Interaction Notes

Note 154

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ON THE INPUT ADMITTANCE OF AN INFINITELY LONG CYLINDRICAL ANTENNA EXCITED BY A FINITE UNIFORM DISTRIBUTED SOURCE*

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ABSTRACT

A uniform symmetrical distributed source is used to derive the input admittance expression for a perfectly conducting, infinitely long, cylindrical antenna. This type of source avoids the singularity which a conventional infinitesimally narrow-gap source introduces in the admittance expression. Analytical expressions are developed for the asymptotic behavior of the admittance for large and small complex frequency amplitudes. Also, a conventional admittance expression for an infinite cylindrical antenna excited by an infinitesimally narrow-gap source is derived in a commonly used form. For comparison, the two admittance expressions are applied to a transient problem. The two time-domain results are found to be significantly different, particularly for early times.

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I dedicate this work to my wife Joan, whose strong will and determination, quest for truth and understanding, and stride toward high goals is an inspiration for me.

I. INTRODUCTION

The properties of an infinite cylindrical antenna have been studied by several investigators. Among them are Hallen,[1] Papas,[2] and Latham and Lee[3] to name but a few. The usual idealized model for the infinite cylindrical antenna consist of an infinitely long, perfectly conducting circular cylinder excited by a delta-function voltage generator. The hypothetical delta-function voltage generator consists of a source voltage V impressed across a circumferential gap of infinitesimal width. The geometry of the model along with cylindrical coordinates (ρ, ϕ, z) are shown in Fig. 1.

The electric field in the gap is given by

$$E_{gap} = -V\delta(z) \tag{1}$$

where $\delta(z)$ is the Dirac-Delta function with dimensions of inverse distance.

The total electromotive force exciting the antenna can be computed from the electric field in the gap by

$$V_{gap} = -\int_{gap} E_{gap}\, dz \tag{2}$$

which is equal to V for the delta-function voltage source.

The input admittance of the antenna is defined as a frequency domain quantity by the ratio

$$Y = \frac{\tilde{I}(o)}{\tilde{V}_{gap}} \tag{3}$$

where $\tilde{I}(z)$ is the total axial current on the antenna.*

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*The tilde, ~, over a quantity indicates the frequency domain expression of the quantity.

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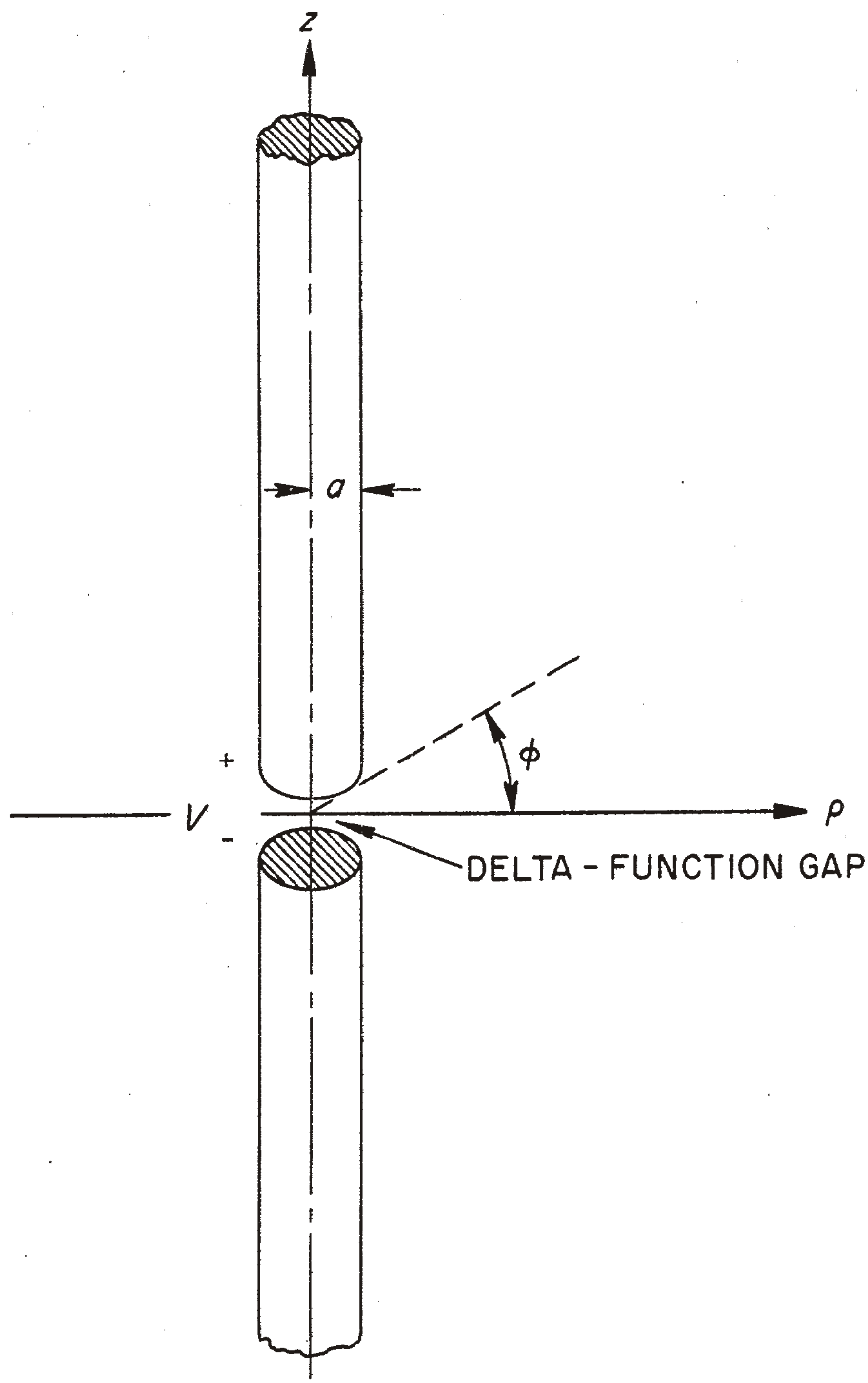

Fig. 1. Infinite Cylindrical Antenna with a Delta-Gap Voltage Source.

Unfortunately, the input admittance of the antenna driven by a delta-function voltage generator is infinite if the radius a is greater than zero.[4] It is plausible to attribute this singularity to the infinite capacitance of the infinitesimally narrow gap at the driving point. Normally, this singularity is removed to obtain a finite input admittance. Two conventionally used processes to remove the singularity in the input admittance are: (1) solve for the antenna current by an iterative procedure which ignores the singularity,[4] and (2) calculate a finite antenna current at $z = \Delta z > 0$. (Ref. 5) The processes of "subtracting-out" the singularity are, however, ambiguous and the accuracy of the admittance solutions may be severely limited over certain frequency ranges.[4] Thus, for transient problems the application of the results obtained from the delta-function voltage driven antenna are not clear cut.

An alternative to the delta-function voltage generator for computing the dipole input admittance a finite distributed source. This source does not introduce a singularity due to an infinitesimally narrow gap. However, the input admittance depends, to a slight extent, on the distribution of the electric field in the gap.[6]

In this note, we consider the input admittance of a perfectly conducting, infinite antenna driven by a distributed source. For simplicity, we assume a uniform distribution for the electric field given by

$$E_s = \begin{cases} -E_o & |z| \le b \\ 0 & \text{elsewhere} \end{cases} \qquad (4)$$

where b is the half-width of the source gap. We also assume that the source is a cylindrical distributed source symmetric about the axis of the antenna and that the frequency of the source field has no z dependence. Furthermore, we specify that the source is an ideal distributed source with zero internal impedance. The geometry of the antenna and distributed source is shown in Fig. 2. The admittance of this antenna should give physically meaningful results when used in transient problems.

II. DERIVATION OF THE ADMITTANCE

The time-harmonic magnetic field in the phi (ϕ) direction ($\tilde{H}_\phi$) of a perfectly conducting, infinitely long, cylindrical antenna driven by a finite ideal cylindrical distributed source with a surface electric field specified by Eq. (4), with $e^{j\omega t}$ suppressed, is given by[7]

$$\tilde{H}_\phi(\rho, z) = \frac{j k \tilde{E}_o}{\pi Z} \int_{-\infty}^{\infty} \frac{e^{j\zeta z} \sin \zeta b \; H_1^{(2)}(\rho [k^2 - \zeta^2]^{1/2})}{\zeta (k^2 - \zeta^2)^{1/2} \; H_o^{(2)}(a [k^2 - \zeta^2]^{1/2})} \, d\zeta \qquad (5)$$

where Z is the free space radiation impedance approximately equal to 120π ohms, k is the propagation constant, and the meanings of a, b, and ρ are given in Fig. 2. which depicts the geometry of the antenna. Equations (31) and (39) in Ref. 7 were used to obtain the expression given by Eq. (5). This same expression can be derived by the application of the superposition technique used in a previous investigation.[8]

In terms of the Laplace transform variable s, the magnetic field becomes

$$\tilde{H}_\phi(\rho, z) = \frac{s \tilde{E}_o}{\pi c Z} \int_{-\infty}^{\infty} \frac{e^{j\zeta z} \sin \zeta b \; K_1(\rho \upsilon)}{\zeta \upsilon K_o(a \upsilon)} \, d\zeta \qquad (6)$$

where $\upsilon = (\zeta^2 + s^2/c^2)^{1/2}$, c is the speed of light in free space, and the relations $s = j\omega$, $k = \omega/c$, $H_o^{(1)}(jx) = -j(2/\pi) K_o(x)$, $H_1^{(1)}(jx) = -(2/\pi) K_1(x)$, and $j(\zeta^2 + s^2/c^2)^{1/2} = (-\zeta^2 - s^2/c^2)^{1/2}$ have been used.

