Interaction Notes

Note 175

April 1974

Exact Solution of the Problem of Quasi-Static Electric Field Penetration Into a Hemispherical Indentation in an Infinite Conducting Plane

John Lam
The Dikewood Corporation
Albuquerque, New Mexico

Abstract

The problem of the penetration of a quasi-static electric field into a hemispherical indentation in an infinite conducting plane is solved exactly. The results are directly applicable to the study of a large class of aircraft antennas; for example, the marker beacon antenna. The solution is obtained by an inversion transformation on the known solution for the problem of a conducting right-angled wedge excited by an electric dipole. A closed form for the electrostatic potential is derived. The value of the potential, the electric surface charge density, the induced dipole moment of the cavity, and the averaged electric field over the length of a thin-wire stub antenna erected from the cavity bottom are calculated. It is found that the penetration electric field strength at the cavity bottom is about 10% of that of the external field, while the averaged field on the symmetry axis varies from 10 to 28%.

I. Introduction

For reasons of aerodynamics, many aircraft receiving antennas are located in cavities indented on the aircraft skin, with the openings covered up by suitable dielectric materials. In designing such antennas it is important to understand the manner in which incoming electromagnetic signals enter such cavities, or how the incident field strengths attenuate with increasing penetration depth. If the operating wavelengths of the antennas are long compared to the cavity dimensions, a quasi-static analysis of the situation is adequate. We are justified to formulate the problem as potential problems for an infinite grounded conducting plane with a cavity, under the excitation of an external static normal electric field and a tangential magnetic field. The dielectric cover can be ignored. The electrostatic and magnetostatic problems are independent, and can be studied separately.

In this work we present the exact solution of the electrostatic problem for the case of a hemispherical indentation. By an adroit inversion transformation the problem is converted into that of a grounded right-angled conducting wedge under the excitation of an electric dipole. Making use of Mac Donald's solution of the wedge problem,[1] we obtain upon re-inversion the exact solution of the electrostatic hemispherical cavity problem in closed form. The associated magnetostatic problem cannot be solved by inversion, and will be studied by a different method in a subsequent report.

The two-dimensional problem of field penetration into a rectangular trough has recently been solved by Marin[2].

II. Formulation and Inversion Transformation

We consider a grounded perfect conductor occupying the half space $z > 0$, except for a hemispherical indentation of radius a on the surface (see Fig. 1). It is excited by a uniform static electric source field

$$\underline{E}_o = E_o \hat{z} \tag{1}$$

which can be derived from the source potential function

$$V_o = - E_o z . \tag{2}$$

The problem consists in finding the total potential function $V(x, y, z)$ such that it is 0 on the conductor surface and approaches V_o at great distances from the indentation.

The geometry of the problem is greatly simplified if we perform an inversion transformation with respect to a sphere of radius 2a centered at a point on the rim of the indentation. Under inversion a point at a distance r from the center is transformed into one at a new distance r' along the same direction such that

$$r r' = (2a)^2 \tag{3}$$

The polar angles remain unchanged. It is easy to see from Fig. 2 that the entire flange of the indentation is mapped by (3) into a half plane, while the hemispherical surface goes over to an orthogonal half plane. The original conductor is mapped into the interior of a right-angled wedge.

To describe the inversion mathematically it is convenient to choose the center of inversion as the origin of our coordinate system. The x-axis is aligned to pass through the center of the hemisphere (see Fig. 2). We

