Interaction Notes

Note 186

August 1974

Natural Modes of Certain Thin-Wire Structures

Lennart Marin
Dikewood Corporation, Westwood Research Branch
Los Angeles, California

Abstract

The electric-field formulation of the electromagnetic scattering problem is used to derive analytical expressions for the natural modes of the following thin-wire structures: a straight wire of variable radius, a wire bent into an L-shape, two collinear wires, two nonstaggered parallel wires, and a thin ring. The natural modes are used to derive expressions for the induced currents on some of these structures.

I. Introduction

The transient analysis of thin-wire structures has been the subject of many numerical investigations. The time dependent transmitting and scattering problems involving an infinitely long, perfectly conducting cylinder can be reduced to the evaluation of an infinite integral[1-4]. The transient scattering from a wire of finite length can be reduced to the solution of a space-time integral equation[5-6] or to the solution of a space integral equation in the frequency domain combined with an inverse Fourier (Laplace) transform[7]. Sofar, all transient analyses of thin-wire structures have been based on the numerical solution of the integral equations and little or no attention has been paid to analytical solutions of these equations. Based on the analytical properties in the complex frequency plane of the field scattered from a finite body[8,9] and certain properties of the electric-field integral equation for thin wires we will derive in this note analytical expressions for the currents on a thin wire induced by an incident pulsed plane wave or excited by a pulsed slice generator.

The natural oscillations of different thin-wire structures have been investigated extensively in the past. Pocklington[10] derived an integral-differential equation for the induced current on thin wires and used this equation to derive approximate expressions for the fundamental natural frequency of a thin straight wire and a thin circular ring. Expressions for the magnitude of the induced current of the fundamental mode were also derived in [10] when the ring is exposed to an incident plane wave propagating along the axis of the ring. The results obtained by Pocklington and Lord Rayleigh[11] are limited only to the weakly damped oscillations of a ring. Oseen extended their results to include the more attenuated part of the spectrum of a perfectly conducting ring as well as the natural frequencies of a resistively loaded ring and those of two perfectly conducting, coaxial rings[12-14]. Oseen also constructed the time history of the induced current from the natural modes when the incident electric field is a step-function modulated, time-harmonic, plane wave. By expanding the field quantities involved in a Fourier series around the circumference of the ring the time history of the current induced in the ring is calculated from the natural modes. The formal appearence of the expressions thus obtained is very similar to the general expressions in [8,15] or those obtained either from the magnetic-field formulation[9] or from the electric-field formulation as presented in Section II of this note.

The method of separation of variables was first used by Abraham to calculate the fundamental natural frequencies of a slender prolate spheroid[16]. By solving the Helmholtz equation in spheroidal coordinates, Page and Adams calculated the wavelengths and logarithmic decrements of the lowest natural oscillations of a prolate spheroid[17-19]. The forced oscillations due to a uniform time-harmonic electric field parallel to the axis of the spheroid were also calculated in [17,19] around the three lowest resonances.

The first treatment of the thin wire was given by Oseen[20] who applied the method of retarded potentials to straight thin wires. Besides deriving asymptotic forms for the natural frequencies of a thin wire Oseen also calculated the current induced on a wire by a transient incident wave. Using a slightly different form of the so-called thin-wire approximation Hallén derived a pure integral equation for the induced current on a thin wire[21,22]. This equation was then used to derive analytical expressions for the natural frequencies and the current distributions of the natural modes. Although the approximate treatment of Oseen and Hallén only yields the weakly damped natural modes the existence of the highly damped modes of a thin wire was already pointed out by Oseen[20].

Even though these early results on transient scattering from certain thin-wire structures are formal in the sense that the scattered field is assumed to be expressed in terms of the natural modes alone, they are very useful in view of the recent results obtained in [9]. The simplicity of various approximate analytic expressions for the induced currents obtained in [14,20] is striking in view of the rather lengthy numerical investigations[23-28] that have been undertaken recently. Indeed, the present investigation has been motivated by the simple results that were obtained in many of the early papers quoted above which, judging from recent numerical investigations, seem to have been forgotten nowaday. Some of the results presented here are therefore not new, they are included here merely to make them more readily available to workers in this area.

The outline of the note is as follows: in Section II the method given in [9] based on the magnetic-field formulation for calculating the transient currents from the natural modes is extended to include the electric-field formulation. In Sections III we present a method of calculating the first-branch natural modes of a thin, straight wire of variable radius. The approach we use follows closely that of Hallén[20], since we find this approach the simplest and most straightforward. In Sections IV and V we consider some special cases which can be of value for certain EMP applications. The natural frequencies of two perpendicular, connected wires (L-wire) are calculated in Section VI. Finally, in Section VII we present the natural frequencies of two parallel wires and those of a thin circular ring (TORUS).

II. Some Properties of the Electric-Field Integral Equation

Consider a perfectly conducting body illuminated by an electromagnetic wave. The induced surface current density $\underline{j}$ on the scattering body satisfies the integral-differential equation

$$\underline{\mathcal{L}}\cdot\underline{j} = s\varepsilon_o \underline{e}^{inc} \tag{1}$$

where $\underline{e}^{inc}$ is the incident tangential electric field on the surface of the scattering body, $\underline{\mathcal{L}}$ a linear operator defined by

$$\underline{\mathcal{L}}\cdot\underline{j} \equiv -\underline{n}\times\underline{n}\times(\underline{\mathcal{L}}_1\cdot\underline{j}) = -\underline{n}\times\underline{n}\times(\underline{\mathcal{L}}_2\cdot\underline{j}) \tag{2}$$

where

$$(\underline{\mathcal{L}}_1\cdot\underline{j})(\underline{r}) = s^2c^{-2}\int_S G(\underline{r},\underline{r}';s)\underline{j}(\underline{r}')dS' - \nabla\int_S G(\underline{r},\underline{r}';s)\nabla'\cdot\underline{j}(\underline{r}')dS',$$

$$(\underline{\mathcal{L}}_2\cdot\underline{j})(\underline{r}) = (s^2c^{-2} - \nabla\nabla\cdot)\int_S G(\underline{r},\underline{r}';s)\underline{j}(\underline{r}')dS', \tag{3}$$

G is the free-space Green's function

$$G(\underline{r},\underline{r}';s) = (4\pi|\underline{r} - \underline{r}'|)^{-1}\exp(-s|\underline{r} - \underline{r}'|/c), \tag{4}$$

c and ε_o are respectively the speed of light and permittivity of free space, s is the complex frequency, and $\underline{n} = \underline{n}(\underline{r})$ is the outward unit normal to the surface S of the scattering body.

From the expressions (2) and (3) it follows immediately that the operator $\underline{\mathcal{L}}$ is a symmetric operator when operating on functions $\underline{j}$ that are tangent to S and square integrable on S, i.e.,

$$\underline{\mathcal{L}}^T \equiv \underline{\mathcal{L}}^{\dagger *} = \underline{\mathcal{L}} \tag{5}$$

where $\underline{\mathcal{L}}^{\dagger}$ is the adjoint operator to $\underline{\mathcal{L}}$, the asterisk denotes complex conjugation and the scalar product between two functions $\underline{j}_1(\underline{r})$ and $\underline{j}_2(\underline{r})$ are defined by $(\underline{j}_1,\underline{j}_2) = \int_S \underline{j}_1(\underline{r})\cdot\underline{j}_2(\underline{r})dS$.

When the incident wave is a delta-function plane wave we know from previous investigations[9] that the induced surface current density is a meromorphic function of s. This implies that the inverse operator $\underline{\mathcal{L}}^{-1}(s)$ is a meromorphic, operator-valued function of s and that the locations of the poles of this inverse operator are given by those values of s, s_n, for which we have a nontrivial solution of the homogeneous equation (1), i.e.,

$$\underline{\mathcal{L}}(s_n)\cdot\underline{j}_n = 0, \qquad \underline{j}_n \neq 0 \qquad \text{and} \qquad \underline{\mathcal{L}}^{\dagger}(s_n)\cdot\underline{f}_n = 0, \qquad \underline{f}_n \neq 0. \tag{6}$$

From (5) and (6) we see that without loss of generality we can choose $\underline{f}_n = \underline{j}_n^*$.

To find the solution of the inhomogeneous equation (1) we follow the procedure used in finding the forced solution of the magnetic-field integral equation[9]. From this solution we derive the following representation of $\underline{\mathcal{L}}^{-1}(s)$ for the special case where $\underline{\mathcal{L}}^{-1}(s)$ has only simple poles,

$$\underline{\mathcal{L}}^{-1}(s) = \sum_{n,m} \{(s - s_n)^{-1}[(\underline{\mathcal{B}}_n\cdot\underline{j}_{nm},\underline{j}_{nm})]^{-1}\underline{j}_{nm}\underline{j}_{nm} + \underline{\mathfrak{R}}_n(s)\} + \underline{\mathcal{E}}(s) \tag{7}$$

where $\underline{\mathcal{B}}_n = (d\underline{\mathcal{L}}/ds)(s_n)$, $\underline{\mathfrak{R}}_n(s)$ is an operator valued polynomial of s and $\underline{\mathcal{E}}(s)$ is an operator-valued, entire function of s. The summation over m takes care of the degeneracy of each natural frequency, i.e., the number of linear independent solutions of (6) for fixed n.

When the incident field has all its sources outside S we show in the Appendix that $(\underline{e}_n^{inc},\underline{j}_n) = 0$ where $\underline{e}_n^{inc}$ denotes the incident tangential electric field on S evaluated at $s = s_n$. Thus, in this case we have the following solution of (1)

$$\underline{j} = \varepsilon_o \sum_{ext} \{s_n(s - s_n)^{-1}[(\underline{\mathcal{B}}_n\underline{j}_{nm},\underline{j}_{nm})]^{-1}(\underline{e}_n^{inc},\underline{j}_{nm})\underline{j}_{nm} + s\underline{\mathfrak{R}}_n(s)\cdot\underline{e}^{inc}\}$$

$$+ \varepsilon_o s\underline{\mathcal{E}}'(s)\cdot\underline{e}^{inc} \tag{8}$$

where $\underline{\mathcal{E}}'(s)$ is an entire operator and $\sum_{ext}$ denotes summation of external modes only. From (8) we can derive the following alternative form for the induced current density on the scattering body,

$$\underline{j} = \varepsilon_o \sum_{ext} \{s_n(s - s_n)^{-1}[(\underline{\mathcal{B}}_n \underline{j}_{nm}, \underline{j}_{nm})]^{-1}(\underline{e}^{inc}, \underline{j}_{nm})\underline{j}_{nm} + s\underline{\mathcal{E}}_n(s)\cdot\underline{e}^{inc}\}$$

$$+ \varepsilon_o s\underline{\mathcal{E}}'(s)\cdot\underline{e}^{inc} \qquad (8')$$

where $\underline{\mathcal{E}}_n(s)$ is an entire operator,

$$s\underline{\mathcal{E}}_n(s)\cdot\underline{e}^{inc} = s\underline{\mathfrak{R}}_n(s)\cdot\underline{e}^{inc}$$

$$+ \varepsilon_o \sum_m (s - s_n)^{-1}[(\underline{\mathcal{B}}_n \underline{j}_{nm}, \underline{j}_{nm})]^{-1}([s\underline{e}^{inc} - s_n \underline{e}_n^{inc}], \underline{j}_{nm})\underline{j}_{nm}.$$

Baum[15] has coined the expressions "class 1" and "class 2" coupling coefficients for the first term in the sums of (8) and (8'), respectively.

The time history of the current due to an arbitrary, incident, transient electromagnetic field can be obtained from (8) by using the same methods as in [9].

In the next sections we will investigate the operator $\underline{\mathcal{L}}$ for several different thin-wire structures and find approximate analytical expressions for the inverse operator $\underline{\mathcal{L}}^{-1}$.

III. A Perfectly Conducting Wire of Variable Radius

In this section we will investigate the integral-differential equation (1) for a thin, perfectly conducting wire with variable radius. For a thin wire the surface current density is predominantly independent of the azimuthal angle. If we consider only this part of the current we can reduce the vector equation (1) to a scalar differential-integral equation for the total current that passes through the cross section of the scattering body. By allowing the radius of the wire to vary along the structure we are able to treat, besides the "ordinary" straight wire with constant radius, such structures as a thin prolate spheroid and two or more connected wires with different radii.

Consider a perfectly conducting, straight wire of length ℓ and variable radius $\rho(z)$, $0 \leq z \leq \ell$, exposed to an incident electromagnetic field whose axial component of the electric field along the axis of the wire is $E_o(z)$, see Fig. 1. A time dependence $\exp(st)$ is assumed and suppressed throughout this section. The total current $I(z)$ on the wire satisfies the following approximate equation, with terms of order ρ_o/ℓ (ρ_o is the mean radius of the wire) being neglected[22],

$$\mathcal{L}I = -s\varepsilon_o E_o. \tag{9}$$

Here, $\mathcal{L}$ is a linear operator given by

$$(\mathcal{L}I)(z) = \left[\frac{d^2}{dz^2} - \frac{s^2}{c^2}\right] \int_0^\ell \frac{\exp(-sR/c)}{4\pi R} I(z')dz' + \frac{1}{2\pi}\left[\frac{\rho'(z)}{\rho(z)} I(z)\right]', \tag{10}$$

$R^2(z,z') = (z - z')^2 + \rho^2(z')$ and the prime in the second term of (10) denotes differentiation with respect to z.

Some comments are now in order concerning (9) and (10). The kernel in the two-dimensional equation (1) has an integrable singularity at $\underline{r} = \underline{r}'$ whereas the integrand in (10) is regular for all values of z'. This discrepancy between (1) and (9) is due to the fact that (9) is only an approximate equation for $I(z)$ and that terms of order ρ_o/ℓ have been neglected in obtaining this equation. We notice however that the kernel in (9) is "almost singular" in that it gets very large for $z = z'$. It should also be mentioned that (9) is not valid close to the end points of the wire and at points where the wire radius changes

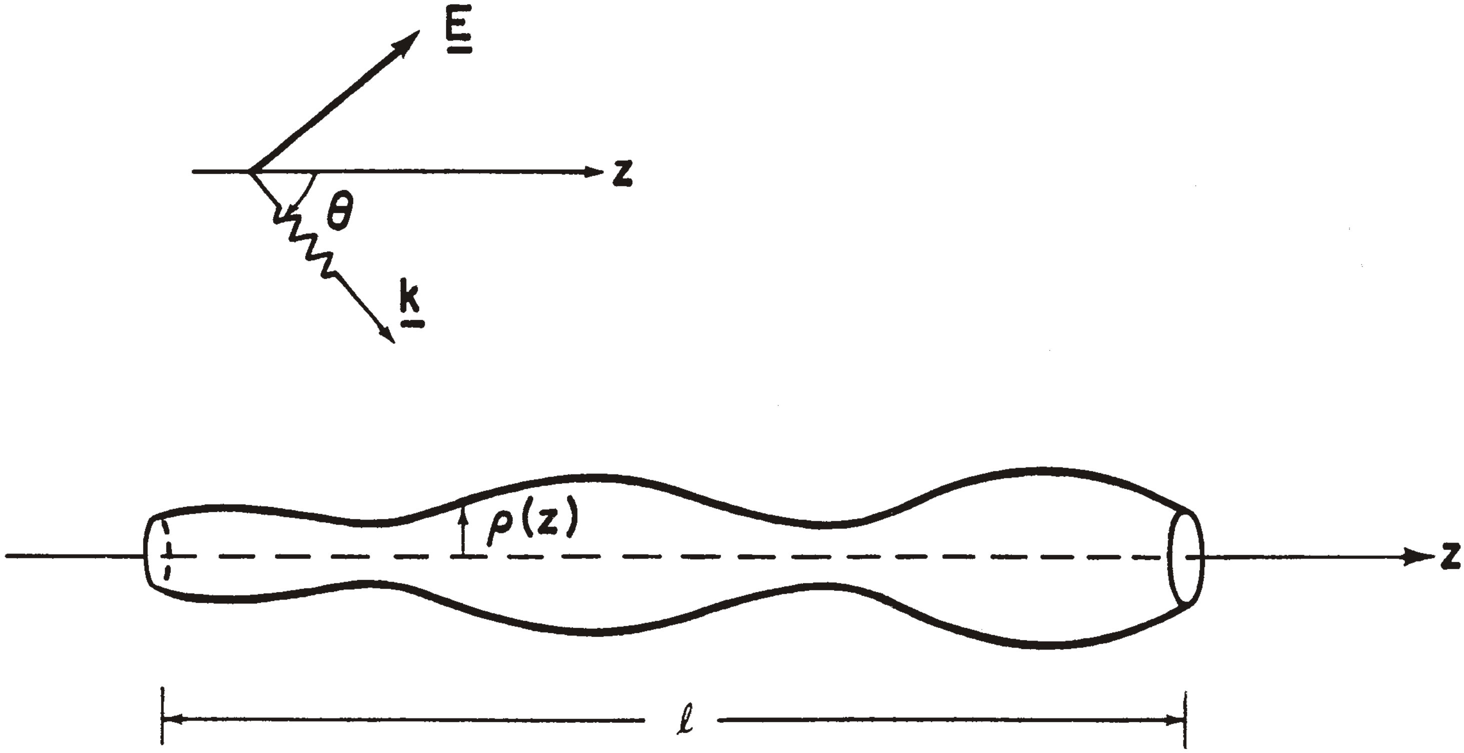

Figure 1. A thin wire of variable radius illuminated by an incident electromagnetic field.

abruptly. The extent of the regions where the approximations involved in obtaining (9) are not valid is of the order of the wire radius. Despite all the approximations and limitations introduced in (9) this equation can be used to obtain approximate expressions for global quantities such as the natural frequencies, the errors in these quantities being of the order of ρ_o/ℓ. The approximate equation (9) can alternatively be obtained from (1) by requireing that the total axial component of the electric field vanishes on the axis of the wire[21].

