Interaction Notes

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Generalization of Babinet's Principle in Terms of the Combined Field to Include Impedance Loaded Aperture Antennas and Scatterers

by

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ABSTRACT

The problem of electromagnetic field coupling through an aperture loaded by a sheet admittance is examined in this report. The combined field, current, charge, vector and scalar potentials are defined. Symmetry decomposition of the field with respect to a plane is discussed. Expressions for the complementary field, current, etc., are derived using the combined field, current, etc., and an expression for the surface admittance of the complementary scatterer is derived. Complementary antennas are defined and the current distributions and the surface admittances of the original and the complementary structures are discussed. Self-complementary antennas are defined for the admittance loaded structures.

FOREWORD

"I think I'll go and meet her," said Alice, for though the flowers were interesting enough she felt that it would be far grander to have a talk with a real Queen.

"You can't possibly do that," said the Rose. "I should advise you to walk the other way."

This sounded nonsense to Alice, so she said nothing, but set off at once toward the Red Queen. To her surprise, she lost sight of her in a moment, and found herself walking in at the front door again.

A little provoked, she drew back, and after looking everywhere for the Queen (whom she spied out at last a long way off), she thought she would try the plan, this time, of walking in the opposite direction.

It succeeded beautifully. She had not been walking a minute before she found herself face to face with the Red Queen, and in full sight of the hill she had been so long aiming at.

Lewis Carroll,
Through the Looking Glass

CONTENTS

CHAPTER 1

INTRODUCTION

Since Booker's now classic work on Babinet's principle[1] for the electromagnetic fields, there appears to have been little work done to extend this principle. Neugebauer[2] has expounded on the problem of diffraction by an aperture in an absorbing screen. The present report extends Babinet's principle to include impedance loaded planar structures. We formulate the problem using the combined field.

Chapter 2 of this report deals with the combined field, potentials, etc. The combined field, current, charge and potentials are defined and the Helmholtz equations for the combined quantities are exhibited. For more detailed information regarding the combined field, the reader is referred to the work by C. E. Baum and B. K. Singaraju.[3, 4, 5]

In Chapter 3, symmetry decomposition of the electric, magnetic, combined fields, etc., is discussed. Symmetry and antisymmetry are defined with respect to a plane. The relationships between the field quantity and the image quantity are derived.

Chapter 4 deals with the boundary conditions. Boundary conditions at a perfectly conducting plane and a sheet impedance plane are derived in terms of the electric, magnetic, and combined fields. Symmetry decomposition of surface currents is discussed and it is shown that the scattered field by plane scatterers is symmetric.

Chapter 5 can be considered as an extension of Chapter 4 to include apertures. In this chapter, aperture fields are defined and boundary conditions for the aperture fields are derived.

Chapter 6 deals with the complementary fields and the generalized Babinet's principle. Complementary fields, admittance, etc., are defined and their transform relationships are derived. Relationships between the aperture fields and the scattered fields of the complementary problem are derived and a relationship for the complementary admittance is obtained. The generalized Babinet's principle is also enunciated.

Chapter 7 deals with the integral equations for sheet impedance loaded scatterers. An integral equation is derived for a general sheet impedance loaded plane scatterer and is specialized to calculate the aperture field.

In Chapter 8, complementary and self-complementary antennas are discussed. Some current and admittance relationships between complementary antennas are derived and self-complementary antennas are defined.

CHAPTER 2

THE COMBINED FIELD AND POTENTIALS

The combined field and potentials play an important role in Babinet's principle. As a consequence the combined field, current, charge, etc., will be reviewed in this section. The fields and potentials considered here are Laplace transformed with respect to the time variable t. The Laplace transformed components are denoted by the symbol tilde ~ above the quantity and the complex frequency is denoted by s.

Maxwell's equations when electric, magnetic charges and currents are present are given by

$$\nabla \times \tilde{\vec{E}} = -s\vec{\mu} \cdot \tilde{\vec{H}} - \tilde{\vec{J}}_m \tag{2.1a}$$

$$\nabla \times \tilde{\vec{H}} = s\vec{\epsilon} \cdot \tilde{\vec{E}} + \tilde{\vec{J}} \tag{2.1b}$$

$$\nabla \cdot \tilde{\vec{D}} = \tilde{\rho} \tag{2.1c}$$

$$\nabla \cdot \tilde{\vec{B}} = \tilde{\rho}_m \tag{2.1d}$$

and the continuity equations are given by

$$\nabla \cdot \tilde{\vec{J}} = -s\tilde{\rho} \tag{2.1e}$$

$$\nabla \cdot \tilde{\vec{J}}_m = -s\tilde{\rho}_m \tag{2.1f}$$

Although in general magnetic currents and charges do not physically exist, they form a useful tool in analyzing aperture problems. For our purposes we let $\vec{\epsilon}$ be equal to ϵ_o and $\vec{\mu}$ equal to μ_o and define the relationships

$$\gamma \equiv \frac{s}{c} \quad \text{(propagation constant)} \tag{2.2a}$$

$$c \equiv \frac{1}{\sqrt{\mu_o \epsilon_o}} \quad \text{(propagation speed in free space)} \tag{2.2b}$$

$$Z_o \equiv \sqrt{\frac{\mu_o}{\epsilon_o}} \quad \text{(wave impedance in free space)} \tag{2.2c}$$

The Helmholtz equations for the electric and magnetic fields can be obtained to be

$$(\nabla^2 - \gamma^2)\tilde{\vec{E}} = s\mu_o \tilde{\vec{J}} + \frac{\nabla\tilde{\rho}}{\epsilon_o} + \nabla \times \tilde{\vec{J}}_m \tag{2.3a}$$

$$(\nabla^2 - \gamma^2)\tilde{\vec{H}} = -\nabla \times \tilde{\vec{J}} + s\epsilon_o \tilde{\vec{J}}_m + \frac{1}{\mu_o}\nabla\tilde{\rho}_m \tag{2.3b}$$

and the radiation condition is given by

$$\lim_{r\to\infty} r\left[\nabla \times \begin{pmatrix} \tilde{\vec{E}}(\vec{r}, s) \\ \tilde{\vec{H}}(\vec{r}, s) \end{pmatrix} + \gamma\vec{e}_r \times \begin{pmatrix} \tilde{\vec{E}}(\vec{r}, s) \\ \tilde{\vec{H}}(\vec{r}, s) \end{pmatrix}\right] = \vec{0} \qquad \mathrm{Re}(s) \geq 0 \tag{2.4}$$

In terms of the scalar and vector potentials, the electric and magnetic fields can be expressed as

$$\tilde{\vec{E}} = -\nabla\tilde{\phi} - s\tilde{\vec{A}} - \frac{1}{\epsilon_o}\nabla \times \tilde{\vec{A}}_m \tag{2.5a}$$

$$\tilde{\vec{H}} = -\nabla\tilde{\phi}_m + \frac{1}{\mu_o}\nabla \times \tilde{\vec{A}} - s\tilde{\vec{A}}_m \tag{2.5b}$$

The combined field vector $\tilde{\vec{F}}_q$ is now introduced; it is defined by

$$\tilde{\vec{F}}_q \equiv \tilde{\vec{E}} + qiZ_o\tilde{\vec{H}} \qquad q = \pm 1 \tag{2.6}$$

Similarly the combined current density is defined as

$$\tilde{\vec{K}}_q \equiv \tilde{\vec{J}} + \frac{qi}{Z_o}\tilde{\vec{J}}_m \qquad q = \pm 1 \tag{2.7}$$

and the combined charge density is defined by

$$\tilde{Q}_q \equiv \tilde{\rho} + \frac{qi}{Z_o}\tilde{\rho}_m \qquad q = \pm 1 \tag{2.8}$$

The ambiguity sign ± 1 associated with the separation index q is used to reconstruct the electric and magnetic quantities from the combined quantities.

Maxwell's equations can now be written in terms of the combined field and current density as

$$[\nabla\times - qi\gamma]\tilde{\vec{F}}_q = qiZ_o\tilde{\vec{K}}_q \tag{2.9}$$

The divergence equations (2.1c) and (2.1d) can be written as

$$\nabla \cdot \tilde{\vec{F}}_q = \frac{1}{\epsilon_o}\tilde{Q}_q \tag{2.10}$$

and the continuity equations reduce to

$$\nabla \cdot \tilde{\vec{K}}_q = -s\tilde{Q}_q \tag{2.11}$$

Combining (2.3a) and (2.3b) the Helmholtz equation for the combined field can be written as

$$(\nabla^2 - \gamma^2)\vec{\tilde{F}}_q = (s\mu_o - qiZ_o)\vec{\tilde{K}}_q + \frac{1}{\epsilon_o}\nabla\tilde{Q}_q \tag{2.12}$$

and the radiation condition is given by

$$\lim_{r\to\infty} r[\nabla\times + \gamma\vec{e}_r\times]\vec{\tilde{F}}_q = \vec{0} \qquad \mathrm{Re}[s] \geq 0 \tag{2.13}$$

In an analogous way we can also define the combined vector and scalar potentials. They are defined by

$$\vec{\tilde{C}}_q \equiv \vec{\tilde{A}} + qiZ_o\vec{\tilde{A}}_m \quad \text{(combined vector potential)} \tag{2.14}$$

$$\tilde{\phi}_q \equiv \tilde{\phi} + qiZ_o\tilde{\phi}_m \quad \text{(combined scalar potential)} \tag{2.15}$$

Then the combined field vector is given in terms of the combined potentials by

$$\vec{F}_q = -\nabla\tilde{\phi}_q + [-s + qic\nabla\times]\vec{\tilde{C}}_q \tag{2.16}$$

and the Lorentz gauge is given by

$$\nabla \cdot \vec{\tilde{C}}_q + \frac{s}{c^2}\tilde{\phi}_q = 0 \tag{2.17}$$

It is easy to show that the combined potentials satisfy

$$[\nabla^2 - \gamma^2]\vec{\tilde{C}}_q = -\mu_o\vec{\tilde{K}}_q \tag{2.18}$$

$$[\nabla^2 - \gamma^2]\tilde{\phi}_q = -\frac{1}{\epsilon_o}\tilde{Q}_q \tag{2.19}$$

and the radiation conditions for the combined potentials are given by

$$\lim_{r\to\infty} r\left[\frac{\partial}{\partial r} + \gamma\right]\tilde{\vec{C}}_q = \vec{0} \qquad \mathrm{Re}(s) > 0 \tag{2.20}$$

$$\lim_{r\to\infty} r\left[\frac{\partial}{\partial r} + \gamma\right]\tilde{\phi}_q = 0 \qquad \mathrm{Re}(s) \geq 0 \tag{2.21}$$

CHAPTER 3

SYMMETRY DECOMPOSITION OF FIELDS, CURRENTS, CHARGES AND POTENTIALS WITH RESPECT TO A PLANE

Consider a symmetry plane taken as the $z = 0$ plane as shown in Fig. 1. Here the $z = 0$ plane is taken as the symmetry plane for convenience only. It is clear that any arbitrary symmetry plane can be transformed to that shown in Fig. 1 by a simple coordinate transformation.