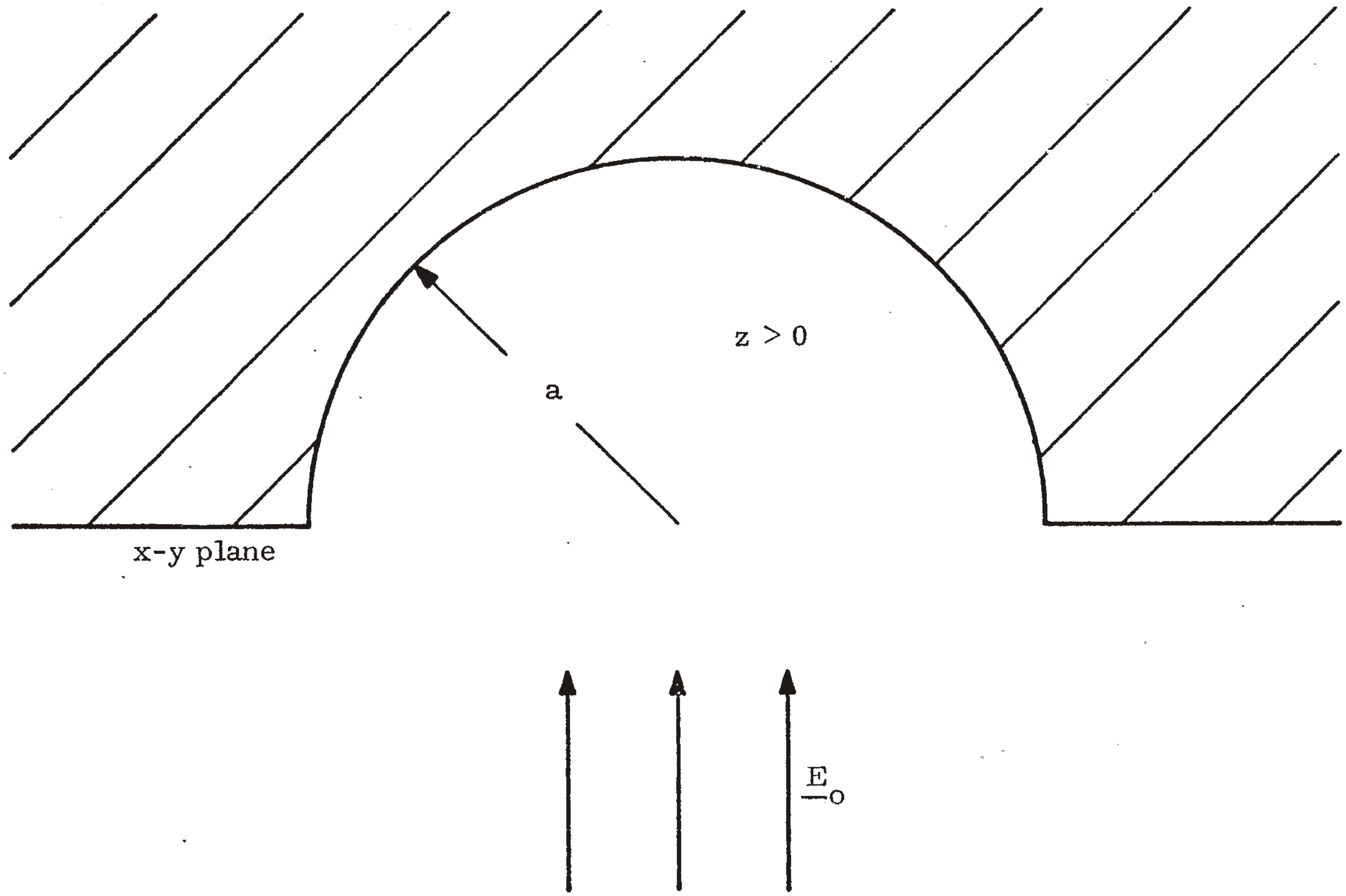

Fig. 1.--Geometry of the problem

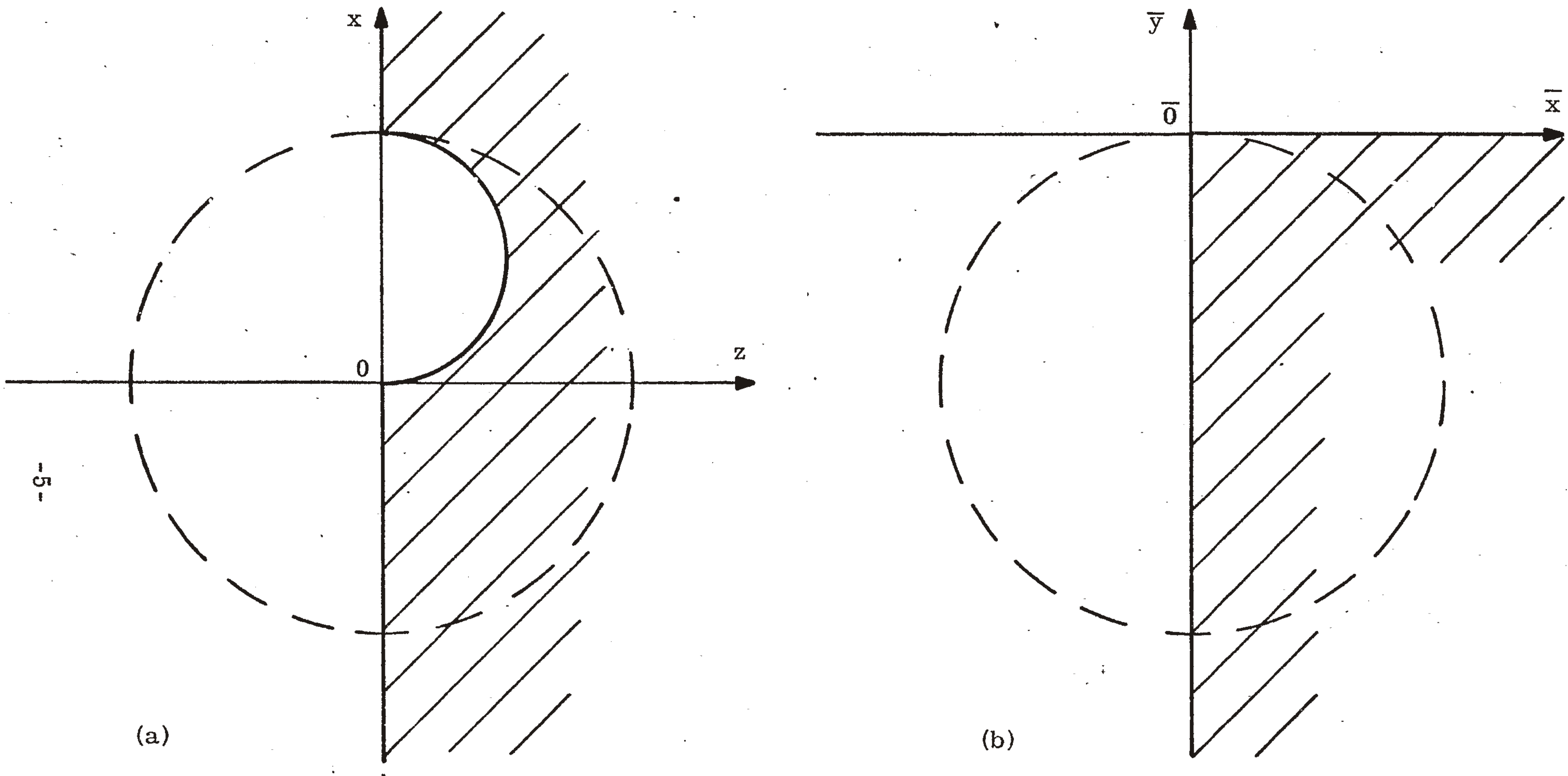

Fig. 2. -- Inversion transformation with respect to a sphere (broken line): (a) before inversion, (b) after inversion

shall denote quantities after the inversion by a prime. Thus, a point (x, y, z) is inverted into a point (x', y', z') such that according to (3)

$$x' = f(r)x, \quad y' = f(r)y, \quad z' = f(r)z, \tag{4}$$

where

$$f(r) = \frac{4a^2}{r^2} = \frac{r'^2}{4a^2}, \quad r = \sqrt{x^2+y^2+z^2}, \quad r' = \sqrt{x'^2+y'^2+z'^2}. \tag{5}$$

It can be shown that the law of inversion for the potential is

$$V(x, y, z) \rightarrow V'(x', y', z') = \frac{r}{2a} V(x, y, z). \tag{6}$$

V' is a solution of the Laplace equation in the inverted coordinates (x', y', z').

