this equation leads to

$$E_{R_n} + E_n - \frac{I(o)Z_L}{2\pi b} = 0 \tag{18}$$

and consequently

$$E_n = \frac{I(o)Z_L}{2\pi b} - \frac{i}{2}\sqrt{\frac{\mu}{\varepsilon}}\left(\frac{-n^2}{kb}K_n + \frac{kb}{2}(K_{n+1} + K_{n-1})\right) I_n. \tag{19}$$

This equation can be solved for I_n, and using the fact that $I(o) = \sum_{n=-\infty}^{\infty} I_n$, $I(o)$ can be expressed in terms of the E_n and K_n:

$$I(o) = \frac{2i\sqrt{\frac{\varepsilon}{\mu}} \sum_{n=-\infty}^{\infty} \frac{E_n}{\left(\frac{-n^2}{kb}K_n + \frac{kb}{2}(K_{n+1}+K_{n-1})\right)}}{1 + \frac{iZ_L}{\pi b}\sqrt{\frac{\varepsilon}{\mu}} \sum_{n=-\infty}^{\infty} \frac{1}{\left(\frac{-n^2}{kb}K_n + \frac{kb}{2}(K_{n+1}+K_{n-1})\right)}} \tag{20}$$

Quantities of interest can be derived directly from this expression. The short circuit current is defined as the limit of I(o) as Z_L approaches zero;

$$I_{sc} = 2i \sqrt{\frac{\varepsilon}{\mu}} \sum_{n=-\infty}^{\infty} \frac{E_n}{\left(\frac{-n^2}{kb} K_n + \frac{kb}{2}(K_{n+1}+K_{n-1})\right)} : \quad (21)$$

open circuit voltage is the limit of $Z_L I(o)$ as Z_L approaches infinity;

$$V_{oc} = 2\pi b \frac{\sum_{n=-\infty}^{\infty} \frac{E_n}{\left(-\frac{n^2}{kb} K_n + \frac{kb}{2}(K_{n+1}+K_{n-1})\right)}}{\sum_{n=-\infty}^{\infty} \frac{1}{\left(\frac{-n^2}{kb} K_n + \frac{kb}{2}(K_{n+1}+K_{n-1})\right)}} : \quad (22)$$

antenna impedance is the ratio V_{oc}/I_{sc}

$$Z = -i\pi b \sqrt{\frac{\mu}{\varepsilon}} \left[\sum_{n=-\infty}^{\infty} \frac{1}{\left(\frac{-n^2}{kb} K_n + \frac{kb}{2}(K_{n+1}+K_{n-1})\right)} \right]^{-1} .$$

(23)

The current $I(\phi)$ at any point on the antenna can be evaluated by solving Equation (19) for I_n, and using Equation (20) to eliminate $I(o)$ from the resulting equation, and finally constructing $I(\phi)$ according to Equation (11). Expressed in terms of the previously derived qualities,

$$I(\phi) = \sum_{n=-\infty}^{\infty} I_n e^{in\phi} = 2i\sqrt{\frac{\varepsilon}{\mu}} \sum_{n=-\infty}^{\infty} \frac{\left(-E_n - \frac{I_{sc} Z_L Z}{2\pi b(Z+Z_L)}\right) e^{in\phi}}{\left(\frac{-n^2}{kb} K_n + \frac{kb}{2}(K_{n+1}+K_{n-1})\right)} \tag{24}$$

The problem thus reduces to finding values for the coefficients K_n. It is in this evaluation that the approach of this paper differs from that found in the published literature. The approach of references (1) and (2) is to represent the Green's function by

$$\frac{e^{ik|\underset{\sim}{r}-\underset{\sim}{r}'|}}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \cong \frac{e^{ikb\sqrt{4\sin^2\frac{(\phi-\phi')}{2} + a^2/b^2}}}{b\sqrt{4\sin^2\frac{(\phi-\phi')}{2} + a^2/b^2}} \tag{25}$$

assume $a \ll b$, and after considerable manipulation evaluate the expression for K_n (Equation 14) by numerical integration This form of the Green's function (18) is equivalent to allowing the current to flow at the center of the conductor, and evaluting the boundary condition for the electric field on the surface of the conductor at the circular locus of points furthermost removed from the plane of the loop. This has the behavior of the true Green's function at all points except where $\phi \approx \phi$, where the singularity of the true Green's function is replaced by a sharply peaked, but finite function.