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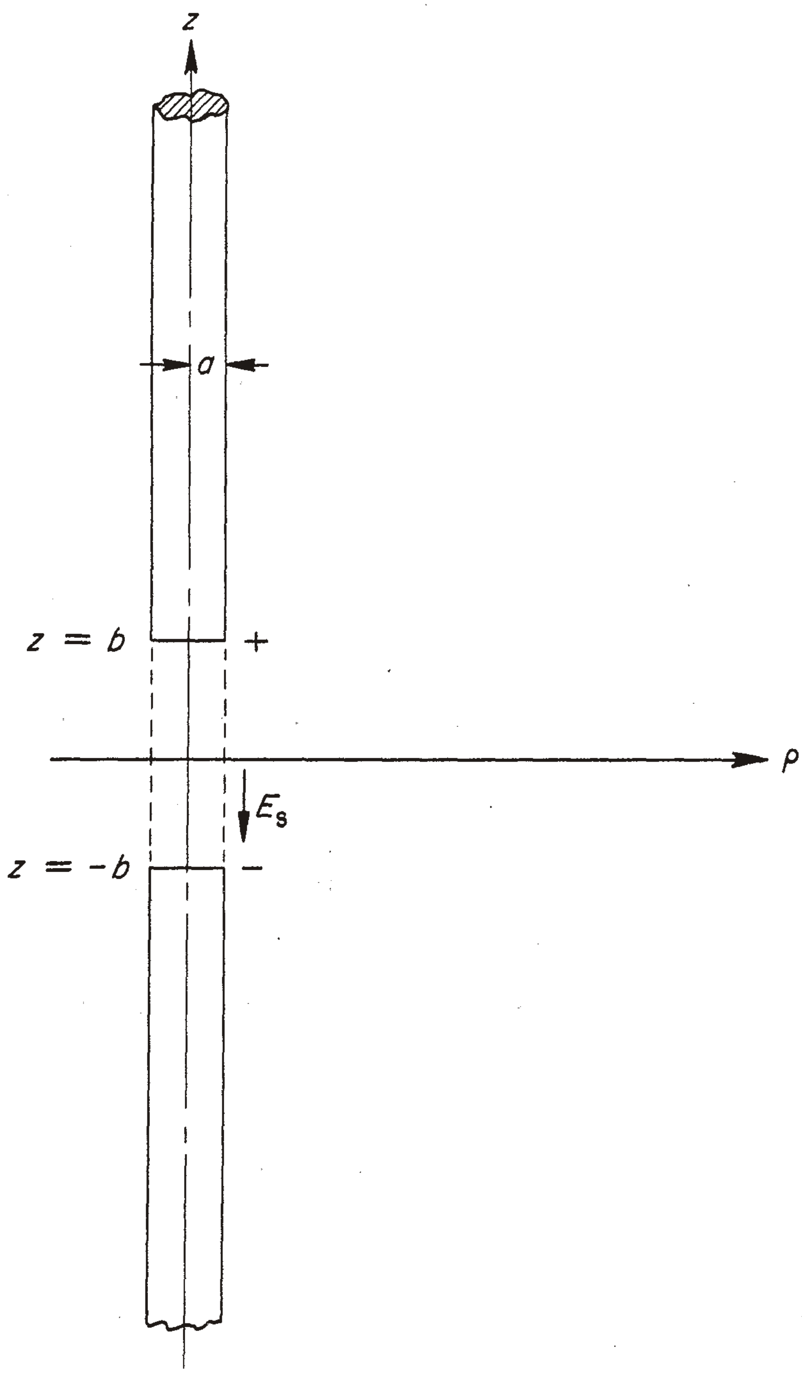


Fig. 2. Infinite Cylindrical Antenna with a Continuous Distributed Source $\tilde{E}_s$.

The total axial current at z = 0, the midpoint between the distributed source gap, is

$$\tilde{I} = \int_0^{2\pi} \tilde{H}_\phi (a,o)\, d\phi$$

$$= \frac{2\, as\, \tilde{E}_s}{c\ Z} \int_{-\infty}^{\infty} \frac{\sin \zeta b}{\zeta} \frac{K_1\ (a\upsilon)}{\upsilon K_0\,(a\upsilon)}\, d\zeta \qquad (7)$$

and the voltage across the gap is

$$\tilde{V}_{gap} = -\int_{-b}^{b} -\tilde{E}_o\, dz = 2b\, \tilde{E}_o \quad . \qquad (8)$$

The substitution of Eqs (7) and (8) into Eq. (3) gives the admittance as

$$Y(a,b) = \frac{as}{bcz} \int_{-\infty}^{\infty} \frac{\sin \zeta b}{\zeta} \frac{K_1\,(a\upsilon)\, d\zeta}{\upsilon\ K_0\,(a\upsilon)} \quad . \qquad (9)$$

For convenience, we can change the variable of integration to $\xi = a\zeta$; Eq. (9) becomes

$$Y(\psi) = \frac{S}{\psi Z} \int_{-\infty}^{\infty} \frac{\sin \psi\xi}{\xi} \frac{K_1\,(u)\, d\xi}{uK_0\,(u)} \qquad (10)$$

where $u = (\xi^2 + S)^{1/2}$, S is a normalized dimensionless Laplace transform variable defined by

$$S = \frac{as}{c} \quad , \qquad (11)$$

and ψ is an antenna parameter given by

$$\psi = b/a \quad . \tag{12}$$

Now for convenience, we define a normalized dimensionless antenna admittance as

$$\Lambda = \frac{ZY}{\pi} = \frac{S}{\pi\psi} \int_{-\infty}^{\infty} \frac{\sin \psi\xi}{\xi} \frac{K_1(u)}{U K_0(u)} d\xi \quad . \tag{13}$$

The integrand in Eq. (13) is an even function of ξ; thus, the admittance can be expressed as

$$\Lambda = \frac{2}{\pi} \frac{S}{\psi} \int_{0}^{\infty} \frac{\sin \psi\xi}{\xi} \frac{K_1(u)}{u K_0(u)} d\xi \quad . \tag{14}$$

III. ASYMPTOTIC BEHAVIOR OF THE ADMITTANCE

In this section, we consider the asymptotic behavior of the admittance for both small and large frequencies and for small ψ.

A. Small Frequency Behavior

As $S \to 0$, the integrand in Eq. (14) becomes large as $\xi \to 0$. It is reasonable to suspect that the major contribution to the integral is made at small values of ξ. For small S and ξ, the integrand takes the form

$$\frac{\sin \psi\xi}{\xi} \frac{K_1(u)}{u K_0(u)} = \frac{-\psi + O(\xi^2)}{u^2 \ln(u)} \sim \frac{-\psi}{u^2 \ln(u)} \tag{15}$$

where the Taylor's series for $\sin \psi\xi$ and the asymptotic expansions of the Bessel functions for small arguments as given by Eqs. (9.6.9) and (9.6.13) in Ref. 9 have been used to obtain Eq. (15).

To determine the asymptotic expansion of Λ for $S \to 0$, we intuitively write Eq. (14) as

$$\Lambda = \frac{2S}{\pi\psi} \int_0^\infty \left\{ \frac{\sin(\psi\xi)}{\xi} \; \frac{K_1(u)}{uK_0(u)} + \frac{\psi}{u^2 \,\ell n\, u} \right\} d\xi$$

$$- \frac{4S}{\pi} \int_0^\infty \frac{d\xi}{(\xi^2 + S^2)\,\ell n\,(\xi^2 + S^2)} = \Lambda_1 + \Lambda_2 \qquad (16)$$

where the asymptotic form of the integrand has been used; Λ_1 and Λ_2 are the first and second integral terms from left to right respectively in Eq. (16).

To determine the order of Λ_1, we write

$$\Lambda_1 = \frac{2S}{\pi\psi} \int_0^\epsilon \frac{\xi^2 \, O(1) \, d\xi}{(\xi^2 + S^2)\,\ell n\,(\xi^2 + S^2)}$$

$$+ \frac{2S}{\pi\psi} \int_\epsilon^\infty \left\{ \frac{O(1)}{\xi(\xi^2+S^2)^{1/2}} + \frac{2\psi}{(\xi^2+S^2)\,\ell n\,(\xi^2+S^2)} \right\} d\xi \qquad (17)$$

where ϵ is a real constant chosen such that $0 < \epsilon \ll 1$. Notice that $\ell n^{-1}(\xi^2+S^2)$ is bounded by $0.5\,\ell n^{-1}(S)$ over the range of integration. The substitution of $0.5\,\ell n^{-1}(S)$ for $\ell n^{-1}(\xi^2+S^2)$ and the evaluation of the integrals in Eq. (17) gives

$$\Lambda_1 = O\left(S\,\ell n^{-1}(S)\right) \quad . \qquad (18)$$

The second integral term can be written as

$$\Lambda_2 = \frac{4}{\pi S} \int_0^\infty \frac{d\xi}{\left(1 + \frac{\xi^2}{S^2}\right) \ell n(S^2) + \ell n\left[\left(1 + \frac{\xi^2}{S^2}\right)\right]}$$

$$= \frac{-2}{\pi} \int_1^\infty \frac{dx}{x\sqrt{x^2-1}\,[\ell n\, S + \ell n\, x]} \qquad (19)$$

where a change of variable of integration $x^2 = 1 + \frac{\xi^2}{S^2}$ has been made. Now let $y = S \ln x$, Eq. (19) becomes

$$\Lambda_2 = \frac{-2q}{\pi} \int_0^\infty \frac{e^{-qy}}{\sqrt{1 - e^{-2qy}} \, [-\ln q + qy]} \, dy \tag{20}$$

where $q = S^{-1}$. As $S \to 0$, $q \to \infty$ and we can apply the formal process provided by Watson's lemma[10] to derive the asymptotic expansion for the integral in Eq. (20). The indirect application of Watson's lemma gives

$$\Lambda_2 \sim \frac{-2}{\pi S} \sum_{n=0}^{\infty} \frac{(-1)^n \, n!}{(\ln S)^{n+1}} \int_0^\infty \frac{e^{-qy} \, (qy)^n}{\sqrt{1 - e^{-2qy}}} \, dy$$

$$= \frac{-2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \, n!}{(\ln S)^{n+1}} \int_0^1 \frac{(\ln (1/x))^n}{\sqrt{1 - x^2}} \, dx \tag{21}$$

$$= \sum_{n=0}^{N-1} \frac{C_n}{(\ln S)^{n+1}} + O\left(\frac{C_N}{(\ln S)^{N+1}}\right)$$

where

$$C_n = \frac{-2 \, (-1)^n \, n!}{\pi} \int_0^1 \frac{(\ln (1/x))^n}{\sqrt{1 - x^2}} \, dx \quad . \tag{22}$$

Evaluation of C_o gives

$$C_o = \frac{-2}{\pi} \int_0^1 \frac{dx}{\sqrt{1 - x^2}} = -1 \quad , \tag{23}$$

and the values of C_1 and C_2 are[11]

$$C_1 = \frac{-2}{\pi}(-1) \int_0^1 \frac{\ln(1/x)\,dx}{\sqrt{1-x^2}} = \ln 2 \tag{24}$$

and

$$C_2 = \frac{-2}{\pi}(2) \int_0^1 \frac{(\ln x)^2\,dx}{\sqrt{1-x^2}} = \frac{-\pi^2}{6} - 2\,(\ln 2)^2 \tag{25}$$

where the values of the integrals in Eqs. (24) and (25) are given by Eqs. (863.41) and (864.33) in Ref. 11 respectively.

