SC-RR-72 0560

RESPONSE OF A TERMINATED TRANSMISSION LINE EXCITED BY A PLANE WAVE FIELD FOR ARBITRARY ANGLES OF INCIDENCE

by

Charles W. Harrison, Jr.
Electromagnetic Effects Division - 1426
Sandia Laboratories
Albuquerque, New Mexico
87115

Clayborne D. Taylor
Mississippi State University
State College, Mississippi 39762

Printed October 1972

This work was supported by the United States Atomic Energy Commission.

SUMMARY

An analysis to determine the response of an impedance-loaded transmission line configuration excited by a plane-polarized electric field of arbitrary spatial orientation has been needed for a long time. The purpose of the present paper is to meet this need by suggesting a method of solution based on the analysis of the simplest of such circuits: an impedance - terminated two-conductor transmission line.

RESPONSE OF A TERMINATED TRANSMISSION LINE
EXCITED BY A PLANE WAVE FIELD FOR
ARBITRARY ANGLES OF INCIDENCE

Introduction

In evaluating the response of transmission line receiving circuits to electromagnetic radiation (EMR) and the electromagnetic pulse (EMP), it is customary to assume that the incident field is polarized parallel to one set of conductors composing the line. The field is then cross polarized with respect to the other set of conductors (normally the terminations). The purpose of the present paper is to develop the theoretical background needed to solve the general case; i.e., when the incident plane wave field arrives at arbitrary angles with respect to the transmission line receiving configurations.

The Current Distribution

Figure 1 represents the transmission line circuit considered in this paper. Parallel wires of radius a, of length s, and spaced at a distance b between centers are terminated in impedances Z_o at $z = 0$ and Z_s at $z = s$. An impedanceless generator of voltage V_o^e is connected in series with the impedance Z_o. The currents in the terminations Z_o and Z_s are, respectively, I_o and I_s. The equal and opposite currents in the line are designated $I(z)$.

The input impedance of the transmission line at $z = 0$ is

$$Z_{in} = Z_c\left(\frac{Z_s + j\,Z_c \tan\beta s}{Z_c + j\,Z_s \tan\beta s}\right) = Z_c\left(\frac{Z_s \cos\beta s + j\,Z_c \sin\beta s}{Z_c \cos\beta s + j\,Z_s \sin\beta s}\right) . \tag{1}$$

Here

$$Z_c = \frac{\zeta_o}{\pi}\ln\left(\frac{b}{a}\right) \tag{2}$$

is the characteristic impedance of the line; $\zeta_o = \omega\mu_o/\beta \simeq 120\pi$ ohms is the characteristic impedance of free space. The radian frequency is ω; $\omega = 2\pi f$. The permeability of space is $\mu_o \simeq 4\pi \times 10^{-7}$ H/m and the radian wave number is $\beta = 2\pi/\lambda$. The free space wavelength is λ. Evidently, the line is considered to be dissipationless, but this is an unnecessary assumption.

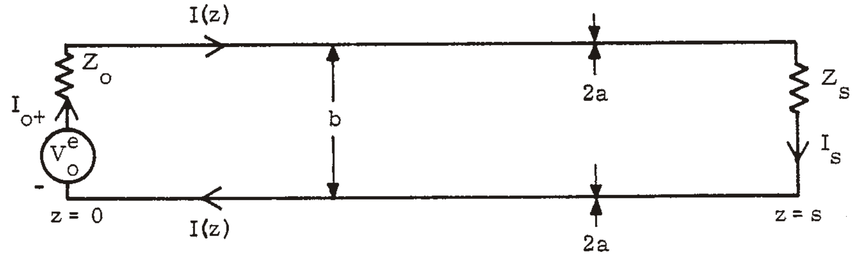

Figure 1. Transmission Line Terminated in Load Impedances Z_o and Z_s and Driven by Voltage V_o^e.