To facilitate the solution of the wedge boundary value problem in the inverted space we introduce a third coordinate system denoted by bars such that

$$\bar{x} = z', \quad \bar{y} = x'-2a, \quad \bar{z} = y' \tag{7}$$

This coordinate transformation consists of a displacement and a rotation, bringing the $\bar{z}$-axis along the edge of the wedge (see Fig. 2). We furthermore introduce cylindrical polar coordinates

$$\bar{x} = \bar{\rho} \cos \phi, \quad \bar{y} = \bar{\rho} \sin \phi \tag{8}$$

so that the two faces of the wedge are given by $\phi = 0$ and $\frac{3}{2}\pi$. Since the Laplace equation is invariant under (7) we have

$$V'(x', y', z') \rightarrow \overline{V}(\bar{x}, \bar{y}, \bar{z}) = V'(x', y', z'). \tag{9}$$

III. Inversion of the Source and the Boundary Condition

Under inversion the external source potential V_o in (2) goes over to a new source potential V_o' according to (6):

$$V_o'(x', y', z') = \frac{r}{2a} V_o(x, y, z) = - E_o \frac{rz}{2a}. \tag{10}$$

In terms of the primed coordinates in (4) this becomes

$$V_o'(x', y', z') = - 8a^3 E_o \frac{z'}{r'^3}. \tag{11}$$

Going further to the barred coordinates in (7) we have by (9)

$$\overline{V}_o(\bar{x}, \bar{y}, \bar{z}) = V_o'(x', y', z') = \frac{-8a^3 E_o \bar{x}}{\left[\bar{x}^2 + (\bar{y}+2a)^2 + \bar{z}^2\right]^{3/2}}. \tag{12}$$

This is the potential of an electric dipole of strength $32\pi\epsilon_o a^3 E_o$ situated right on the wedge surface $\phi = \frac{3}{2}\pi$ at a distance 2a from the edge, that is, at the center of inversion, and pointing in the negative $\bar{x}$-direction.

Because the source dipole is right on the boundary surface it is advantageous to lift it off the surface before attempting the solution. After the solution we can then allow it to fall back to the surface. This limiting process is correct only if we consider the strength of the lifted dipole as being reduced by one half. This is because when the dipole is on the wedge surface, it is really half buried beneath the surface. As soon as it is lifted, the solid angle subtended to it by free space jumps from 2π to 4π. Let the coordinates of the lifted source dipole be

$$\bar{x}' = 2a\cos\phi', \qquad \bar{y}' = 2a\sin\phi', \qquad \bar{z}' = 0. \tag{13}$$

(In the barred system we denote source coordinates by primes.) Then the effective source potential is

$$\bar{V}_o^{\,eff}(\bar{x},\bar{y},\bar{z}) = \frac{-4a^3E_o(\bar{x}-\bar{x}')}{\left[(\bar{x}-\bar{x}')^2+(\bar{y}-\bar{y}')^2+\bar{z}^2\right]^{3/2}} \tag{14}$$

as compared to (12). At the end of the calculations we must let ϕ' in (13) tend to $\frac{3}{2}\pi$.

Furthermore, since a dipole is made up of two equal and opposite charges, we need only solve the problem for one charge and then apply superposition. Specifically we need only consider the source charge potential

$$\bar{v}_o(\bar{x},\bar{y},\bar{z}) = \frac{-4a^3E_o}{\left[(\bar{x}-\bar{x}')^2+(\bar{y}-\bar{y}')^2+\bar{z}^2\right]^{1/2}}\,. \tag{15}$$

The solution of the problem for the source dipole potential can then be obtained by differentiation since

$$\bar{V}_o^{\,eff} = \frac{\partial}{\partial\bar{x}'}\,\bar{v}_o \tag{16}$$

We next consider the inversion of boundary conditions. From (6) it is clear that the inversion image of a zero potential surface is again a zero potential surface. Thus the original grounded plane with a hemispherical indentation is inverted into a grounded right-angled wedge. In other words, the homogeneous Dirichlet boundary value problem is inverted into another homogeneous Dirichlet problem. We mention in passing that the homogeneous Neumann problem is not so fortunate. As can be easily seen from (6), under inversion a homogeneous Neumann boundary condition in general degrades to a mixed condition involving an oblique derivative. Consequently the associated magnetostatic problem for our hemispherical geometry is not tractable with the method of inversion.

IV. Solution of the Wedge Problem

By (15) we see that our original electrostatic problem has been reduced to that of a source charge of strength $-16\pi\epsilon_o a^3 E_o$ at the co-ordinates (13) over a grounded right-angled conducting wedge defined by the angles $\phi = 0$ and $\frac{3}{2}\pi$. After the solution of the wedge problem we are to differentiate the solution with respect to the source coordinate $\overline{x}'$ according to (16), and then set the source angle ϕ' equal to $\frac{3}{2}\pi$. Finally we have to re-invert the solution according to (6).

The wedge problem is by no means trivial. Fortunately the solution for any wedge angle has already been published by Mac Donald[1]. He first obtained the solution as an infinite series and then summed it up in closed form by an integral. For exterior wedge angle α and a charge q at $(\overline{\rho}', \phi', \overline{z}')$ his expression for the total potential reads

$$\overline{v}(\overline{\rho}, \phi, \overline{z}) = \frac{1}{4\pi\epsilon_o} \frac{q}{\alpha\sqrt{2\overline{\rho}\overline{\rho}'}} \int_\eta^\infty \frac{d\zeta}{\sqrt{\cosh\zeta - \cosh\eta}} \times$$

$$\times \left[\frac{\sinh\frac{\pi}{\alpha}\zeta}{\cosh\frac{\pi}{\alpha}\zeta - \cos\frac{\pi}{\alpha}(\phi-\phi')} - \frac{\sinh\frac{\pi}{\alpha}\zeta}{\cosh\frac{\pi}{\alpha}\zeta - \cos\frac{\pi}{\alpha}(\phi+\phi')} \right],$$

$$0 < \phi,\ \phi' < \alpha \qquad (17)$$

where $$\cosh\eta = \frac{\overline{\rho}^2 + \overline{\rho}'^2 + (\overline{z} - \overline{z}')^2}{2\overline{\rho}\overline{\rho}'} \qquad (18)$$

The same result can also be derived directly by Sommerfeld's generalized method of images[3].