To obtain an upper bound for C_3, we first integrate by parts:

$$C_3 = \frac{-12}{\pi} \int_0^1 \frac{(\ln x)^3\,dx}{\sqrt{1-x^2}}$$

$$= \frac{-12}{\pi}(\ln x)^3 \arcsin x \Big|_0^1 + \frac{12}{\pi} \int_0^1 \frac{3(\ln x)^2 \arcsin x}{x}\,dx \tag{26}$$

$$= \frac{36}{\pi} \int_0^1 \frac{(\ln x)^2 \arcsin x}{x}\,dx \le \frac{36}{\pi} \int_0^1 \frac{\pi}{2}(\ln x)^2\,dx = 36 \quad .$$

Collecting the results of Eqs. (18), (21), (23), (24), (25), and (26) gives the asymptotic expansion of Λ for $S \to 0$ as

$$\Lambda = \sum_{n=0}^{2} \frac{C_n}{(\ln S)^{n+1}} + O\left((\ln S)^{-4}\right)$$

$$= \frac{-1}{\ln S} + \frac{\ln 2}{(\ln S)^2} - \frac{(2\ln^2 2 + \pi^2/6)}{(\ln S)^3} + O\left((\ln S)^{-4}\right) \quad . \tag{27}$$

Now substituting S = jka in the first term of Eq. (27) gives

$$\Lambda \sim \frac{-1}{\mathrm{Ln}\,(jka)} = \frac{-1}{\ell n(ka) + j\,\frac{\pi}{2}} \tag{28}$$

which is in good agreement with the result obtained by Schelkunoff.[6]

B. Large Frequency Behavior

As $S \to \infty$, Eq. (14) can be written as

$$\Lambda = \frac{2S}{\pi\psi} \int_0^\infty \frac{\sin \psi\xi}{\xi\; u} \left[1 + \frac{1}{2u} + \frac{1}{8u^2} + O(u^{-3})\right] d\xi \tag{29}$$

where the asymptotic expansions of the Bessel functions for large arguments as given by Eq. (9.7.2) in Ref. 9 have been used. Now we make a change of variable of integration $x = \xi/S$, Eq. (29) becomes

$$\Lambda = \frac{2}{\pi\psi} \left\{ \int_0^\infty \frac{\sin(\psi Sx)}{x\sqrt{x^2+1}}\, dx + \frac{1}{2S} \int_0^\infty \frac{\sin(\psi Sx)}{x\,(x^2+1)}\, dx \right.$$

$$\left. + \frac{1}{8S^2} \int_0^\infty \frac{\sin(\psi Sx)}{x\,(x^2+1)^{3/2}}\, dx + \frac{O(1)}{S^3} \int_0^\infty \frac{\sin(\psi Sx)\, dx}{x\,(x^2+1)^2} \right\} . \tag{30}$$

The first integral can be written as[12]

$$\int_0^\infty \frac{\sin(\psi Sx)\, dx}{x\sqrt{x^2+1}} = \int_0^{\psi S} K_0(t)\, dt$$

$$= \frac{\pi}{2} - \int_{\psi S}^\infty K_0(t)\, dt \quad . \tag{31}$$

As $|S| \to \infty$, the asymptotic form of Eq. (31) is[12]

$$\int_0^\infty \frac{\sin(\psi S x)\ dx}{x\sqrt{x^2+1}} = \frac{\pi}{2} + O\left(\frac{e^{-\psi S}}{\sqrt{\psi S}}\right) \tag{32}$$

where the asymptotic form of $K_o(t)$ for $t \to \infty$ has been used to obtain the order term in Eq. (32). The second integral term in Eq. (30) is listed by Eq. (859.005) in Ref. 11 as

$$\frac{1}{2S}\int_0^\infty \frac{\sin(\psi S x)\ dx}{x(x^2+1)} = \frac{1}{2S}\left(\frac{\pi}{2} - \frac{\pi}{2}e^{-\psi S}\right) \quad . \tag{33}$$

The third integral term in Eq. (30) can be expressed as[12]

$$\frac{1}{8S^2} = \int_0^\infty \frac{\sin(\psi S x)\ dx}{x(x^2+1)^{3/2}} = \frac{1}{8S^2}\int_0^{\psi S} t\,K_1(t)\ dt$$
$$= \frac{1}{8S^2}\left[-\pi - \int_{\psi S}^\infty t\,K_1(t)\ dt\right] \quad . \tag{34}$$

As $|S| \to \infty$, the substitution of the asymptotic form of $K_1(t)$ as $t \to \infty$ gives[12]

$$\frac{1}{8S^2}\int_0^\infty \frac{\sin(\psi S x)\ dx}{x(x^2+1)^{3/2}} \sim \frac{-\pi}{8S^2} - \Gamma(3/2, \psi S)$$
$$= \frac{-\pi}{8S^2} + O\left(\frac{\sqrt{\psi}\,e^{-\psi S}}{S^{3/2}}\right) \quad . \tag{35}$$

The fourth integral in Eq. (30) is given by Eq. (859.014) in Ref. 11 as

$$\int_0^\infty \frac{\sin(\psi S x)\ dx}{x(x^2+1)^2} = \frac{\pi}{2}\left(1 - \frac{2+\psi S}{2}e^{-\psi S}\right) \quad . \tag{36}$$

Collecting the results of Eqs. (30), (32), (33), (35), and (36) gives the asymptotic expansion of Λ for $|S| \to \infty$ with $|\arg S| < \frac{\pi}{2}$ as

$$\Lambda = \frac{1}{\psi} + \frac{1}{2S\psi} - \frac{1}{4S^2\psi} + O\left(|S|^{-3}\right) \tag{37a}$$

where ψ is assumed finite. For $|\arg S| = \pi/2$, the asymptotic form of Λ for $S = j\omega \to j\infty$ is

$$\Lambda = \frac{1}{\psi} + O\left(|S|^{-1/2}\right) \quad . \tag{37b}$$

C. Small ψ Behavior

To determine the asymptotic form of Λ for $\psi \to 0$ with S restricted to finite values, we first break the integral in Eq. (14) into two parts:

$$\Lambda = \frac{2S}{\pi\psi} \int_0^{\mu} \frac{\sin \psi\xi}{\xi} \frac{K_1(u)}{u\,K_o(u)}\, d\xi + \frac{2S}{\pi\psi} \int_{\mu}^{\infty} \frac{\sin \psi\xi}{\xi} \frac{K_1(u)}{u\,K_o(u)}\, d\xi$$

$$= I_1 + I_2 \tag{38}$$

where μ is defined as a real constant selected such that $1 \ll \mu < 20$ and I_1 and I_2 are the first and second integral terms in Eq. (38).

We can obtain an upper bound for I_1 by writing

$$I_1 \le \frac{2S}{\pi\psi} \frac{K_1(S)}{S\,K_o\,(S)} \int_0^{\mu} \frac{\sin \psi\xi}{\xi}\, d\xi$$

$$= \frac{2\,K_1\,(S)}{\pi\,K_o\,(S)} \left(\mu + O(\psi)\right) \tag{39}$$

and

$$I_1 = O(1) \quad . \tag{40}$$

The second integral term in Eq. (38) can be written as

$$I_2 = \frac{2S}{\pi\psi} \int_{\psi\mu}^{\infty} \frac{\sin x}{x} \frac{K_1 (v)}{v\, K_o (v)} dx \tag{41}$$

where a change of variable of integration $x = \psi\xi$ has been made and

$$v = \left(\frac{x^2}{\psi^2} + S^2\right)^{1/2} \quad . \tag{42}$$

The substitution of the asymptotic expansions for the Bessel functions for $v \to \infty$ gives

$$I_2 = \frac{2S}{\pi\psi} \int_{\psi\mu}^{\infty} \frac{\sin x}{x\ v} [1 + \frac{1}{2v} + 0(v^{-2})]\ dx \quad . \tag{43}$$

Now rewrite I_2 as

$$I_2 = \frac{2S}{\pi\psi} \int_{o}^{\infty} (\cdot\ \cdot\ \cdot\ \cdot)\ dx - \frac{2S}{\pi\psi} \int_{o}^{\psi\mu} (\cdot\ \cdot\ \cdot\ \cdot)\ dx \quad . \tag{44}$$

As $\psi \to 0$, the asymptotic form of the second integral in Eq. (44) is

$$- \frac{2S}{\pi\psi} \int_{o}^{\psi\mu} \frac{\sin x}{x\ v} [1 + \frac{1}{2v} + 0(v^{-2})] \sim - \frac{2S}{\pi\psi} \int_{o}^{\psi\mu} [\frac{1}{v} + \frac{1}{2v^2} + 0(v^{-3})]\ dx = 0(1) \tag{45}$$

and

$$\begin{aligned} I_2 &= \frac{2S}{\pi\psi} \int_{o}^{\infty} \frac{\sin x}{x\ v} [\ 1 + \frac{1}{2v} + 0(v^{-2})]\ dx + 0(1) \\ &= \frac{2S}{\pi\psi} \int_{o}^{\infty} \frac{\sin \psi\xi}{\xi u} [\ 1 + \frac{1}{2u} + 0(u^{-2})]\ d\xi + 0(1) \quad . \end{aligned} \tag{46}$$

Applying the results of Eqs. (29), (30), (31), and (33) gives

$$I_2 \sim \frac{2}{\pi\psi} \int_0^{\psi S} K_0(t)\, dt + \frac{1}{\pi\psi S} \int_0^{\infty} \frac{\sin(\psi S x)}{x\,(x^2+1)}\, dx$$

$$+ \frac{O(1)}{\psi} \int_0^{\psi S} t\, K_1(t)\, dt \tag{47}$$

$$\sim \frac{-2}{\pi\psi} \int_0^{\psi S} \ln t\, dt + \frac{1}{\pi\psi S}\left(\frac{\pi\psi S}{2}\right) + \frac{O(1)}{\psi} \int_0^{\psi S} t\left(\frac{1}{t}\right) dt$$

where the asymptotic forms of the Bessel functions for small argument as given by Eqs. (9.6.8) and (9.6.9) in Ref. 9 have been used. Evaluating the integrals in Eq. (47) gives

$$I_2 = -\frac{2S}{\pi} \ln(\psi S) + O(1) \quad . \tag{48}$$

Collecting the results of Eqs. (40) and (48) gives the asymptotic behavior of Λ for $\psi \to 0$ with S finite as

$$\Lambda \sim -\frac{2S}{\pi} \ln(\psi S) \quad . \tag{49}$$

IV. THE MODIFIED DELTA-GAP ADMITTANCE

In order to compare the admittance derived in Section II with a conventional admittance expression, the admittance derived from the admittance of an infinite cylindrical antenna excited by a delta-gap source is considered in this section. The current at the driving point of a cylindrical antenna excited by a delta-gap voltage source $\tilde{V}_\delta$ is

$$\tilde{I}_\delta = \frac{as\tilde{V}_\delta}{cz} \int_{-\infty}^{\infty} \frac{e^{isb}\, K_1(a\upsilon)}{\upsilon\, K_0(a\upsilon)}\, ds \tag{50}$$

where the singularity has been "subtracted out" by calculating the current at a distance "b" from the feed point and υ is the quantity used in Eq. (6).

It follows that the normalized admittance is given by

$$\Lambda_\delta = \frac{z\,\tilde{I}_\delta}{\pi\nabla} = \frac{S}{\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi\psi}\,K_1(u)}{u\,K_o(u)}\,d\xi$$

$$= \frac{2S}{\pi} \int_0^{\infty} \cos \xi\psi \, \frac{K_1(u)}{u\,K_o(u)}\,d\xi \quad . \tag{51}$$

The behavior of Λ_δ for $|S| \to 0$ can be derived in a similar manner as used to determine the small S behavior of Λ, i.e., as $|S| \to 0$ the major contribution to the integral in Eq. (51) is made at small values of ξ. Thus, we write Λ_δ as

$$\Lambda_\delta = -\frac{2S}{\pi} \int_0^{\infty} \frac{\cos \xi\psi}{(\xi^2+S^2)\ell n\, S}\,d\xi$$

$$+ \frac{2S}{\pi} \int_0^{\infty} \frac{\cos \xi\psi}{\xi^2+S^2} \left\{\frac{u\,K_o(u)}{K_1(u)} + \ell n^{-1}(s)\right\} d\xi$$

$$= \frac{-1}{\ell n\, S} + O(|S|) \tag{52}$$

where the asymptotic form of the integrand in Eq. (51) for small u has been used.