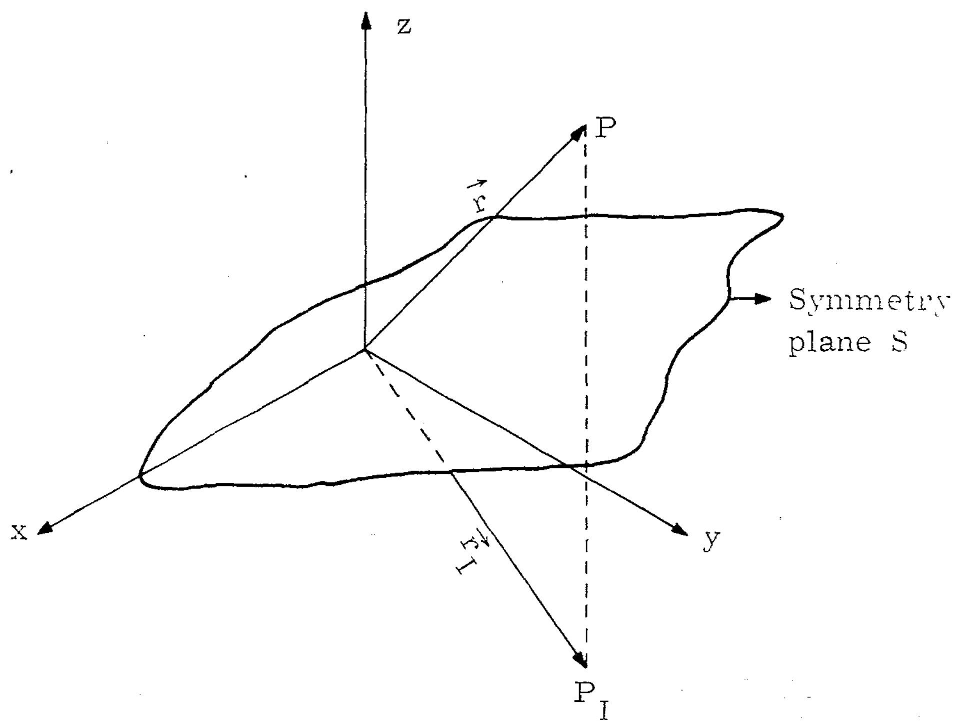


Figure 1. Symmetry Plane

In Fig. 1 P is some arbitrary point on one side of the symmetry plane with the position vector $\vec{r}$, P_I then is the image of P with respect to the symmetry plane S with the position vector $\vec{r}_I$. Defining a reflection dyadic $\overset{\rightrightarrows}{R}$ such that

$$\overleftrightarrow{R} = (R_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{3.1}$$

then

$$\vec{r}_I = \overleftrightarrow{R} \cdot \vec{r} , \qquad \vec{r} = \overleftrightarrow{R} \cdot \vec{r}_I \tag{3.2}$$

It is clear that $\overleftrightarrow{R}$ is its own inverse and hence

$$\overleftrightarrow{R} \cdot \overleftrightarrow{R} = \overleftrightarrow{I} \tag{3.3}$$

where $\overleftrightarrow{I}$ is the identity dyadic.

Corresponding to the reflection of $\vec{r}$ to $\vec{r}_I$, we can also make reflections of the field quantities. These image fields can be written as

$$\tilde{\vec{E}}_I(\vec{r}) = \overleftrightarrow{R} \cdot \tilde{\vec{E}}(\vec{r}_I) \tag{3.4a}$$

$$\tilde{\vec{D}}_I(\vec{r}) = \overleftrightarrow{R} \cdot \tilde{\vec{D}}(\vec{r}_I) \tag{3.4b}$$

$$\tilde{\rho}_I(\vec{r}) = \tilde{\rho}(\vec{r}_I) \tag{3.4c}$$

$$\tilde{\vec{J}}_I(\vec{r}) = \overleftrightarrow{R} \cdot \tilde{\vec{J}}(\vec{r}_I) \tag{3.4d}$$

for the electric quantities, and for the magnetic quantities they are given by

$$\tilde{\vec{B}}_I(\vec{r}) = -\overleftrightarrow{R} \cdot \tilde{\vec{B}}(\vec{r}_I) \tag{3.5a}$$

$$\tilde{\vec{H}}_I(\vec{r}) = -\overleftrightarrow{R} \cdot \tilde{\vec{H}}(\vec{r}_I) \tag{3.5b}$$

$$\tilde{\rho}_{m_I}(\vec{r}) = -\tilde{\rho}_m(\vec{r}_I) \qquad (3.5c)$$

$$\tilde{\vec{J}}_{m_I}(\vec{r}) = -\vec{R} \cdot \tilde{\vec{J}}_m(\vec{r}_I) \qquad (3.5d)$$

where the subscript (I) denotes the image quantities. In Fig. 2, these relationships are exhibited.

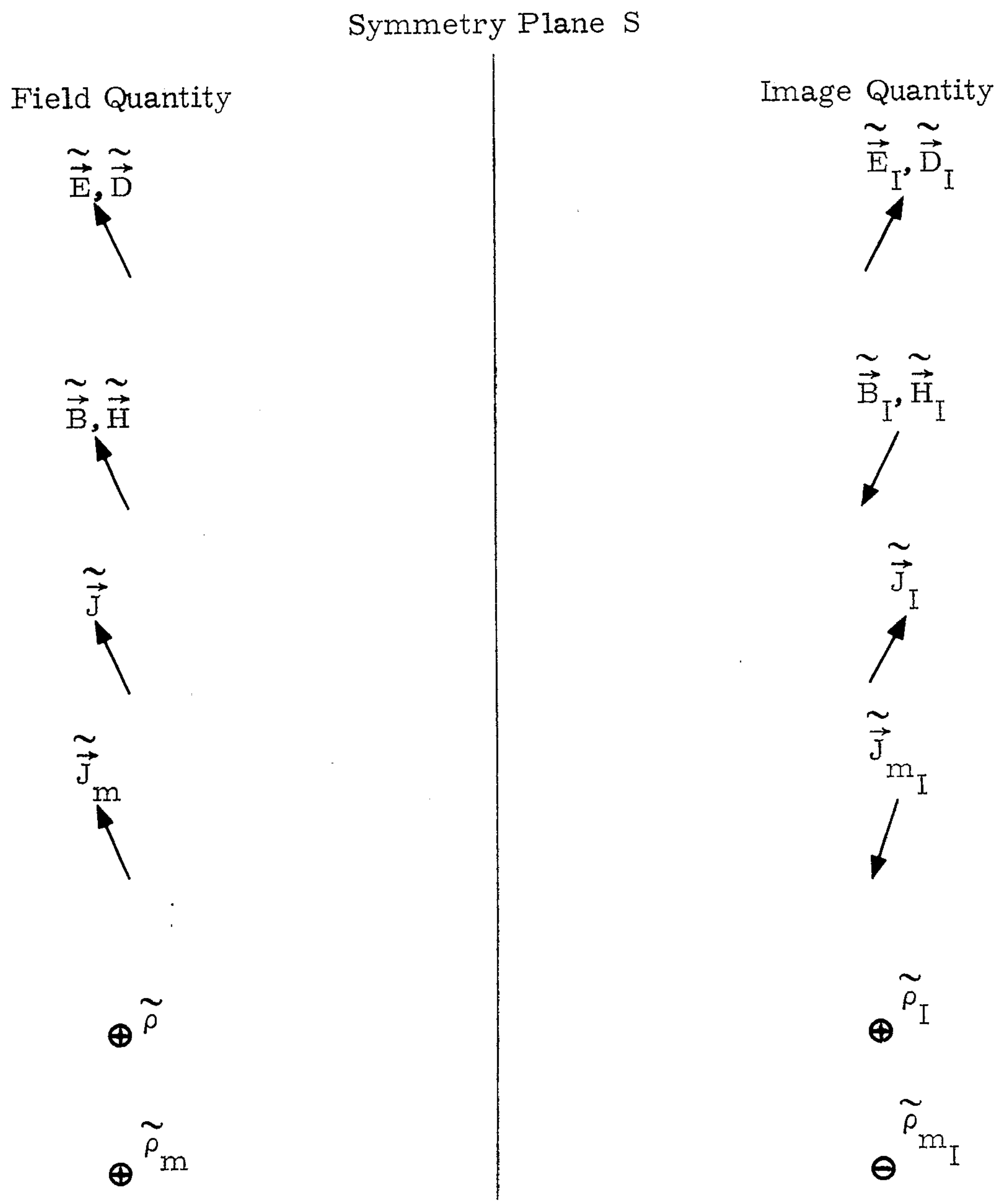


Figure 2. Reflection of the Field Quantities Through a Symmetry Plane

We now define a symmetric (antisymmetric) quantity as one half the sum (difference) of the original quantity and its image quantity. Representing the symmetric and antisymmetric quantities with the subscripts sy and as, respectively, the symmetric and antisymmetric parts for the fields, currents and charges are given by

$$\tilde{\vec{E}}_{\substack{sy\\as}}(\vec{r}) = \frac{1}{2}\left[\tilde{\vec{E}}(\vec{r}) \pm \vec{\vec{R}} \cdot \tilde{\vec{E}}(\vec{r}_I)\right] \tag{3.6a}$$

$$\tilde{\vec{D}}_{\substack{sy\\as}}(\vec{r}) = \frac{1}{2}\left[\tilde{\vec{D}}(\vec{r}) \pm \vec{\vec{R}} \cdot \tilde{\vec{D}}(\vec{r}_I)\right] \tag{3.6b}$$

$$\tilde{\vec{B}}_{\substack{sy\\as}}(\vec{r}) = \frac{1}{2}\left[\tilde{\vec{B}}(\vec{r}) \mp \vec{\vec{R}} \cdot \tilde{\vec{B}}(\vec{r}_I)\right] \tag{3.6c}$$

$$\tilde{\vec{H}}_{\substack{sy\\as}}(\vec{r}) = \frac{1}{2}\left[\tilde{\vec{H}}(\vec{r}) \mp \vec{\vec{R}} \cdot \tilde{\vec{H}}(\vec{r}_I)\right] \tag{3.6d}$$

$$\tilde{\vec{J}}_{\substack{sy\\as}}(\vec{r}) = \frac{1}{2}\left[\tilde{\vec{J}}(\vec{r}) \pm \vec{\vec{R}} \cdot \tilde{\vec{J}}(\vec{r}_I)\right] \tag{3.6e}$$

$$\tilde{\vec{J}}_{m_{\substack{sy\\as}}}(\vec{r}) = \frac{1}{2}\left[\tilde{\vec{J}}_m(\vec{r}) \mp \vec{\vec{R}} \cdot \tilde{\vec{J}}_m(\vec{r}_I)\right] \tag{3.6f}$$

$$\tilde{\rho}_{\substack{sy\\as}}(\vec{r}) = \frac{1}{2}\left[\tilde{\rho}(\vec{r}) \pm \tilde{\rho}(\vec{r}_I)\right] \tag{3.6g}$$

$$\tilde{\rho}_{m_{\substack{sy\\as}}}(\vec{r}) = \frac{1}{2}\left[\tilde{\rho}_m(\vec{r}) \mp \tilde{\rho}_m(\vec{r}_I)\right] \tag{3.6h}$$

where the first sign is associated with the symmetric part, while the later is associated with the antisymmetric part. Decomposition of the fields, currents and charges into symmetric and antisymmetric parts simplifies analysis of problems where a symmetry plane exists. Some interesting physical structures have a plane of symmetry and as a consequence, symmetry decomposition becomes a valuable tool in analyzing these structures. From (3.4a) through (3.5d) we can write