To obtain the solution of our source dipole problem we substitute in (17) and (18)

$$q = -16\pi\epsilon_o a^3 E_o, \quad \alpha = \frac{3}{2}\pi, \quad \bar{\rho}' = 2a, \quad \bar{z}' = 0, \tag{19}$$

differentiate $\bar{v}$ with respect to $\bar{x}' = 2a\cos\phi'$, and then put $\phi' = \frac{3}{2}\pi$. Thus we obtain the total potential

$$\bar{V}(\bar{x},\bar{y},\bar{z}) = \frac{8a^2E_o}{9\pi\sqrt{\bar{\rho}a}} \int_{\eta}^{\infty} \frac{d\zeta}{\sqrt{\cosh\zeta - \cosh\eta}} \; \frac{\sinh\frac{2}{3}\zeta\,\sin\frac{2}{3}\phi}{\left[\cosh\frac{2}{3}\zeta + \cos\frac{2}{3}\phi\right]^2} \tag{20}$$

where

$$\cosh\eta = \frac{\bar{\rho}^2 + 4a^2 + \bar{z}^2}{4\bar{\rho}a} \tag{21}$$

V. Potential

By (6) and (9) the total potential of our original hemispherical cavity problem is

$$V(x, y, z) = \frac{2a}{r} \overline{V}(\overline{x}, \overline{y}, \overline{z}) \tag{22}$$

We must re-express all coordinates $(\overline{x}, \overline{y}, \overline{z})$ in terms of (x, y, z) by using (4) and (7). Furthermore, for reason of symmetry, we make a shift of the coordinate origin to bring it from the rim to the center of the hemispherical cavity. This is achieved with the replacement

$$x \rightarrow x + a \tag{23}$$

The z-axis now coincides with the symmetry axis of the cavity.

The algebra of the coordinate back-transformation is lengthy but straightforward. The final result is

$$V(x, y, z) = aE_o \frac{8\sqrt{2}}{9\pi} \frac{a \sin \frac{2}{3}\phi}{\sqrt[4]{(r^2-a^2)^2 + 4a^2z^2}} \int_{\eta}^{\infty} \frac{d\zeta}{\sqrt{\cosh \zeta - \cosh \eta}} \times$$
$$\times \frac{\sinh \frac{2}{3}\zeta}{\left[\cosh \frac{2}{3}\zeta + \cos \frac{2}{3}\phi\right]^2}, \tag{24}$$

where

$$\cosh \eta = \frac{r^2 + a^2}{\sqrt{(r^2-a^2)^2 + 4a^2z^2}},$$

$$r = \sqrt{x^2 + y^2 + z^2},$$

$$\phi = \tan^{-1}\left(\frac{a^2 - r^2}{2az}\right), \qquad 0 < \phi < \frac{3}{2}\pi. \tag{25}$$

It is essential that we select the proper branches of the arc tangent. In particular we have

$$\phi = \begin{cases} 0 & z > 0,\ r = a \\ 1/2\ \pi & z = 0,\ r < a \\ \pi & z < 0,\ r = a \\ 3/2\ \pi & z = 0,\ r > a\ . \end{cases} \tag{26}$$

Thus the factor $\sin \frac{2}{3}\phi$ in (24) alone satisfies the boundary condition on the conductor.

Numerical values of the total potential in the cavity are tabulated in Table 1 and plotted in Fig. 3. It is found that for r greater than about 2.3a the total potential V differs from the incident potential V_o by less than 1%. The equipotential surfaces are sketched in Fig. 4.

Table 1. Total potential in the cavity in units of aE_o ($\rho = \sqrt{x^2 + y^2}$).

z/a \ ρ/a	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.9	0.011	0.010	0.008	0.006	0.002					
0.8	0.024	0.023	0.021	0.018	0.013	0.007				
0.7	0.040	0.039	0.036	0.032	0.027	0.019	0.011	0.001		
0.6	0.059	0.058	0.055	0.050	0.043	0.035	0.024	0.013		
0.5	0.082	0.081	0.078	0.072	0.064	0.054	0.041	0.027	0.011	
0.4	0.110	0.109	0.105	0.098	0.089	0.077	0.062	0.045	0.025	0.004
0.3	0.143	0.142	0.137	0.130	0.119	0.106	0.088	0.067	0.043	0.015
0.2	0.182	0.180	0.176	0.168	0.156	0.141	0.122	0.097	0.067	0.031
0.1	0.227	0.225	0.220	0.212	0.200	0.184	0.163	0.137	0.102	0.056
0.0	0.278	0.276	0.271	0.263	0.251	0.236	0.215	0.188	0.153	0.104