The large S behavior of Λ_δ can be derived by using the large u asymptotic form of the integrand in Eq. (51). The result is

$$\Lambda_\delta \sim \frac{2S}{\pi} \int_0^{\infty} \frac{\cos \xi\psi}{\sqrt{\xi^2+S^2}}\,d\xi = \frac{2S}{\pi} K_o(\psi S) \tag{53}$$

for $|\psi S| \to \infty$. Thus, the asymptotic form of Λ_δ for $|S| \to \infty$ with $\psi > 0$ is

$$\Lambda_\delta \sim \frac{2S}{\pi} K_o(\psi S) \sim \sqrt{\frac{2S}{\pi\psi}} \; e^{-\psi S} \quad . \tag{54}$$

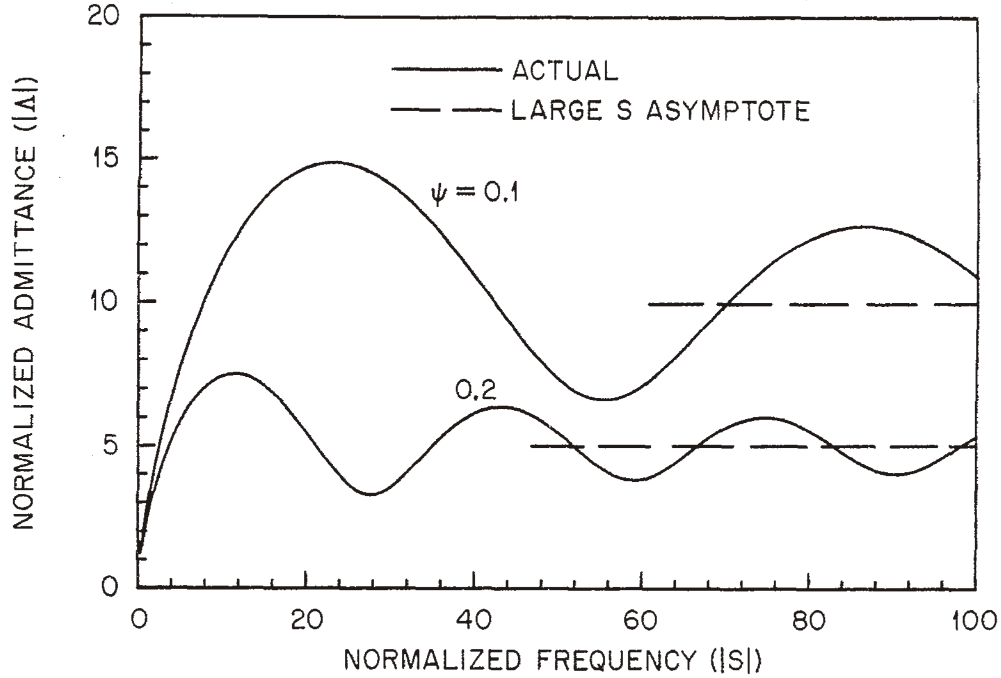
By comparing Eq. (54) with Eqs. (37a) and (37b), it is clear that the large S asymptotic behaviors of Λ and Λ_δ are significantly different. As $|S| \to \infty$, the value of Λ approaches a constant equal to ψ^{-1} whereas Λ_δ approaches zero for $|\arg S| \neq \pi/2$ or infinity for $|\arg S| = \pi/2$.

V. RESULTS

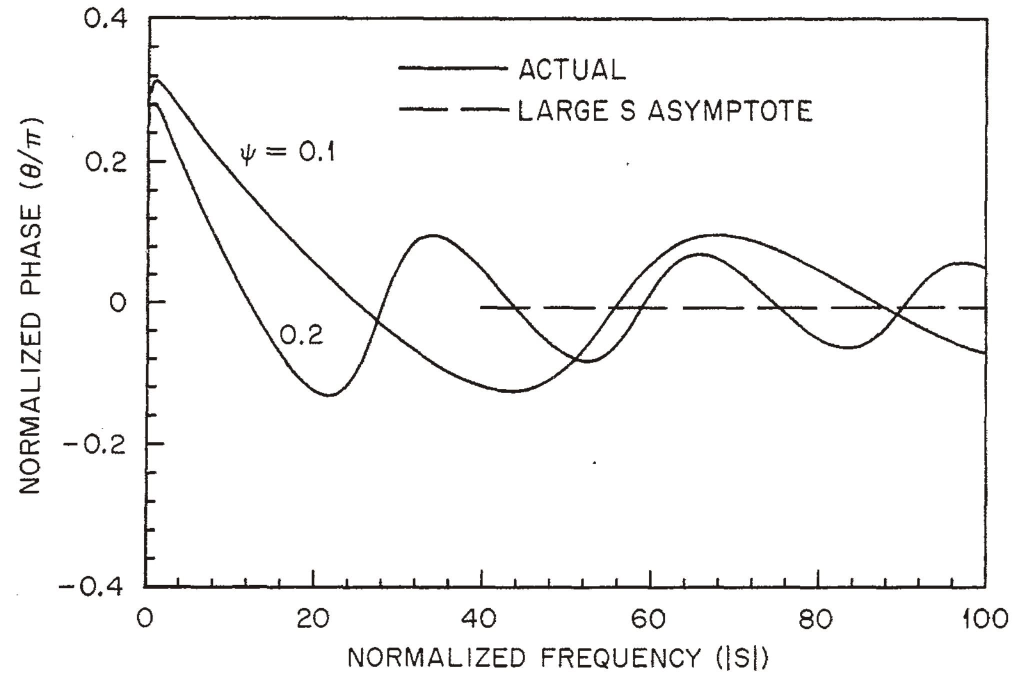
The normalized admittance Λ is plotted as a function of complex frequency in Figs. 3 through 12a with ψ as a parameter. The solid curves were obtained by numerically integrating the integral in Eq. (14) with a relative error of less than 0.1%. The large and small S asymptotic forms of Λ are shown as dashed lines. The small S asymptotic form was obtained from the first three terms in the expansion given by Eq. (27), and the large $|S|$ asymptotic form was obtained from the first three terms in Eq. (37a) for $|\arg S| < \pi/2$ and from the first term in Eq. (37b) for $|\arg S| = \pi/2$.

The normalized modified delta-gap admittance Λ_δ is shown in Fig. 12b as a function of real S. These curves were obtained by numerically integrating Eq. (51) with a relative error of less than 0.1%. A comparison between the delta-gap and finite-gap admittance functions shows a slight difference for real S less than 0.5 as shown in Fig. 12. In Fig. 13, Λ_δ is plotted as a function of real S over a larger frequency range. For larger values of real S, there is a significant difference as shown by comparing Figs. 11 and 13.

ORNL-DWG 73-12551



(a) AMPLITUDE OF THE ADMITTANCE



(b) PHASE OF THE ADMITTANCE

Fig. 3. Normalized Admittance as a Function of Complex $S = |S|\ e^{j\pi/2}$ with ψ as a Parameter.

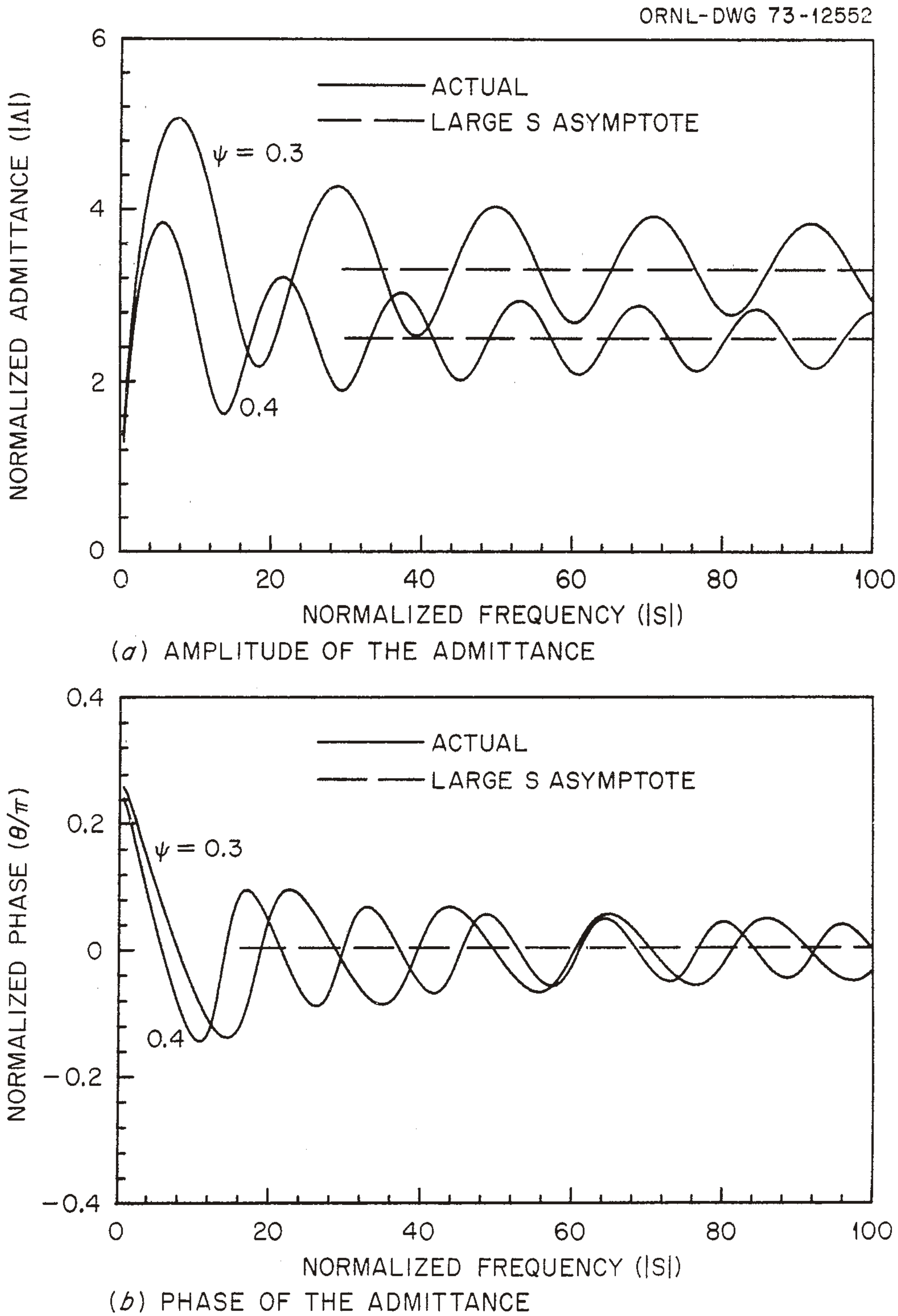


Fig. 4. Normalized Admittance as a Function of Complex $S = |S|\ e^{j\pi/2}$ with ψ as a Parameter.