3. SOLUTION

The basis of the present evaluation of the coefficients K_n is the following formula for the Green's function in spherical coordinates;

$$\frac{e^{ik|\underset{\sim}{r}-\underset{\sim}{r}'|}}{|\underset{\sim}{r}-\underset{\sim}{r}'|} = 4\pi ik \sum_{\ell=0}^{\infty} j_\ell(kr_<)h_\ell^{(1)}(kr_>) \sum_{m=-\ell}^{\ell} Y_\ell^{m^*}(\theta',\phi')Y_\ell^m(\theta,\phi), \tag{26}$$

where $j_\ell(z)$ is the spherical Bessel function, order ℓ, $h_\ell^{(1)}(z)$ is the spherical Hankel function of the first kind, order ℓ, and $r_<(r_>)$ is the smaller (larger) of the two magnitudes, $|\underset{\sim}{r}|$ and $|\underset{\sim}{r}'|$. Orienting the spherical coordinate system such that the axis of the wire is described by the circle $\theta = \frac{\pi}{2}$, $r=b$, this yields an expression

$$K_n = ik \sum_{\ell=|n|,|n|+2,\ldots}^{\infty} (2\ell+1) j_\ell(kb) h_\ell^{(1)}(k(b+a))$$

$$\frac{2^{2n}}{\pi} \left[\frac{\Gamma\left(\frac{\ell+n+1}{2}\right)}{\Gamma\left(\frac{\ell-n+2}{2}\right)} \right]^2 \frac{(\ell-n)!}{(\ell \; n)!} \qquad (27)$$

This expression represents the case where the current is on the axis of the wire and the boundary conditions are satisfied at the points on the surface of the wire at a distance a+b from the center. The effects of such small geometrical differences should be small. To more precisely describe, in this representation, the Green's function of Equation (18), one should write the argument of the Hankel function as $k\sqrt{b^2+a^2}$ and evaluate one of the spherical harmonics at $\theta = \arctan(b/a)$. This deviation from $\theta = \pi/2$ gives corrections to the series (18) of second order (a^2/b^2) and higher orders, which can usually be ignored.

This form (Equation 27) can be slightly simplified by combining the gamma function and factorial terms;

$$K_n = \frac{-ik}{\pi} \sum_{\ell=|n|,|n|+2,...}^{\infty} (2\ell+1)\; j_\ell(kb) h_\ell^{(1)}(k(b+a)) \cdot$$

$$\frac{\Gamma\left(\frac{\ell+n+1}{2}\right)}{\Gamma\left(\frac{\ell+n+2}{2}\right)} \quad \frac{\Gamma\left(\frac{\ell-n+1}{2}\right)}{\Gamma\left(\frac{\ell-n+2}{2}\right)} \tag{28}$$

This series converges, albeit slowly, and has been numerically evaluated in order to compare present theory with the published values of admittance ($Y = 1/Z$) of Reference 4. Good quantitative agreement has been achieved by summing ℓ up to 200 (at most 100 terms in series) for n values from 0 to 19. Terms for $n \geq 20$ were ignored in the evaluation of Y, as they were in Reference 4. Evaluation of this series is simplified by the use of the asymptotic expressions for the Bessel and Hankel functions when $\ell >> kb$;

$$j_\ell(kb) h_\ell^{(1)}(k(b+a)) \sim \frac{-i}{k(b+a)(2\ell+1)} \left(\frac{b}{b+a}\right)^{\ell} . \tag{29}$$

An investigation of the asymptotic behavior (in n) of the expression (28) will reveal results at variance with the conclusions of previous authors (1, 2). Inserting the appropriate limiting forms for the Bessel, Hankel and gamma functions[6], one finds

$$K_n \underset{\text{large } n}{\sim} -\frac{1}{\pi b}\left(\frac{b}{b+a}\right)^{|n|+1} \sum_{\lambda=0}^{\infty} \frac{\Gamma(\lambda+1/2)}{\sqrt{\lambda+|n|}} \frac{(b/b+a)^{2\lambda}}{\lambda!}$$

$$\sim -\frac{1}{\pi b} e^{-|n|\frac{a}{b}} \sum_{\lambda=0}^{\infty} \frac{\Gamma(\lambda+1/2)}{\sqrt{\lambda+|n|}} \frac{(b/b+a)^{2\lambda}}{\lambda!}, \tag{30}$$

a function which decreases monotonically with increasing $|n|$. (The second form above depends on the inequality $a << b$, but does not require $|n|a < b$.) The coefficients, α_n, of the current expansion, thus have an asymptotic form

$$\alpha_n \sim \frac{n^2}{\pi k b^2} e^{-|n|\frac{a}{b}} \sum_{\lambda=0}^{\infty} \frac{\Gamma(\lambda+1/2)}{\sqrt{\lambda+|n|}} \frac{(b/b+a)^{2x}}{\lambda!} \tag{31}$$