The current I_o is given by

$$I_o = \frac{V_o^e}{Z_o + Z_{in}} = \frac{V_o^e}{D}\left(Z_c \cos \beta s + j\, Z_s \sin \beta s\right) , \tag{3}$$

where

$$D = Z_c\left(Z_o + Z_s\right) \cos \beta s + j\left(Z_c^2 + Z_o Z_s\right) \sin \beta s \,. \tag{4}$$

The voltage V_o across the sending end of the transmission line is

$$V_o = \frac{V_o^e\, Z_{in}}{Z_o + Z_{in}} = \frac{V_o^e\, Z_c}{D}\left(Z_s \cos \beta s + j\, Z_c \sin \beta s\right) . \tag{5}$$

The current I(z) is given by the formula[1]

$$I(z) = I_o \cos \beta z - j\, \frac{V_o}{Z_c} \sin \beta z \,. \tag{6}$$

Substituting (3) and (5) in (6) yields

$$I(z) = \frac{V_o^e}{D}\left[Z_c \cos \beta(s-z) + j\, Z_s \sin \beta(s-z)\right] . \tag{7}$$

The current I_s is obtained by setting z = s in (7). Thus,

$$I_s = \frac{V_o^e\, Z_c}{D} \tag{8}$$

Transmission line theory is valid only when $\beta b << 1$. It follows that the currents I_o and I_s are essentially uniform in the terminations.

The Vector Potential in the Far Zone of the Radiating Circuit

Figure 2 illustrates the orientation of the driven transmission line with respect to the coordinate systems employed in calculating the radiation vectors of the circuit. An inspection of the figure reveals that the various distances from the current elements to a point $P(R, \theta, \phi)$ in the far zone of the transmission line are

$$
\begin{aligned}
R_s &= R_o - s\cos\theta \\
R_z &= R_o - z'\cos\theta \\
R_{z1} &= R_z + \frac{b}{2}\sin\theta\cos\phi \\
R_{z2} &= R_z - \frac{b}{2}\sin\theta\cos\phi .
\end{aligned}
\qquad (9)
$$

The two components of the vector potential A_x^r and A_z^r are to be determined at point P. By noting carefully the assumed directions of current flow in the transmission line, it is readily verified that

$$
\begin{aligned}
A_x^r &= \frac{\mu_o b}{4\pi} \frac{V_o^e Z_c}{D} \frac{e^{-j\beta R_o}}{R_o} e^{j\beta s\cos\theta} \\
&- \frac{\mu_o b}{4\pi} \frac{V_o^e}{D} \frac{e^{-j\beta R_o}}{R_o} (Z_c \cos\beta s + j Z_s \sin\beta s) \\
&= \frac{V_o^e \mu_o b}{4\pi D} \frac{e^{-j\beta R_o}}{R_o} F_1(\theta, s) ,
\end{aligned}
\qquad (10)
$$

where

$$
F_1(\theta, s) = Z_c e^{j\beta s\cos\theta} - Z_c\cos\beta s - j Z_s \sin\beta s \qquad (11)
$$

and

$$
\begin{aligned}
A_z^r &= \frac{\mu_o}{4\pi}\left[\int_o^s I(z')\frac{e^{-j\beta R_{z1}}}{R_{z1}}dz' - \int_o^s I(z')\frac{e^{-j\beta R_{z2}}}{R_{z2}}dz'\right] \\
&\simeq -j\frac{\mu_o \beta b\sin\theta\cos\phi}{4\pi}\frac{e^{-j\beta R_o}}{R_o}\int_o^s I(z')e^{j\beta z'\cos\theta}dz' .
\end{aligned}
\qquad (12)
$$