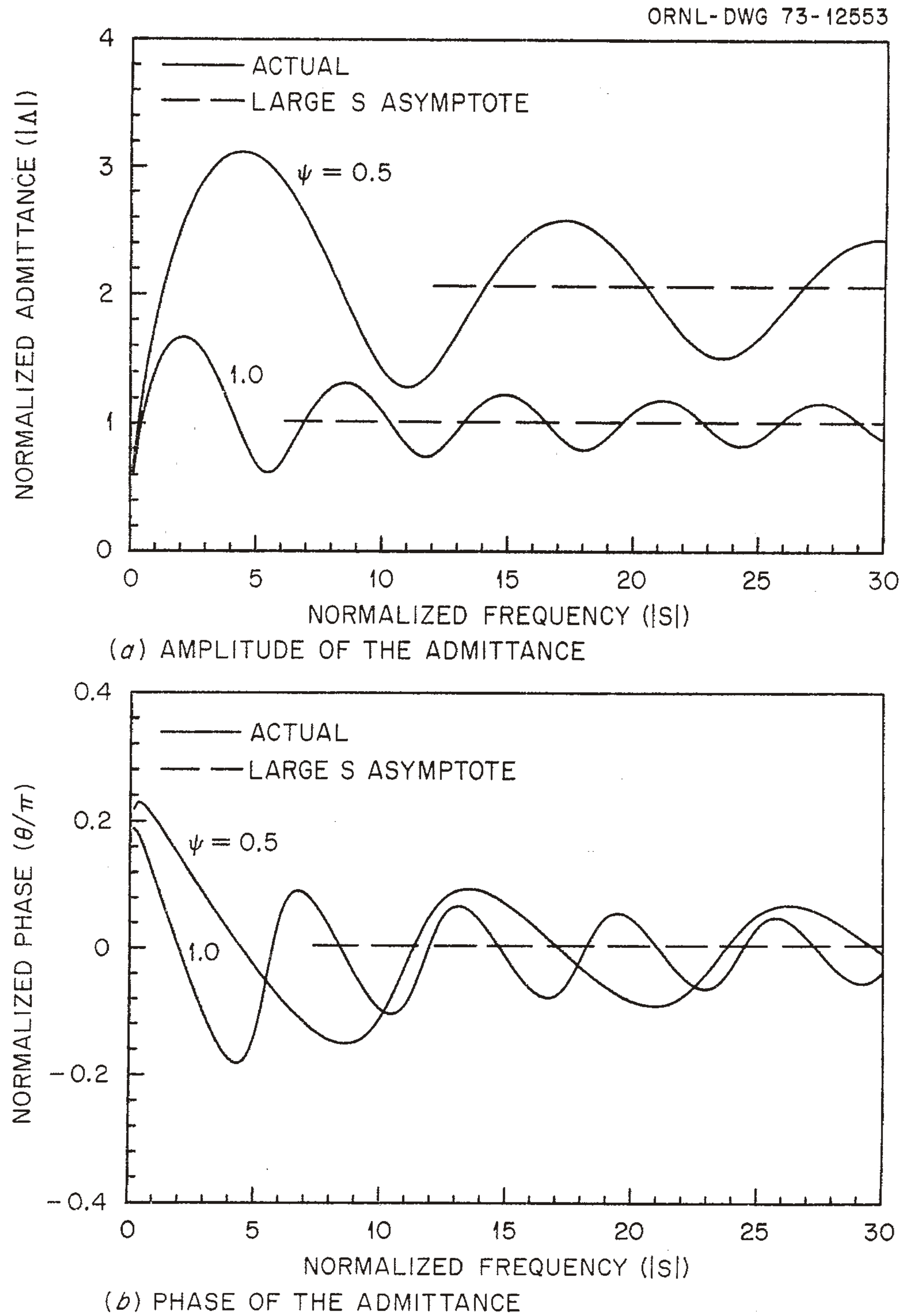


Fig. 5. Normalized Admittance as a Function of Complex $S = |S| e^{j\pi/2}$ with ψ as a Parameter.

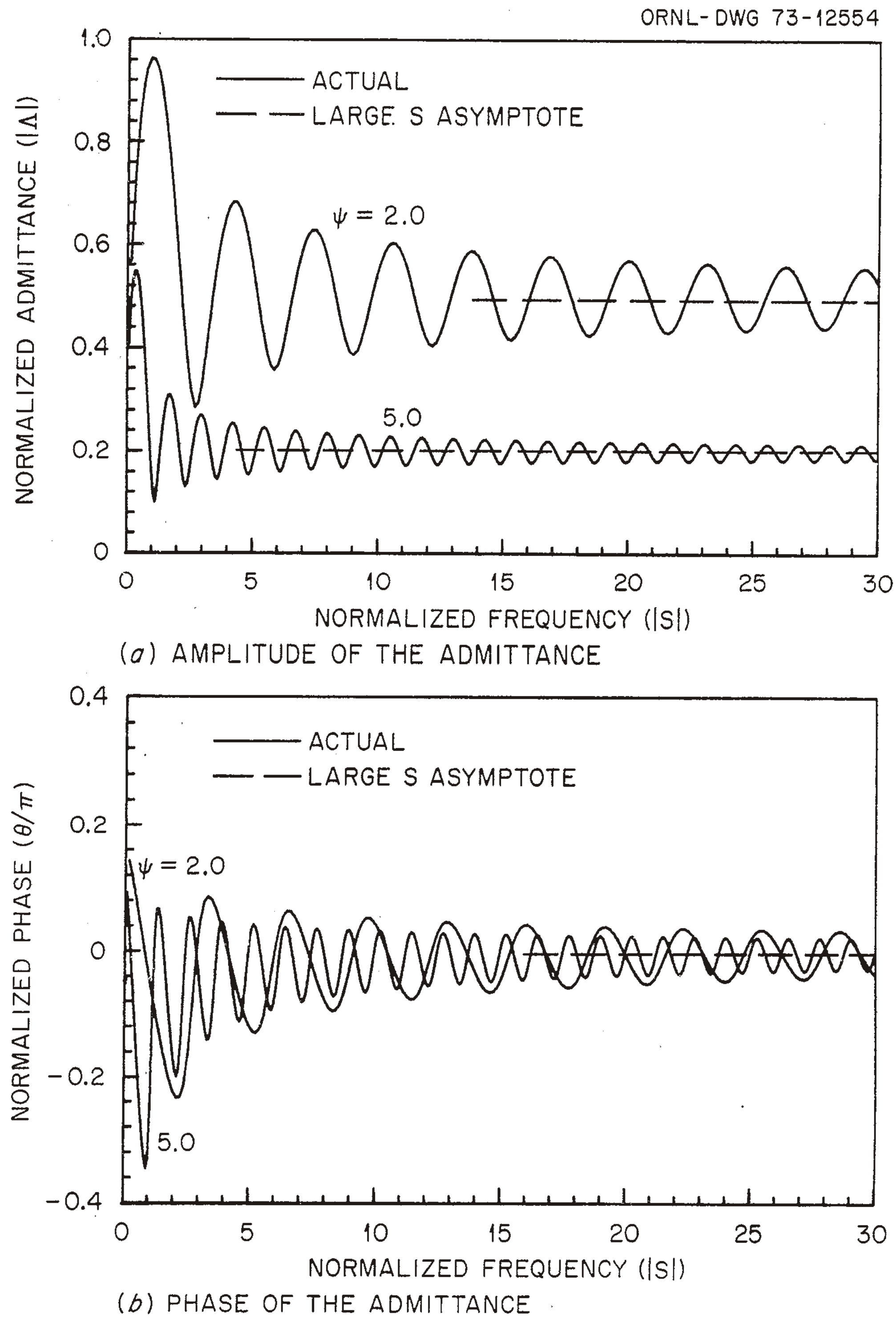


Fig. 6. Normalized Admittance as a Function of Complex $S = |S|\ e^{j\pi/2}$ with ψ as a Parameter.

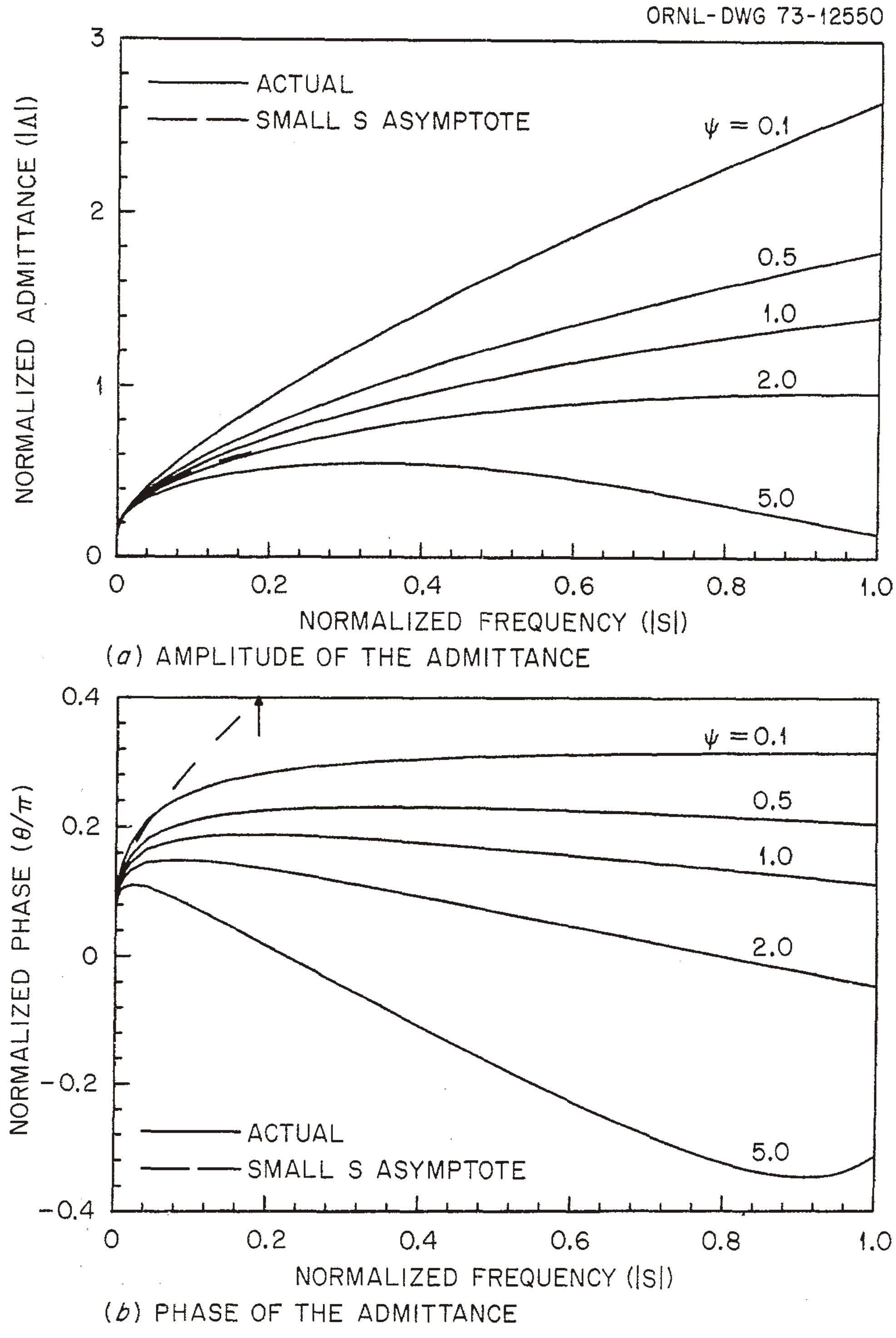


Fig. 7. Normalized Admittance as a Function of Complex $S = |S| e^{j\pi/2}$ with ψ as a Parameter.