The natural modes of a thin wire are given by the nontrivial solutions of the homogeneous equation (9). To find these solutions it is advantageous first to integrate (9) and obtain the following integral equation[21]

$$(\mathcal{L}_1 I)(z) = -Z_o^{-1} \int_0^z E_o(z')\sinh[s(z - z')/c]dz' \tag{11}$$

where

$$(\mathcal{L}_1 I)(z) = \int_0^\ell \frac{\exp(-sR/c)}{4\pi R} I(z')dz' + \int_0^z \frac{\rho'(z')I(z')}{2\pi\rho(z')} \cosh \frac{s(z-z')}{c} dz'$$

$$+ A_1 \sinh \frac{sz}{c} + B_1 \cosh \frac{sz}{c} , \tag{12}$$

A_1, B_1 are constants of integration to be determined from the end conditions $I(0) = I(\ell) = 0$, and Z_o is the wave impedance of free space, $Z_o \approx 377$ ohms. Next, we follow Hallén[21] and approximate the first integral expression in (12) and neglect terms of order ρ_o/ℓ to get

$$\int_0^\ell \frac{\exp(-sR/c)}{R} I(z')dz' \approx \Omega I(z) + \ln\left[\frac{4(\ell-z)z\rho_o^2}{\ell^2\rho^2(z)}\right]I(z)$$
$$+ \int_0^\ell \frac{I(z')\exp(-s|z-z'|/c)-I(z)}{|z-z'|} dz' + O(\Omega^{-1}) + O(\frac{\rho}{\ell}) \tag{13}$$

where $\Omega = 2\ln(\ell/\rho_o)$. Equations (11)-(13) combined with the end condition $I(0) = 0$ results in the following equation

$$I(z) + \frac{1}{\Omega}\left\{\ln\left[\frac{4(\ell-z)z\rho_o^2}{\ell^2\rho^2(z)}\right] + \int_0^\ell \frac{I(z')\exp[-s|z-z'|/c]-I(z)}{|z-z'|}\,dz'\right.$$

$$\left. - \cosh\frac{sz}{c}\int_0^\ell \frac{I(z')\exp[-sz'/c]}{z'}\,dz' + 2\int_0^z \frac{\rho'(z')I(z')}{\rho(z')}\cosh\frac{s(z-z')}{c}\,dz'\right\}$$

$$= A\sinh\frac{sz}{c} - \frac{4\pi}{Z_o\Omega}\int_0^z E_o(z')\sinh\frac{s(z-z')}{c}\,dz' \tag{14}$$

which with the end condition $I(\ell) = 0$ gives

$$\int_0^\ell \frac{I(z')\exp[-s(\ell-z')/c]}{\ell-z'}\,dz' - \cosh\frac{s\ell}{c}\int_0^\ell \frac{I(z')\exp[-sz'/c]}{z'}\,dz'$$

$$+ 2\int_0^\ell \frac{\rho'(z')I(z')}{\rho(z')}\cosh\frac{s(\ell-z')}{c}\,dz' = A\Omega\sinh\frac{s\ell}{c} - \frac{4\pi}{Z_o}\int_0^\ell E_o(z')\sinh\frac{s(\ell-z')}{c}\,dz' \tag{15}$$

where $A = 4\pi\Omega^{-1}A_1$. It is in principle possible to eliminate the unknown constant A between (14) and (15) and thus obtain one equation for $I(x)$. However, we prefer to keep the form with two equations and an unknown constant.

The natural frequencies s_n and the current distribution $I_n(z)$ of the natural modes can be obtained by finding the nontrivial solutions of the homogeneous equations (14) and (15), i.e., by putting $E_o(z) \equiv 0$ in these equations. If we neglect terms of order ρ_o/ℓ, the nontrivial solutions of the set of equations (14) and (15) are the same as those of the homogeneous equation (10).* Expanding s_n and $I_n(z)$ in a power series of Ω^{-1} and assuming that the lowest order term in the expansion of $I_n(z)$ is independent of Ω we arrive, after some tedious but straightforward algebraic manipulations, at the following expressions by equating terms in the power series expansion in Ω^{-1}, c.f. [21]

$$s_n = (n\pi c/\ell)\left[i - \Omega^{-1}E(2n\pi)/(n\pi) - i\ell^{-1}\int_0^\ell \cos(2n\pi z/\ell)\ln[\rho(z)/\rho_o]\,dz + O(\Omega^{-2})\right], \tag{16}$$

---

*Note that the nontrivial solutions of the homogeneous, approximate equations (14) and (15) only give the exterior resonances whereas the nontrivial solutions of the exact homogeneous equation (1) give both the exterior and interior resonances, c.f. [29].

$$I_n(z) = \sin(n\pi z/\ell) + \Omega^{-1}[i_n(z) + i_n'(z)] + O(\Omega^{-2}) \tag{17}$$

where

$$\begin{aligned} i_n(z) = i\Big[&\exp(-in\pi z/\ell)E[2n\pi(\ell - z)/\ell] - \exp(in\pi z/\ell)E(2n\pi z/\ell) \\ &+ (2z - \ell)\cos(2n\pi z/\ell)E(2n\pi)/\ell\Big]/2 \\ &- \sin(n\pi z/\ell)\ln[4(\ell - z)z/\ell^2], \end{aligned} \tag{18}$$

$$\begin{aligned} i_n'(z) = \sin(n\pi z/\ell)\Big[&2\ln[\rho(z)/\rho_o] - \int_0^z [1 - \cos(2n\pi z'/\ell)][\rho'(z')/\rho(z')]dz'\Big] \\ + \cos(n\pi z/\ell)\Big[&z/\ell \int_0^\ell \sin(2n\pi z'/\ell)[\rho'(z')/\rho(z')]dz' \\ &- \int_0^z \sin(2n\pi z'/\ell)[\rho'(z')/\rho(z')]dz'\Big], \end{aligned} \tag{19}$$

$$\begin{aligned} E(z) &= \int_0^z [1 - \exp(-i\zeta)]\zeta^{-1}d\zeta \\ &= \gamma + \ln(z) - Ci(z) + iSi(z), \end{aligned} \tag{20}$$

γ is the Euler constant ($\gamma = 0.577\ldots$), and $Ci(z)$ and $Si(z)$ are the cosine and sine integrals, respectively[30].

Since $I_n(z)$ is a solution of the set of homogeneous equations (14) and (15) $I_n(z)$ can be determined only within a multiplicative constant. In (17) we have made a perticular choice of this constant, so that the leading term in the current distribution is equal to $\sin(n\pi z/\ell)$. The induced current on the wire is, of course, independent of the choice of the value of this constant, c.f. (8).

This systematic way of finding the natural frequencies of a thin wire can only be applied to those frequencies that are close to the imaginary axis in the complex s-plane. The highly damped natural modes of a thin wire cannot be found by using these perturbation techniques. From previous numerical calculations[23,24] of transient scattering problems we know that the contribution to the induced current from the highly damped modes is important only for early times. For these early times it is felt that an expression of the induced current based on a traveling wave expansion[29] is more advantageous from the computational point of view than an expansion in natural modes (i.e., standing waves).

After determining the natural modes of a thin wire, let us now see how we can use them in constructing the forced solution of (9). We first assume that all poles are simple and non-degenerate (this assumption has been substantiated in all cases that have so far been investigated). Following the general theory in Section II we have the following approximate solution of (10),

$$I(z,s) = \sum_n \left[\frac{\varepsilon_o s_n}{s-s_n} \frac{C_n}{B_n} I_n(z) + (\mathfrak{R}_n E_o)(z,s)\right] + (\mathcal{E} E_o)(z,s) \tag{21}$$

where

$$C_n = \int_0^\ell E_o(z,s_n) I_n(z) dz,$$

$$B_n = (\mathcal{B}_n I_n, I_n),$$

$\mathfrak{R}_n$ is a operator-valued polynomial of s and $\mathcal{E}_n$ is an entire operator-valued function of s. The operator $\mathcal{B}_n$ is given by $\mathcal{B}_n = \mathcal{B}(s_n)$ and $\mathcal{B}(s) = (d/ds)\mathcal{L}(s)$. From (10) we get

$$(\mathcal{B}_n I_n)(z) = \left[\frac{d^2}{dz^2} - \frac{s_n^2}{c^2}\right] \int_0^\ell \frac{I_n(z') \exp(-s_n R/c)}{4\pi c} dz'$$

$$+ \frac{2s_n}{c^2} \int_0^\ell \frac{I_n(z') \exp(-s_n R/c)}{4\pi R} dz' . \tag{22}$$

The representation (22) for $\mathcal{B}_n$ can be simplified somewhat to

$$(\mathcal{B}_n I_n)(z) \approx \frac{s_n}{2\pi c^2} \left\{ (\Omega - 1) I_n(z) + \ln\left[\frac{4(\ell-z)z\rho_o^2}{\ell^2\rho^2(z)}\right] I_n(z) \right.$$

$$\left. + \int_0^\ell \frac{I_n(z')\exp(-s_n|z-z'|/c) - I_n(z)}{|z-z'|} dz' \right\}. \quad (23)$$

An expression of $(\mathcal{B}_n I_n)(z)$ accurate up to order Ω^o is then obtained by substituting the expressions (16)-(19) for s_n and $I_n(z)$ into (23). Thus,

$$(\mathcal{B}_n I_n)(z) = [in\Omega/(2c\ell)]\{\sin(n\pi z/\ell) + \Omega^{-1}[b_n(z) + b_n'(z)] + O(\Omega^{-2})\} \quad (24)$$

where

$$b_n(z) = (z/\ell)\cos(n\pi z/\ell)E(2n\pi), \quad (25)$$

$$b_n'(z) = \sin(n\pi z/\ell) \int_0^\ell [\sin(2n\pi z'/\ell) + \cos(2n\pi z'/\ell) - 1][\rho'(z')/\rho(z')]dz'$$

$$+ \cos(n\pi z/\ell)\left[z/\ell \int_0^\ell \sin(2n\pi z'/\ell)[\rho'(z')/\rho(z')]dz'\right.$$

$$\left. - \int_0^z \sin(2n\pi z'/\ell)[\rho'(z')/\rho(z')]dz'\right]. \quad (26)$$

A simple integration of the expressions given by (17) and (24) then yields

$$B_n = (\mathcal{B}_n I_n, I_n) = [in\Omega/(4c)][1 + \Omega^{-1}(b_n + b_n') + O(\Omega^{-2})] \quad (27)$$

where

$$b_n = [1 + i/(4n\pi)]E(2n\pi) - 2\ln 2, \quad (28)$$

$$b_n' = \int_0^\ell 4n\pi(\ell - z)\ell^{-2} \sin(2n\pi z/\ell)\ln[\rho(z)/\rho_o]dz. \quad (29)$$

In constructing the time history of the current on the antenna we also need to calculate the scalar product C_n as given by (21). We will here give explicit expressions for C_n in two special cases, namely, (1) when the wire is excited by a slice generator (the antenna problem) and (2) when a plane wave impinges on the wire (the scattering problem).

In the antenna problem we assume that the wire is excited by a slice generator located at $z = b$ and whose output voltage is $V(s)$, so that

$$E_o(z,s) = V(s)\delta(z - b). \tag{30}$$

In this case we get the following expression for C_n, C_n^{ant}

$$C_n^{ant} = V(s_n)\left[\sin(n\pi b/\ell) + \Omega^{-1}[i_n(b) + i_n'(b)]\right] \tag{31}$$

where $i_n(z)$ and $i_n'(z)$ are given by (18) and (19), respectively.

In the scattering case we assume that the direction of propagation of the incident plane wave makes an angle θ with the positive z-direction so that

$$E_o(z,s) = E_o(s)\sin\theta \exp(-szc^{-1}\cos\theta). \tag{32}$$

After some lengthy algebraic manipulations we arrive at the following expressions for C_n^{sc},

$$C_n^{sc} = E_o\ell\left[(n\pi)^{-1}\sin\theta\{1 - \cos[n\pi(1 - \cos\theta)]\} + \Omega^{-1}(c_n + c_n') + O(\Omega^{-2})\right] \tag{33}$$

where

$$c_n = \Big[[1 + 2(1 + \cos^2\theta)/(in\pi \sin^2\theta)]E(2n\pi) - 4 \ln 2$$

$$+ \cos\theta\{E[n\pi(1 + \cos\theta)] - E[n\pi(1 - \cos\theta)]\}$$

$$- [\sin^2\theta/(1 - \cos\theta)]E^*[n\pi(1 - \cos\theta)] \tag{34}$$

$$- [\sin^2\theta/(1 + \cos\theta)]E^*[n\pi(1 + \cos\theta)]\Big]\Big[\{1 - \exp[in\pi(1 - \cos\theta)]\}/(2n\pi \sin\theta)\Big]$$

$$+ \Big[E[n\pi(1 + \cos\theta)] - E[n\pi(1 - \cos\theta)]\Big]\Big[\{1 + \exp[in\pi(1 - \cos\theta)]\}/(2n\pi \sin\theta)\Big],$$

$$c_n' = 2(1 + \cos^2\theta)\{1 - \exp[in\pi(1 - \cos\theta)]\}/(n\pi\ell \sin^3\theta) \int_0^\ell \cos(2n\pi z/\ell)\ln[\rho(z)/\rho_o]dz$$

$$- 2\exp[in\pi(1 - \cos\theta)]/(\ell \sin\theta) \int_0^\ell \sin(2n\pi z/\ell)\ln[\rho(z)/\rho_o]dz \tag{35}$$

$$+ i \sin\theta\,(1 + \cos\theta)/[2\ell(1 - \cos\theta)] \int_0^\ell \exp[-in\pi(1 + \cos\theta)z/\ell]\ln[\rho(z)/\rho_o]dz$$

$$- i \sin\theta\,(1 - \cos\theta)/[2\ell(1 + \cos\theta)] \int_0^\ell \exp[in\pi(1 - \cos\theta)z/\ell]\ln[\rho(z)/\rho_o]dz\ .$$

To sum up this section we have derived, from an approximate form of the electric-field integral equation, asymptotic expressions for the natural frequencies, the current distribution of the natural modes, and the excitation coefficient of these modes when the driving field is either a slice generator on the wire or an incident plane wave. In the next two sections we will study two particural cases somewhat more in detail, namely, (1) a straight thin wire of constant radius and (2) two connected straight thin wires having constant but different radii.