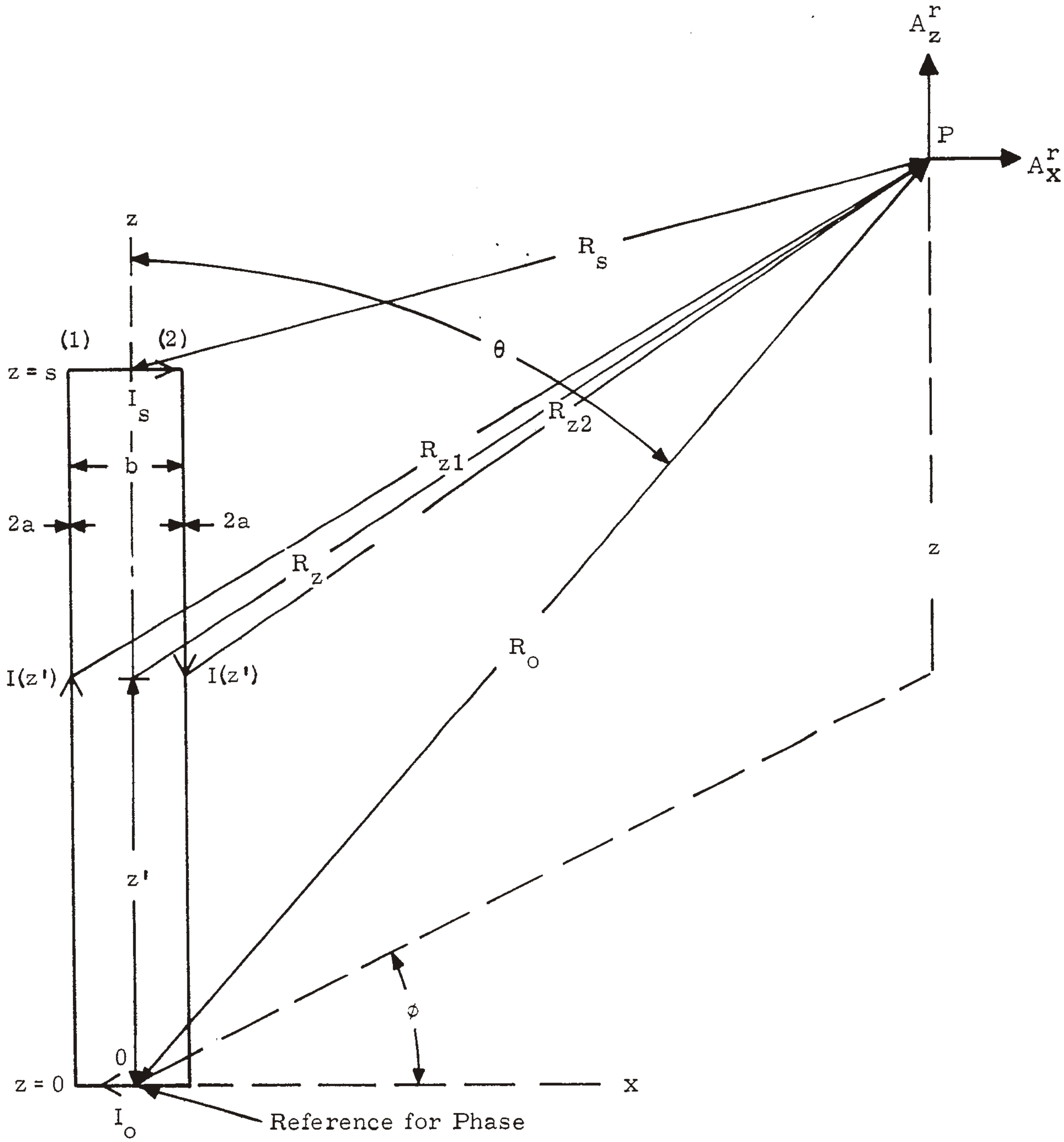

Figure 2. Exposition of the Coordinate System Utilized in Calculating the Vector Potential in the Far Zone of the Driven Transmission Line.

The substitution (7), with z primed, into (13) yields

$$A_z^r = \frac{V_o^e \mu_o b \cos\phi}{4\pi D \sin\theta} \frac{e^{-j\beta R_o}}{R_o} F_2(\theta, s) , \tag{13}$$

where

$$\begin{aligned} F_2(\theta, s) = Z_c &\left(\cos\theta\, e^{j\beta s \cos\theta} - \cos\theta \cos\beta s - j \sin\beta s\right) \\ + Z_s &\left(e^{j\beta s\cos\theta} - j\cos\theta \sin\beta s - \cos\beta s\right) . \end{aligned} \tag{14}$$

The integration of (12), leading to (13), involves the use of only the standard integrals

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$$

and

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) .$$

The Spherical Components of the
Vector Potential and the Radiation Field

The spherical components of the vector potentials A_θ^r and A_ϕ^r are related to the Cartesian components A_x^r and A_z^r by the formulas[2]

$$A_\theta^r = A_x^r \cos\theta \cdot \cos\phi - A_z^r \sin\theta \tag{15}$$

$$A_\phi^r = - A_x^r \sin\phi . \tag{16}$$

Also, the spherical components of the electric field in the far zone of the transmission line are given by[3]

$$E_\theta^r = - j\,\omega A_\theta^r \tag{17}$$

$$E_\phi^r = - j\,\omega A_\phi^r . \tag{18}$$

From the foregoing it is now easily demonstrated that

$$E_\theta^r = -j \frac{\beta b \zeta_o V_o^e e^{-j\beta R_o}}{4\pi D R_o} \left[F_1 (\theta, s) \cos\theta - F_2 (\theta, s)\right] \cos\phi \tag{19}$$

$$E_\phi^r = j \frac{\beta b \zeta_o V_o^e e^{-j\beta R_o}}{4\pi D R_o} F_1 (\theta, s) \sin\phi \; . \tag{20}$$