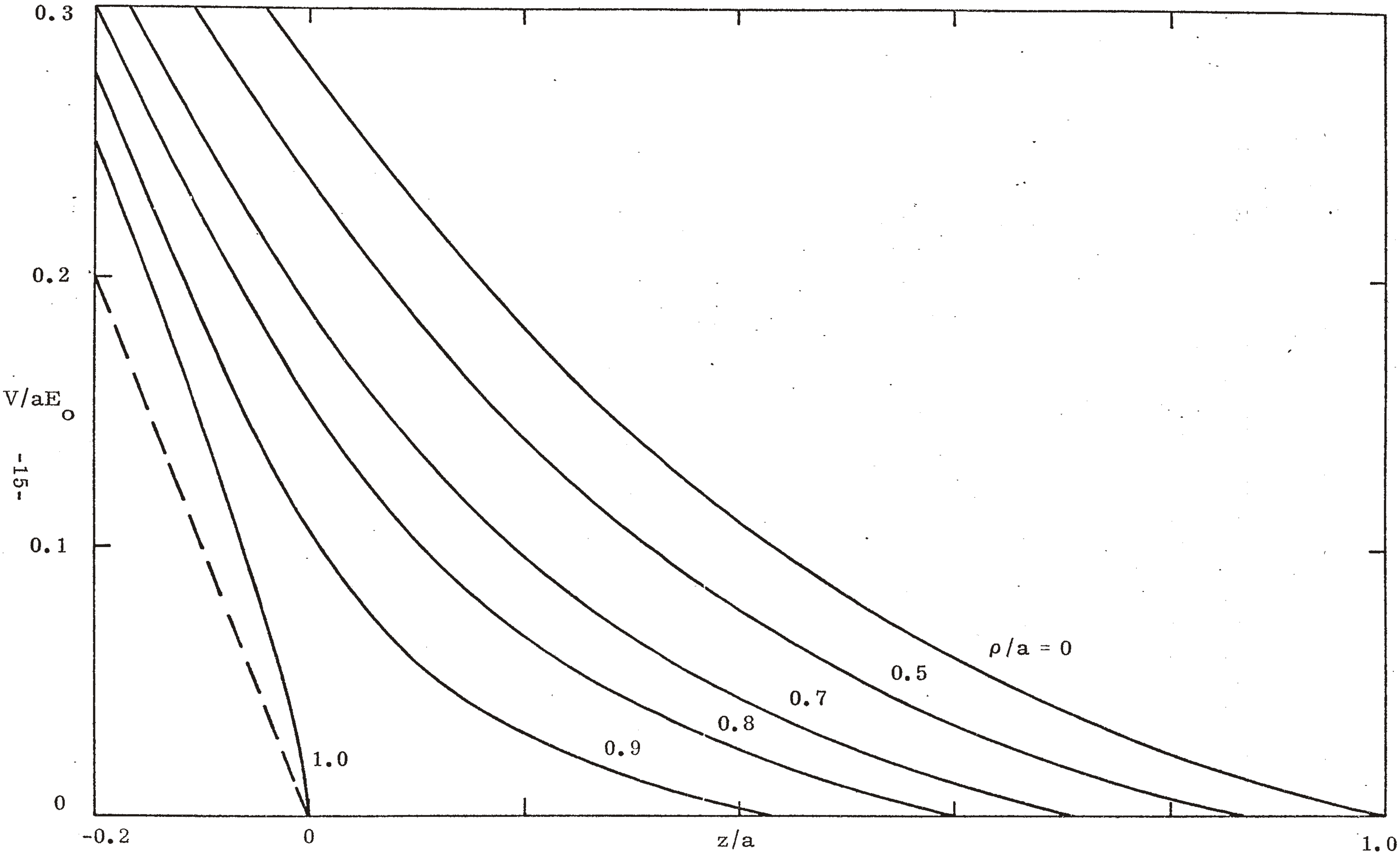

Fig. 3.-- Total potential inside the cavity. The curves are labelled by ρ/a ($\rho = \sqrt{x^2+y^2}$). The broken line on the far left is the incident potential $V_o = -E_o z$.

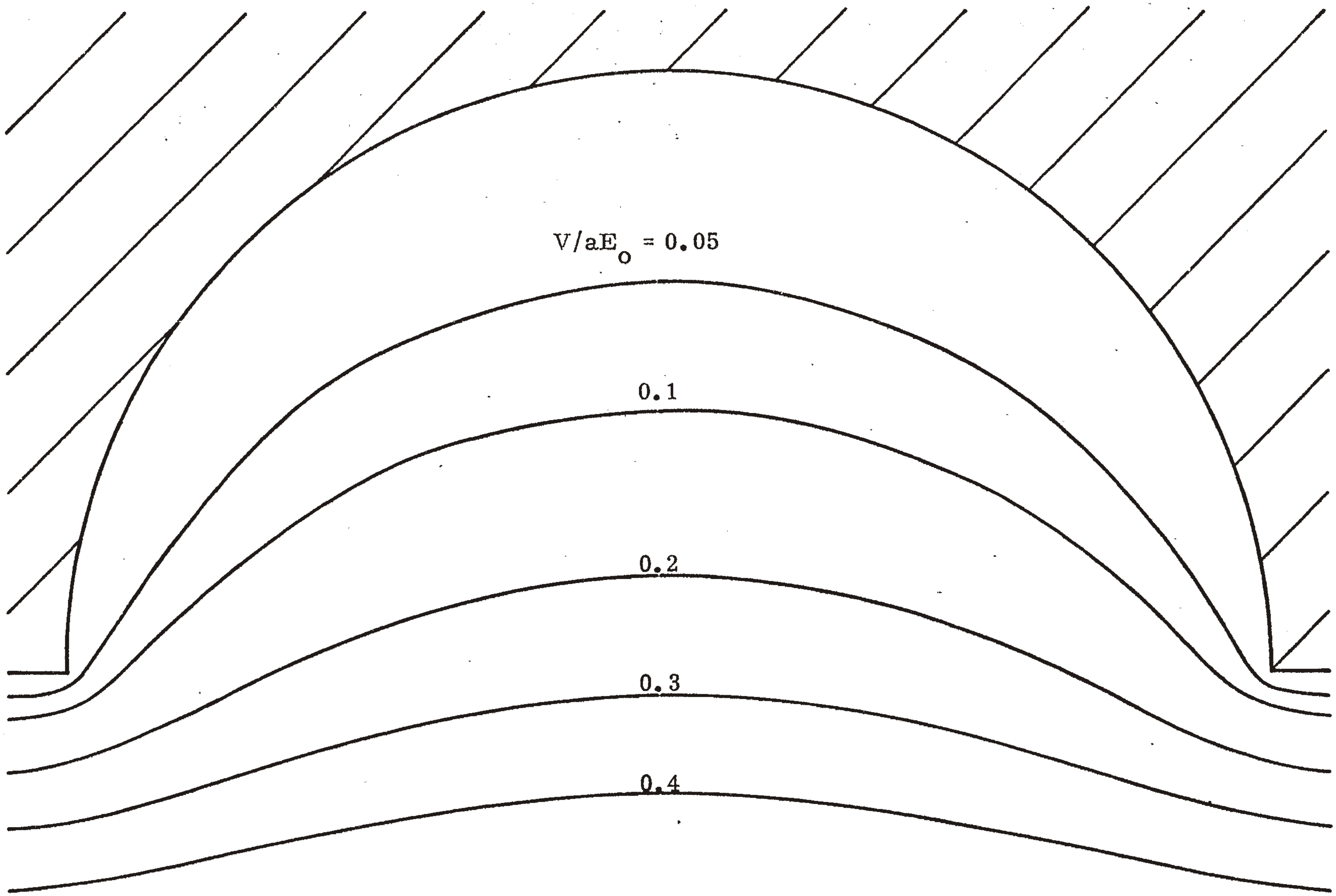

Fig. 4.-- Equipotential Surfaces

VI. Surface Charge Density

The surface charge density σ on the conductor is proportional to the normal electric field. Thus on the hemispherical surface we have