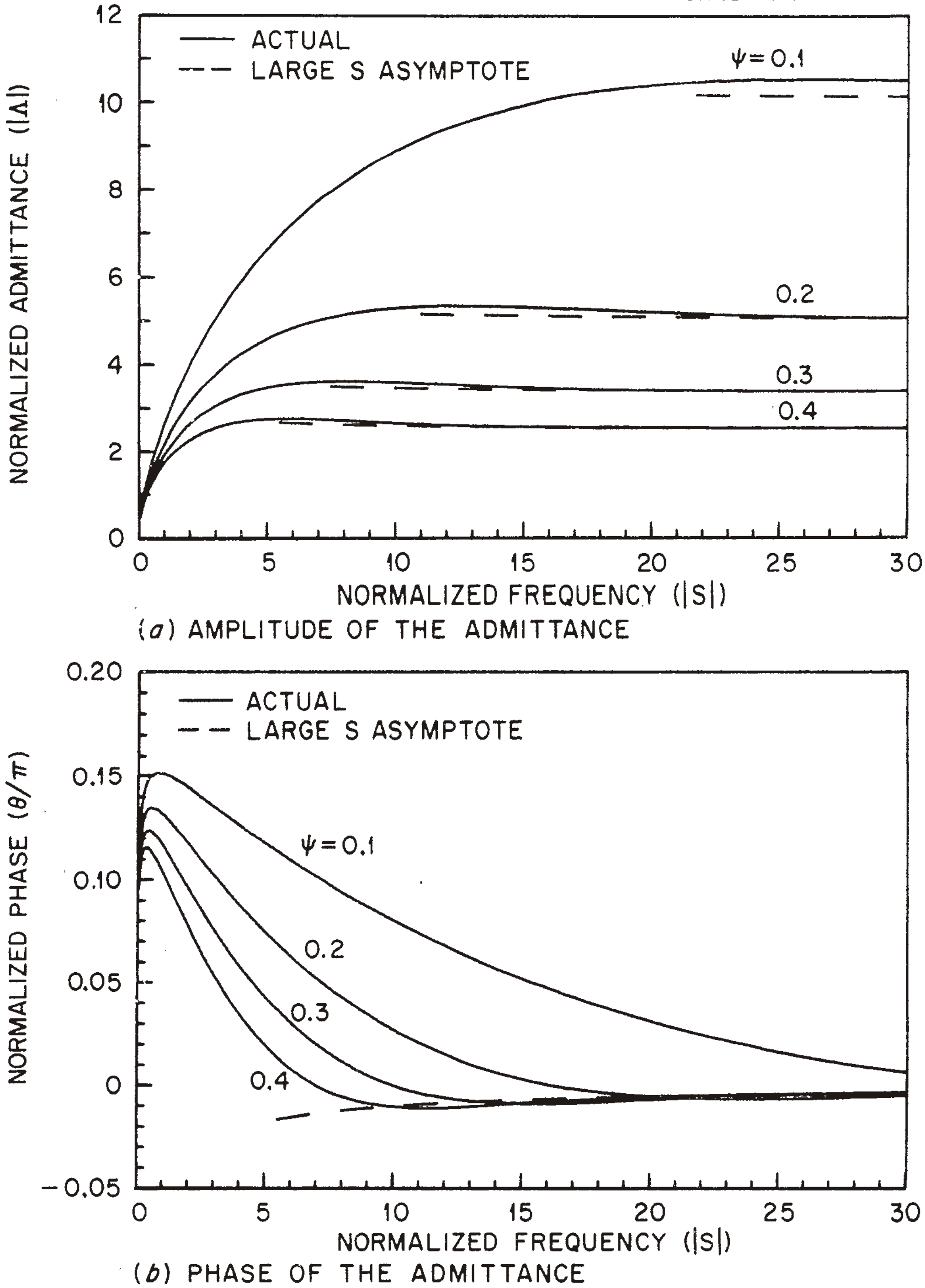


Fig. 8. Normalized Admittance as a Function of Complex $S = |S|e^{j\pi/4}$ with ψ as a Parameter.

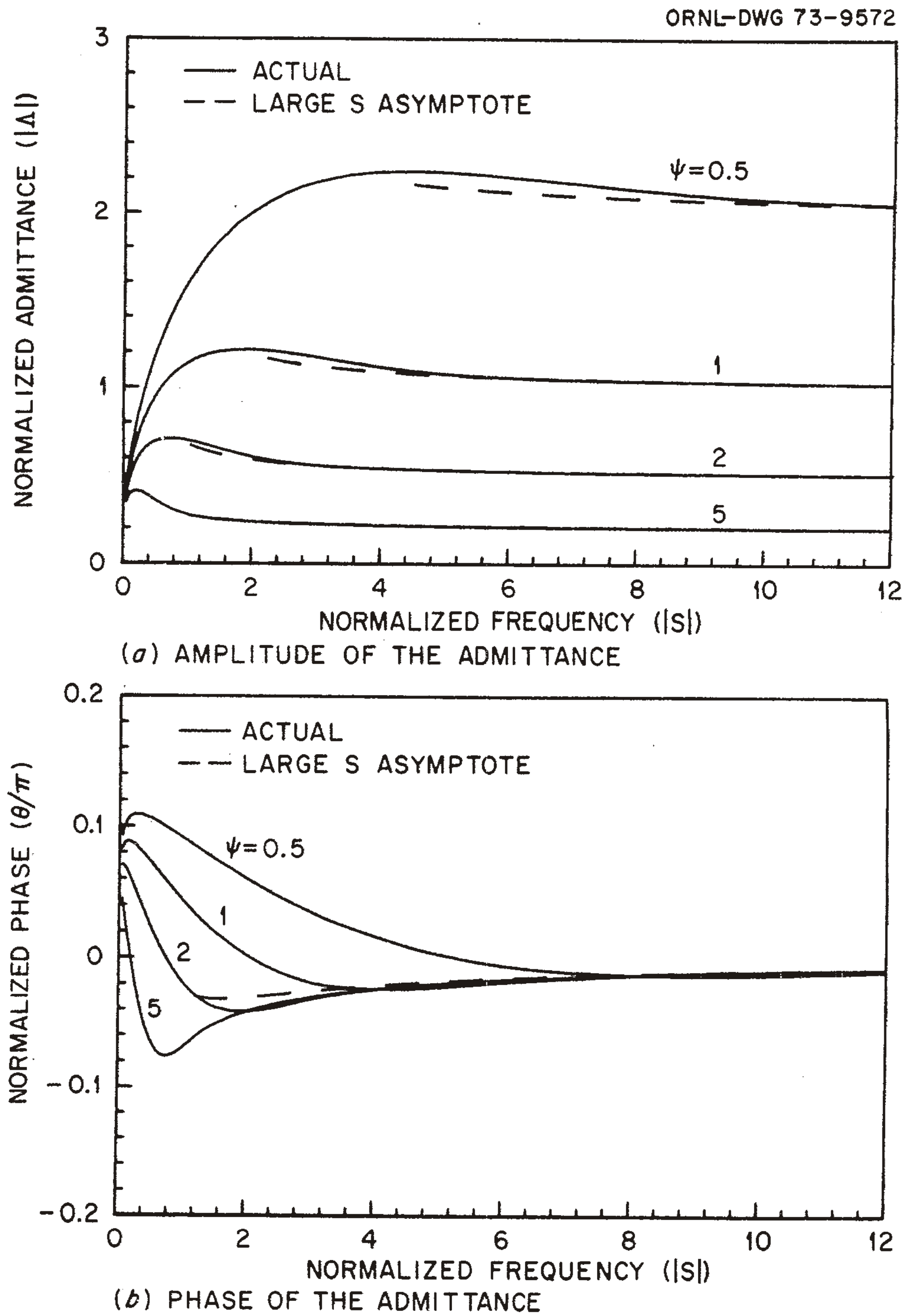


Fig. 9. Normalized Admittance as a Function of Complex $S = |S|\ e^{j\pi/4}$ with ψ as a Parameter.