IV. A Straight Thin Wire

The straight thin wire has been the subject of exhaustive investigations due to its relative simplicity. Approximate expressions for the natural frequencies were obtained by Oseen as early as 1914. He also derived approximate expressions for the current distribution and the coupling coefficient of a few modes when the incident field is a plane wave whose direction of propagation is perpendicular to the wire. In calculating the transient response Oseen used the eigenvalues and eigenfunctions of the operator $\mathcal{L}$ in (9), i.e., he finds the nontrivial solutions of the equation

$$(\mathcal{L}I)(s,z) = \lambda(s)\Omega(z)I(s,z) \tag{36}$$

where $\Omega(z) = \frac{1}{2}\ln[4z(\ell - z)/a^2]$ is a positive weighting function (the term characteristic modes has been coined later for solutions of equations similar to (36)). The results thus obtained are formal in the sense that it is assumed without a proof that the functions $I_n(s,z)$ given by the nontrivial solutions of (36) form a complete set and that the inverse eigenvalues $\lambda_n^{-1}(s)$ are meromorphic functions of s. Nevertheless, in view of the rather cumbersome numerical calculations performed lately, the simple results obtained by Oseen[20] have motivated us to undertake this approximate, analytical investigation of the natural modes of certain thin-wire structures.

The approach used here is conventional in that we are solving the integral equation directly with perturbation techniques. This approach seems to be the most effecient way in the general case. In the case of a straight wire, Fourier transform methods combined with the theory of Wiener and Hopf may be used to find the natural frequencies[31]. However, the complicated expressions thus obtained forces one to resort to approximations which may be very accurate for thin wires when $|\mathrm{Re}\{s\ell/c\}| < 1$ and $|\mathrm{Im}\{sa/c\}| \ll 1$.[31] It is therefore possible to use this approach in calculating certain natural frequencies. But using these approximate expressions to determine the analytical properties of the scattered field is bound to fail due to the approximations introduced, and we therefore disagree with the conclusions drawn in [32] that the delta gap introduces a branch cut in the scattered field. Also, the existence of the branch cut is in direct contradiction to the general results obtained in [9].

The rather crude way used to evaluate inverse Fourier transform in [32] also gives rise to an unphysical result for the late time behavior of the current on a center-fed antenna (that the total current behaves like the outgoing wave from the gap and that it decays like the inverse logarithm of time).

To get some quantitative information about the accuracy of the asymptotic expansions discussed in this note we have in Fig. 2 graphed three different representations of the fundamental natural frequency of a straight, thin wire, namely, (1) an asymptotic form which is correct up to order Ω^{-1} (c.f. [20], [21], (17)), (2) an asymptotic form which is correct up to order Ω^{-2} (c.f. [21]), and (3) the numerical results obtained in [23]. We note that for $a/\ell = 0.01$, the natural frequenies calculated from these different ways all differ about 20% from each other. We also note that the asymptotic form which is correct up to order Ω^{-2} gives a too large value of $|Re\{s_n\}|$, whereas the asymptotic form which is correct up to order Ω^{-1} yields a somewhat too large value of $Im\{s_n\}$. Since the convergence of the asymptotic expansion is doubtful[21] and judging from the results presented in Fig. 2 it is questionable if the accuracy of the approximate solution would be improved by including the Ω^{-2}-order term when $a/\ell = 0.01$. For that reason and also because of the fact that the complexity of calculating the Ω^{-2}-order term is rather large for more complicated structures, we choose to include only terms of order Ω^{-1} in all other cases that will be treated below.

The seven lowest first-branch natural frequencies calculated from (17) are graphed in Fig. 3 for different values of a/ℓ. We note that the imaginary part of the complex frequency is almost constant and that the absolute value of the real part of the complex frequency decays rather slowly with a/ℓ.

Before concluding this section we want to point out that in the case of a straight wire all quantities denoted by a prime in the previous section are zero, i.e., $i_n'(z) \equiv 0$, $b_n' \equiv 0$, $c_n' \equiv 0$. Furthermore, the results obtained already in [9, 20, 21] answer two of the questions raised later in [23], i.e., that the current distribution of the first-branch natural indeed is complex and that the delta gap does not introduce a branch cut in the induced current.

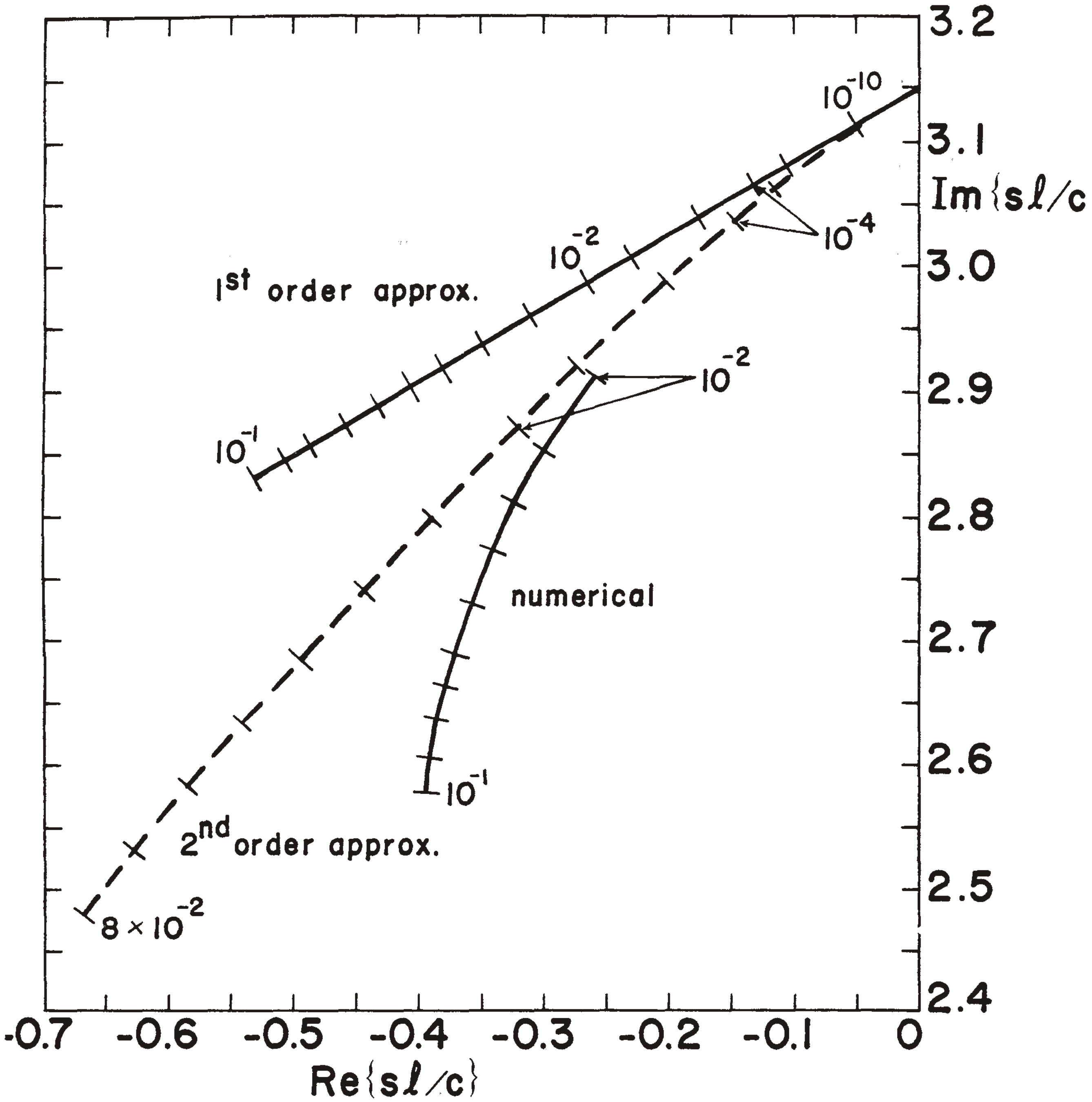

Figure 2. The fundamental natural frequency of a thin wire. The natural frequencies for $a/\ell = 10^{-10}$, 10^{-5}, 10^{-4}, 10^{-3}, 0.005, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1 are indicated in the figure.

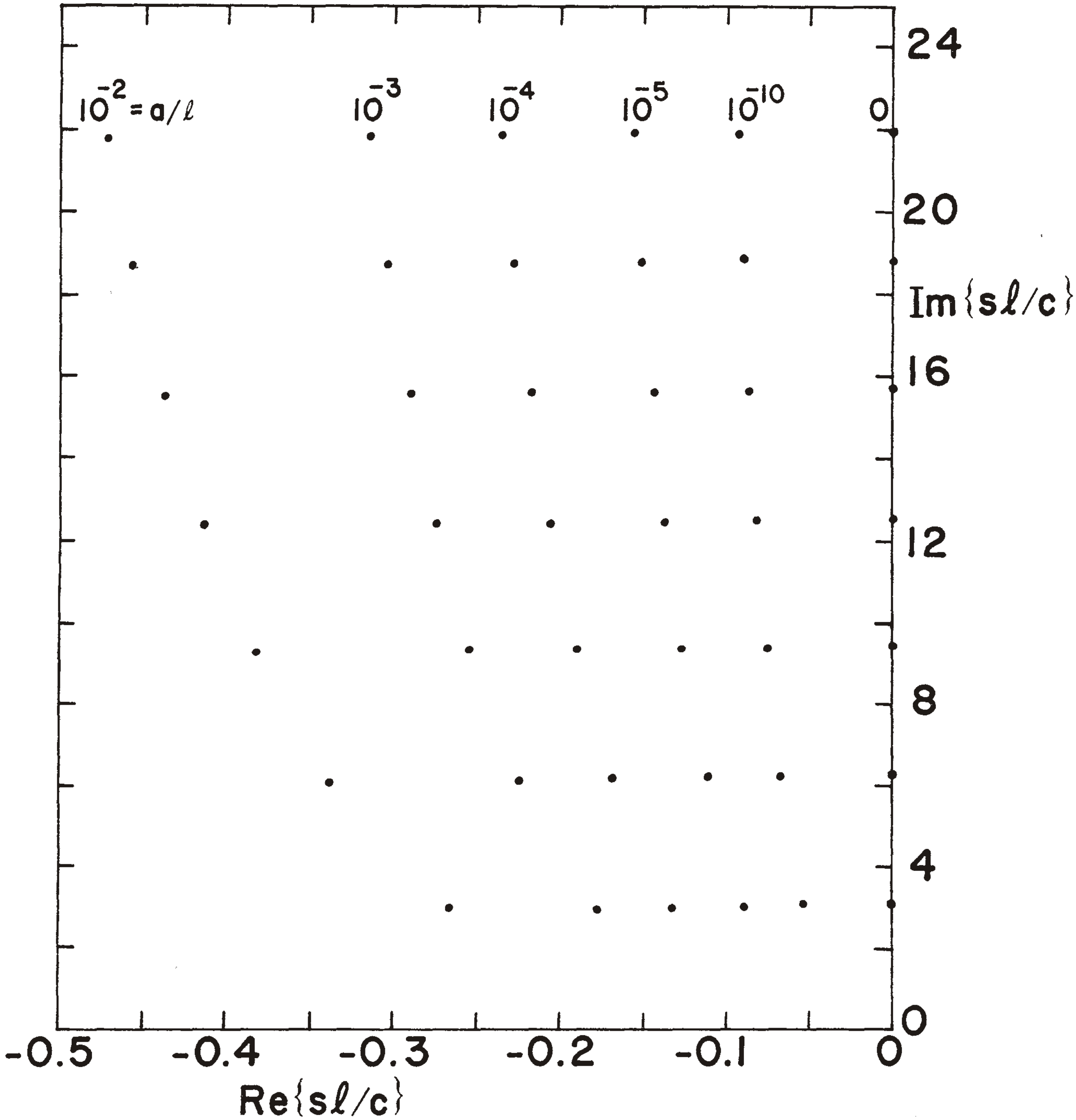

Figure 3. Natural frequencies of a thin wire.

V. Two Connected Straight Wires of Different Radii

Another case that can be treated with the general theory developed in Section III is that of two connected straight wires of different radii. The two wires are smoothly connected to each other in a junction region whose length Δ is much smaller than the lengths of the two wires but much larger than their radii, i.e.,

$$a_1 << \Delta, \qquad a_2 << \Delta, \qquad d >> \Delta, \qquad \ell - d >> \Delta \tag{37}$$

where a_1 and a_2 are the radii and d and $\ell - d$ are the lengths of the two wires (see Fig. 4). We define the mean radius ρ_o and the quantity Ω in the following way:

$$\rho_o = [da_1 + (\ell - d)a_2]/\ell, \qquad \Omega = 2 \ln(\ell/\rho_o). \tag{38}$$

The natural frequencies of this type of structure is approximately given by

$$s_n = (n\pi c/\ell)[i - \Omega^{-1}E(2n\pi)/(n\pi) + i\Omega^{-1} \ln(a_2/a_1)\sin(2n\pi d/\ell)/(2n\pi)$$
$$+ O(\Omega^{-2}) + O(\Delta/\ell)]. \tag{39}$$

For this special scattering structure we obtain the following explicit expressions for the quantities denoted by a prime in (17), (23), (26), and (33):

$$i_n'(z) = 2[\ln(a_1/\rho_o)U(d - z) + \ln(a_2/\rho_o)U(z - d)]\sin(n\pi z/\ell)$$
$$- \ln(a_2/a_1)\Big[[1 - \cos(2n\pi d/\ell)]\sin(n\pi z/\ell)U(z - d)$$
$$- [z/\ell - U(z - d)]\sin(2n\pi d/\ell)\cos(n\pi z/\ell)\Big] + O(\Delta/\ell), \tag{40}$$

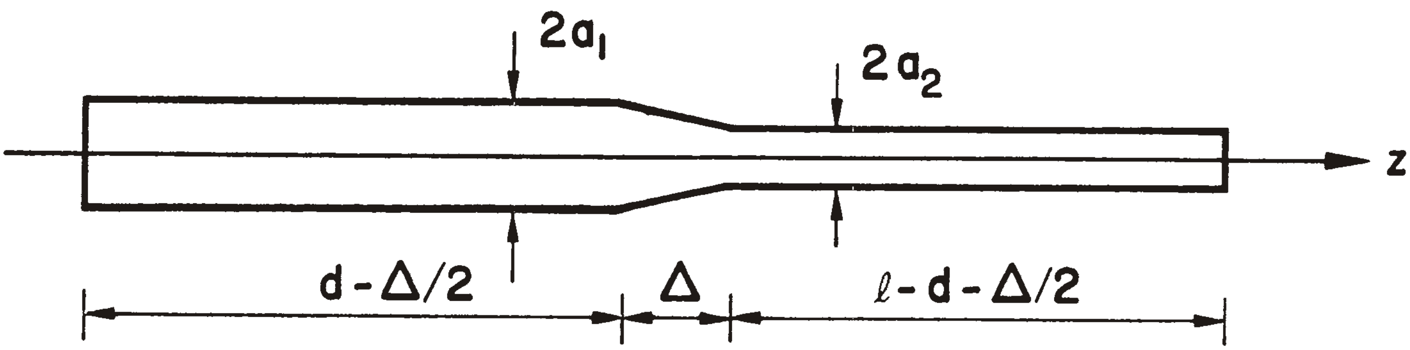

Figure 4. Two connected straight wires of different radii.

$$b'_n = 2\ \ln(a_1/\rho_o) + 2\ \ln(a_2/a_1)[(1 - d/\ell)\cos(2n\pi d/\ell) + 1/(2n\pi)\sin(2n\pi d/\ell)], \quad (41)$$

$$\begin{aligned} c'_n = \Big[&\ln(a_1/\rho_o) - \exp[in\pi(1 - \cos\theta)]\ln(a_2/\rho_o)\Big]/(n\pi \sin\theta) \\ &+ \ln(a_1/a_2)\Big[2(1 + \cos^2\theta)\{1 - \exp[in\pi(1 - \cos\theta)]\}(2n^2\pi^2 \sin^3\theta)^{-1}\sin(2n\pi d/\ell) \\ &- 2\exp[in\pi(1 - \cos\theta)](2n\pi \sin\theta)^{-1}[1 - \cos(2n\pi d/\ell)] \\ &- \sin\theta \exp[-in\pi(1 + \cos\theta)d/\ell]/[2n\pi(1 - \cos\theta)] \\ &- \sin\theta \exp[in\pi(1 - \cos\theta)d/\ell]/[2n\pi(1 + \cos\theta)]\Big] \end{aligned} \quad (42)$$

where $U(z)$ is the Heaviside unit step function.