$$\sigma(\theta) = \epsilon_o \frac{\partial V}{\partial r}\bigg|_{r=a}, \qquad 0 < \theta < \frac{\pi}{2}, \tag{27}$$

where $\theta = \cos^{-1}\frac{z}{r}$. Since the factor $\sin\frac{2}{3}\phi$ in (24) vanishes on the conductor, it is clear that we need to differentiate it only. Working out the derivative we obtain

$$\sigma(\theta) = \sigma_o \frac{16}{27\pi} \frac{1}{\cos^{3/2}\theta} \int_\eta^\infty \frac{d\zeta}{\sqrt{\cosh\zeta - \cosh\eta}} \frac{\sinh\frac{2}{3}\zeta}{\left[\cosh\frac{2}{3}\zeta + 1\right]^2}, \tag{28}$$

where

$$\sigma_o = -\epsilon_o E_o, \qquad \cosh\eta = \frac{1}{\cos\theta}. \tag{29}$$

σ_o is the uniform surface charge density on the conducting plane if the cavity were absent.

Similarly on the flange we have

$$\sigma(\rho) = \epsilon_o \frac{\partial V}{\partial z}\bigg|_{z=0}, \qquad \rho = \sqrt{x^2 + y^2} > a, \tag{30}$$

or explicitly

$$\sigma(\rho) = \sigma_o \frac{32\sqrt{2}}{27\pi} \frac{a^3}{(\rho^2 - a^2)^{3/2}} \int_\eta^\infty \frac{d\zeta}{\sqrt{\cosh\zeta - \cosh\eta}} \frac{\sinh\frac{2}{3}\zeta}{\left[\cosh\frac{2}{3}\zeta - 1\right]^2} \tag{31}$$

where

$$\cosh \eta = \frac{\rho^2 + a^2}{\rho^2 - a^2} . \tag{32}$$

The integrals (28) and (31) are evaluated numerically and σ plotted in Fig. 5. As is expected σ diverges at the edge of the cavity. To find out the nature of the divergence we note that the lower limits of the integrals go to infinity at the edge and that

$$\sigma \sim \cosh^{3/2} \eta \int_{\eta}^{\infty} \frac{d\zeta}{\sqrt{\cosh \zeta - \cosh \eta}} \; \frac{\sinh \frac{2}{3} \zeta}{\left[\cosh \frac{2}{3} \zeta \pm 1\right]^2} . \tag{33}$$

Putting $\zeta = \xi + \eta$ and using

$$\cosh \eta \sim \frac{1}{2} e^{\eta}, \quad \cosh (\xi+\eta) \sim \sinh (\xi+\eta) \sim \frac{1}{2} e^{\xi+\eta} \tag{34}$$

etc. as $\eta \rightarrow \infty$ we find

$$\sigma \sim e^{\frac{1}{3}\eta} \int_{0}^{\infty} \frac{e^{-\frac{2}{3}\xi} d\xi}{\sqrt{e^{\xi} - 1}} . \tag{35}$$

Therefore near the edge

$$\sigma (\theta) \sim \frac{1}{\cos^{1/3}\theta}, \qquad \sigma (\rho) \sim \frac{1}{(\rho^2 - a^2)^{1/3}} , \tag{36}$$

in agreement with the general result of Meixner[4].