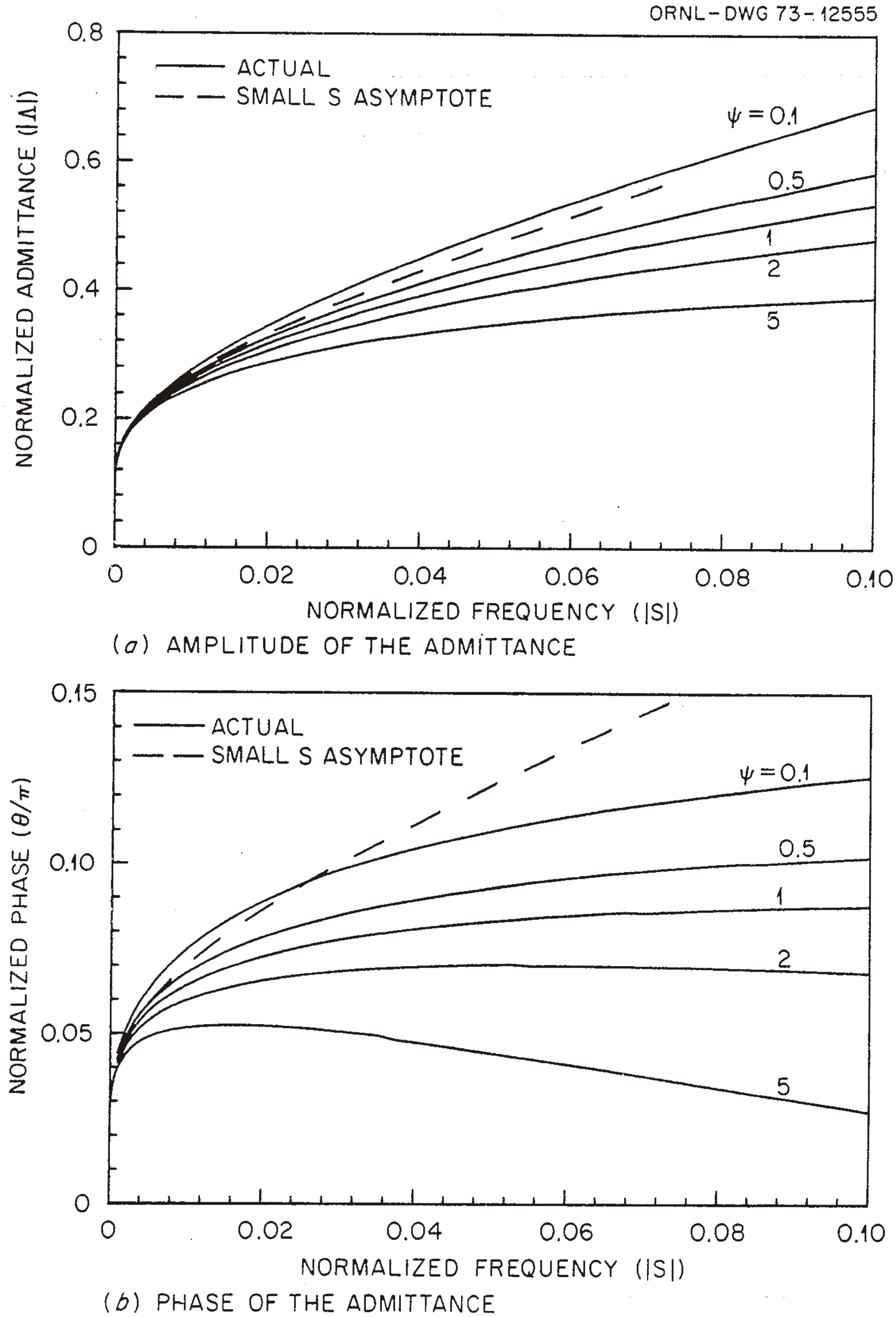
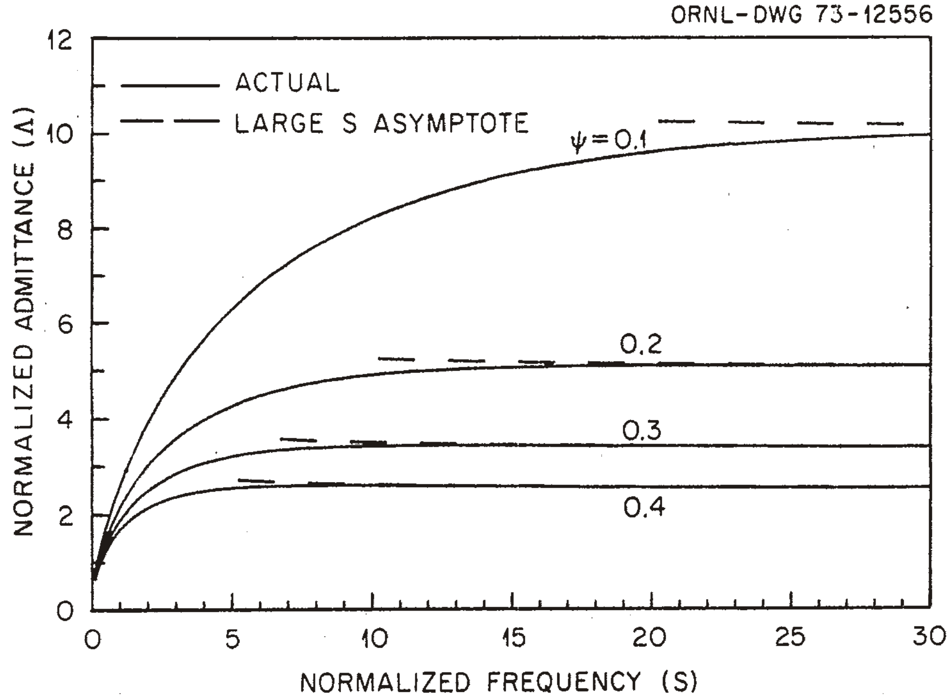
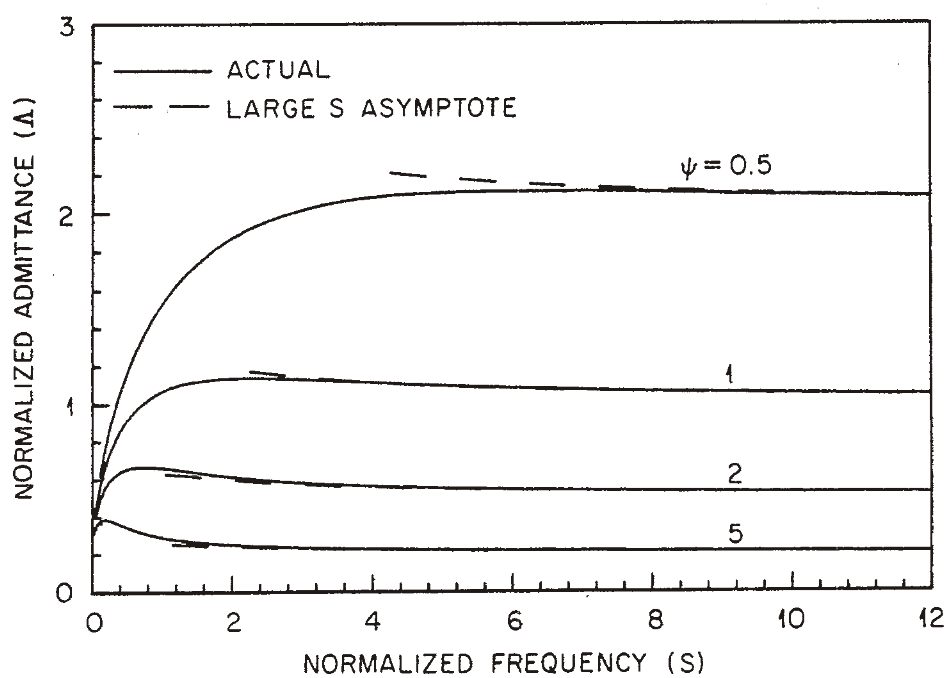


Fig. 10. Normalized Admittance as a Function of Complex $S = |S| \; e^{j\pi/4}$ with ψ as a Parameter.

ORNL-DWG 73-12556



(a) SMALL ψ



(b) LARGE ψ

Fig. 11. Normalized Admittance as a Function of Real S with ψ as a Parameter.

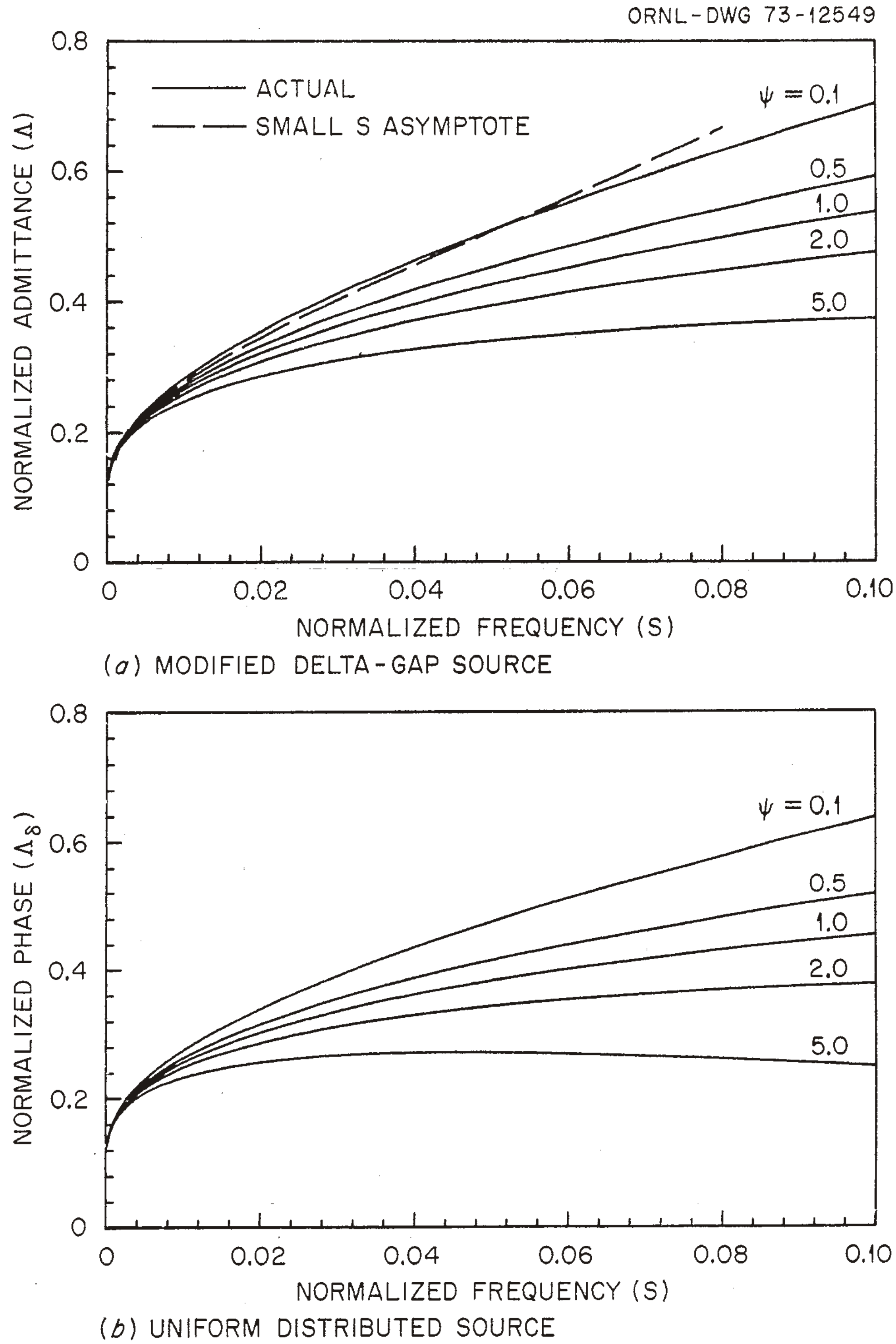


Fig. 12. Normalized Admittance as a Function of Real S with ψ as a Parameter.

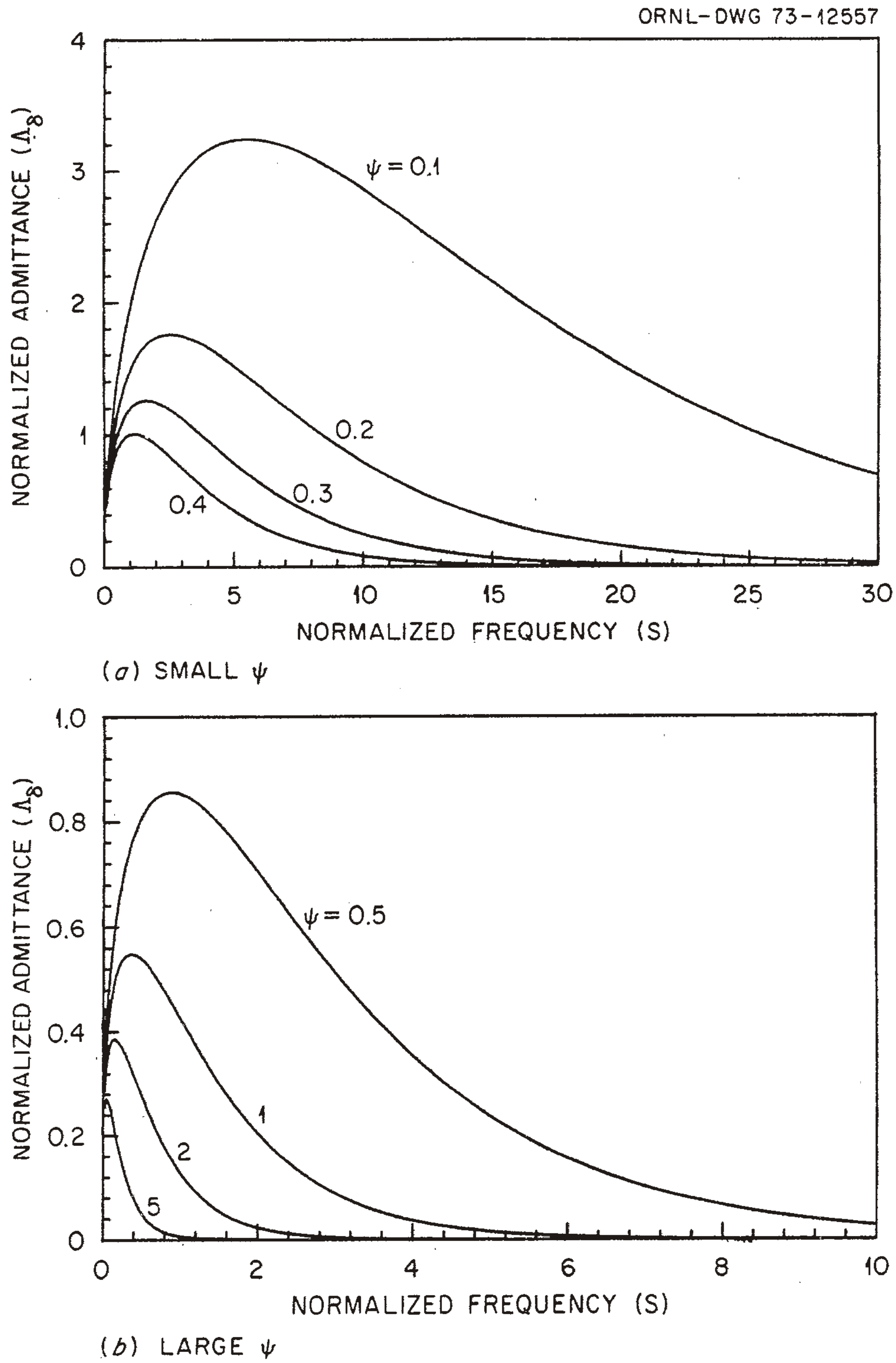


Fig. 13. Normalized Modified Delta-Gap Admittance as a Function of Real S with ψ as a Parameter.