$$\tilde{\vec{E}}_{\substack{sy\\as}}(\vec{r}_I) = \pm\vec{\vec{R}} \cdot \tilde{\vec{E}}_{\substack{sy\\as}}(\vec{r}) \tag{3.7a}$$

$$\tilde{\vec{D}}_{\substack{sy\\as}}(\vec{r}_I) = \pm\vec{\vec{R}} \cdot \tilde{\vec{D}}_{\substack{sy\\as}}(\vec{r}) \tag{3.7b}$$

$$\tilde{\vec{H}}_{\substack{sy\\as}}(\vec{r}_I) = \mp\vec{\vec{R}} \cdot \tilde{\vec{H}}_{\substack{sy\\as}}(\vec{r}) \tag{3.7c}$$

$$\tilde{\vec{B}}_{\substack{sy\\as}}(\vec{r}_I) = \mp\vec{\vec{R}} \cdot \tilde{\vec{H}}_{\substack{sy\\as}}(\vec{r}) \tag{3.7d}$$

$$\tilde{\vec{J}}_{\substack{sy\\as}}(\vec{r}_I) = \pm\vec{\vec{R}} \cdot \tilde{\vec{J}}_{\substack{sy\\as}}(\vec{r}) \tag{3.7e}$$

$$\tilde{\vec{J}}_{m_{\substack{sy\\as}}}(\vec{r}_I) = \mp\vec{\vec{R}} \cdot \tilde{\vec{J}}_{m_{\substack{sy\\as}}}(\vec{r}) \tag{3.7f}$$

$$\tilde{\rho}_{\substack{sy\\as}}(\vec{r}_I) = \pm\tilde{\rho}_{\substack{sy\\as}}(\vec{r}) \tag{3.7g}$$

$$\tilde{\rho}_{m_{\substack{sy\\as}}}(\vec{r}_I) = \mp\tilde{\rho}_{m_{\substack{sy\\as}}}(\vec{r}) \tag{3.7h}$$

Similarly the combined field, current density, charge density and potentials can also be decomposed into symmetric and antisymmetric parts as

$$\tilde{\vec{F}}_{q_{\substack{sy\\as}}}(\vec{r}) = \frac{1}{2}\left[\tilde{\vec{F}}_q(\vec{r}) \pm \vec{\vec{R}} \cdot \tilde{\vec{F}}_{-q}(\vec{r}_I)\right] \tag{3.8a}$$

$$\tilde{\vec{K}}_{q_{\substack{sy\\as}}}(\vec{r}) = \frac{1}{2}\left[\tilde{\vec{K}}_q(\vec{r}) \pm \vec{\vec{R}} \cdot \tilde{\vec{K}}_{-q}(\vec{r}_I)\right] \tag{3.8b}$$

$$\tilde{Q}_{q_{\substack{sy\\as}}}(\vec{r}) = \frac{1}{2}\left[\tilde{Q}_q(\vec{r}) \pm \tilde{Q}_{-q}(\vec{r}_I)\right] \tag{3.8c}$$

$$\tilde{\vec{C}}_{q_{\substack{sy\\as}}}(\vec{r}) = \frac{1}{2}\left[\tilde{\vec{C}}_q(\vec{r}) \pm \vec{\vec{R}} \cdot \tilde{\vec{C}}_{-q}(\vec{r}_I)\right] \tag{3.8d}$$

$$\tilde{\phi}_{q_{\substack{sy\\as}}}(\vec{r}) = \frac{1}{2}\left[\tilde{\phi}_q(\vec{r}) \pm \tilde{\phi}_{-q}(\vec{r}_I)\right] \tag{3.8e}$$

Symmetry decomposition of the fields, currents, charges, etc., does simplify a certain class of problems; this will become clear in the later part of this report. A more detailed discussion of the symmetry decomposition is available in a report by C. E. Baum.[6]

CHAPTER 4

GENERAL BOUNDARY CONDITIONS

Our main aim in this report is to obtain a generalized Babinet's principle appropriate for impedance loaded apertures. Before we examine this generalization, we will obtain some boundary conditions appropriate for our work. We will primarily be interested in aperture antennas and coupling through apertures. The apertures are assumed to be in a perfectly conducting infinitely thin plane and the surface of the aperture could be covered with a thin sheet impedance surface.

4.1 Boundary Conditions on a Perfectly Conducting Plane

Let S represent a perfectly conducting plane surface. This conducting plane could be in the x-y plane in the cartesian coordinate system; however, this specialization is not essential. Let $\vec{n}$ represent a unit normal to this conducting plane. Let an arbitrary electromagnetic wave be incident on this conducting plane which induces surface current density $\tilde{\vec{J}}_s$ and surface charge density $\tilde{\rho}_s$ on the conducting plane. It is well known that on the conducting surface (with n pointing away from the side of interest)

$$\vec{n} \times \tilde{\vec{E}} = \vec{0} \tag{4.1.1a}$$

$$\vec{n} \times \tilde{\vec{H}} = \tilde{\vec{J}}_s \tag{4.1.1b}$$

$$\vec{n} \cdot \tilde{\vec{D}} = \tilde{\rho}_s \tag{4.1.1c}$$

$$\vec{n} \cdot \tilde{\vec{B}} = 0 \tag{4.1.1d}$$

Letting $\vec{n} \equiv \vec{e}_z$, a unit vector in the z direction, we define a dyadic $\vec{\vec{\tau}}$ such that

$$\vec{\vec{\tau}} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.1.2}$$

As a result of (4.1.2), we can replace $\vec{n}\times$ by $\vec{\vec{\tau}}\cdot$. We define another dyadic $\vec{\vec{T}}$ as

$$\vec{\vec{T}} \equiv -\vec{n}\times\vec{n}\times = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \tag{4.1.3}$$

and a dyadic $\vec{\vec{n}}$ as

$$\vec{\vec{n}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.1.4}$$

The dyadics $\vec{\vec{T}}$ and $\vec{\vec{n}}$ as given by (4.1.3) and (4.1.4) are simply the projection dyadics as defined by Halmos.[7] They have eigenvalues which are either 0 or 1. They also have the property $\vec{\vec{T}} \cdot \vec{\vec{T}} = \vec{\vec{T}}$, $\vec{\vec{n}} \cdot \vec{\vec{n}} = \vec{\vec{n}}$ and $\vec{\vec{T}} + \vec{\vec{n}} = \vec{\vec{I}}$. These projection dyadics play an important part in analyzing plane waves through anisotropic media. In (4.1.4) $\vec{\vec{n}}$ can be considered in some sense as a complement of $\vec{\vec{T}}$. We can write (4.1.1a) through (4.1.1d) as

$$\vec{\vec{T}} \cdot \tilde{\vec{E}} = \vec{0} \tag{4.1.5a}$$

$$\vec{\vec{T}} \cdot \tilde{\vec{H}} = -\vec{\vec{\tau}} \cdot \tilde{\vec{J}}_s = -\vec{n} \times \tilde{\vec{J}}_s \tag{4.1.5b}$$

$$\vec{n} \cdot \tilde{\vec{D}} = \tilde{\rho}_s \tag{4.1.5c}$$

$$\vec{n} \cdot \tilde{\vec{H}} \doteq \vec{0} \tag{4.1.5d}$$

In terms of the combined field and current we can write (4.1.5a), (4.1.5b) as

$$\vec{T} \cdot \tilde{\vec{F}}_q = -\frac{qiZ_o}{2} \vec{\tau} \cdot \left[\tilde{\vec{K}}_{s_q} + \tilde{\vec{K}}_{s_{-q}}\right] \tag{4.1.6}$$

where $\tilde{\vec{K}}_{s_q}$ represents the combined surface current. Similarly (4.1.5d) can be written as

$$\vec{n} \cdot \left[\tilde{\vec{F}}_q - \tilde{\vec{F}}_{-q}\right] = \vec{0} \tag{4.1.7}$$

and (4.1.5c) as

$$\vec{n} \cdot \left[\tilde{\vec{F}}_q + \tilde{\vec{F}}_{-q}\right] = \frac{1}{2\epsilon_o}\left[\tilde{Q}_q + \tilde{Q}_{-q}\right] \tag{4.1.8}$$

These equations then represent the boundary conditions for the fields and the combined field at the surface of a perfectly conducting plane.

4.2 Sheet Impedance Boundary Conditions

We now consider the case in which the plane S described in section 4.1 has a finite admittance. We represent this sheet admittance by $\overleftrightarrow{Y}_s$, a dyadic. Since the sheet is infinitesimally thin, all components of $\overleftrightarrow{Y}_s$ associated with the z coordinate (normal to the plane) are zero. Letting $\vec{n}$ be the outward normal, we represent the magnetic field on either side of S by $\vec{H}_+$ and $\vec{H}_-$ as shown in Fig. 3.

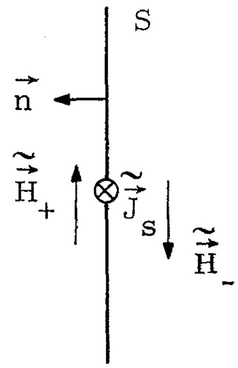


Figure 3. Impedance Sheet

In terms of the electric field $\tilde{\vec{E}}$ on the impedance sheet, the surface current $\tilde{\vec{J}}_s$ is given by

$$\tilde{\vec{J}}_s = \tilde{\vec{\vec{Y}}}_s \cdot \tilde{\vec{E}} = \vec{\vec{Y}}_s \cdot \vec{\vec{T}} \cdot \tilde{\vec{E}} \tag{4.2.1}$$

Calculating the surface current in terms of $\tilde{\vec{H}}_+$ and $\tilde{\vec{H}}_-$ yields

$$\vec{n} \times \left[\tilde{\vec{H}}_+ - \tilde{\vec{H}}_-\right] = \tilde{\vec{J}}_s \tag{4.2.2}$$

Equating (4.2.1) and (4.2.2),

$$\vec{n} \times \left[\tilde{\vec{H}}_+ - \tilde{\vec{H}}_-\right] = \tilde{\vec{\vec{Y}}}_s \cdot \vec{\vec{T}} \cdot \tilde{\vec{E}} \tag{4.2.3}$$

This can also be rewritten as

$$-\vec{\vec{T}} \cdot \left[\tilde{\vec{H}}_+ - \tilde{\vec{H}}_-\right] = \vec{n} \times \tilde{\vec{\vec{Y}}}_s \cdot \vec{\vec{T}} \cdot \tilde{\vec{E}} = \vec{\vec{\tau}} \cdot \tilde{\vec{\vec{Y}}}_s \cdot \vec{\vec{T}} \cdot \tilde{\vec{E}} \tag{4.2.4}$$

It is clear from (4.2.4) that the transverse components of the electric and magnetic fields are coupled through the impedance sheet. Noting that

$$\tilde{\vec{H}}_{\pm} = \frac{-qi}{2Z_o}\left[\tilde{\vec{F}}_{\pm_q} - \tilde{\vec{F}}_{\pm_{-q}}\right] \tag{4.2.5}$$

$$\tilde{\vec{E}}_{\pm} = \frac{1}{2}\left[\tilde{\vec{F}}_{\pm_q} + \tilde{\vec{F}}_{\pm_{-q}}\right] \tag{4.2.6}$$

(4.2.4) can be written as

$$\frac{qi}{Z_o}\vec{\vec{T}} \cdot \left[\tilde{\vec{F}}_{+_q} - \tilde{\vec{F}}_{+_{-q}} - \tilde{\vec{F}}_{-_q} + \tilde{\vec{F}}_{-_{-q}}\right] = \vec{\vec{\tau}} \cdot \tilde{\vec{\vec{Y}}}_s \cdot \vec{\vec{T}} \cdot \left[\tilde{\vec{F}}_{+_q} + \tilde{\vec{F}}_{-_{-q}}\right] \tag{4.2.7}$$

or alternately,

$$\frac{-qi}{Z_o}\vec{n} \times \left[\tilde{\vec{F}}_{+_q} - \tilde{\vec{F}}_{+_{-q}} - \tilde{\vec{F}}_{-_q} + \tilde{\vec{F}}_{-_{-q}}\right] = \tilde{\vec{\vec{Y}}}_s \cdot \vec{\vec{T}} \cdot \left[\tilde{\vec{F}}_{+_q} + \tilde{\vec{F}}_{-_{-q}}\right] \tag{4.2.8}$$

In (4.2.7) and (4.2.8), because of the definition of $\tilde{\vec{\vec{Y}}}_s$, $\vec{\vec{T}}$ need not necessarily be introduced.