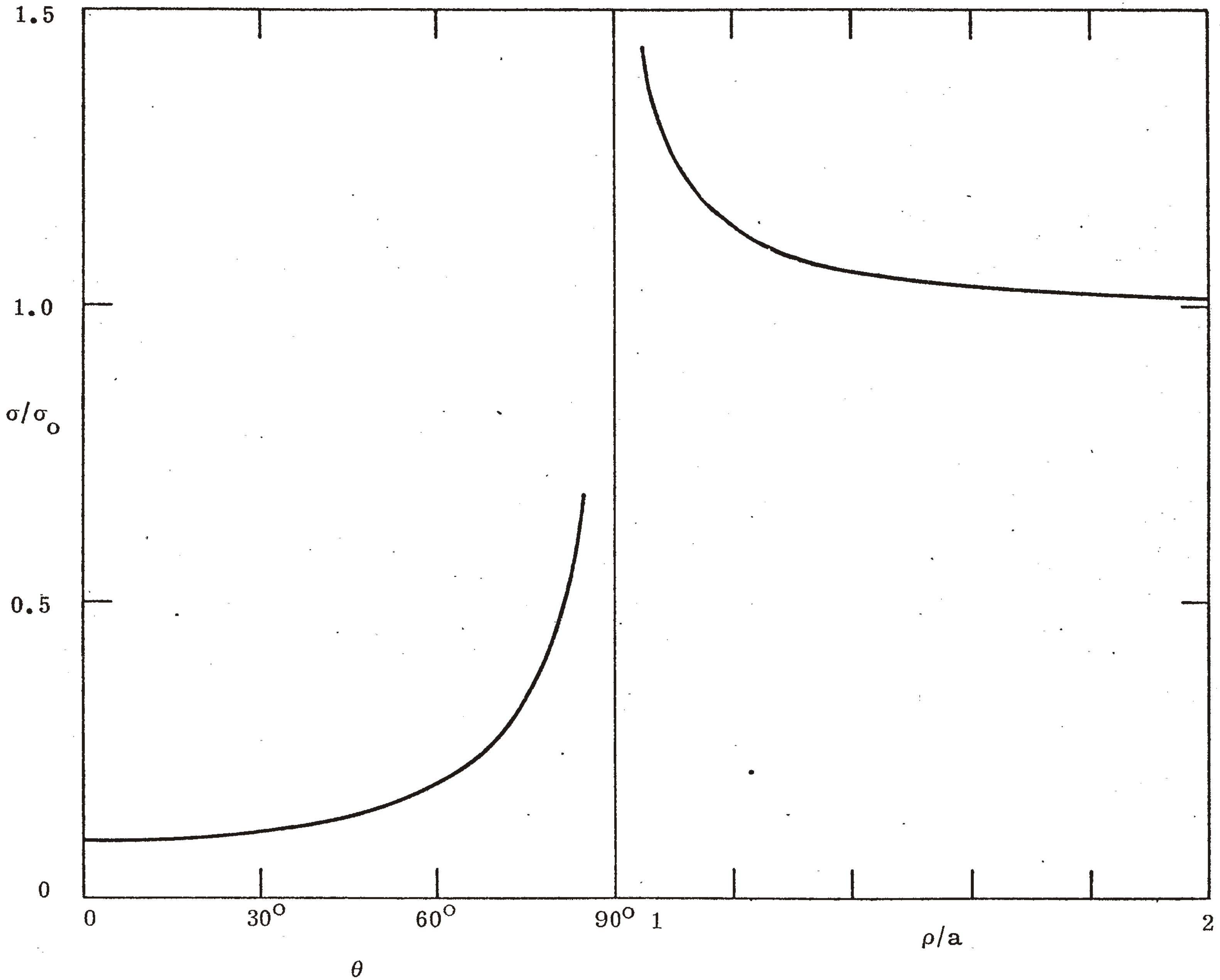

Fig. 4. -- Surface charge density over $\sigma_o = -\epsilon_o E_o$.

VII. Asymptotic Expansion and Induced Dipole Moment

We want to show that at large distances from the cavity the total potential (24) reduces to the incident source potential plus an induced potential of a dipole form. Since our problem has axial symmetry, it is sufficient to carry out the asymptotic expansion along the z-axis.

For $r = -z$ ($z < 0$), η as defined in (25) is 0. We can write (24) as

$$V(0,0,z) = aE_o \frac{4\sqrt{2}}{3\pi} \frac{a}{\sqrt{z^2+a^2}} \frac{\partial}{\partial \phi} I(\phi) , \tag{37}$$

where

$$I(\phi) = \int_0^\infty \frac{d\zeta}{\sqrt{\cosh \zeta - 1}} \frac{\sinh \frac{2}{3}\zeta}{\cosh \frac{2}{3}\zeta + \cos \frac{2}{3}\phi} . \tag{38}$$

Setting $t = \cosh \frac{1}{3}\zeta$ we obtain

$$I(\phi) = 3\int_1^\infty \frac{dt}{\sqrt{4t^3 - 3t - 1}} \frac{t}{t^2 - \sin^2 \frac{1}{3}\phi} . \tag{39}$$

The integration is elementary since

$$\sqrt{4t^3 - 3t - 1} = (2t+1)\sqrt{t-1} , \tag{40}$$

and we find

$$I(\phi) = \frac{\pi}{2}\sqrt{\frac{3}{2}}\left(\frac{1}{\sqrt{\frac{3}{2}}\sqrt{1 - \sin\frac{1}{3}\phi} + 1 - \sin\frac{1}{3}\phi} + \right.$$

$$\left. + \frac{1}{\sqrt{\frac{3}{2}}\sqrt{1 + \sin\frac{1}{3}\phi} + 1 + \sin\frac{1}{3}\phi} \right) . \tag{41}$$

Let us take the limit $z \to -\infty$. Then the angle ϕ as defined in (25) tends to $\frac{3}{2}\pi$, and we put

$$\phi = \frac{3}{2}\pi - 3\epsilon \ , \tag{42}$$

where

$$3\epsilon = \sin^{-1} \frac{2a|z|}{z^2+a^2} \gtrsim 0 \ . \tag{43}$$

Thus

$$\sin\frac{1}{3}\phi = \sin\left(\frac{1}{2}\pi - \epsilon\right) = 1 - \frac{1}{2}\epsilon^2 + \frac{1}{24}\epsilon^4 - 0(\epsilon^6) ,$$

$$\frac{1}{\sqrt{\frac{3}{2}}\sqrt{1 - \sin\frac{1}{3}\phi} + 1 - \sin\frac{1}{3}\phi} = \frac{2}{\sqrt{3}}\frac{1}{\epsilon} - \frac{2}{3} + \frac{\sqrt{3}}{4}\epsilon - \frac{2}{9}\epsilon^2 + 0(\epsilon^3) ,$$

$$\frac{1}{\sqrt{\frac{3}{2}}\sqrt{1 + \sin\frac{1}{3}\phi} + 1 + \sin\frac{1}{3}\phi} = \frac{1}{2+\sqrt{3}} + \frac{1}{2}\frac{1+\frac{1}{4}\sqrt{3}}{(2+\sqrt{3})^2}\epsilon^2 + 0(\epsilon^4) . \tag{44}$$

Substituting (44) in (41) and noting from (42) that $d\phi = -3d\epsilon$, we obtain

$$V(0,0,z) = aE_o \frac{2}{9}\frac{a}{\sqrt{z^2+a^2}}\left[\frac{2}{\epsilon^2} - \frac{3}{4} + \sqrt{3}\left(\frac{4}{9} - \frac{1+\frac{1}{4}\sqrt{3}}{(2+\sqrt{3})^2}\right)\epsilon + 0(\epsilon^2)\right] . \tag{45}$$

From (43) we find that

$$\epsilon = \frac{2}{3}\frac{a}{|z|} - \frac{2}{9}\left(\frac{a}{|z|}\right)^3 + \ldots \ . \tag{46}$$

Substituting in (45) we finally obtain for $z \to -\infty$

$$V(0,0,z) = -E_o z + \frac{4\sqrt{3}}{27}\left(\frac{4}{9} - \frac{1+\frac{1}{4}\sqrt{3}}{(2+\sqrt{3})^2}\right)\frac{E_o a^3}{z^2} + 0\left(\frac{1}{z^3}\right) . \tag{47}$$

The first term is the incident potential. The second term is the leading contribution of the induced potential. As is generally expected, it is of a dipole form. There is a gap of two terms between them. The existence of this gap attests to the correctness of our solution and the accuracy of our asymptotic expansion.