To compare the effects of applying Λ and Λ_δ to transient problems, consider the indicial admittances

$$A(ct/a) = L^{-1}\left\{\frac{\Lambda(S)}{S}\right\} \tag{55}$$

and

$$A_\delta(ct/a) = L^{-1}\left\{\frac{\Lambda_\delta(S)}{S}\right\} \tag{56}$$

where $L^{-1}\{f(s)\}$ is the inverse Laplace transform of $f(s)$.

Since the early time responses are of interest, it is convenient to expand the admittance functions as

$$\begin{aligned} \Lambda = \frac{2S}{\pi\psi}\int_0^\infty d\xi\,\frac{\sin\xi\psi}{\xi u}\left[\frac{K_1(u)}{K_0(u)} - \left(1 + \frac{1}{2u} - \frac{1}{8u^2}\right)\right] \\ + \frac{1}{\psi} - \frac{1}{8\psi S^2} + \frac{1}{2S\psi}\left(1 - e^{-\psi S}\right) + \frac{K_0(\psi S)}{4\pi\, S} \\ - \left(\frac{2}{\pi} - \frac{1}{4\pi\, S^2}\right)\int_S^\infty K_0(\psi\tau)\,d\tau \end{aligned} \tag{57}$$

and

$$\begin{aligned} \Lambda_\delta = \frac{2S}{\pi}\left\{\int_0^\infty d\xi\,\frac{\cos\xi\psi}{u}\left[\frac{K_1(u)}{K_0(u)} - \left(1 + \frac{1}{2u} - \frac{1}{8u^2}\right)\right]\right. \\ \left. + K_0(\psi S) + \frac{\pi}{4S}e^{-\psi S} - \frac{\psi}{8S}K_1(\psi S)\right\} \quad , \end{aligned} \tag{58}$$

where the large u asymptotic expansions of the Bessel function have been used. The indicial admittance A(T) can be expressed as

$$A(T) = \frac{2}{\pi\psi}\lambda(T) + \left\{\frac{1}{\psi} + \frac{T}{2\psi} - \frac{T^2}{8\psi}\right\} U(T)$$

$$+ \frac{1}{\psi}\left\{\frac{T^2 \sec^{-1}(T/4)}{8\pi} - \frac{4\sqrt{T^2 - \psi^2}}{8\pi}\right. \quad (59)$$

$$\left. - \frac{(T-\psi)}{2} - \frac{2}{\pi}\sec^{-1}(T/\psi)\right\} U(T-\psi) \quad ,$$

where $U(T-\psi)$ is the unit step function given by

$$U(T-\psi) = \begin{cases} 0 & T < \psi \\ 1 & T \geq \psi \end{cases} \quad (60)$$

and T is a normalized time variable given by

$$T = \frac{ct}{a} \quad . \quad (61)$$

The function $\lambda(T)$ is given by

$$\lambda(T) = L^{-1}\left\{\int_0^\infty d\xi \, \frac{\sin \xi\psi}{\xi u}\left[\frac{K_1(u)}{K_o(u)} - \left(1 + \frac{1}{2u} - \frac{1}{8u^2}\right)\right]\right\} \quad . \quad (62)$$

The indicial admittance $A_\delta(T)$ can be expressed as

$$A_\delta(T) = \frac{2}{\pi}\lambda_\delta(T)$$

$$+ \left\{\frac{2}{\sqrt{T^2 - \psi^2}} + \frac{1}{2} - \frac{\sqrt{T^2 - \psi^2}}{4\pi}\right\} U(t-\psi) \quad (63)$$

where

$$\lambda_o(T) = L^{-1} \left\{ \int_0^\infty d\xi \, \frac{\cos \xi\psi}{u} \left[\frac{K_1(u)}{K_o(u)} - \left(1 + \frac{1}{2u} - \frac{1}{8u^2} \right) \right] \right\} \quad . \tag{64}$$

Note that the time response of $A_\delta(T)$ is delayed until $T \geq \psi$. This is due to the propagation time required for the antenna current to reach the distance b from the origin.

Now consider the early time asymptotic form of A(T). As $T \to 0$,

$$\frac{2}{\pi\psi} \lambda(T) \sim \frac{10 \; T^2}{64\psi} \; U(T) \tag{65}$$

where the theorem in Ref. 8 has been used. Thus,

$$A(T) \sim \frac{1}{\psi} \, U(T) \quad . \tag{66}$$

And as $T-\psi \to 0$,

$$\frac{2}{\pi} \lambda_\delta(T) \sim \frac{5\psi}{64} \, (T-\psi) \; U(T-\psi) \tag{67}$$

and

$$A_\delta(T) \sim \frac{2}{\pi \sqrt{T^2 - \psi^2}} \; U(T-\psi) \quad . \tag{68}$$

As can be seen from Eqs. (66) and (68), there is a significant difference between the early time behaviors of A(T) and $A_\delta(T)$. The indicial admittances are shown for $\psi = 1.0$ in Fig. 14.

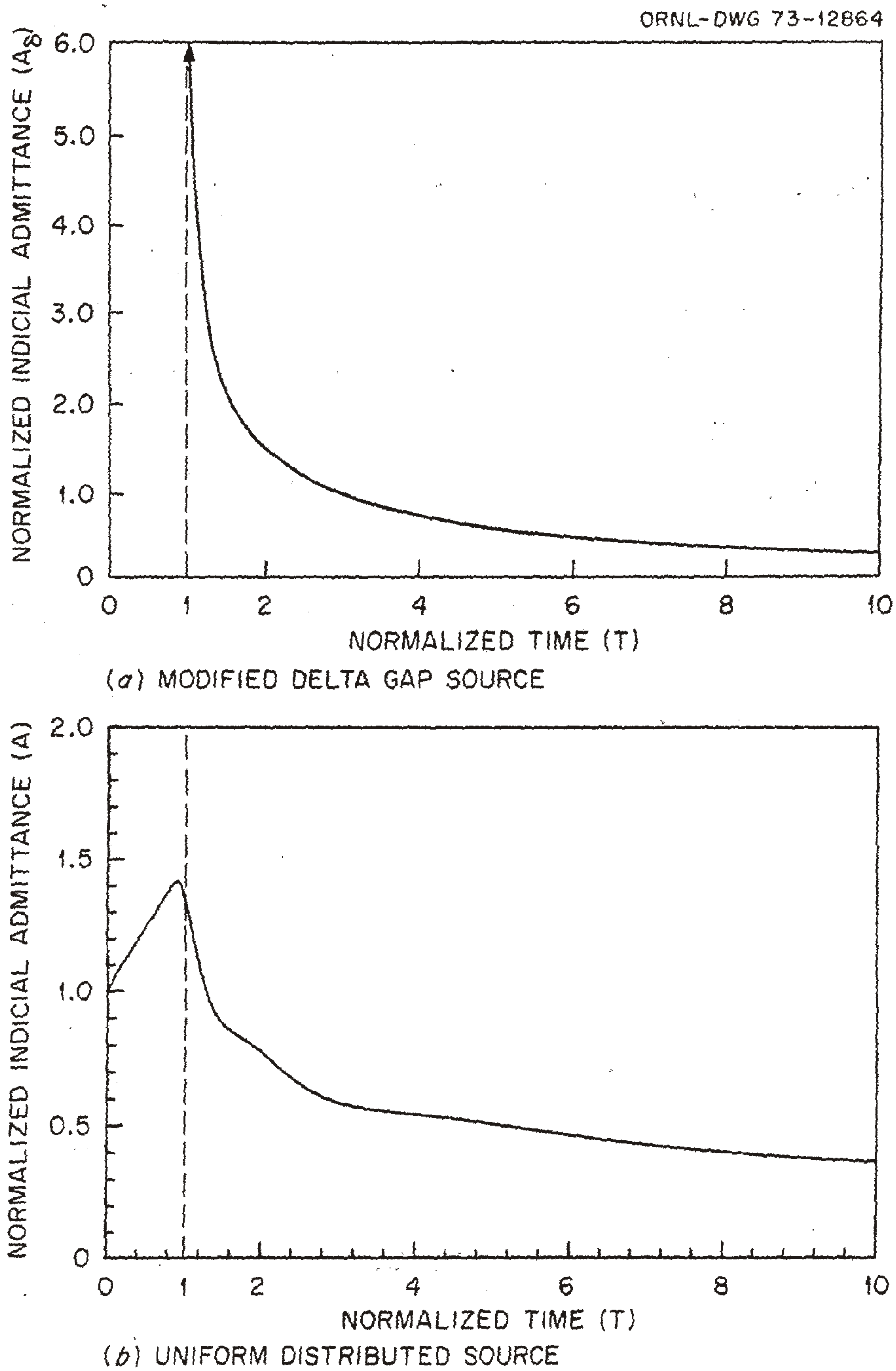


Fig. 14. The Normalized Indicial Admittances of an Infinite Cylindrical Antenna Excited by a Delta-Gap and a Finite Gap Source.

VI. SUMMARY AND CONCLUSIONS

In this note, an expression has been derived for the input admittance of an infinitely long, perfectly conducting, cylindrical antenna driven by a uniform distributed source. It was found that the small complex frequency asymptotic behavior of the admittance is inversely proportional to the logarithm of frequency. Also, it was found that the admittance asymptotically approaches a constant as the amplitude of the frequency becomes large.

The admittance of an infinite cylindrical antenna excited by a delta-gap source has been derived in a modified form which supposedly removes the singularity. The finite-gap and delta-gap admittances are in good agreement for small frequency amplitudes but are significantly different for large frequency amplitudes. This implies that the early time results in a transient problem employing the antenna admittance can be significantly dependent on the admittance function chosen for the calculation. The indicial admittances of the two admittance expressions have been considered to examine their transient characteristics. It was found that the time history of the delta-gap indicial admittance is initially singular whereas the finite-gap indicial admittance is initially equal to the inverse of ψ. The two indicial admittances are in reasonably good agreement for late times, $ct/a > 10$.

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