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ON SOME SIMPLE, NUMERICALLY EFFICIENT TECHNIQUES FOR HANDLING ELECTRIC-FIELD INTEGRAL EQUATIONS

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ABSTRACT

The difficulties with the usual formulation of Pocklington's equation for numerical solution of thin-wire problems are examined and an integro-difference equation is proposed which circumvents the difficulties and whose solution can be shown to converge asymptotically at the same rate as that of Hallén's equation with the same basis set and point-matching. Furthermore, it is shown that testing Pocklington's equation with piecewise sinusoids results in a similar integro-difference equation but which has a solution equal to that of the corresponding point-matched Hallén equation. For any choice of basis functions, the integro-difference equation has the simple kernel, the fast convergence, the simplicity of point-matching, and the adequate treatment of rapidly varying incident fields, but none of the additional unknowns normally associated with Hallén's equation. Furthermore, for the special choice of piecewise sinusoids as the basis functions, the method reduces to Richmond's piecewise sinusoidal reaction matching technique, or Galerkin's method.

The treatment is generalized to the case of currents which exist on co-planar surfaces and conclusions similar to the above are reached.

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INTRODUCTION

In order to handle more complicated geometries using moment methods, it is necessary to optimize the numerical solution procedures from the point of view of speed and convergence. This leads one to a study of the properties of various integral equation formulations and of the choice of basis and testing functions [1] in solution methods, both with an end toward improving the numerical efficiency of given computations. Also desirable are techniques which are conceptually simple to apply (so as to minimize programming time) and which have a wide range of applicability.

One difficulty which frequently arises in the numerical solution of an integral equation is the appearance of derivatives outside the vector potential integrals on the induced currents. For thin wires, this problem, encountered in Pocklington's equation, is usually handled in one of three ways. First, the E-field integro-differential equation may be converted to a Hallén type equation plus boundary conditions on the current. This procedure has the disadvantages of introducing additional unknowns into the problem (associated with the homogeneous solutions of the differential operator) and of producing a new integral equation which does not incorporate the boundary conditions on the unknown current. However, the Hallén type equation offers good convergence for almost all commonly used basis functions. In the second scheme, the kernel of the E-field integral equation is made regular by approximations which result in the so-called reduced kernel, and the differentiation is brought inside the integral and onto the unknown current by integration by parts. When collocation (point-matching) is used with this technique and a basis representation for current is chosen which permits slope discontinuities in current, e.g., piecewise constant or piecewise linear representation , convergence is relatively slow. Convergence can usually be improved by a somewhat more complicated choice of basis functions having no slope discontinuities. Finally, a relatively expensive testing procedure, such as Galerkin's method, may be used to treat the derivatives and to accelerate convergence.

In this paper, we present a method for treating the differential operator which is simple, economical, and which renders the solution of Pocklington's equation asymptotically convergent at the same rate as that of Hallén's equation with point-matching. The procedure is also approximately equivalent in the method of moments to testing with either piecewise linear or piecewise sinusoidal functions. Furthermore, it is demonstrated that testing of Pocklington's equation with piecewise sinusoids results in a slight modification of the method which is exactly equivalent to Hallén's equation with point-matching for any choice of basis functions and to Richmond's sinusoidal reaction matching procedure [2] when piecewise sinusoids are also chosen as the basis functions for the current. Even though the method enjoys the high rate of convergence of other more sophisticated techiques, it is computationally equivalent to, and as simple to apply as, point-matching.

Considered briefly are extensions of the basic idea to integral equations for non-parallel wires and planar surfaces.

STRAIGHT WIRE

We consider a thin cylindrical dipole of radius a formed by a perfectly conducting tube of length L and driven at the center by a delta-gap generator of V volts. Requiring the z-component of the tangential electric field to vanish on the conductor results in Pocklington's equation,

(1a) $$\left[\frac{d^2}{dz^2} + k^2\right] A_z(z) = -j\omega\mu\varepsilon V\,\delta(z), \qquad -L/2 \leq z \leq L/2 ,$$

where

(1b) $$A_z(z) = \frac{\mu}{4\pi}\int_{-L/2}^{+L/2} I(z')G(z-z')dz'$$

and the kernel is given by

(1c) $$G(w) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-jk\sqrt{w^2+4a^2\sin^2\phi'/2}}}{\sqrt{w^2+4a^2\sin^2\phi'/2}}\,d\phi' .$$

In the above expressions I(z) is the unknown total axial current, μ and ε are the permeability and permittivity, respectively, of the medium surrounding the tube, k is the wave-number or 2π/(wavelength in the medium), and ω is the angular frequency of the suppressed time dependence $\exp(j\omega t)$.