Let us briefly investigate some of the properties of the current distribution of each mode near the junction between the two wires. From (40) we see that the current distribution itself is a continuous function in this approximation whereas its derivative has a discontinuity at the junction. This discontinuity is given by

$$\begin{aligned} \delta_n &= \left(\frac{dI_n}{dz}\right)_{z=d+\Delta/2} - \left(\frac{dI_n}{dz}\right)_{z=d-\Delta/2} \\ &= \frac{2n\pi}{\Omega\ell} \ln\frac{a_2}{a_1} \cos\frac{n\pi d}{\ell} + O(\Omega^{-2}) + O(\tfrac{\Delta}{\ell}) \\ &= \frac{2}{\Omega} \ln\frac{a_2}{\rho_o}\left(\frac{dI_n}{dz}\right)_{z=d+} - \frac{2}{\Omega} \ln\frac{a_1}{\rho_o}\left(\frac{dI_n}{dz}\right)_{z=d-} + O(\Omega^{-2}) + O(\tfrac{\Delta}{\ell}). \end{aligned} \quad (43)$$

Thus, we observe that the discontinuity of dI_n/dz is proportional to Ω^{-1}; so, in this asymptotic sense the discontinuity is small. The continuity equation implies that the so-called linear charge density, i.e., the surface charge density multiplied by 2π times the radius of the wire, of each mode is discontinuous at the junction. We also note that the surface charge density of each mode is discontinuous at the junction. We therefore disagree with the junction conditions used in [33]. The conditions that actually should be used require that (1) the total current and (2) the scalar potential ϕ be continuous at the junction. It is easy to show that the current distribution of each mode satisfies these junction conditions in the approximation used to obtain (40). Equation (43) suggests that instead of using the junction condition of ϕ being continuous we can use the following approximate condition: the quantity

$$\frac{dI(\xi_i)}{d\xi_i}\left[1 - \frac{2}{\Omega} \ln \frac{a_i}{\rho_o}\right]$$

is continuous at the junction. Here, we use the arclength ξ_i of the i^{th} wire as measured from the junction and the direction of the current in the i^{th} wire is along the unit vector $\hat{\xi}_i$ (see Fig. 5). We note that if all wires have the same radii then the junction condition simply becomes I and $dI/d\xi$ being continuous at the junction.

Finally, we note that in the approximation used in this section, the junction condition obtained from (43) agrees with that derived by King[34].

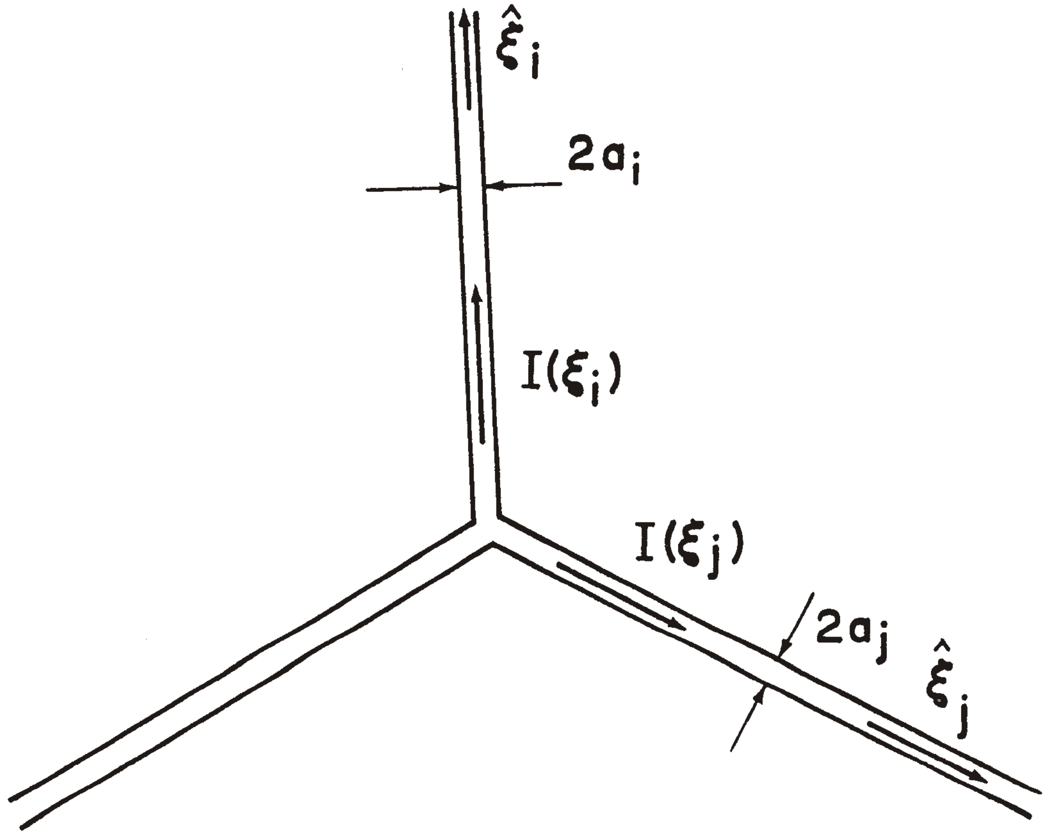

Figure 5. A junction of thin wires.

VI. The L-Wire

Another structure that lends itself to an analytical analysis is a thin wire bent into an L-shape. In Fig. 6 we have depicted two, connected, straight wires of constant radius whose axes are perpendicular to each other. Note that although the radius of each wire is constant it is not the same for the two different wires. The dimensions of the wire structure are shown in Fig. 6.

To find the natural frequencies of this structure we introduce the quantity ξ measuring the arc length of the structure from the point P_1 such that

$$\xi = \begin{cases} x + \ell_1, & \xi \text{ belongs to wire 1} \\ y + \ell_1, & \xi \text{ belongs to wire 2} \end{cases} \tag{44}$$

i.e., $\xi = 0$ at P_1 and $\xi = \ell_1 + \ell_2 = \ell$ at P_2. Denoting the total current by $I(\xi)$ we derive the following equation for $I(\xi)$:

$$\int_0^\ell \left[\frac{\partial^2}{\partial\xi\partial\xi'} G(\xi,\xi') + \frac{s^2}{c^2} G(\xi,\xi')\hat{\xi}\cdot\hat{\xi}'\right] I(\xi')d\xi' = s\varepsilon_o\hat{\xi}\cdot\underline{E}(\xi) \tag{45}$$

where $\hat{\xi}$ is the unit tangent vector at ξ. This equation can be integrated to yield the following equation[35]

$$\int_0^\ell K(\xi,\xi')I(\xi')d\xi' - \cosh\frac{s\xi}{c}\int_0^\ell G(0,\xi')I(\xi')d\xi'$$

$$= A\sinh\frac{s\xi}{c} - Z_o^{-1}\int_0^\xi \hat{\xi}'\cdot\underline{E}_o(\xi')\sinh\frac{s(\xi-\xi')}{c}\,d\xi' \tag{46}$$

where

$$K(\xi,\xi') = G(\xi,\xi')\hat{\xi}\cdot\hat{\xi}' - \int_0^\xi \frac{\partial G(\eta,\xi')}{\partial\eta}\hat{\eta}\cdot\hat{\xi}$$

$$+ \frac{\partial G(\eta,\xi')}{\partial\eta} + G(\eta,\xi')\frac{\partial(\hat{\eta}\cdot\hat{\xi}')}{\partial\eta}\cosh\frac{s(\xi-\eta)}{c}\,d\eta. \tag{47}$$

Neglecting the contributions from terms of the order of the wire radii to their

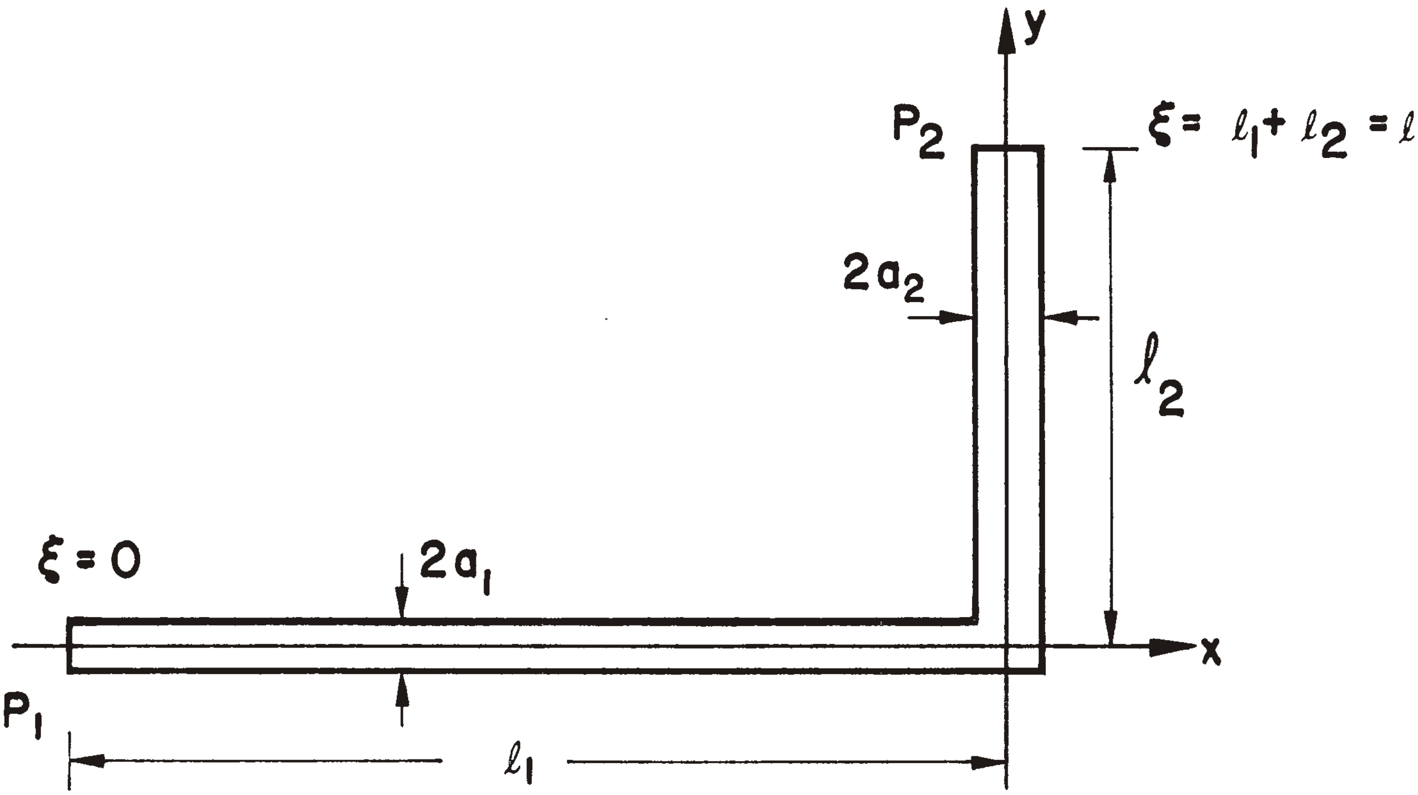

Figure 6. Two, connected, perpendicular, straight wires (L-wire).

lengths we can use the so-called thin wire approximation of $G(\xi,\xi')$,

$$G(\xi,\xi') = (4\pi R)^{-1}\exp(-sR/c), \qquad R^2(\xi,\xi') = (\xi - \xi')^2 + \rho^2(\xi). \tag{48}$$

Note that the integral equation (46) has been written in such a form that any solution of (46) satisfies the end condition $I(0) = 0$.

For the L-wire we can split (46) into the following set of integral equations

$$\int_0^{\ell_1} G(\xi,\xi')I(\xi')d\xi' - \int_{\ell_1}^{\ell}\int_0^{\xi} \frac{\partial G(\eta,\xi')}{\partial\xi'} \cosh\frac{s(\xi-\eta)}{c}\, d\eta\, I(\xi')d\xi'$$

$$- \cosh\frac{s\xi}{c}\int_0^{\ell_1} G(0,\xi')I(\xi')d\xi'$$

$$= A \sinh\frac{s(\xi-\xi')}{c} - Z_o^{-1}\int_0^{\xi} \hat{\xi}'\cdot\underline{E}_o(\xi')\sinh\frac{s(\xi-\xi')}{c}\, d\xi', \qquad 0 < \xi < \ell_1$$

$$\tag{49}$$

$$\int_{\ell_1}^{\ell} G(\xi,\xi')I(\xi')d\xi' - \int_0^{\ell_1}\int_{\ell_1}^{\xi} \frac{\partial G(\eta,\xi')}{\partial\xi'} \cosh\frac{s(\xi-\eta)}{c}\, d\eta\, I(\xi')d\xi'$$

$$- \int_{\ell_1}^{\ell}\int_0^{\ell_1} \frac{\partial G(\eta,\xi')}{\partial\xi'} \cosh\frac{s(\xi-\eta)}{c}\, d\eta\, I(\xi')d\xi' + \cosh\frac{s(\xi-\ell_1)}{c}\int_0^{\ell_1} G(\ell_1,\xi')I(\xi')d\xi'$$

$$- \cosh\frac{s(\xi-\ell_1)}{c}\int_{\ell_1}^{\xi} G(\ell_1,\xi')I(\xi')d\xi' - \cosh\frac{s\xi}{c}\int_0^{\ell_1} G(0,\xi')I(\xi')d\xi'$$

$$= A \sinh\frac{s\xi}{c} - Z_o^{-1}\int_0^{\xi} \hat{\xi}'\cdot\underline{E}_o(\xi')\sinh\frac{s(\xi-\xi')}{c}\, d\xi', \qquad \ell_1 < \xi < \ell.$$

To find the natural frequencies of the L-wire we use the other end condition $I(\ell) = 0$. By employing the same methods as in Section III we derive the following expression for the natural frequencies of the L-wire:

$$s_n = \frac{in\pi c}{\ell} + \frac{1}{\Omega}\left[i(-1)^n \sin\frac{n\pi(\ell_1-\ell_2)}{\ell} \ln\frac{\ell_1}{\ell_2} \right.$$

$$- \frac{1}{2}\left\{1 + \exp\left(\frac{-2in\pi\ell_1}{\ell}\right)\right\} E\left(\frac{2n\pi\ell_2}{\ell}\right) - \frac{1}{2}\left\{1 + \exp\left(\frac{-2in\pi\ell_2}{\ell}\right)\right\} E\left(\frac{2n\pi\ell_1}{\ell}\right)$$

$$\left. - \frac{n}{4\ell}\int_0^{\ell_1}\int_0^{\ell_2} \frac{\exp[-in\pi(u^2+v^2)^{\frac{1}{2}}/\ell]}{(u^2+v^2)^{\frac{1}{2}}} \left\{\cos\frac{n\pi(\ell+u+v)}{\ell} + \cos\frac{n\pi(\ell_1-\ell_2-u+v)}{\ell}\right\} du\, dv \right]$$

$$+ O(\Omega^{-2}) \tag{50}$$

where $\Omega = 2\ln[\ell^2/(\ell_1 a_1 + \ell_2 a_2)]$. The double integral in (50) can be expressed in terms of elementary functions and the function $E(x)$ defined in (20), and the integrals defining the kernel in (49) can be expressed in terms of elementary functions.

The function $\nu_n(\ell_1,\ell_2,\ell)$ within the square bracket in (50) which is multiplied with Ω^{-1} to give the natural frequencies is graphed in Figs. 7a-7c for $n = 1,2,3$ and $0 \le \ell_1 < \ell/2$. From symmetry it is clear that $\nu_n(\ell_1,\ell_2,\ell) = \nu_n(\ell_2,\ell_1,\ell)$ so the range plotted covers all possible cases. A comparison was also made between the approximate results obtained here and the numerical results reported in [28] and an agreement within 20% was found for $a/\ell = 0.01$. This difference is approximately the same as the variation of the natural frequencies with ℓ_1. When making estimates of different quantities based on an approximate thin-wire analysis it is therefore worthwhile to simplify the model judiously and yet maintain a certain accuracy.