4.3 Symmetry Decomposition of the Surface Currents and the Resulting Boundary Conditions on the $z = 0$ Plane

Considering a plane scattering surface S, representing the incident and the scattered fields by the subscripts inc and sc, respectively, we can decompose these fields into a symmetric part and an antisymmetric part denoted by the subscripts sy and as as

$$\tilde{\vec{E}}_{inc} = \tilde{\vec{E}}_{sy_{inc}} + \tilde{\vec{E}}_{as_{inc}} \tag{4.3.1}$$

$$\tilde{\vec{E}}_{sc} = \tilde{\vec{E}}_{sy_{sc}} + \tilde{\vec{E}}_{as_{sc}} \tag{4.3.2}$$

Similarly the surface current density $\tilde{\vec{J}}_s$ on the scatterer on the $z = 0$ plane can also be written as

$$\tilde{\vec{J}}_s = \tilde{\vec{J}}_{s_{sy}} + \tilde{\vec{J}}_{s_{as}} \tag{4.3.3}$$

where

$$\tilde{\vec{J}}_{s_{sy}} = \frac{1}{2}\left[\tilde{\vec{J}}_s + \tilde{\vec{J}}_s\right] \tag{4.3.4a}$$

and

$$\tilde{\vec{J}}_{s_{as}} = \frac{1}{2}\left[\tilde{\vec{J}}_s - \tilde{\vec{J}}_s\right] \tag{4.3.4b}$$

Noting that for an infinitesimally thin plane scatterer $n \cdot \tilde{\vec{J}}_s = 0$, we have

$$\tilde{\vec{J}}_{s_{as}} = \vec{0} \tag{4.3.4c}$$

This implies that for a plane scatterer the antisymmetric parts of the scattered electric and magnetic fields are zero. Hence

$$\tilde{\vec{F}}_{sc_{as_q}} = \vec{0} \tag{4.3.5}$$

$$\tilde{\vec{F}}_{sc_q} = \tilde{\vec{F}}_{sy_{sc_q}} \tag{4.3.6}$$

where we have used the combined fields. This conclusion implies that in the case of plane scatterers the scattered field has only a symmetric part.

As in the case of sheet impedance, $\tilde{\vec{H}}_+$ and $\tilde{\vec{H}}_-$ can be defined and can be split into symmetric and antisymmetric parts as

$$\tilde{\vec{H}}_+ = \tilde{\vec{H}}_{+_{sy}} + \tilde{\vec{H}}_{+_{as}} \tag{4.3.7}$$

$$\tilde{\vec{H}}_- = \tilde{\vec{H}}_{-_{sy}} + \tilde{\vec{H}}_{-_{as}} \tag{4.3.8}$$

Since the surface current $\tilde{\vec{J}}_s$ does not have an antisymmetric part, $\tilde{\vec{H}}_{\pm_{as}}$ is not affected by $z = 0$ scattering, $\tilde{\vec{H}}_{\pm_{as}}$ is continuous through $z = 0$. Hence at $z = 0$

$$\tilde{\vec{H}}_+ - \tilde{\vec{H}}_- = \tilde{\vec{H}}_{sy_+} - \tilde{\vec{H}}_{sy_-} \tag{4.3.9}$$

However

$$\tilde{\vec{H}}_{sy_+} = -\vec{\vec{R}} \cdot \tilde{\vec{H}}_{sy_-} \quad ; \qquad \tilde{\vec{H}}_{sy_-} = -\vec{\vec{R}} \cdot \tilde{\vec{H}}_{sy_+} \tag{4.3.10}$$

Hence, at $z = 0$

$$\tilde{\vec{H}}_+ - \tilde{\vec{H}}_- = 2\vec{\vec{T}} \cdot \tilde{\vec{H}}_{sy_+} = -2\vec{\vec{T}} \cdot \tilde{\vec{H}}_{sy_-} \tag{4.3.11}$$

this can also be written as

$$2\vec{\vec{T}} \cdot \tilde{\vec{H}}_{sy_+} = -\vec{n} \times \tilde{\vec{J}}_s = -\vec{\vec{\tau}} \cdot \tilde{\vec{J}}_s \tag{4.3.12}$$

Writing the surface current in terms of the electric field,

$$\tilde{\vec{J}}_s = \tilde{\vec{\vec{Y}}}_s \cdot \left[\tilde{\vec{E}}_{inc_{sy}} + \tilde{\vec{E}}_{sc}\right] \qquad z = 0 \tag{4.3.13}$$

where $\tilde{\vec{E}}_{sc}$ has a symmetric part only. From (4.3.12) and (4.3.13) we infer that

$$2\vec{\vec{T}} \cdot \tilde{\vec{H}}_{sy_+} = -\vec{\vec{\tau}} \cdot \tilde{\vec{\vec{Y}}}_s \cdot \left[\tilde{\vec{E}}_{inc_{sy}} + \tilde{\vec{E}}_{sc}\right] \qquad z = 0 \qquad (4.3.14)$$

If $\tilde{\vec{\vec{Y}}}_s = \vec{\vec{0}}$, i.e., a perfect insulator,

$$\vec{\vec{T}} \cdot \tilde{\vec{H}}_{sy_+} = 0 \qquad z = 0 \qquad (4.3.15)$$

and if $\tilde{\vec{\vec{Y}}}_s = \vec{\vec{\infty}}$ we can also write

$$\vec{\vec{n}} \cdot \left[\tilde{\vec{H}}_{inc} + \tilde{\vec{H}}_{sc}\right] = \vec{0} \qquad z = 0 \qquad (4.3.16)$$

We can write the boundary conditions (4.3.14), (4.3.15) and (4.3.16) in terms of the combined field as

$$\frac{qi}{Z_o} \vec{\vec{T}} \cdot \left[\tilde{\vec{F}}_{+_{q_{sy}}} - \tilde{\vec{F}}_{+_{-q_{sy}}}\right]$$

$$= \frac{1}{2} \vec{\vec{\tau}} \cdot \tilde{\vec{\vec{Y}}}_s \cdot \left[\left\{\tilde{\vec{F}}_{+_{q_{inc_{sy}}}} + \tilde{\vec{F}}_{-_{q_{inc_{sy}}}}\right\}\right.$$

$$\left. + \left\{\tilde{\vec{F}}_{+_{q_{sc}}} + \tilde{\vec{F}}_{-_{q_{sc}}}\right\}\right] \qquad z = 0 \qquad (4.3.17)$$

If $\tilde{\vec{\vec{Y}}}_s = \vec{\vec{0}}$

$$\vec{\vec{T}} \cdot \left[\tilde{\vec{F}}_{+_{q_{sy}}} - \tilde{\vec{F}}_{-_{q_{sy}}}\right] = \vec{0} \qquad z = 0 \qquad (4.3.18)$$

and if $\tilde{\vec{\vec{Y}}}_s = \vec{\vec{\infty}}$,

$$\vec{\vec{n}} \cdot \left[\left\{ \tilde{\vec{\vec{F}}}_{q_{inc}} - \tilde{\vec{\vec{F}}}_{-q_{inc}} \right\} + \left\{ \tilde{\vec{\vec{F}}}_{q_{sc}} - \tilde{\vec{\vec{F}}}_{-q_{sc}} \right\} \right] = \vec{0} \qquad z = 0 \qquad (4.3.19)$$

CHAPTER 5

SYMMETRY RELATIONS AND BOUNDARY CONDITIONS FOR AN APERTURE IN A PERFECTLY CONDUCTING PLANE AND FOR THE COMPLEMENTARY SCREEN

Consider a perfectly conducting infinite plane S with an aperture A in the plane. The aperture region A may be impedance loaded. In Fig. 4, P represents the perfectly conducting region while A represents an aperture. The plane $S = P \cup A$ is assumed to be in the $z = 0$ plane. We now develop certain symmetry relations and boundary conditions for the perfectly conducting plane with an aperture and for the complementary screen.

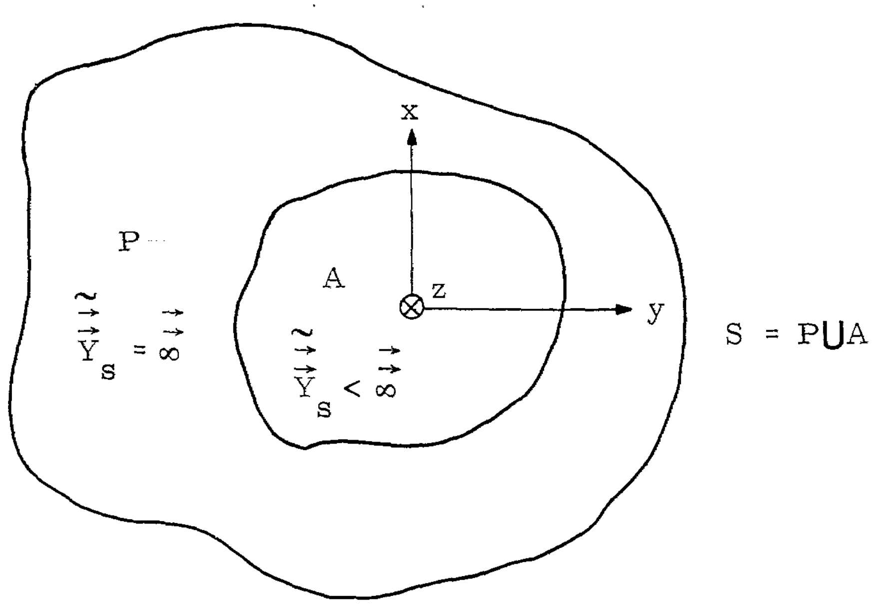


Figure 4. Impedance Loaded Aperture

5.1 Perfectly Conducting Plane

We now consider the case when A is covered with a perfectly conducting surface as P, i.e., $P \cup A$ is perfectly conducting. The incident wave is assumed to be incident on S from $z < 0$ direction. Representing the scattered field for this case by the subscript c instead of sc (indicating aperture "closed") we have

$$\tilde{\vec{E}}_{inc} + \tilde{\vec{E}}_c = 0 \qquad \text{for all } z > 0 \tag{5.1.1}$$