This contrasts with the asymptotic form derived by Hallén (1) and reiterated by Storer (2)

$$\alpha_n \sim \frac{n^2}{\pi k b^2}\left\{\ln\frac{2b}{a} - \gamma \ldots - \ln n\right\}, \tag{32}$$

where $\gamma = 0.5772\ldots$ in Euler's constant. This form shows α_n becomes very small (or even, in quite special circumstances, zero) for $n \sim \frac{2b}{a} e^{-\gamma}$, thus yielding large contributions to the current ($\sim \frac{1}{\alpha_n}$). Beyond this point, the form changes sign and diverges with increasing n, allowing a convergent series for the current (Equation 17). Equation (31) shows α_n increasing with n for small (however still asymptotic) n due to the n^2 behavior, but finally a turnaround at $n \sim \frac{2b}{a}$. where α_n begins its asymptotic decrease due to the exponential behavior. The diverse nature of these two results is a consequence of the treatment given to the two inequalities, $n >> 1$ and $\frac{a}{b} << 1$. The earlier results (Equation 24) exhibit expressions which are good for large n and small $\frac{a}{b}$, only so long as $na << b$. This limitation ($na << b$) has been avoided in deriving the form (31). The two limiting forms agree, as they should, for $na << b$, but when this inequality is not satisfied the proper asymptotic behavior is given by Equation (30).

4. DISCUSSION

A result of the eventual convergence of Equation (31) is that in general, the expression for the current, Equation (7), diverges, whereas the supposed incorrect result (32) yields a convergent current expression, and furthermore gives results which compare well with experiment. The explanation of this paradox can be found by considering the exact kernel and comparing its results with those exhibited using the approximate kernel (Equation (25), (28)). Wu[3] has written the exact kernel and then reduced it by assuming the current to exist only on the surface of the wire, and to be constant around the perimeter of the wire for fixed ϕ. His expression, which is adequate for the purposes of this discussion, yields

$$K_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \; e^{in\phi} K(\phi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \; e^{in\phi} \cdot$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \; \frac{e^{ik\sqrt{(2b \sin \phi/2)^2 + (2a \sin \theta/2)^2}}}{\sqrt{(2b \sin \phi/2)^2 + (2a \sin \theta/2)^2}} \; . \qquad (33)$$

Wu's Green's function behaves generally like that of Equation (18), with the quite noticeable exception of the singularity at $\phi=0$. The singularity is logarithmic, and thus integrable, (in the principle value sense), and yields finite values for the Fourier coefficients, K_n for finite n. This singularity is indeed what allows the more exact kernel to yield convergent results for the antenna current. Considering only the region of expression (33) which differs from the K_n's derived from the approximate kernel (20), are written

$$\Delta K_n \sim \frac{1}{(2\pi)^2} \int_{-\delta}^{\delta} d\phi \int_{-\varepsilon}^{\varepsilon} d\theta \; e^{in\phi} \; \frac{1}{\sqrt{b^2\phi^2 + a^2\theta^2}} \qquad (34)$$

where δ and ε are small but finite angular intervals. If they are chosen such that $b\delta << a\varepsilon$ then one can show

$$\Delta K_n = \frac{4}{(2\pi)^2 a} \left\{ \log\left(2 \frac{a\varepsilon}{b\delta}\right) \frac{\sin n\delta}{n} + \frac{S_1(n\delta)}{n} \right\} \qquad (35)$$

where $S_1(x) = \int_0^x dx \frac{\sin x}{x}$ is the sine integral which approaches $\pi/2$ for large x. For large values of n, thus

$\Delta K_n \sim \frac{1}{n}$.

This addition (Equation 35) to the function K_n is always small compared to the K_n of Hallén and Storer except (possibly) at the n value when the expression in parentheses of Equation (32) vanishes, or nearly vanishes. Thus the only change of these old results due to the inclusion of the singularity in the Green's function is to eliminare the (near) divergence mentioned by Hallén and Storer. This was pointed out by Wu[3] (in fact the presence of this "divergence" is what prompted Wu to derive his results using the exact Kernel).

For low n values, ΔK_n makes only minor corrections to the K_n expressions developed in this paper (Equation 21). Asymptotically, however, the $\frac{1}{n}$ behavior of ΔK_n will cause it to become dominant compared to the

$e^{-|n|\frac{a}{b}}$ behavior of K_n (Equation 22)). The ΔK_n term forces the asumptotic behavior of the α_n of this paper to $\alpha_n \sim n$ and allows the current in a circular antenna driven by a delta function generator,

$$I(\phi) \propto \sum_n \frac{e^{in\phi}}{\alpha_n} \tag{36}$$

to be finite everywhere except at $\phi=0$, the position of the load, where the delta function load induces the equivalent of an infinite current.