For thin wires, one often uses the reduced kernel approximation [3],

$$G(w) \approx K(w) = \frac{e^{-jk\sqrt{w^2+a^2}}}{\sqrt{w^2+a^2}} \tag{1d}$$

The vector potential A_z is evaluated on the surface of the tube and is assumed to be circumferentially independent. We therefore exhibit only the z dependence in writing (1a).

In the simplest procedure, we point-match Eq. (1a) to obtain the set of equations

$$\left[\frac{d^2}{dz^z} + k^2\right] A_z(z)\bigg|_{z=z_m} = \frac{-j\omega\mu\varepsilon V}{\Delta z}\,\delta_{mo}\ , \qquad m=0, \pm 1, \cdots, \pm N\ , \tag{2}$$

where $z_m = m\Delta z$, $\Delta z = L/(2N+2)$,

and δ_{mo} is the Kronecker delta function. To obtain the right side of (2), the delta function of (1a) is approximated by a rectangular pulse of unit area distributed over the subdomain m=0, i.e., over the interval $[-\Delta z/2, \Delta z/2]$, and the resulting driving field is sampled at the center of the region.

Hallén's equation is obtained from (1a) in the usual way and enforced at the match points z_m to arrive at

$$A_z(z_m) = B\cos kz_m + C\sin kz_m - \frac{j\omega\mu\varepsilon}{2k} V \sin k|z_m|\ , \tag{3a}$$

$$m=0,\ \pm 1,\ \cdots, \pm (N+1)\ .$$

We also now require the current boundary conditions

(3b) $$I(\pm L/2) = 0 \quad .$$

When the unknown current is expanded in terms of a set of basis functions $f_n(z)$,

(4) $$I(z) = \sum_{n=-N}^{N} I_n f_n(z) \quad ,$$

and substituted into (2) or (3), a matrix equation results for the unique determination of the unknown current expansion coefficients I_n. We assume in the following that the expansion (4) is chosen so that (3b) is automatically satisfied.

Returning to (1), we may choose to replace the derivative operator with the difference approximation

(5) $$\left[\frac{d^2}{dz^2} + k^2\right] A_z(z) \Big|_{z=z_m} \simeq \left[\frac{\Delta^2}{\Delta z^2} + k^2\right] A_z(z) \Big|_{z=z_m}$$

$$= \frac{A_z(z_{m+1}) - 2(1-k^2\Delta z^2/2)A_z(z_m) + A_z(z_{m-1})}{\Delta z^2} \quad .$$

This form has been used by Tesche [4] and others with good success. One line of reasoning suggesting this choice begins with a comparison of Eqs.(2) and(3). In Fig.1b we represent qualitatively both sides of Eq.(3) in the case where the expansion functions are assumed to be of the subdomain type. The resulting vector potential $A_z(z)$ ripples somewhat, but at precisely the match points z_m the potential is required to be equal to the right hand side of (3a) which is a smooth function. For the purposes of illustration we regard the constants B and C as already having been determined by the two equations $m=\pm(N+1)$ in (3a). Ideally, of course, the left- and right-hand sides should be equal for all z, but this is unlikely in an approximate solution. We note further that because of the rippled form of the vector potential, as represented by the left-hand side of (3a) with subdomain basis functions, derivatives of A_z will be completely inaccurate (compared to those of the "correct" vector potential) unless the subdomain size is extremely small. Yet, Eq. (2) may be interpreted as a constraint on a weighted sum of the vector potential and its second derivative at the match point. The inaccuracy of the second derivative of the potential produces resultant fields which are extremely peaked at the match points. This is shown by Fig. 1c which illustrates qualitatively both sides of Eq. (1) (subject to (1d)) with the pulse approximation for the delta function and a point-matching solution. Because of the unphysical behavior of the left-hand side of (1) due to the slope discontinuities in current, the solution of Pocklington's equation (2) tends to converge slowly as compared to that of Hallén's equation (3) in which the unnatural slope discontinuities are effectively smoothed. The choice of (5) to approximate the harmonic operator might be expected to result in improved convergence for Pocklington's equation since the derivative is calculated by "numerical differentiation" which, because of its gross sampling, is insensitive to the small scale ripples introduced by the unphysical representation of the current (subdomain basis functions).