On the $z = 0$ plane we have

$$\vec{T} \cdot \left[\tilde{\vec{E}} = \vec{T} \cdot \tilde{\vec{E}}_{inc} + \tilde{\vec{E}}_c \right] = \vec{0} \tag{5.1.2}$$

As shown in section 4.3, the surface current $\tilde{\vec{J}}_s$ induced on the plane S is symmetric. This implies that the scattered field $\tilde{\vec{E}}_c$ is also symmetric. Hence from (5.1.2)

$$\vec{T} \cdot \tilde{\vec{E}}_{inc} = \vec{T} \cdot \tilde{\vec{E}}_{inc_{sy}} = -\vec{T} \cdot \tilde{\vec{E}}_c \tag{5.1.3}$$

Similarly

$$\vec{n} \cdot \tilde{\vec{H}}_{inc} = \vec{n} \cdot \tilde{\vec{H}}_{inc_{sy}} = -\vec{n} \cdot \tilde{\vec{H}}_c \tag{5.1.4}$$

Equations (5.1.3) and (5.1.4) are simply the boundary conditions on a perfectly conducting surface. In terms of the combined field we can write (5.1.3) and (5.1.4) as

$$\vec{T} \cdot \left[\tilde{\vec{F}}_{inc_q} + \tilde{\vec{F}}_{inc_{-q}} \right] = -\vec{T} \cdot \left[\tilde{\vec{F}}_{c_q} + \tilde{\vec{F}}_{c_{-q}} \right] \tag{5.1.5}$$

$$\vec{n} \cdot \left[\tilde{\vec{F}}_{inc_q} - \tilde{\vec{F}}_{inc_{-q}} \right] = -\vec{n} \cdot \left[\tilde{\vec{F}}_{c_{-q}} - \tilde{\vec{F}}_{c_{-q}} \right] \tag{5.1.6}$$

5.2 Perfectly Conducting Plane with an Aperture

We now consider a perfectly conducting plane with an aperture in the plane. The plane is assumed to be in the $z = 0$ plane with the incident wave from $z < 0$ direction. Denoting the fields due to the aperture by the subscript a, the fields on the $+z$ side of the aperture by a subscript $+$ and those on the $-z$ by $-$, we can write

$$\tilde{\vec{E}}_+ = \tilde{\vec{E}}_a \tag{5.2.1}$$

$$\tilde{\vec{H}}_+ = \tilde{\vec{H}}_a \tag{5.2.2}$$

$$\tilde{\vec{E}}_- = \tilde{\vec{E}}_{inc} + \tilde{\vec{E}}_c + \tilde{\vec{E}}_a \tag{5.2.3}$$

$$\tilde{\vec{H}}_- = \tilde{\vec{H}}_{inc} + \tilde{\vec{H}}_c + \tilde{\vec{H}}_a \tag{5.2.4}$$

where $\tilde{\vec{E}}_c$ and $\tilde{\vec{H}}_c$ are the quantities defined in section 5.1, and

$$\tilde{\vec{E}}_c + \tilde{\vec{E}}_a = \tilde{\vec{E}}_{sc_-} \tag{5.2.5}$$

$$\tilde{\vec{H}}_c + \tilde{\vec{H}}_a = \tilde{\vec{H}}_{sc_-} \tag{5.2.6}$$

Since the scattered fields are symmetric, we can write

$$\tilde{\vec{E}}_{sy_-} = \tilde{\vec{E}}_{inc_{sy}} + \tilde{\vec{E}}_c + \tilde{\vec{E}}_a \tag{5.2.7}$$

$$\tilde{\vec{H}}_{sy_-} = \tilde{\vec{H}}_{inc_{sy}} + \tilde{\vec{H}}_c + \tilde{\vec{H}}_a \tag{5.2.8}$$

$$\tilde{\vec{E}}_{as_-} = \tilde{\vec{E}}_{inc_{as}} \tag{5.2.9}$$

$$\tilde{\vec{H}}_{as_-} = \tilde{\vec{H}}_{inc_{as}} \tag{5.2.10}$$

$$\tilde{\vec{E}}_{sy_+} = \tilde{\vec{E}}_a \tag{5.2.11}$$

$$\tilde{\vec{H}}_{sy_+} = \tilde{\vec{H}}_a \tag{5.2.12}$$

$$\tilde{\vec{E}}_{sc_{as_-}} = \vec{0} = \tilde{\vec{H}}_{sc_{as_+}} \tag{5.2.13}$$

It is clear from the above equations that for the scattered fields, we only have to be concerned with the symmetric part. It is also clear that $\tilde{\vec{E}}_c$ can be simply calculated by replacing the aperture and the screen by a perfectly conducting screen. Hence we need to calculate the aperture fields only. On P we can write

$$\vec{\vec{T}} \cdot \left[\tilde{\vec{E}}_{inc_{sy}} + \tilde{\vec{E}}_c\right] = 0 \qquad \text{on P} \tag{5.2.14}$$

$$\vec{\vec{n}} \cdot \left[\tilde{\vec{H}}_{inc_{sy}} + \tilde{\vec{H}}_c\right] = 0 \qquad \text{on P} \tag{5.2.15}$$

From (5.2.7), (5.2.8), (5.2.14) and (5.2.15) we can write

$$\vec{\vec{T}} \cdot \tilde{\vec{E}}_{sy_-} = \vec{\vec{T}} \cdot \tilde{\vec{E}}_a \qquad \text{on A} \tag{5.2.16}$$

$$\vec{\vec{n}} \cdot \tilde{\vec{H}}_{sy_-} = \vec{\vec{n}} \cdot \tilde{\vec{H}}_a \qquad \text{on A} \tag{5.2.17}$$

In terms of the combined field (5.2.16) and (5.2.17) can be written as

$$\vec{\vec{T}} \cdot \left[\tilde{\vec{F}}_{sy_q} + \tilde{\vec{F}}_{sy_{-q}}\right] = \vec{\vec{T}} \cdot \left[\tilde{\vec{F}}_{a_q} + \tilde{\vec{F}}_{a_{-q}}\right] \qquad \text{on A} \tag{5.2.18}$$

$$\vec{\vec{n}} \cdot \left[\tilde{\vec{F}}_{sy_q} - \tilde{\vec{F}}_{sy_{-q}}\right] = \vec{\vec{n}} \cdot \left[\tilde{\vec{F}}_{a_q} - \tilde{\vec{F}}_{a_{-q}}\right] \qquad \text{on A} \tag{5.2.19}$$

The aperture fields satisfy the condition

$$\vec{\vec{T}} \cdot \vec{\tilde{E}}_a = \vec{0} \quad \text{on } P \tag{5.2.20}$$

$$\vec{\vec{n}} \cdot \vec{\tilde{H}}_a = \vec{0} \quad \text{on } P \tag{5.2.21}$$

or in terms of the combined fields

$$\vec{\vec{T}} \cdot \left[\vec{\tilde{F}}_{a_q} + \vec{\tilde{F}}_{a_{-q}}\right] = \vec{0} \quad \text{on } P \tag{5.2.22}$$

$$\vec{\vec{n}} \cdot \left[\vec{\tilde{F}}_{a_q} - \vec{\tilde{F}}_{a_{-q}}\right] = \vec{0} \quad \text{on } P \tag{5.2.23}$$

These are the boundary conditions which any electromagnetic field must satisfy at a perfectly conducting surface.

Now considering the aperture region A, the boundary conditions are quite different from those on the conducting plane S. If the aperture region is covered by an infinitesimally thin admittance sheet of admittance $\vec{\vec{Y}}_s$, the surface current on A is given by

$$\vec{\tilde{J}}_s = \vec{\vec{\tilde{Y}}}_s \cdot \left[\vec{\tilde{E}}_{inc_{sy}} + \vec{\tilde{E}}_{sc}\right] \tag{5.2.24}$$

However, from (4.3.11), (4.3.12), (5.2.8) and (5.2.12) we have

$$\vec{\tilde{J}}_s = 2\vec{n} \times \left[\vec{\tilde{H}}_+ - \vec{\tilde{H}}_-\right] = 2\vec{n} \times \vec{\tilde{H}}_{sy_+} = 2\vec{n} \times \left[\vec{\tilde{H}}_{c_+} + \vec{\tilde{H}}_{a_+}\right] \tag{5.2.25}$$

Equating (5.2.24) and (5.2.25)

$$2\vec{n} \times \left[\vec{\tilde{H}}_{c_+} + \vec{\tilde{H}}_{a_+}\right] = \vec{\vec{\tilde{Y}}}_s \cdot \left[\vec{\tilde{E}}_{inc_{sy}} + \vec{\tilde{E}}_c + \vec{\tilde{E}}_a\right] \tag{5.2.26}$$

Noting that $\tilde{\overleftrightarrow{Y}}_s \cdot$ is equivalent to $\tilde{\overleftrightarrow{Y}}_s \cdot \overleftrightarrow{T} \cdot$, and $\overleftrightarrow{T} \cdot \left[\tilde{\vec{E}}_{inc_{sy}} + \tilde{\vec{E}}_c\right]$ is zero on the aperture A,

$$2\vec{n} \times \left[\tilde{\vec{H}}_{c_+} + \tilde{\vec{H}}_{a_+}\right] = \tilde{\overleftrightarrow{Y}}_s \cdot \tilde{\vec{E}}_a = \tilde{\vec{J}}_s \qquad (5.2.27)$$

or alternately

$$2\overleftrightarrow{T} \cdot \left[\tilde{\vec{H}}_{inc_{sy}} - \tilde{\vec{H}}_{a_+}\right] = \vec{\tau} \cdot \tilde{\overleftrightarrow{Y}}_s \cdot \tilde{\vec{E}}_a \qquad (5.2.28)$$

Defining the short circuit current density $\tilde{\vec{J}}_{s_{s.c.}}$ as

$$\tilde{\vec{J}}_{s_{s.c.}} = -2\vec{n} \times \tilde{\vec{H}}_{inc} = -2\vec{n} \times \tilde{\vec{H}}_{inc_{sy}} \qquad (5.2.29)$$

we can rewrite (5.2.28) as

$$\tilde{\vec{J}}_{s_{s.c.}} + 2\vec{n} \times \tilde{\vec{H}}_{a_+} = \tilde{\overleftrightarrow{Y}}_s \cdot \tilde{\vec{E}}_{a_+} = \tilde{\vec{J}}_s \qquad (5.2.30)$$

For the special case of $\tilde{\overleftrightarrow{Y}}_s \equiv \overleftrightarrow{0}$ we obtain

$$\tilde{\vec{J}}_{s_{s.c.}} = 2\vec{n} \times \tilde{\vec{H}}_{a_+} = \vec{0} \qquad (5.2.31)$$

If $\tilde{\overleftrightarrow{Y}}_s = \overleftrightarrow{\infty}$, i.e., a perfectly conducting infinite sheet, we have $\tilde{\vec{J}}_s = 0$, which implies that the perturbation terms $\tilde{\vec{E}}_a$ and $\tilde{\vec{H}}_a$ are identically zero.