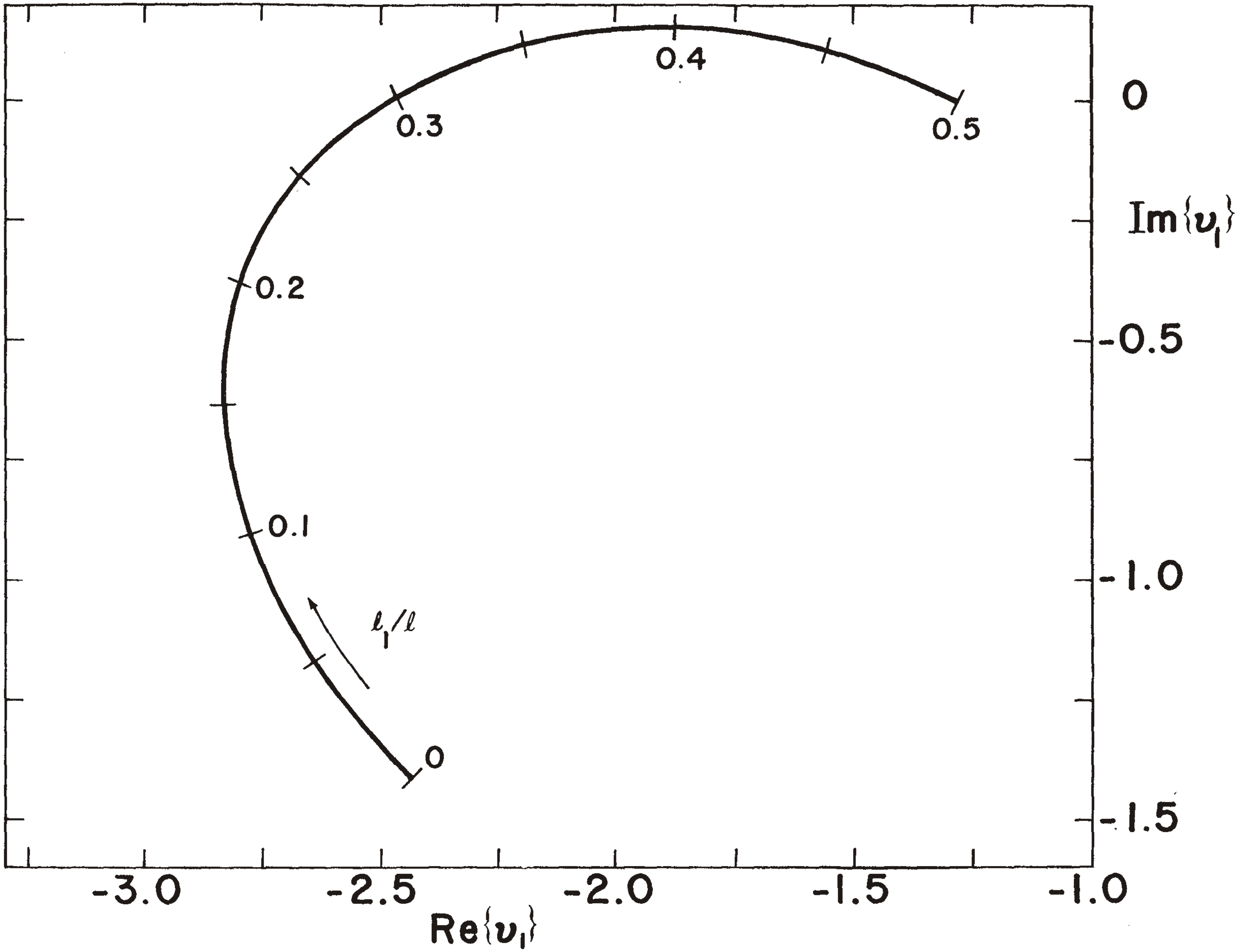

Figure 7a. The functions $\nu_1(\ell_1,\ell_2)$. The first natural frequency is $s_1 = \frac{i\pi c}{\ell} + \frac{c}{\Omega\ell}\,\nu_1(\ell_1,\ell_2)$.

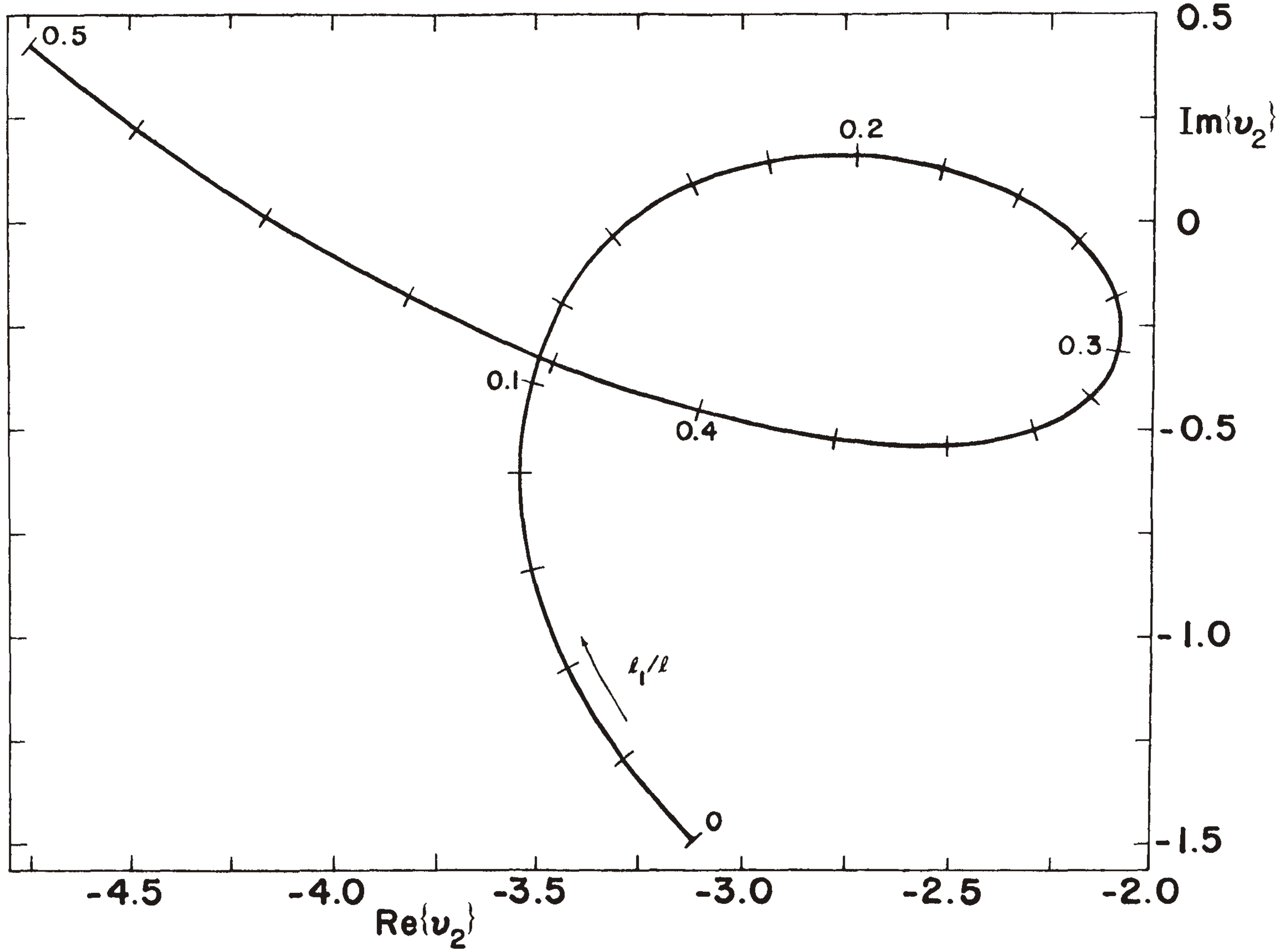

Figure 7b. The function $\nu_2(\ell_1,\ell_2)$. The second natural frequency is $s_2 = \frac{i2\pi c}{\ell} + \frac{c}{\Omega\ell}\,\nu_2(\ell_1,\ell_2)$.

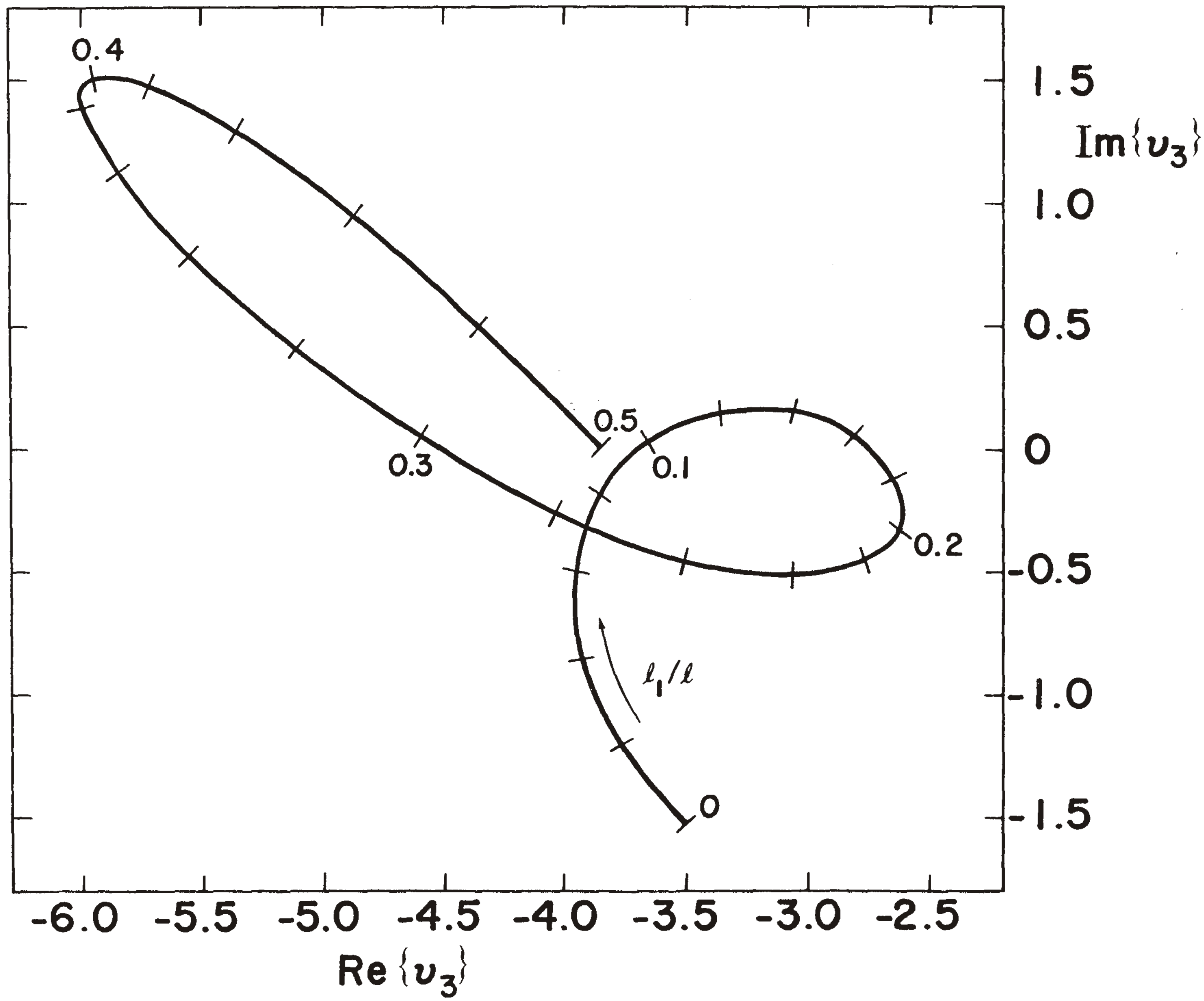

Figure 7c. The function $\nu_3(\ell_1,\ell_2)$. The third natural frequency is $s_3 = \frac{i3\pi c}{\ell} + \frac{c}{\Omega\ell}\nu_3(\ell_1,\ell_2)$.

VII. Some Other Structures

The method described in Section III can also be used to calculate the natural frequencies of two wires whose axes are parallel and those of a thin wire bent into a circular ring. We will, in this section, calculate some of the natural frequencies of the following structures: (1) two collinear wires, (2) two parallel nonstaggered wires, and (3) a thin ring (TORUS).

A. Two Collinear Wires

The natural frequencies of two collinear wires (each wire has length ℓ, radius a and the two wires are separated by a distance b, see Fig. 8) are given by[20]

$$s_n^{\pm} = \frac{in\pi c}{\ell} + \frac{1}{\Omega}\frac{c}{\ell}\,\sigma_n^{\pm}\left(\frac{b}{\ell}\right) + O(\Omega^{-2}), \qquad \Omega = 2\ln\frac{\ell}{a} \tag{51}$$

where

$$\begin{aligned}\sigma_n^{\pm}(\xi) = {} & -E(2n\pi) \mp i\sin[n\pi(1+\xi)]\ln\frac{(1+\xi)^2}{\xi(2+\xi)} \\ & \mp \frac{1}{2}\exp[in\pi(1+\xi)]\Big[E[2n\pi(2+\xi)] - 2E[2n\pi(1+\xi)] + E[2n\pi\xi]\Big].\end{aligned} \tag{52}$$

In this expression the + sign (- sign) corresponds to the case where the current distribution is equal and in the same (opposite)direction on the two wires. The functions $\sigma_n^{\pm}(\xi)$ are graphed in Figs. 9a-9c for $n = 1,2,3$ and different values of ξ. From these curves we note that the loci of the functions $\sigma_n^{\pm}(\xi)$ form spirals with the center at $\sigma_n^{\pm}(\infty) = -E(2n\pi)$. For large values of ξ we have asymptotically

$$\sigma_n^{\pm}(\xi) \sim -E(2n\pi) \mp \frac{1}{2}\xi^{-2}\exp[in\pi(1+\xi)]. \tag{53}$$

This rather fast convergence for large values of b/ℓ of the natural frequencies towards their values for two noninteracting wires can be understood as a weak electromagnetic interaction between the two collinear wires. The loci of the natural frequencies as calculated from the theory presented in this note are roughly the same as those calculated numerically for two, solid, collinear

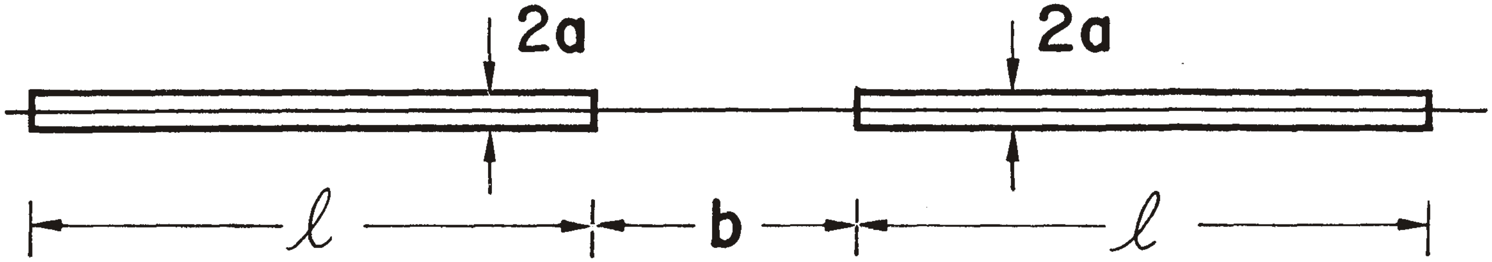

Figure 8. Two collinear wires.