The dipole term in (47) defines an induced dipole moment $\underline{p}$ of the hemispherical cavity which characterizes the far zone behavior of the induced potential. We have

$$\underline{p} = \alpha_e \underline{E}_o , \quad \alpha_e = P\epsilon_o a^3 \tag{48}$$

where

$$P = -4\pi \frac{4\sqrt{3}}{27}\left(\frac{4}{9} - \frac{1+\frac{1}{4}\sqrt{3}}{(2+\sqrt{3})^2}\right) \simeq -1.10 . \tag{49}$$

This is to be compared with the finding in the more readily soluble problem of a hemispherical boss, where $P = 4\pi$.

VIII. Averaged Electric Field along the Axis

We suppose that a thin-wire stub antenna of length L is erected from the bottom of the cavity along the symmetry z-axis. Then the electrical signal it receives is proportional to the averaged electric field along its length calculated in its absence. It is clear that the electric field along the z-axis has only a z-component. Thus

$$\overline{E}_z(L) = \frac{1}{L} \int_{a-L}^{a} E_z\,(0,\ 0,\ z)\,dz\ ,$$

$$= \frac{-1}{L} \int_{a-L}^{a} \frac{\partial}{\partial z} V\,(0,\ 0,\ z)\,dz\ ,$$

$$= \frac{1}{L} V\,(0,\ 0,\ a-L)\ . \qquad (50)$$

This expression is easily evaluated from Table 1, and the results are plotted in Fig. 6. Its value varies from about 0.1 E_o at the bottom to about 0.28 E_o at the cavity opening.

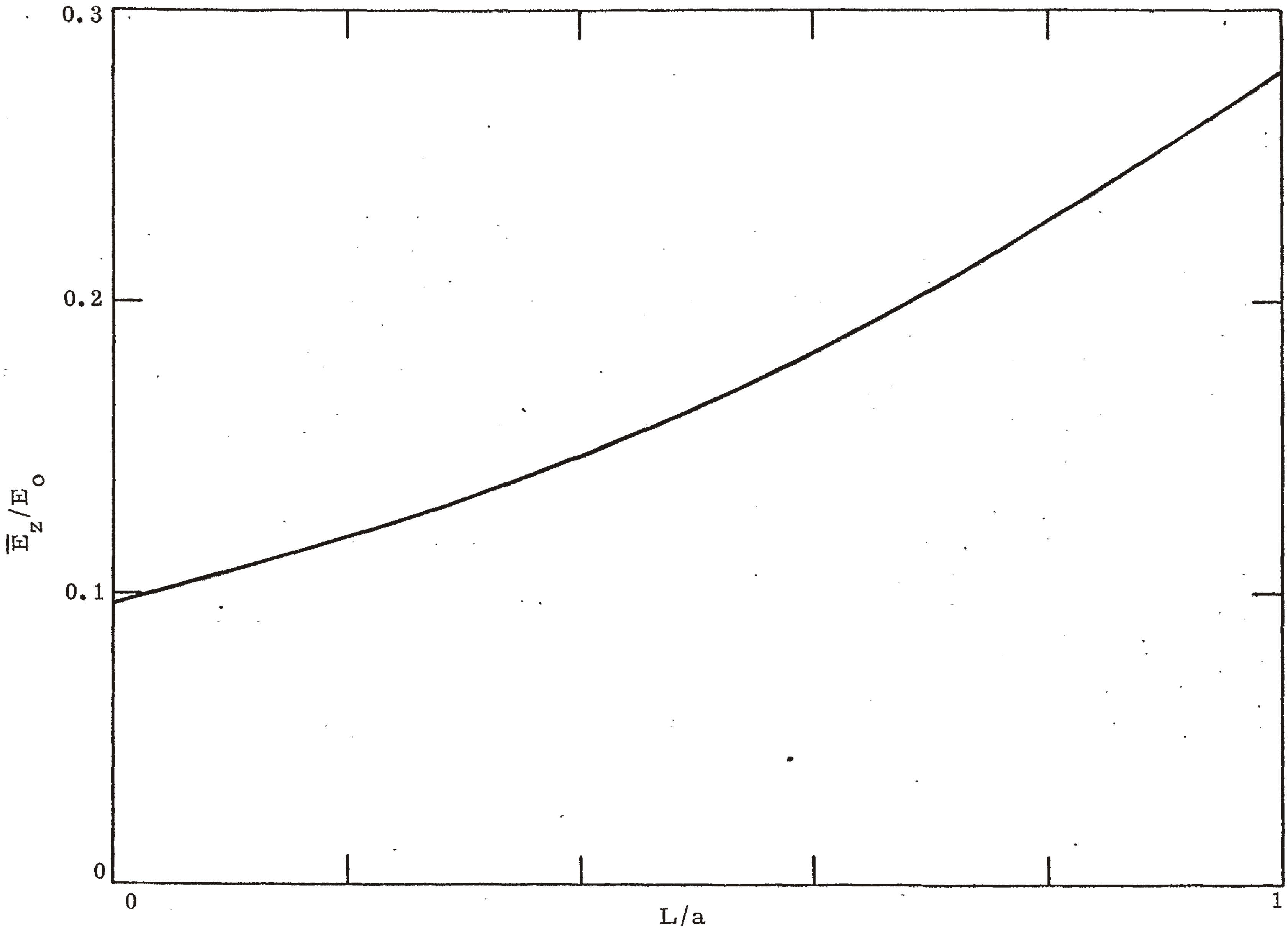

Fig. 6.--Averaged electric field along the symmetry axis in the cavity.

Acknowledgment

It is a pleasure to thank Dr. K. S. H. Lee for many discussions during the course of this work.

References

[1] H. M. MacDonald, "The electrical distribution on a conductor bounded by two spherical surfaces cutting at any angle," Proc. Lond. Math. Soc. Ser. 1, 26, 156-172 (1895).

[2] L. Marin, "Quasi-static field penetration into a two-dimensional rectangular well in a ground plane," Interaction Note 171, March 1974.

[3] A. Sommerfeld, "Über verzweigte Potentiale im Raum," Proc. Lond. Math. Soc. Ser. 1, 28, 395-429 (1897).

[4] J. Meixner, "Die Kantenbedingung in der Theorie der Beugung elektromagnetischer Wellen an vollkommen leitenden ebenen Schirmen," Ann. d. Phys. Ser. 6, 6, 2-9 (1949).