In terms of the combined field, we can write (5.2.28) as

$$2\vec{\vec{T}} \cdot \left[\tilde{\vec{F}}_{inc_q} - \tilde{\vec{F}}_{inc_{-q}} - \tilde{\vec{F}}_{a_q} + \tilde{\vec{F}}_{a_{-q}}\right] = \frac{1}{qiZ_o} \vec{\vec{\tau}} \cdot \tilde{\vec{\vec{Y}}}_s \cdot \tilde{\vec{E}}_a \qquad (5.2.32)$$

For the antisymmetric part of the incident field we have on S

$$\vec{\vec{T}} \cdot \left[\tilde{\vec{H}}_{inc_{as}} + \tilde{\vec{H}}_{c_+}\right] = 0 \qquad (5.2.33)$$

However,

$$\vec{\vec{T}} \cdot \tilde{\vec{H}}_{c_+} = -\vec{\vec{T}} \cdot \tilde{\vec{H}}_{c_-} \qquad (5.2.34)$$

hence (5.2.33) becomes

$$\vec{\vec{T}} \cdot \left[\tilde{\vec{H}}_{c_-} - \tilde{\vec{H}}_{c_+}\right] = 2\vec{\vec{T}} \cdot \tilde{\vec{H}}_{inc_{as}} \qquad (5.2.35)$$

The antisymmetric part of the field is continuous through the aperture.

CHAPTER 6

COMPLEMENTARY FIELDS, SOME OF THEIR RELATIONS AND BABINET'S PRINCIPLE

In this chapter we will first define the complementary fields. These complementary quantities are usually defined from the duality of the Maxwell's equations. Using these complementary fields, we will derive the generalized Babinet's principle and the complementary antenna relationship.

6.1 Complementary Fields, Potentials, Charges and Currents

The transformation of fields, potentials, charges and currents between electric and magnetic quantities forms the basis for Babinet's principle. The original fields, potentials, charges, combined fields, etc., are represented by

$$\tilde{\vec{E}},\ \tilde{\vec{H}},\ \tilde{\vec{J}},\ \tilde{\vec{J}}_m,\ \tilde{\rho},\ \tilde{\rho}_m,\ \tilde{\vec{F}}_q,\ \tilde{\vec{K}}_q,\ \tilde{Q}_q,\ \tilde{\vec{C}}_q,\ \tilde{\phi}_q\ .$$

and the transformed or complementary quantities are represented by

$$\tilde{\vec{E}}',\ \tilde{\vec{H}}',\ \tilde{\vec{J}}',\ \tilde{\vec{J}}'_m,\ \tilde{\rho}',\ \tilde{\rho}'_m,\ \tilde{\vec{F}}'_q,\ \tilde{\vec{K}}'_q,\ \tilde{Q}'_q,\ \tilde{\vec{C}}'_q,\ \tilde{\phi}'_q$$

We also define complementary sheet admittance $\tilde{\overleftrightarrow{Y}}'_s$ corresponding to the sheet admittance $\tilde{\overleftrightarrow{Y}}_s$. These complementary quantities are simply generalizations of the terms defined by Booker. We then have

$$\tilde{\vec{F}}'_q = \tilde{\vec{E}}' + qiZ_o\tilde{\vec{H}}' = -qi\tilde{\vec{F}}_q = -qi\tilde{\vec{E}} + Z_o\tilde{\vec{H}} \tag{6.1.1}$$

and this yields

$$\tilde{\vec{E}}' = Z_o \tilde{\vec{H}} \quad \text{or} \quad \tilde{\vec{H}} = \frac{1}{Z_o} \tilde{\vec{E}}' \tag{6.1.1a}$$

$$\tilde{\vec{H}}' = -\frac{1}{Z_o} \tilde{\vec{E}} \quad \text{or} \quad \tilde{\vec{E}} = -Z_o \tilde{\vec{H}}' \tag{6.1.1b}$$

Similarly

$$\tilde{\vec{K}}'_q = \tilde{\vec{J}}' + \frac{qi}{Z_o} \tilde{\vec{J}}'_m = -qi\tilde{\vec{K}}_q = -qi\tilde{\vec{J}} + \frac{1}{Z_o} \tilde{\vec{J}}_m \tag{6.1.2}$$

$$\tilde{\vec{J}}' = \frac{1}{Z_o} \tilde{\vec{J}}_m \quad \text{or} \quad \tilde{\vec{J}}_m = Z_o \tilde{\vec{J}}' \tag{6.1.2a}$$

$$\tilde{\vec{J}}'_m = -Z_o \tilde{\vec{J}} \quad \text{or} \quad \tilde{\vec{J}} = -\frac{1}{Z_o} \tilde{\vec{J}}'_m \tag{6.1.2b}$$

$$\tilde{Q}'_q = \tilde{\rho}' + \frac{qi}{Z_o} \tilde{\rho}'_m = -qi\tilde{Q}_q = -qi\tilde{\rho} + \frac{1}{Z_o} \tilde{\rho}_m \tag{6.1.3}$$

$$\tilde{\rho}' = \frac{1}{Z_o} \tilde{\rho}_m \quad \text{or} \quad \tilde{\rho}_m = Z_o \tilde{\rho}' \tag{6.1.3a}$$

$$\tilde{\rho}'_m = -Z_o \tilde{\rho} \quad \text{or} \quad \tilde{\rho} = -\frac{1}{Z_o} \tilde{\rho}'_m \tag{6.1.3b}$$

$$\tilde{\vec{C}}'_q = \tilde{\vec{A}}' + qiZ_o \tilde{\vec{A}}'_m = -qi\tilde{\vec{C}}_q = -qi\tilde{\vec{A}} + Z_o \tilde{\vec{A}}_m \tag{6.1.4}$$

$$\tilde{\vec{A}}' = Z_o \tilde{\vec{A}}_m \quad \text{or} \quad \tilde{\vec{A}}_m = \frac{1}{Z_o} \tilde{\vec{A}}' \tag{6.1.4a}$$

$$\tilde{\vec{A}}'_m = -\frac{1}{Z_o} \tilde{\vec{A}} \quad \text{or} \quad \tilde{\vec{A}} = -Z_o \tilde{\vec{A}}'_m \tag{6.1.4b}$$

$$\tilde{\phi}'_q = \tilde{\phi}' + qiZ_o\tilde{\phi}'_m = -qi\tilde{\phi}_q = -qi\tilde{\phi} + Z_o\tilde{\phi}_m \tag{6.1.5}$$

$$\tilde{\phi}' = Z_o\tilde{\phi}_m \qquad \text{or} \qquad \tilde{\phi}_m = \frac{1}{Z_o}\tilde{\phi}' \tag{6.1.5a}$$

$$\tilde{\phi}'_m = -\frac{1}{Z_o}\tilde{\phi} \qquad \text{or} \qquad \tilde{\phi} = -Z_o\tilde{\phi}'_m \tag{6.1.5b}$$

From these generalized transformations of Booker, we can conclude that, given a combined quantity Γ, the transformed or complementary quantity Γ' is related to it by

$$\Gamma' = -qi\Gamma , \qquad \Gamma = qi\Gamma' \tag{6.1.6}$$

This gives us a simple transformation by which the complementary quantity can be obtained from the original quantity, and conversely.

The boundary conditions for the transformed quantities also transform as described in (6.1.1) through (6.1.5). Simply stated this implies that if tangential components of $\tilde{\vec{E}}$ are specified, the tangential components of $\tilde{\vec{H}}'$ are immediately known. Similarly if the normal components of $\tilde{\vec{E}}$ are specified, the normal components of $\tilde{\vec{H}}'$ are also specified.

6.2 Transformation of Admittance

Let us consider an aperture as shown in fig. 4. The aperture A is covered with a sheet admittance $\tilde{\vec{\vec{Y}}}_s$. On the aperture, from (5.2.28)

$$2\vec{\vec{T}} \cdot \left[\tilde{\vec{H}}_{inc} - \tilde{\vec{H}}_{a_+}\right] = \vec{\vec{\tau}} \cdot \tilde{\vec{\vec{Y}}}_s \cdot \tilde{\vec{E}}_a = \vec{n} \times \tilde{\vec{\vec{Y}}}_s \cdot \tilde{\vec{E}}_a \tag{6.2.1}$$

Using (6.1.1a) and (6.1.1b) and transforming (6.2.1) to the complementary field quantities

$$\frac{2}{Z_o}\vec{\vec{T}} \cdot \left[\tilde{\vec{E}}'_{inc} - \tilde{\vec{E}}'_{a_+}\right] = -Z_o\vec{\vec{\tau}} \cdot \tilde{\vec{\vec{Y}}}_s \cdot \tilde{\vec{H}}'_a \qquad (6.2.2)$$

or

$$\frac{2}{Z_o}\vec{n} \times \left[\tilde{\vec{E}}'_{inc} - \tilde{\vec{E}}'_{a_+}\right] = Z_o\vec{\vec{\tau}} \cdot \tilde{\vec{\vec{Y}}}_s \cdot \tilde{\vec{H}}'_a \qquad (6.2.3)$$

Consider the complement of S with sheet admittance $\vec{\vec{Y}}'_s$ and sheet current $\vec{J}'_s$,

$$2\vec{n} \times \tilde{\vec{H}}'_{sc_+} = \tilde{\vec{J}}'_s \qquad (6.2.4)$$

or

$$2\vec{\vec{\tau}} \cdot \tilde{\vec{H}}'_{sc_+} = \tilde{\vec{\vec{Y}}}'_s \cdot \vec{\vec{T}} \cdot \left[\tilde{\vec{E}}'_{inc} + \tilde{\vec{E}}'_{sc}\right] \qquad (6.2.5)$$

Multiplying (6.2.2) by $(Z_o/2)\tilde{\vec{\vec{Y}}}'_s$

$$-\frac{Z_o^2}{2}\tilde{\vec{\vec{Y}}}'_s \cdot \vec{\vec{\tau}} \cdot \tilde{\vec{\vec{Y}}}_s \cdot \tilde{\vec{H}}'_a = \tilde{\vec{\vec{Y}}}'_s \cdot \vec{\vec{T}} \cdot \left[\vec{E}'_{inc} - \vec{E}'_{a_+}\right] \qquad (6.2.6)$$

Comparing the terms in (6.2.5) and (6.2.6) we have

$$\tilde{\vec{E}}'_{sc} = -\tilde{\vec{E}}'_{a_+} \qquad (6.2.7a)$$

$$\tilde{\vec{H}}'_a = -\tilde{\vec{H}}'_{sc_+} \qquad (6.2.7b)$$

$$\tilde{\vec{\vec{Y}}}'_s \cdot \vec{\vec{\tau}} \cdot \tilde{\vec{\vec{Y}}}_s = \frac{4}{Z_o^2}\vec{\vec{\tau}} \qquad (6.2.7c)$$

If $\tilde{\overset{\rightrightarrows}{Y}}_s$ is a scalar represented by $\tilde{Y}_s$ and inverse by $\tilde{Z}_s$, then

$$\tilde{Y}'_s = \frac{4}{Z_o^2 \tilde{Y}_s} = \frac{4}{Z_o^2} \tilde{Z}_s \tag{6.2.8}$$

This has been derived as a local relationship for every point on complementary objects (antennas and scatterers). It is interesting to note that this relationship has been previously found to hold for the input admittances of complementary antennas.[8]