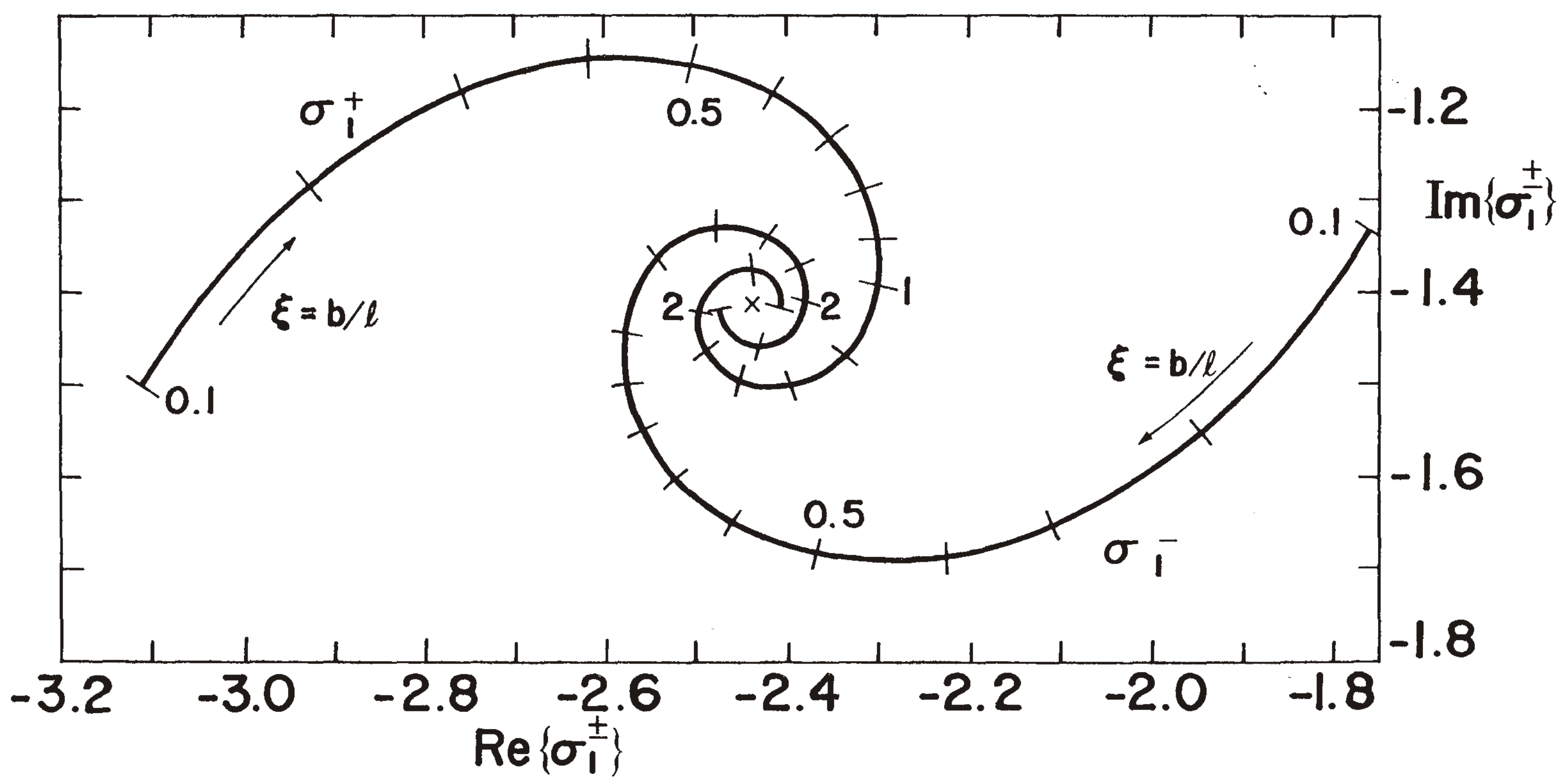

Figure 9a. The functions $\sigma_1^{\pm}(\frac{b}{\ell})$ for $0.1 \le b/\ell \le 3$. The first natural frequency is $s_1^{\pm} = \frac{i\pi c}{\ell} + \frac{c}{\Omega\ell}\,\sigma_1^{\pm}$. The $\times$ denotes $\sigma_1^{\pm}(\infty)$.

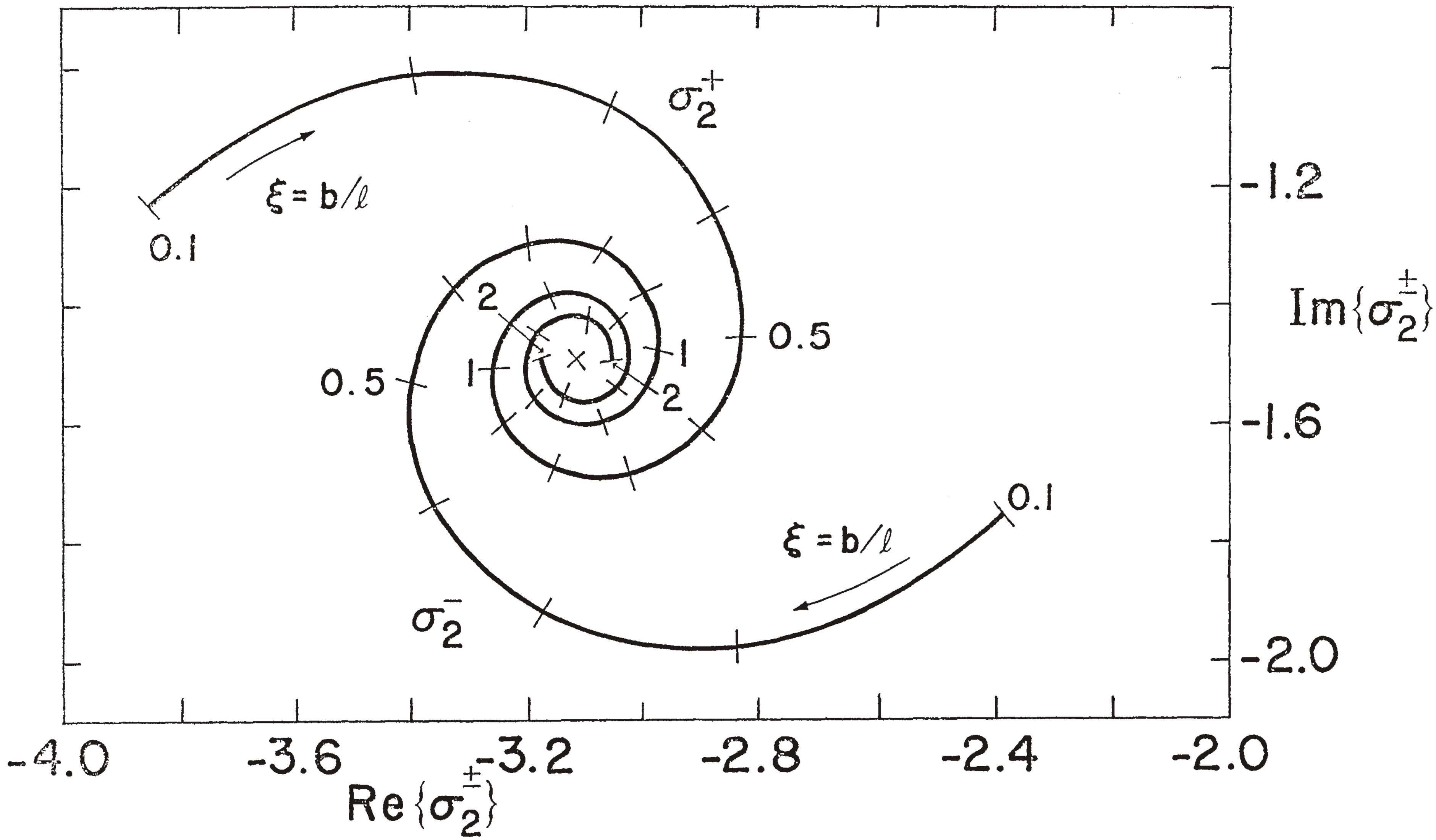

Figure 9b. The functions $\sigma_2^{\pm}(\frac{b}{\ell})$ for $0.1 \le b/\ell \le 3$. The first natural frequency is $s_2^{\pm} = \frac{i2\pi c}{\ell} + \frac{c}{\Omega\ell}\sigma_2^{\pm}$. The $\times$ denotes $\sigma_2^{\pm}(\infty)$.

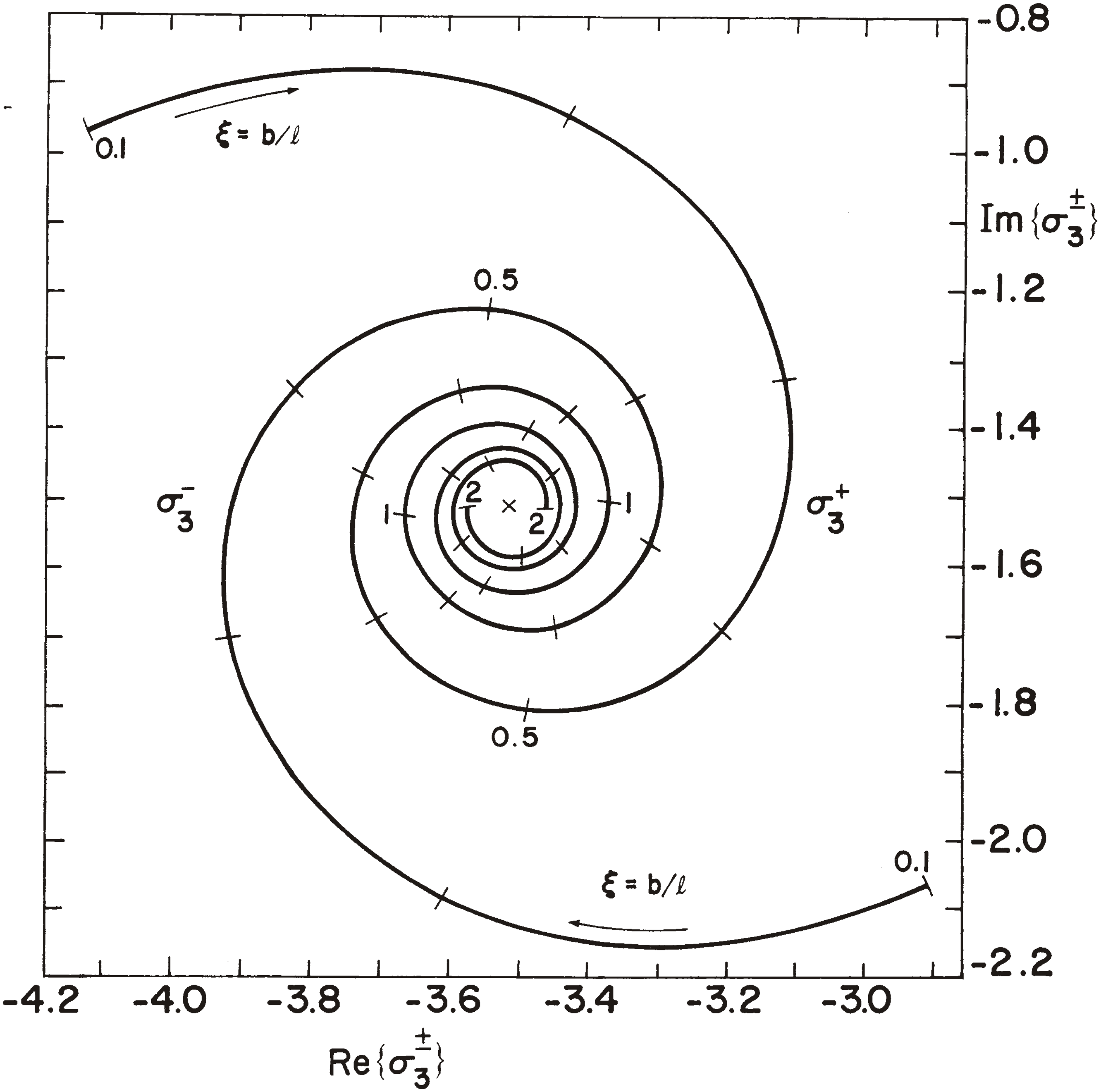

Figure 9c. The functions $\sigma_3^{\pm}(\frac{b}{\ell})$ for $0.1 \le b/\ell \le 2$. The third natural frequency is $s_3^{\pm} = \frac{i3\pi c}{\ell} + \frac{c}{\Omega\ell}\,\sigma_3^{\pm}$. The $\times$ denotes $\sigma_3^{\pm}(\infty)$.

cylinders[25]. The difference between the loci can be attributed to the fact that the approximate theory presented in this note only accounts for modes with a small damping constant.

B. Two Parallel, Nonstaggered Wires

The natural frequencies of two parallel, nonstaggered wires (each wire has length ℓ, radius a and the two wires are separated by a distance d, see Fig. 10) are found to be

$$s_n^{\pm} = \frac{in\pi c}{\ell} + \frac{1}{\Omega}\frac{c}{\ell}\tau_n^{\pm}(\frac{d}{\ell}) + O(\Omega^{-2}), \qquad \Omega = 2\ln(\frac{\ell}{a}) \tag{54}$$

where

$$\tau_n^{\pm}(\xi) = -E(2n\pi) \pm \left[2E[n\pi\xi] - E[n\pi(\sqrt{1+\xi^2}-1)] - E[n\pi(\sqrt{1+\xi^2}+1)]\right] \tag{55}$$

and again the $+$ sign ($-$ sign) corresponds to the case where the current distribution is equal and in the same (opposite) direction on the two wires. The functions $\tau_n^{\pm}(\xi)$ are graphed in Figs. 11a-11c for $n = 1,2,3$ and different values of ξ. For large values of ξ we have asymptotically

$$\tau_n^{\pm}(\xi) \sim -E(2n\pi) \pm \frac{2i}{n\pi\xi}\left[(-1)^n \exp(-in\pi\sqrt{1+\xi^2}) - \exp(-in\pi\xi)\right]. \tag{56}$$

From Figs. 11a-11c and (53), (56) we note that the electromagnetic interaction between the two parallel nonstaggered wires is much stronger than that of two collinear wires, as expected.

For small values of ξ we note that $\tau_n^{+}(\xi) \sim -2E(2n\pi) + 2in\pi\xi$ showing that in this case the two parallel wires act like one wire whose $\Omega = \ln(\ell/a)$. For $\ell >> d >> a$ and in the approximation we use, this result agrees with that of Schelkunoff, $\Omega_1 = \ln(\ell/a) + \ln(\ell/d)$. On the other hand, $\tau_n^{-}(\xi) \sim -2in\pi\xi$ for small values of ξ so that $s_n^{-} \sim in\pi c/\ell$ and in this case the two wires act like an open ended transmission line and the natural modes are the standing waves on this transmission line. We also note that for small values of d/ℓ the natural frequency s_n^{-} can be written as

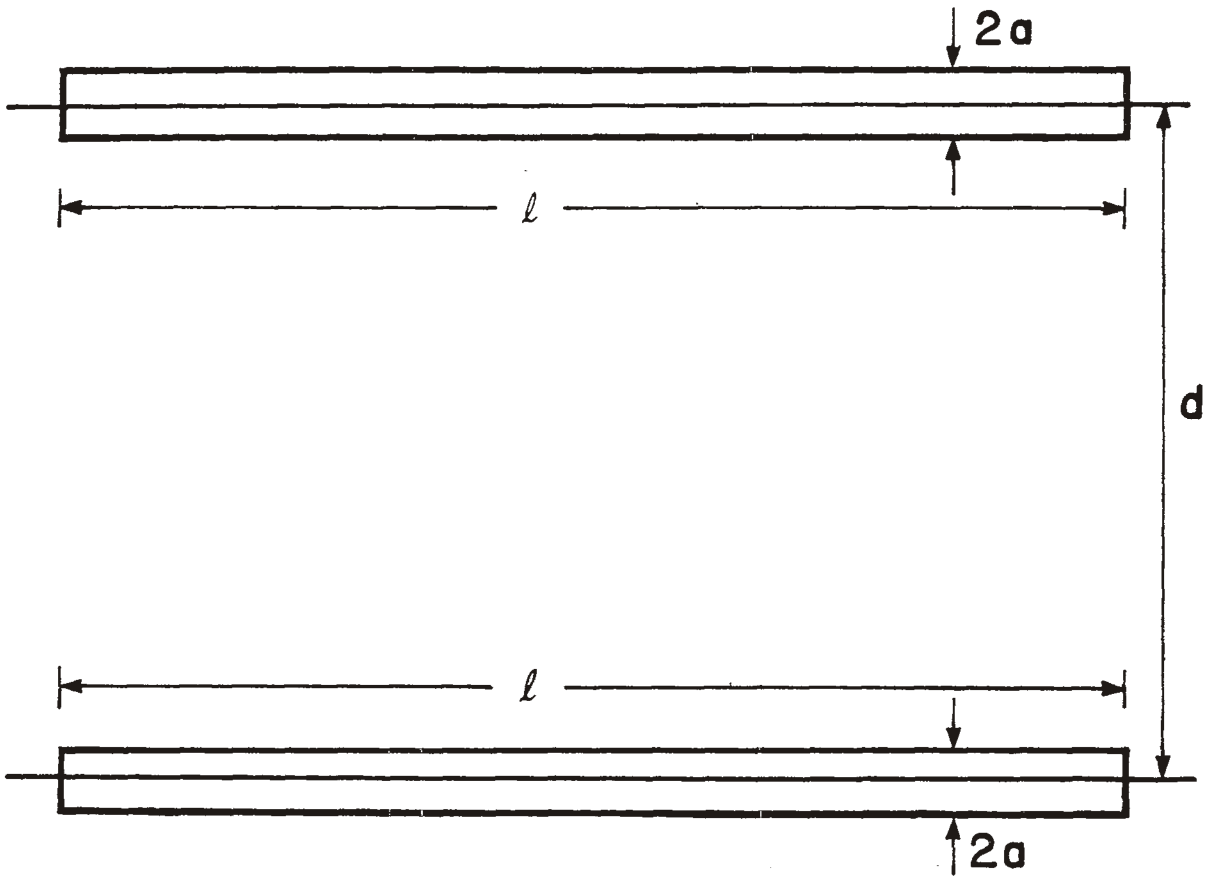

Figure 10. Two parallel, nonstaggered wires.