Current in an Unloaded Dipole Receiving Antenna:
Application of the Reciprocal Theorem

Let an unloaded dipole receiving antenna be placed in the far field of the driven transmission line with its axis parallel to E_θ^r. The current at the center of the dipole is

$$I_d(0) = \frac{-2h_{e\theta}\left(\frac{\pi}{2}\right) E_\theta^r}{Z_A} \; , \tag{21}$$

where Z_A is the impedance of the antenna and $2h_{e\theta}\left(\frac{\pi}{2}\right)$ is the complex effective length of the dipole when it is parallel to the incident field component E_θ^r given by (19). According to the Rayleigh-Carson reciprocol theorem, the impedanceless generator V_o^e can be moved to the center of the dipole from its position in series with the impedance Z_o at $z = 0$ in the transmission line (Figure 1), and the current $I_\theta(0)$ will then equal the current $I_d(0)$ given by (21). Thus, with (19),

$$I_\theta(0) = j \frac{2h_{e\theta}\left(\frac{\pi}{2}\right) V_o^e \beta b \zeta_o e^{-j\beta R_o}}{4\pi Z_A D R_o} \left[F_1 (\theta, s) \cos\theta - F_2 (\theta, s)\right] \cos\phi \tag{22}$$

The electric field actually maintained by the driven dipole in the vicinity of the transmission line is

$$E_\theta^{inc} = \frac{j \zeta_o V_o^e e^{-j\beta R_o}}{2\pi Z_A R_o} \beta h_{e\theta}\left(\frac{\pi}{2}\right) \; . \tag{23}$$

Solving (23) for V_o^e and substituting its value in (22) yield

$$I_\theta(0) = \frac{E_\theta^{inc} b}{D} \left[F_1(\theta, s) \cos\theta - F_2(\theta, s)\right] \cos\phi \tag{24}$$

The current $I_\theta(0)$ flows in Z_o when the receiving dipole is parallel to E_θ^r given by (19).

In a similar manner,

$$I_\phi(0) = -\frac{E_\phi^{inc} b}{D} F_1(\theta, s) \sin\phi . \tag{25}$$

Here $I_\phi(0)$ is the current flowing in Z_o when the receiving dipole is parallel to E_ϕ^r given by (20).

In the general case it is necessary to resolve E^{inc} into components E_θ^{inc} and E_ϕ^{inc}. Equations (24) and (25) are then used to determine the currents in Z_o caused by the field components. The total current in the impedance is then the complex sum of the individual currents.

The currents $I_\theta(s)$ and $I_\phi(s)$ in the load impedance Z_s is obtained by simple interchange of the ends of the transmission line.

Comments on the Validity, Reliability, and Authenticity of the Theory

Let $\theta = \pi/2$ and $\phi = \pi$ in (24). Then $E^{inc}_{\theta=\frac{\pi}{2}} = -E_z^{inc}$. Equation (24) becomes

$$I_\theta(0) = j \frac{E_z^{inc} b}{D} \left[Z_c \sin\beta s + j Z_s (1 - \cos\beta s)\right] . \tag{26}$$

Except for notational differences, (26) is identical to (2) in Reference 4. Note that since $\beta b << 1$, $\sin\left(\frac{\beta b}{2} \sin\phi\right) \simeq \frac{\beta b}{2} \sin\phi$.

Let $\theta = 0$ and $\phi = \frac{\pi}{2}$ in (25). One obtains

$$I_\phi(0) = \frac{E^{inc} b}{D} \left(Z_c \cos\beta s + j Z_s \sin\beta s - Z_c e^{j\beta s}\right) , \tag{27}$$

where use has been made of (11).

This result is easily verified by a direct solution of the problem. Refer to Figure 3.