If $\tilde{Y}_s$ is a dyadic, by definition it does not have an inverse in the 3 dimensional sense. Taking $\vec{e}_z = \vec{n}$, i.e., in the z direction, we have

$$\tilde{\overset{\rightrightarrows}{Y}}'_s = \frac{4}{Z_o^2} \overset{\rightrightarrows}{\tau} \cdot \left(\tilde{\overset{\rightrightarrows}{Y}}_s^{-1}\right)_2 \cdot \left(\overset{\rightrightarrows}{\tau}^{-1}\right)_2 \tag{6.2.9}$$

where the two dimensional inverse is used. It is interesting to note that $\tilde{\overset{\rightrightarrows}{Y}}'_s$ is a constant multiple of the similarity transformation of $\left(\tilde{\overset{\rightrightarrows}{Y}}_s^{-1}\right)_2$. Defining $\overset{\rightrightarrows}{\tau}'$ as

$$\overset{\rightrightarrows}{\tau}' \equiv qi\overset{\rightrightarrows}{\tau} = \begin{bmatrix} 0 & -qi & 0 \\ qi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.2.10}$$

we can write (6.2.9) as

$$\tilde{\overset{\rightrightarrows}{Y}}'_s = \frac{4}{Z_o^2} \overset{\rightrightarrows}{\tau}' \cdot \left(\tilde{\overset{\rightrightarrows}{Y}}_s^{-1}\right)_2 \cdot \overset{\rightrightarrows}{\tau}' \tag{6.2.11}$$

This is the complementary antenna principle for an arbitrary sheet admittance. The two dimensional inverse is defined for 3×3 matrices (dyadics) with no third (z) row or column elements as

$$\overleftrightarrow{B} = \begin{bmatrix} \begin{bmatrix} B_2 \end{bmatrix} & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.2.12a}$$

$$\left(\overleftrightarrow{B}^{-1}\right)_2 = \begin{bmatrix} \begin{bmatrix} B_2 \end{bmatrix}^{-1} & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.2.12b}$$

6.3 Generalized Babinet's Principle

Consider an aperture as shown in fig. 4 with A loaded by an admittance sheet and P being perfectly conducting. On S, i.e., the whole plane,

$$\overleftrightarrow{T} \cdot \left[\tilde{\vec{E}}_{inc} + \tilde{\vec{E}}_c\right] = \vec{0} = \overleftrightarrow{T} \cdot \left[\tilde{\vec{E}}_{inc_{sy}} + \tilde{\vec{E}}_c\right] \tag{6.3.1}$$

$$\overleftrightarrow{n} \cdot \left[\tilde{\vec{H}}_{inc} + \tilde{\vec{H}}_c\right] = \vec{0} = \overleftrightarrow{n} \cdot \left[\tilde{\vec{H}}_{inc_{sy}} + \tilde{\vec{H}}_c\right] \tag{6.3.2}$$

On A we have

$$2\overleftrightarrow{T} \cdot \left[\tilde{\vec{H}}_{inc} - \tilde{\vec{H}}_{a_+}\right] = \overleftrightarrow{\tau} \cdot \overleftrightarrow{Y}_s \cdot \tilde{\vec{E}}_a \tag{6.3.3}$$

$$\overleftrightarrow{T} \cdot \left[\tilde{\vec{H}}_{c_-} - \tilde{\vec{H}}_{c_+}\right] = 2\overleftrightarrow{T} \cdot \tilde{\vec{H}}_{inc_{as}} \tag{6.3.4}$$

Now consider the complementary screen where the perfectly conducting surface S is replaced by free space and the admittance sheet $\overleftrightarrow{Y}_s$ by its complementary sheet $\overleftrightarrow{Y}'_s$, the complementary problem is as shown

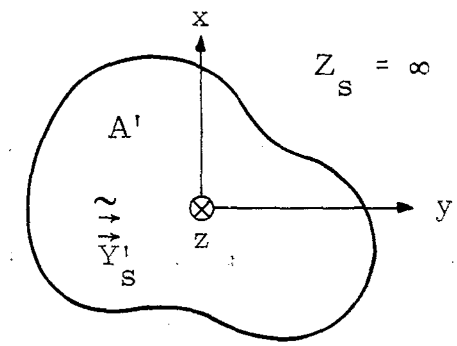


Figure 5. The Complementary Screen

in fig. 5. In the case of the well-known Babinet's principle for open apertures in a perfectly conducting infinite sheet, the original screen in conjunction with its complementary screen forms an infinite, perfectly conducting screen. It is interesting to note that this does not hold in the case of impedance loaded apertures except when $\tilde{\vec{\vec{Y}}}_s = \tilde{\vec{\vec{0}}}$. The boundary conditions for the complementary screen A' shown in fig. 5 are obtained by transforming (6.3.3) and (6.3.4) according to (6.1.1a) and (6.1.1b) to be

$$\frac{2}{Z_o}\vec{\vec{T}} \cdot \left[\tilde{\vec{E}}'_{inc} - \tilde{\vec{E}}'_{a_+}\right] = -Z_o \vec{\vec{\tau}} \cdot \tilde{\vec{\vec{Y}}}_s \cdot \tilde{\vec{H}}'_a \tag{6.3.5}$$

or

$$\frac{2}{Z_o}\vec{n} \times \left[\tilde{\vec{E}}'_{inc} - \tilde{\vec{E}}'_{a_+}\right] = Z_o \tilde{\vec{\vec{Y}}}_s \cdot \tilde{\vec{H}}'_a \tag{6.3.6}$$

and

$$\vec{\vec{T}} \cdot \left[\tilde{\vec{E}}'_{c_-} - \tilde{\vec{E}}'_{c_+}\right] = 2\vec{\vec{T}} \cdot \tilde{\vec{E}}'_{inc_{as}} \tag{6.3.7}$$

Using (6.2.7a) and (6.2.7b), (6.3.6) can be rewritten as

$$\frac{2}{Z_o}\vec{n} \times \left[\tilde{\vec{E}}'_{inc} + \tilde{\vec{E}}'_{sc}\right] = -Z_o \tilde{\vec{\vec{Y}}}_s \cdot \tilde{\vec{H}}'_{sc_+} \qquad (6.3.8)$$

Hence (6.3.7) and (6.3.8) form the boundary conditions for the complementary problem on A.

An interesting result is: if an incident field $\tilde{\vec{E}}_{inc}$ and $\tilde{\vec{H}}_{inc}$ is incident on a screen from $z < 0$ direction, if the transverse scattered fields for $z > 0$ for the original screen are represented by $\tilde{\vec{E}}_a$ and $\tilde{\vec{H}}_a$ and for the complementary screen by $\tilde{\vec{E}}'_a$ and $\tilde{\vec{H}}'_a$, we have

$$\tilde{\vec{E}}_{inc_{sy}} = \tilde{\vec{E}}_a + Z_o \tilde{\vec{H}}'_a \qquad (6.3.9)$$

and

$$\tilde{\vec{H}}_{inc_{sy}} = \tilde{\vec{H}}_a - \frac{1}{Z_o}\tilde{\vec{E}}'_a \qquad (6.3.10)$$

or in the combined form

$$\tilde{\vec{F}}_{inc_{sy_q}} = \tilde{\vec{F}}_{a_q} - qi\tilde{\vec{F}}'_{a_q} \qquad (6.3.11)$$

It is important to note that Babinet's principle as derived here is consistent with that derived by Booker and other authors.[8,9] The only differences that have to be taken into consideration in treating impedance loaded apertures are the boundary conditions one will impose on the fields in the aperture. These are the conditions derived in section 5.2.

The generalized Babinet's principle as presented here states the following: If an aperture covered by an admittance sheet is present in a perfectly conducting screen, the complementary problem is given by a disc whose admittance is given by the complementary formula (6.2.9). The present formulation also allows one to treat an annular slot which was very difficult to treat until now. Some integral formulations for the aperture field will be treated in the next chapter.

CHAPTER 7

INTEGRAL EQUATIONS

With the increase in the general use of digital computers, there have evolved several effective techniques for solving integral equations. As a consequence, integral equations have become more popular in solving scattering problems. In this chapter we will discuss integral equations as applicable to apertures in plane conducting screens.

7.1 Integral Representation for the Combined Field

Considering source free space in which scatterers $S_1, \cdots, S_n$ are immersed, if we denote the incident electric and magnetic fields by the subscript inc, the field at some point p where $p \notin S_i$ can be written as

$$\tilde{\vec{E}}_p = \tilde{\vec{E}}_{inc} + \int_{S_1, \cdots, S_n} \Big[-s\mu_o \tilde{G}_o (\vec{n} \times \tilde{\vec{H}}) + (\vec{n} \times \tilde{\vec{E}}) \times \nabla' \tilde{G}_o + (\vec{n} \cdot \tilde{\vec{E}}) \nabla' \tilde{G}_o \Big] ds' \tag{7.1.1}$$

$$\tilde{\vec{H}}_p = \tilde{\vec{H}}_{inc} + \int_{S_1, \cdots, S_n} \Big[s\epsilon_o \tilde{G}_o (\vec{n} \times \tilde{\vec{E}}) + (\vec{n} \times \tilde{\vec{H}}) \times \nabla' \tilde{G}_o + (\vec{n} \cdot \tilde{\vec{H}}) \nabla' \tilde{G}_o \Big] ds' \tag{7.1.2}$$

where

$$\tilde{G}_o(r, r') = \frac{1}{4\pi} \frac{e^{-\gamma |\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} \tag{7.1.3}$$

In the above, the unprimed coordinates represent the field coordinates while the primed coordinates represent the scatterer, and ∇' operates on the prime coordinates only. Now using (2.6) to define the combined field $\tilde{\vec{F}}_q$, $q = \pm 1$, we can combine (7.1.1) and (7.1.2) as

$$\tilde{\vec{F}}_{p_q} = \tilde{\vec{F}}_{inc_q} + \int_{S_1, \cdots, S_n} \left[\frac{s}{c} qi \left(\vec{n} \times \tilde{\vec{F}}_q \right) \tilde{G}_o + \left(\vec{n} \times \tilde{\vec{F}}_q \right) \times \nabla' \tilde{G}_o + \left(\vec{n} \cdot \tilde{\vec{F}}_q \right) \nabla' \tilde{G}_o \right] ds' \qquad (7.1.4)$$

or alternately

$$\tilde{\vec{F}}_{p_q} = \tilde{\vec{F}}_{inc_q} + \int_{S_1, \cdots, S_n} \left[\left(\vec{n} \times \tilde{\vec{F}}_q \right) \left(\frac{s}{c} qi \tilde{G}_o + \times \nabla' \tilde{G}_o \right) + \left(\vec{n} \cdot \tilde{\vec{F}}_q \right) \nabla' \tilde{G}_o \right] ds' \qquad (7.1.5)$$

Using (7.1.5) and Maue's integral equation[10] one can express the combined current distribution on the scatterer in the form of one integral equation. A more detailed study of the combined field integral equation and its implications is discussed in a companion report.[5]

7.2 Integral Equations for Plane Scatterers with an Aperture

Over the years there have evolved numerous techniques for solving coupling through apertures. Although the equivalence principle has remained the basic starting point, integral equations with many variations have been applied to obtain solutions.[8, 9, 10] Our technique employed here involves the Babinet's principle.