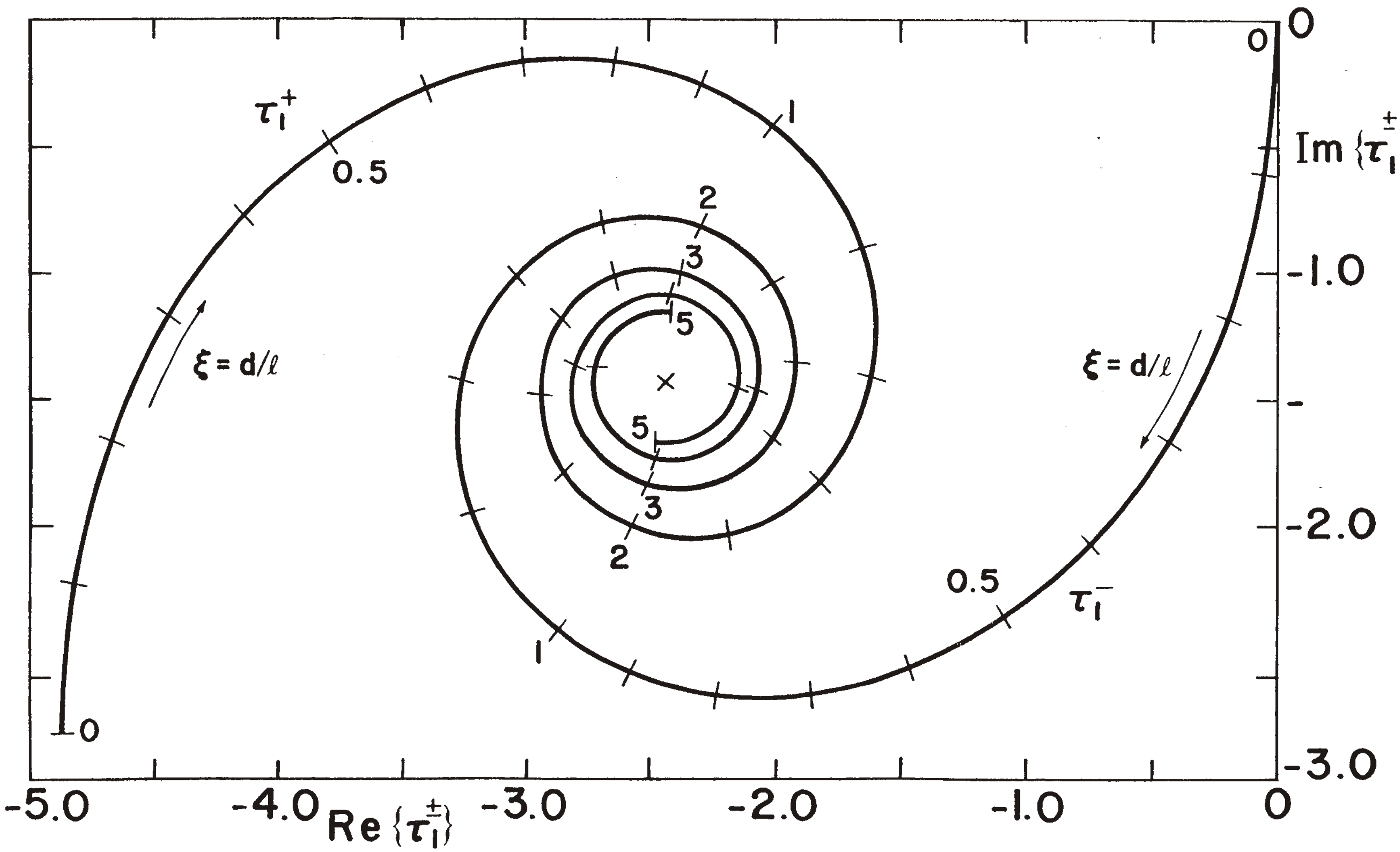

Figure 11a. The functions $\tau_1^{\pm}(\frac{d}{\ell})$ for $0 \leq d/\ell \leq 5$. The first natural frequency is $s_1^{\pm} = \frac{i\pi c}{\ell} + \frac{c}{\Omega\ell}\,\tau_1^{\pm}(\frac{d}{\ell})$. The $\times$ denotes $\tau_1^{\pm}(\infty)$.

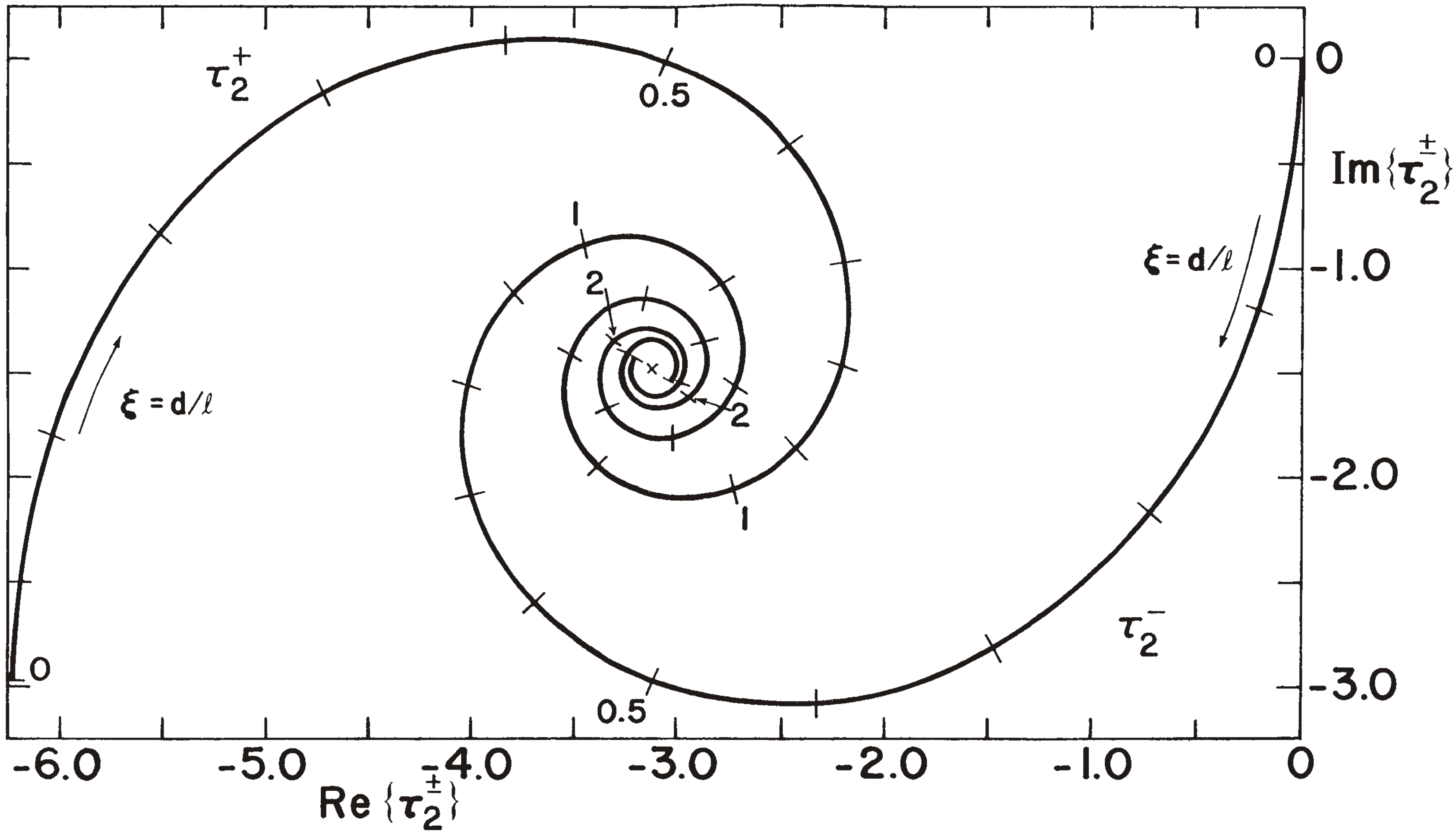

Figure 11b. The functions $\tau_2^{\pm}(\frac{d}{\ell})$ for $0 \le d/\ell \le 3$. The second natural frequency is $s_2^{\pm} = \frac{i2\pi c}{\ell} + \frac{c}{\Omega\ell}\,\tau_2^{\pm}(\frac{d}{\ell})$. The $\times$ denotes $\tau_2^{\pm}(\infty)$.

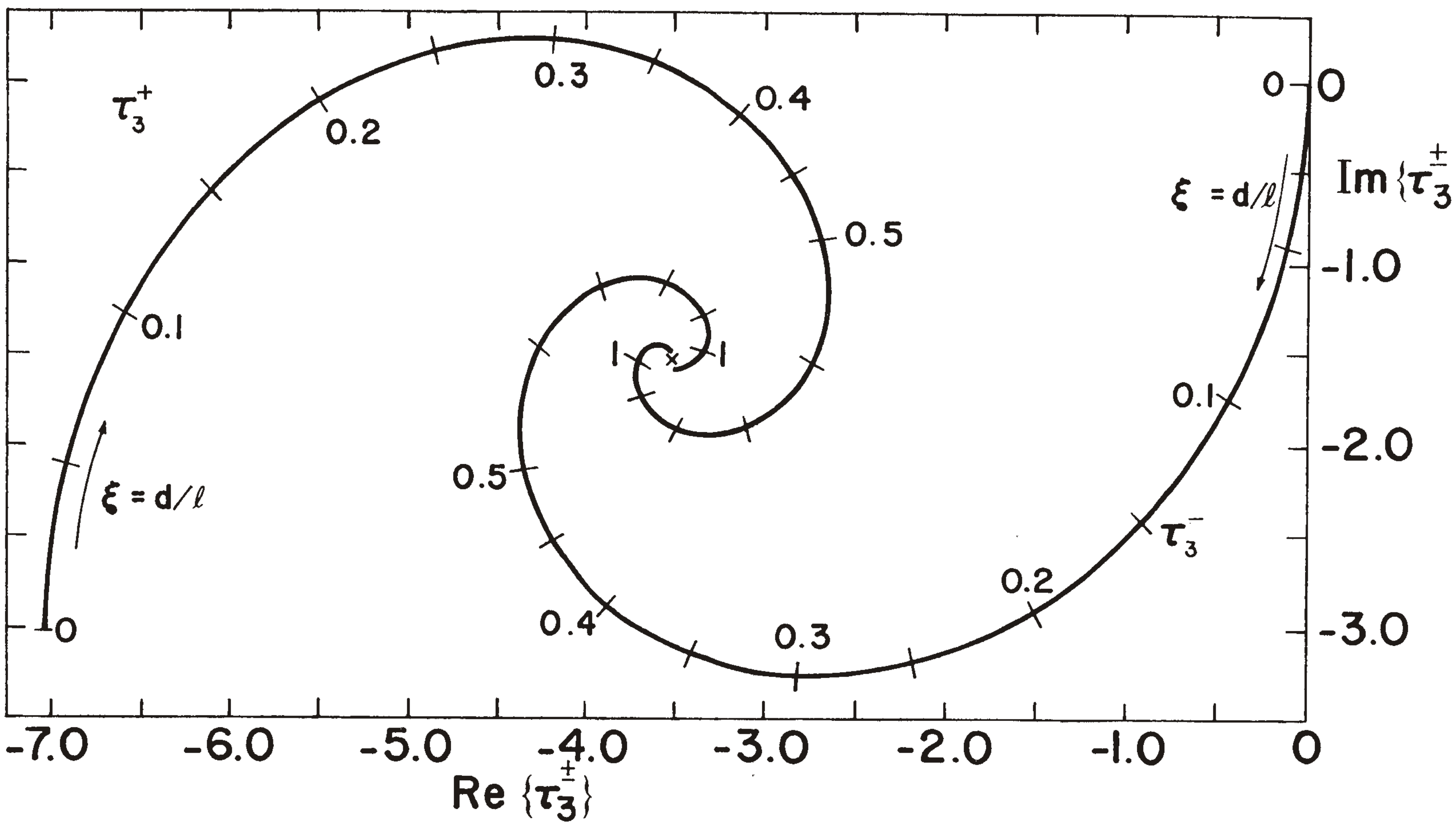

Figure 11c. The functions $\tau_3^{\pm}(\frac{d}{\ell})$ for $0 \le d/\ell \le 1.2$. The third natural frequency is $s_3^{\pm} = \frac{i3\pi c}{\ell} + \frac{c}{\Omega\ell}\,\tau_3^{\pm}(\frac{d}{\ell})$. The $\times$ denotes $\tau_3^{\pm}(\infty)$.

$$s_n^- = in\pi c/(\ell + \ell_c), \qquad \ell_c = d/\ln(\ell/a) + O(\Omega^{-2}) \tag{57}$$

where $\ell_c/2$ is the so-called end correction for an open ended transmission-line, i.e., the extra length that accounts for the end capacitance between the two wires[36]. The expression (57) agrees within 20% with the results obtained by Wu for $a/\ell = 0.05$.

The results obtained in [27] by numerical means for the natural frequencies of a wire above a ground plane should be compared with the results obtained here for s_n^-. Both the results in [27] and those of this note show a spiraling behavior. However, due to the approximations introduced in this note we can only account for modes with small damping constants and this fact is attributed to the discrepancy between the results presented in Figs. 11a-11c and those presented in [27].