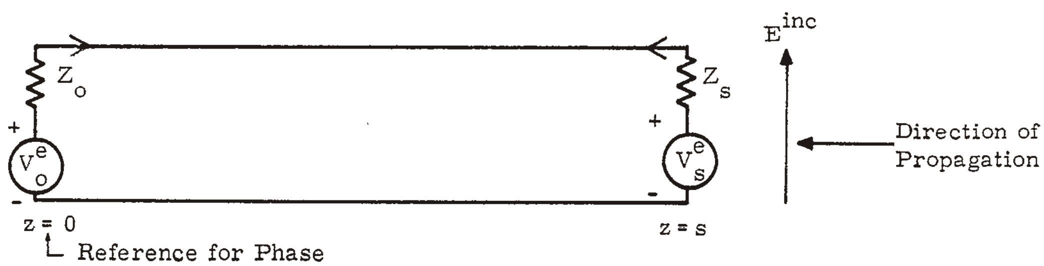

Figure 3. Diagram for Determining the Current in the Load Impedances of a Transmission Line Driven by Generators V_o^e and V_s^e.

Clearly the voltage V_s^e at $z = s$ is given by the relation

$$V_s^e = E^{inc}\, b\, e^{j\beta s} \tag{28}$$

inasmuch as the reference for phase is taken at $z = 0$. Also,

$$V_o^e = E^{inc}\, b\,. \tag{29}$$

From (3), the current in Z_o resulting from the voltage V_o^e is

$$I_o' = \frac{E^{inc}\, b}{D} (Z_c \cos \beta s + j\, Z_s \sin \beta s)\,. \tag{30}$$

Also, from (8), one may infer that the current in Z_o due to V_s^e may be written

$$I_o'' = \frac{E^{inc}\, b\, e^{j\beta s}\, Z_c}{D}\,. \tag{31}$$

Now the current in Z_o due to the simultaneous operation of both generators (considering their polarity) is

$$I_\phi(0) = I_o' - I_o'' = \frac{E^{inc}\, b}{D} \left(Z_c \cos \beta s + j\, Z_s \sin \beta s - Z_c\, e^{j\beta s}\right)\,. \tag{32}$$

This result agrees precisely with (27). It should be mentioned that (32) may also be obtained from the literature.[5]

Conclusions

A general theory has been advanced for determining the current in a specified load impedance in a complicated transmission line network. The procedure is to calculate the radiation field of the configuration when driven by a generator, and then invoke reciprocity. If the incident electric field is arbitrarily oriented in space, the components along θ and ϕ are determined. Each component sets up a current in the load impedance. The total current is found by superposition. A given problem may be tedious, but the technique of solution is straightforward and parallels the methodology of the present paper.

The authors have published several papers on transmission line topics. In these the direction of the incident field has been specified as tangential to the longitudinal conductors; i.e., cross polarized with respect to the terminations so they do not have to be considered in the theory.

In a recent paper[6] the incident field is assumed to be parallel to a missile and its external transmission line. Attention is invited to the fact that the complete interaction of the scattered field from the missile with the transmission line is properly considered. This means, among other things, that the line terminations are taken into account in the theory.

REFERENCES

1. R. W. P. King, H. R. Mimno, and A. H. Wing, "Transmission Lines, Antennas and Wave Guides," McGraw-Hill Book Co. Inc., 1945, p 19, Eq (18.14).

2. S. A. Schelkunoff, "A General Radiation Formula," Proc. IRE, Vol 27, No. 10, October 1939, p 662, Eq 8.

3. R. W. P. King, "Theory of Linear Antennas," Harvard University Press, 1956, p 21, Section 9, Eq 7.

4. C. W. Harrison, Jr., "Reducing the Response of Single-Phase Transmission Lines to Electrical Noise," IEEE Transactions on Electromagnetic Compatibility, Vol EMC-14, No. 2, May 1972, pp 79-81.

5. R. W. P. King and C. W. Harrison, Jr., "Excitation of an External Terminated Longitudinal Conductor on a Rocket by a Transverse Electromagnetic Field," IEEE Transactions on Electromagnetic Compatibility, Vol EMC-14, No. 1, February 1972, pp 1-3.

6. R. W. P. King and C. W. Harrison, Jr., "Transmission Line Coupled to a Cylinder in an Incident Field," IEEE Transactions on Electromagnetic Compatibility, Vol EMC-14, No. 3, pp 97-105, August 1972.