APPENDIX

LOOP SURROUNDED BY CONDUCTING MEDIUM

If the infinite medium in which the loop is buried is assumed to have a non-zero conductivity, simple modifications in the expressions of this paper allow for a proper description. If the conductivity of the surrounding medium is σ, the differential equations for the potentials (in the frequency domain) read

$$(\nabla^2 + k'^2)\underset{\sim}{A}_R(\underset{\sim}{r}) = -\mu\, \underset{\sim}{J}(r) \qquad \text{(A-1)}$$

$$(\nabla^2 + k'^2)\phi_R(\underset{\sim}{r}) = -\frac{1}{\varepsilon}\,\rho(\underset{\sim}{r}) \qquad \text{(A-2)}$$

where $k'^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega$. With the above equations, the Lorentz condition (Equation 8) is equivalent to the charge continuity equation: either can be used to yield the same integral expression for electric field in terms of currents (e.g., Equation 9). With a conductive surrounding medium, two possible versions of the Lorentz condition suggest themselves:

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{A}_R(\underset{\sim}{r}) - i\mu\varepsilon\omega\, \phi_R(\underset{\sim}{r}) = 0 \;; \qquad \text{(A-3)}$$

and

$$\nabla \cdot \mathbf{A}_R(\mathbf{r}) - (i\mu\varepsilon\omega - \mu\sigma)\, \phi_R(\mathbf{r}) = 0. \qquad \text{(A-4)}$$

Equation A-3 is equivalent to a continuity equation where charge is conserved in the source region, i.e., no charge can be transferred between the antenna and the surrounding medium. This represents an antenna with a very thin coating of perfect insulator on the wire. Equation A-4 is equivalent to a continuity equation in which total charge is conserved (of course), but charge can leak from the antenna to the surrounding medium and back, thus a bare wire in a conducting medium. To account for both these cases, the Lorentz condition will be written as follows

$$\nabla \cdot \mathbf{A}_R(\mathbf{r}) - \frac{ik_A^2}{\omega}\, \phi_R(\mathbf{r}) = 0, \qquad \text{(A-5)}$$

where $k_A^2 = k^2 = \mu\varepsilon\omega^2$ for insulated antenna,

and $k_A^2 = k'^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega$ for bare wire antenna.

These modifications result in modifications to the equation of this paper as follows:

$$I'_{sc} = \frac{2i\omega\mu}{k_A} \sum_{n=-\infty}^{\infty} \frac{E_n}{\left(\frac{-n^2}{k_A b} K'_n + \frac{k_A b}{2} (K'_{n+1} + K'_{n-1})\right)} ; \qquad \text{(A-6)}$$

$$V'_{oc} = 2\pi b \frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{E_n}{\left(\frac{-n^2}{k_A b} K'_n + \frac{k_A b}{2} (K'_{n+1}+K'_{n-1})\right)}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{\left(\frac{-n^2}{k_A b} K'_n + \frac{k_A b}{2} (K'_{n+1}+K'_{n-1})\right)}} \; ; \qquad \text{(A-7)}$$

$$Z' = -i\pi b \frac{\omega\mu}{k_A} \left[\sum_{n=-\infty}^{\infty} \frac{1}{\left(\frac{-n^2}{k_A b} K'_n + \frac{k_A b}{2} (K'_{n+1}+ K'_{n-1})\right)} \right]^{-1} ; \qquad \text{(A-8)}$$

$$I(\phi) = 2i \frac{\omega\mu}{k_A} \sum_{n=-\infty}^{\infty} \frac{\left(E_n - \frac{I'_{sc} Z_L Z'}{2\pi b (Z'+Z_L)}\right)}{\left(\frac{-n^2}{k_A b} K'_n + \frac{k_A b}{2} (K'_{n+1}+K'_{n-1})\right)} e^{in\phi} ; \qquad \text{(A-9)}$$

where

$$K'_n = -i \frac{k'}{\pi} \sum_{\substack{\ell=|n|, \\ |n|+2,\ldots}}^{\infty} (2\ell+1) J_\ell(k'b) h_\ell^{(1)}(k'(b+a)) \cdot$$

$$\frac{\Gamma\left(\frac{\ell+n+1}{2}\right) \Gamma\left(\frac{\ell-n+1}{2}\right)}{\Gamma\left(\frac{\ell+n+2}{2}\right) \Gamma\left(\frac{\ell-n+2}{2}\right)} . \qquad \text{(A-10)}$$

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