C. The Ring

Let us next consider the natural frequencies of a thin wire bent into a ring. The radius of the ring is b and the wire radius is a (see Fig. 12). To find the natural modes of a thin ring we first expand the current in a Fourier series

$$I(\phi) = \sum_{m=-\infty}^{\infty} I_m \exp(im\phi). \tag{58}$$

The natural frequencies s_{nm} are given by $s_{nm} = c\zeta_{nm}/(2b)$ where the ζ_{nm} are the solutions to the transcendental equations[14]

$$8m^2 L_{2m}(\zeta) - \zeta^2[L_{2m+2}(\zeta) + L_{2m-2}(\zeta)] = 0, \qquad m = 1,2,3,\ldots \tag{59}$$

where

$$L_m(\zeta) = \int_0^{\pi/2} (\delta^2 + \sin^2\phi)^{-\frac{1}{2}} \exp[-\zeta(\delta^2 + \sin^2\phi)^{\frac{1}{2}}] \cos m\phi \, d\phi \tag{60}$$

and $\delta = a/(2b)$. The lowest root of each one of the equations (59) is given by[11,14]

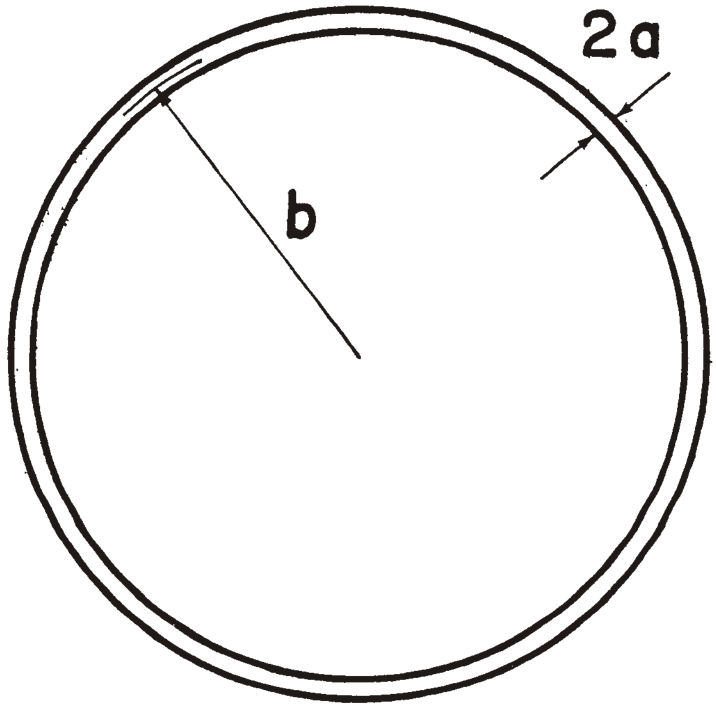

Figure 12. A thin wire bent into a ring.

$$\zeta_{1m} = 2m\left[i + \frac{\pi}{2\Omega}\left[F_{2m-1}(2m) - F_{2m+1}(2m)\right] + O(\Omega^{-2})\right], \qquad (61)$$

where

$$\Omega = 2\ \ln(8b/a), \qquad (62)$$

$$F_m(\zeta) = \pi^{-1}\int_0^{\pi} \exp[i(\zeta \sin\phi - m\phi)]\,d\phi, \qquad (63)$$

and $F_m(\zeta) = J_m(\zeta) - iE_m(\zeta)$ where $J_m(\zeta)$ is the Bessel function of the first kind and order m and $E_m(\zeta)$ is the Weber function[30]. Observing that the functions $F_m(\zeta)$ satisfy the following recursion relationship

$$F_{m-1}(\zeta) - F_{m+1}(\zeta) = 2F'_m(\zeta) \qquad (64)$$

and using the method of stationary phase we get the following asymptotic form of ζ_{1m} for large values of m, (c.f. [11]),

$$\zeta_{1m} \sim 2im - \frac{1}{\Omega}\left(\frac{m}{3}\right)^{1/3}\Gamma\left(\frac{2}{3}\right)(\sqrt{3} + i). \qquad (65)$$

Another limiting form for the natural frequencies can be obtained by observing that (59) has the following asymptotic solution for large $|\zeta_{nm}|$, (c.f. [14])

$$\zeta_{nm} \sim \frac{1}{2}\ln\delta + \pi i\left(\frac{1}{4} + m \pm 2n\right). \qquad (66)$$

This last formula is of limited value, since it requires that the natural frequencies are very large (so that their wavelength is of the order of the wire radius). Nevertheless, it is interesting since it shows that for each m all natural frequencies have a finite damping constant (although large).

Before concluding this section we briefly compare the first-branch natural frequencies with the fundamental resonance for each m in the ring case. In the case of the wire we have for large n,

$$\lambda_n^w \sim \frac{2\ell}{n}, \qquad \delta_n^w \sim \frac{\ln(2n\pi\gamma)}{2\pi \ln(\ell/a)} \sim 0.16 \frac{\ln n}{\ln(\ell/a)} \tag{67}$$

and in the case of the ring

$$\lambda_{1m}^r \sim \frac{\ell}{m}, \qquad \delta_{1m}^r \sim \frac{\pi\sqrt{3}}{4} \Gamma(\tfrac{2}{3}) \frac{m^{1/3}}{\ln(4\ell/\pi a)} \sim 1.84 \frac{m^{1/3}}{\ln(\ell/a)} \tag{68}$$

where λ denotes the wavelength of the oscillations and δ the logarithmic decrement.

VIII. Concluding Remarks

The general method of obtaining the time history of the currents and charges on a perfectly conducting body based the magnetic-field formulation[9] has been extended in this note to incorporate the electric-field formulation. Asymptotic expressions for the natural modes have been derived for certain thin-wire structures. The analyses are based on an integrated form of the electric-field formulation for thin wires (the so-called Hallén integral equation.) Based on the results obtained in this note the time domain response to an incident transient wave can easily be obtained. The natural-mode representation of the induced current is most useful for late and intermediate times because in this case only a moderate number of natural modes are sufficient to accurately describe the induced current. For earlier times other approaches, such as the time-domain integral equations, are more useful and it would be very valuable to get some approximate analytical solutions of those equations for early times, i.e., times of up to the order of two to three times the transit time along the structure. For a simple straight wire, the early-time response can alternatively be obtained by using a traveling wave method[29,31,37].

There are several applications of the method presented in this note. One case is the EMP response of a long trailing wire antenna of the AABNCP. To get the currents and voltages induced on the terminals of this antenna one can model the effects of the aircraft itself with a set of intersecting wires. Also, in certain space systems one has two or more bodies connected with long rods. In calculating the currents induced on those structures one can model the bodies attached to the rods with some lumped network-parameters (impedances and generators) and then use the thin-wire scheme described in this note. The incident field can either be an incident, pulsed plane wave (EMP) or moving charged particles (SGEMP).

One other structure that can be treated with the method described in this note is the thin-wire model of an aircraft. In this case we find it most profitable to use the integrated form (11) of the electric-field equation for each straight-wire section of the aircraft model. The unknown constants thus introduced can be determined by requiring that the current vanishes at the free ends of each wire and that the scalar potential is continuous at the junction between the wires and that no net current flows into the junction.

When all wires have the same radius, the first one of these "junction conditions" can be replaced by the condition that the derivative of the current is continuous at the junction.

In determining the late-time behavior of the current on the RES simulator it is helpful to have the natural frequencies of a resistively loaded antenna. The methods described in this note can also be used with ease to find the natural modes of an impedance loaded thin wire. For such a structure one can derive the following approximate equation[2] (c.f. (9) and (10))

$$(\mathcal{L}I)(z) - s\varepsilon_o Z(z,s) I(z) = -s\varepsilon_o E_o(z) \tag{69}$$

where $Z(z,s)$ is the impedance per unit length of the wire, and the operator $\mathcal{L}$ is defined by (10). Equation (69) can be integrated to yield a form very similar to (11), and this equation can be solved with perturbation techniques. The first approximation is equivalent to an impedance-loaded-transmission-line model of the antenna[38,39]. In the next approximation we take into account the damping of the modes due to the radiation losses as in the perfectly-conducting-wire case.

Appendix

In this appendix we will show a feature of the scalar product $(\underline{e}_n^{inc}, \underline{j}_{nm})$ that enables us to distinguish exterior modes from interior ones.

Let $\underline{j}_{nm}$ denote the surface current density on S of an interior mode. From this current density we can calculate the electromagnetic field $\underline{E}_{nm}, \underline{H}_{nm}$ of this mode in V, the region inside S. Furthermore, let $\underline{E}_n^{inc}$ and $\underline{H}_n^{inc}$ denote the incident electromagnetic field evaluated at $s = s_n$. Some vector algebraic manipulations combined with the Maxwell equations give

$$\begin{aligned}
(\underline{e}_n^{inc}, \underline{j}_{nm}) &= \int_S \underline{E}_n^{inc} \cdot (\underline{n} \times \underline{H}_{nm}) dS \\
&= \int_S \underline{n} \cdot (\underline{H}_{nm} \times \underline{E}_n^{inc}) dS \\
&= \int_V \nabla \cdot (\underline{H}_{nm} \times \underline{E}_n^{inc}) dV \\
&= \int_V (\underline{E}_n^{inc} \cdot \nabla \times \underline{H}_{nm} - \underline{H}_{nm} \cdot \nabla \times \underline{E}_n^{inc}) dV \\
&= \int_V (\varepsilon_o s_n \underline{E}_n^{inc} \cdot \underline{E}_{nm} + \mu_o s_n \underline{H}_n^{inc} \cdot \underline{H}_{nm}) dV \\
&= \int_V (\underline{E}_{nm} \cdot \nabla \times \underline{H}_n^{inc} - \underline{H}_n^{inc} \cdot \nabla \times \underline{E}_{nm} - \underline{i}_n^{inc} \cdot \underline{E}_n) dV \\
&= \int_V \nabla \cdot (\underline{H}_n^{inc} \times \underline{E}_{nm}) dV - \int_V \underline{i}_n^{inc} \cdot \underline{E}_{nm} dV \\
&= \int_S \underline{n} \cdot (\underline{H}_n^{inc} \times \underline{E}_{nm}) dS - \int_V \underline{i}_n^{inc} \cdot \underline{E}_{nm} dV \\
&= -\int_S \underline{H}_n^{inc} \cdot (\underline{n} \times \underline{e}_{nm}) dS - \int_V \underline{i}_n^{inc} \cdot \underline{E}_{nm} dV. \qquad \text{(A1)}
\end{aligned}$$

Since S is a perfectly conducting surface we have $\underline{n} \times \underline{e}_{nm} = 0$, and

$$(\underline{e}_n^{inc}, \underline{j}_{nm}) = -\int_V \underline{i}_n^{inc} \cdot \underline{E}_{nm} dV \qquad \text{(A2)}$$

where $\underline{i}^{inc}$ denotes the sources of the incident field. If the incident field has its sources outside S, as is the case of a plane wave, we get

$$(\underline{e}_n^{inc}, \underline{j}_{nm}) = 0. \tag{A3}$$

Equation (A3) implies that all interior modes are orthogonal to any incident field provided that all sources of the incident field are outside the perfectly conducting, scattering body.

Acknowledgement

It is fun to talk to Dr. K.S.H. Lee about the topic of this note; much light was shed on obscure points in discussions with him and in trying to answer the sometimes provocative questions raised by him. I also wish to thank Prof. E. Hallén who taught me Electromagnetics and whose work on linear antennas motivated this note. Thanks also go to Dr. C.E. Baum for his unwaning interest in EM theory and SEM.

References

1. P.O. Brundell, "Transient Electromagnetic Waves Around a Cylindrical Transmitting Antenna," Ericsson Technics, Vol. 16, No. 1, pp. 137-162, 1960.
2. T.T. Wu, "Transient Response of a Dipole Antenna," J. Math. Phys., Vol. 2, No. 6, pp. 892-894, 1961.
3. R.W. Latham and K.S.H. Lee, "Waveforms Near a Cylindrical Antenna," Sensor and Simulation Note 89, June 1969.
4. P.R. Barnes, "The Axial Current Induced on an Infinitely Long, Perfectly Conducting, Circular Cylinder in Free Space by a Transient Electromagnetic Plane Wave," Interaction Note 64, March 1971.
5. T.K. Liu, "Direct Time Domain Analysis of Linear EMP Radiators," Sensor and Simulation Note 154, July 1972.
6. A.J. Poggio, "The Space-Time Domain Magnetic Vector Potential Integral Equation," IEEE Trans. Antennas Propagat., Vol. AP-19, No. 5, pp. 702-704, 1971.
7. R.W. Sassman, "The Current Induced on a Finite, Perfectly Conducting, Solid Cylinder in Free Space by an Electromagnetic Pulse," Interaction Note 11, July 1967.
8. C.E. Baum, "On the Singularity Expansion Method for the Solution of Electromagnetic Interaction Problems," Interaction Note 88, December 1971.
9. R.W. Latham and L. Marin, "Analytical Properties of the Field Scattered by a Perfectly Conducting, Finite Body," Interaction Note 92, January 1972.
10. H.C. Pocklington, "Electrical Oscillations in Wires," Proc. Cambridge Phil. Soc., Vol. 9, pp. 324-332, 1897.
11. Lord Rayleigh, "Electrical Vibrations on a Thin Anchor-Ring," Proc. Roy. Soc., A, Vol.87, pp.193-202, 1912.
12. C.W. Oseen, "Über die elektromagnetischen Schwingungen an einem dünnen Ankerring," Physik. Zeitschr., Vol. 14, pp. 1222-1226, 1913.
13. C.W. Oseen, "Über die elektromagnetischen Schwingungen an dünnen Ringen," Arkiv Mat. Astr. Fysik, Vol. 9, No. 12, pp. 1-30, 1914.
14. C.W. Oseen, "Über das elektromagnetische Spektrum eines dünnen Ringes," Arkiv Mat. Astr. Fysik, Vol. 9, No. 28, pp. 1-34, 1914.

15. C.E. Baum, "On the Singularity Expansion Method for the Case of First Order Poles," Interaction Note 129, October 1972.

16. M. Abraham, "Die electrischen Schwingungen um einen stabförmigen Leiter, behandelt nach der Maxwell'schen Theorie," Ann. Phys. (Leipzig), Vol. 66, pp. 435-472, 1898.

17. L. Page and N. Adams, "The Electrical Oscillations of a Prolate Spheroid, Paper I," Phys. Rev., Vol. 53, pp. 819-831, 1938.

18. L. Page, "The Electrical Oscillations of a Prolate Spheroid, Paper II," Phys. Rev., Vol. 65, pp. 98-110, 1944.

19. L. Page, "The Electrical Oscillations of a Prolate Spheroid, Paper III," Phys. Rev., Vol. 65, pp. 111-116, 1944.

20. C.W. Oseen, Über die elektrischen Schwingungen an dünnen Stäben," Arkiv Mat. Astr. Fysik, Vol. 9, No. 30, pp. 1-27, 1914.

21. E. Hallén, "Über die elektrischen Schwingungen in drahtförmigen Leitern," Uppsala Univ. Årsskrift, No. 1, pp. 1-102, 1930.

22. E. Hallén, "Theoretical Investigations into the Transmitting and Receiving Qualities of Antennae," Nova Acta Reg. Soc. Sci. Upsaliensis, Vol. 11, No. 4, pp. 1-44, 1938.

23. F.M. Tesche, "On the Singularity Expansion Method as Applied to Electromagnetic Scattering from Thin-Wires," Interaction Note 102, April 1972.

24. L. Marin, "Natural-Mode Representation of Transient Scattering From Rotationally Symmetric, Perfectly Conducting Bodies and Numerical Results for a Prolate Spheroid," Interaction Note 119, September 1972.

25. L. Marin, "Natural Modes of Two Collinear Cylinders," Sensor and Simulation Note 176, May 1973.

26. F.M. Tesche, "Application of the Singularity Expansion Method to the Analysis of Impedance Loaded Antennas," Sensor and Simulation Note 177, May 1973.

27. T.H. Shumpert, "EMP Interaction With a Thin Cylinder Above a Ground Plane Using the Singularity Expansion Method," Sensor and Simulation Note 182, June 1973.

28. D.R. Wilton, K.R. Umashankar, "Parametric Study of an L-Shaped Wire Using the Singularity Expansion Method," Interaction Note 152, November 1973.

29. E. Hallén, Electricitetslära, Stockholm, 1953.

30. M. Abramowitz and I.N. Stegun, Editors, "Handbook of Mathematical Functions," National Bureau of Standards, AMS-55, 1964.

31. E. Hallén, Exact Treatment of Antenna Wave Reflection at the End of a Tube-Shaped Cylindrical Antenna," IRE Trans. Antennas Propagat., Vol. AP-4, No. 4, pp. 479-491, 1956.

32. S.W. Lee and B. Leung, "The Natural Resonance Frequency of a Thin Cylinder and its Application to EMP Studies," Interaction Note 96, February 1972.

33. F.M. Tesche, "Numerical Considerations for the Calculation of Currents Induced on Intersecting Wires Using the Pocklington Integro-Differential Equation," Interaction Note 150, January 1974.

34. R.W.P. King, Transmission-line Theory, Dover, New York, 1965.

35. K.K. Mei, "On the Integral Equations of Thin Wire Antennas," IEEE Trans. Antennas Propagat., Vol. AP-13, No. 3, pp. 374-378, 1965.

36. T.T. Wu, "Theory of the Dipole Antenna and Two-Wire Transmission Line," J. Math. Phys., Vol. 2, pp. 550-574, 1961.

37. O. Einarsson, "The Current Distribution on Cylindrical Antennas of Arbitrary Length," Trans. Roy. Inst. Techn., No. 216, Stockholm, Sweden, 1963.

38. C.E. Baum, "Resistively Loaded Radiating Dipole Based on a Transmission-Line Model for the Antenna," Sensor and Simulation Note 81, April 1969.

39. T.T. Wu and R.W.P. King, "The Cylindrical Antenna With Nonreflecting Resistive Loading," IEEE Trans. Antennas Propagat., Vol. AP-13, No. 3, pp. 369-373, 1965.