Using (5.2.1) through (5.2.4), since $\tilde{\vec{E}}_{inc}$ and $\tilde{\vec{H}}_{inc}$ are known, $\tilde{\vec{E}}_c$ and $\tilde{\vec{H}}_c$ are known. As a consequence, if the field is assumed to be incident on the aperture from $z < 0$ direction, the total symmetric field for $z < 0$ can be considered as the superposition of the field present when the plane is completely perfectly conducting, i.e., the aperture portion is covered with perfectly conducting sheet and those fields known as the aperture fields. The total scattered field for $z > 0$ is simply the aperture field. We note from (5.2.7) through (5.2.13) that the scattered field is symmetric which also implies that $\tilde{\vec{E}}_a$ and $\tilde{\vec{H}}_a$ are also symmetric. These aperture fields satisfy

$$\vec{\vec{T}} \cdot \tilde{\vec{E}}_a = \vec{0} \qquad \text{on } P \tag{7.2.1}$$

$$\vec{n} \cdot \tilde{\vec{H}}_a = \vec{0} \qquad \text{on } P \tag{7.2.2}$$

On the aperture A, from (5.2.25), (5.2.27) and (5.2.30)

$$\tilde{\vec{J}}_s = \tilde{\vec{J}}_{s_{s.c.}} + 2\vec{n} \times \tilde{\vec{H}}_{a_+} \tag{7.2.3}$$

where $\tilde{\vec{J}}_{s_{s.c.}}$ is given by (5.2.29). We note in (7.1.8) that $2\vec{n} \times \tilde{\vec{H}}_{a_+}$ is simply a perturbation term when the aperture is present.

It is interesting to note that (7.2.3) also includes the contributions due to the sheet impedance terms. When the aperture is not loaded, primarily there exist three techniques. The first technique involves the calculation of the current on the perfectly conducting plane and in turn calculating the field at a given point. Because of the limits involved in the integration process, this technique is seldom used. The second procedure involves the equivalence principle in replacing the screen S by a magnetic current sheet on A and solving for the field quantities. The third technique involves the use of complementary field quantities, where currents on the complementary screen and the resulting fields are calculated. The fields due to the original screen are then calculated using the field transformations discussed in section 6.1. It is clear that techniques 2 and 3 are very similar and in the complementary transform sense are identical.

A complete discussion of the integral equations for the aperture problems is considered outside the scope of this report. It is well known that if the disk is infinitesimally thin, numerical instabilities occur in the evaluation of the integral equations. A new integral equation was formulated recently[11] which circumvents the problems of numerical instability. For electrically small apertures the technique of power series[12, 13] (Taylor or Rayleigh series) can still be used. In all of these techniques, the edge condition should be imposed which in some formulations is incorporated as a line integral around the contour of the scatterer or in terms of series expansion. Because of the complexities involved in these equations, we will delegate a detailed discussion to a future report. Once the current on the scatterer is calculated, the scattered fields are given by

$$\tilde{\vec{E}}'_{sc_+}(\vec{r}) = \int_{S^+} \left[-s\mu_o \tilde{G}_o(\vec{r},\vec{r}\,')\tilde{\vec{J}}'_s(r') + \frac{\nabla' \cdot \tilde{\vec{J}}'_s(\vec{r}\,')}{s\epsilon_o} \nabla'\tilde{G}_o(\vec{r},\vec{r}\,') \right] dS' \qquad (7.2.4)$$

and

$$\tilde{\vec{H}}'_{sc_+}(\vec{r}) = \int_{S^+} \left[\tilde{\vec{J}}'_s(\vec{r}\,') \times \nabla'\tilde{G}_o(\vec{r},\vec{r}\,') \right] dS' \qquad (7.2.5)$$

where $\tilde{\vec{J}}'_s(\vec{r})$ is the current density on the complementary disk while $\tilde{\vec{E}}'_{sc_+}(\vec{r})$ and $\tilde{\vec{H}}'_{sc_+}(\vec{r})$ are the scattered fields for $z > 0$. Since these are the scattered fields for the complementary scatterer, the aperture fields can be simply obtained by using the field transformations.

CHAPTER 8

COMPLEMENTARY AND SELF-COMPLEMENTARY ANTENNAS: THEIR CURRENT DISTRIBUTIONS AND IMPEDANCE RELATIONS

Booker in his work[1] has shown that a resonant half wave slot and a half wave dipole are complementary. We have already derived the complementary antenna impedance relationship in (6.2.9) for a general plane surface impedance. We now use this relationship to show some properties of complementary antennas.

8.1 Complementary Antennas

Consider a perfectly conducting infinite plane sheet with a thin slot covered by a sheet admittance $\tilde{\vec{\vec{Y}}}_s$ as shown below.

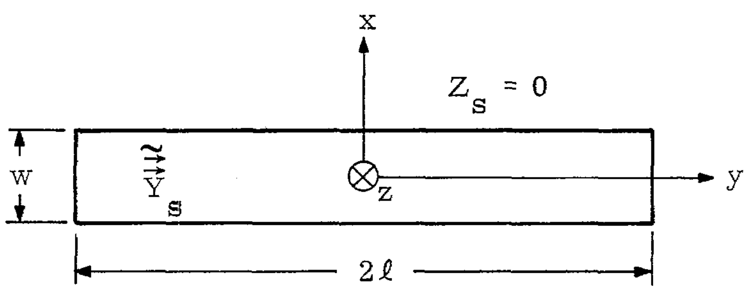


Figure 6. An Impedance Slot in an Infinite Plane

The complementary scatterer for the slot is given by

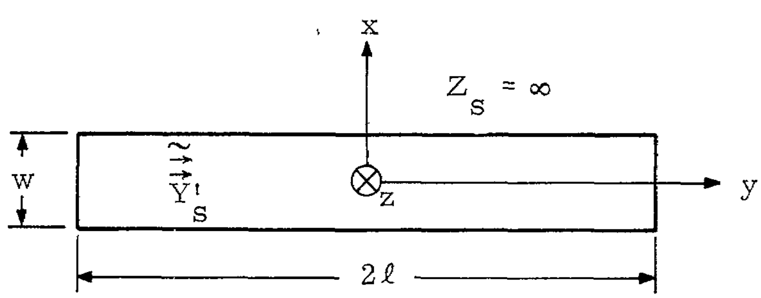


Figure 7. Complementary Strip

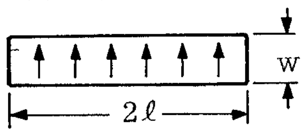
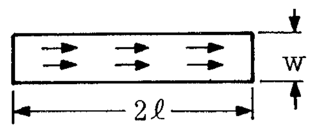
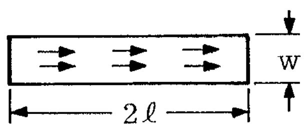
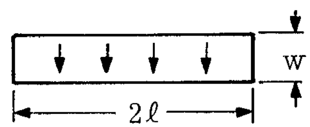
Direction of the Electric Field	Direction of the Current in the Slot	Direction of the Current in the Complementary Strip	Dominant Surface Admittance Term in the Slot	Dominant Surface Admittance Term in the Complementary Strip
x			$\tilde{Y}_{xx}$	$\tilde{Y}'_{yy}$
y ($\vec{\vec{Y}}_S$ assumed diagonal in second case)			$\tilde{Y}_{yy}$	$\tilde{Y}'_{xx}$

Table 1. Dominant Currents and Admittances for Thin Slots

If for instance we want to determine the input admittance of the slot of fig. 6, we consider a dipole of length 2ℓ, width w with the surface admittance $\overset{\approx}{Y}'_s$ defined by (6.2.9). We determine the input impedance of the dipole and using (6.2.1) determine the input admittance for the slot.

It has been shown by many authors[14] that an equivalence exists between thin strips and wires in the static approximation. If the strip is of width w, it can be replaced by a thin wire of radius w/4. As a consequence of this equivalence between strips and wires which also establishes an equivalence between slots and wires, we can calculate the diffraction from a slit by using the diffracted field due to a wire. For instance if we want to determine the diffracted field due to an annular ring in a conducting plane, we can calculate the field due to an equivalent wire loop and transform it such that we obtain the diffracted field for the original problem.

8.2 Self-Complementary Antennas

We define a self-complementary antenna as one which when rotated in its plane by an angle θ produces the complementary antenna. This of course would imply that any rotation by an angle 2θ would keep the geometry of the antenna the same. If we denote the surface admittance of a point p of the antenna by $\overset{\approx}{Y}_s(\rho, \phi)$ and of the complementary antenna at the same point by $\overset{\approx}{Y}'_s(\rho, \phi)$ where (ρ, ϕ, z) is a usual cylindrical coordinate system, the self-complementarity would require that

$$\overset{\approx}{Y}'_s(\rho, \phi + \theta) = \overset{\rightrightarrows}{R}' \cdot \overset{\approx}{Y}_s(\rho, \phi) \cdot (\overset{\rightrightarrows}{R}'^{-1})_2 \qquad (8.1)$$

where $\overset{\rightrightarrows}{R}'$ is a rotation matrix (dyadic) which maps the point p (ρ, ϕ) in the original antenna into its equivalent point p_o $(\rho, \phi + \theta)$ of the complementary antenna. One of these simple structures is shown in fig. 8. More general cases include rotation matrices with the dyadic $\overset{\approx}{Y}'_s$ (or $\overset{\approx}{Y}_s$). Self-complementary structures have long been used to build transmission lines, antennas, etc. The main difference here is the loading which is incorporated in the structure.

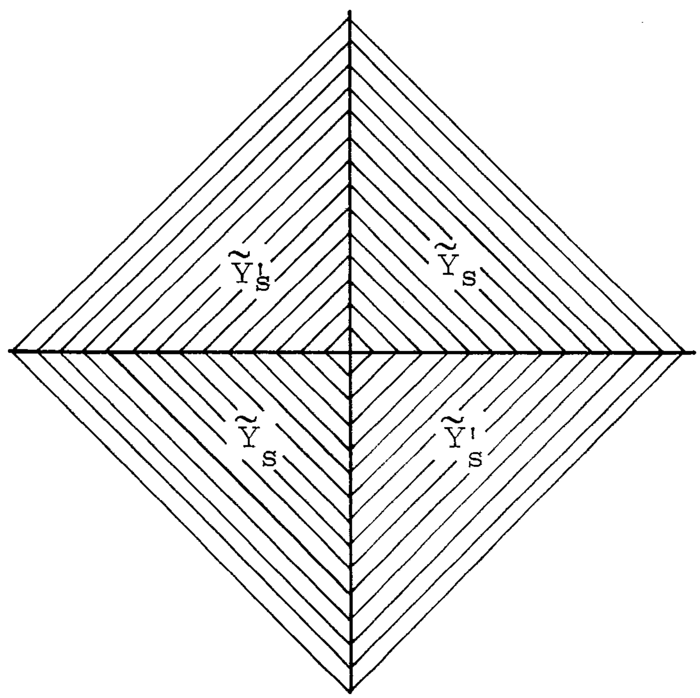


Figure 8. Example of Self-Complementary Screen for Scalar Admittance Case

An interesting example of an admittance loaded self-complementary antenna is one whose surface admittance is $\tilde{Y}_s = 2/Z_o$ in certain places and/or perfectly uniconducting in some particular direction in other places. Some of the interesting properties of these antennas will be studied in a later report.

CHAPTER 9

EPILOGUE

She was standing before an arched doorway over which were the words "QUEEN ALICE" in large letters, and on each side of the arch there was a bell handle; one was marked "Visitors' Bell," the other "Servants' Bell."

"I'll wait till the song's over," thought Alice, "and then I'll ring the--the--which bell must I ring?" she went on, very much puzzled by the names. "I'm not a visitor, and I'm not a servant. There ought to be one marked 'Queen,' you know."

Just then the door opened a little way, and a creature with a long beak put its head out for a moment and said, "No admittance till the week after next!" and shut the door again with a bang.

Lewis Carroll,
Through the Looking Glass

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