According to the procedure, we replace the derivative operator in (2) with approximation (5), which results in the integro-difference equation

$$A_z(z_{m+1})-2(1-k^2\Delta z^2/2)A_z(z_m) + A_z(z_{m-1}) = -j\omega\mu\varepsilon V\Delta z\delta_{mo}, \tag{6}$$

$$m=0, \pm 1, \cdots, \pm N ,$$

whose difference equation solution is easily verified to be

$$A_z(z_m) = B \cos m\Theta + C \sin m\Theta - j\frac{\omega\mu\varepsilon V}{2k} \cdot \frac{\sin|m\Theta|}{\sqrt{1-(k\Delta z/2)^2}} , \tag{7}$$

$$m=0, \pm 1, \cdots , \pm(N+1) ,$$

where $\Theta=\cos^{-1}(1-k^2\Delta z^2/2)$.

Eq. (7) corresponds closely to Hallén's equation (3a) as can be seen by noting that for N large $k\Delta z<<1$ and we may write

$$\cos\Theta \simeq 1-\frac{\Theta^2}{2!} = 1- \frac{k^2\Delta z^2}{2}$$

from which we conclude that

$$\lim_{N\to\infty} \Theta = k\Delta z .$$

In a practical sense, the limit is approached rather quickly as illustrated in Table 1. When this limit is substituted into (7), Hallén's (point-matched) equation (3a) is obtained. When a basis set is chosen and a solution of (7) is obtained, we observe that it must necessarily satisfy (6) and vice versa. We conclude that a solution of (6) will very quickly converge to the corresponding solution of Hallén's equation with point-matching. Furthermore, comparison of (6) and (3a) shows that no extra calculations or integrations are needed in the integro-difference equation formulation compared to Hallén's formulation and, in addition, the two unknown constants B and C no longer appear. Figure 2 illustrates the convergence of the admittance of a dipole antenna as calculated from Eqs. (3) and (6). Because the dipole is near resonant length, the susceptance behaves rather poorly, even for the Hallén solution. Nevertheless, the convergence of the solution of the integro-difference equation (6) to the Hallén solution is evident.. As verification of the statement concerning the equality of solutions to (6) and (7), Eq. (7) was also solved and the current distributions were found to be equal within machine accuracy. The basis set chosen in this case was a set of piecewise linear or triangle functions defined by

(8)
$$T_m(z) = \begin{cases} \dfrac{\Delta z - |z - z_m|}{\Delta z}, & z_{m-1} \le z \le z_{m+1}, \\ 0, & \text{otherwise.} \end{cases}$$

However, we emphasize that the conclusions above are independent of the choice of basis functions. Also shown for comparison in Figure 2 is the convergence of the admittance obtained from a solution of (2) with piecewise sinusoidal basis functions,

(9)
$$S_m(z) = \begin{cases} \dfrac{\sin k(\Delta z - |z - z_m|)}{\sin k\Delta z}, & z_{m-1} \le z \le z_{m+1}, \\ 0, & \text{otherwise,} \end{cases}$$

which approximate the piecewise linear functions (8) for small Δz. Because of the aforementioned problems concerned with using (2), the convergence in this case is rather poor. The use of this basis set and its attendant problems have been investigated by Pearson and Butler [5].

We can achieve an even closer correspondence between the integro-difference equation and Hallén's equation, if we replace the factor $(1-k^2\Delta z^2/2)$ appearing in Eq. (6) by cos $k\Delta z$, from which it differs negligibly for small Δz (N large). The resulting integro-difference equation is

(10) $$A_z(z_{m+1}) - 2\cos k\Delta z\, A_z(z_m) + A_z(z_{m-1}) = -j\omega\mu\varepsilon V\Delta z\,\delta_{mo} ,$$

$$m=0, \pm 1, \cdots, \pm N ,$$

for which we can write the difference equation solution immediately:

(11) $$A_z(z_m) = B\cos k\, z_m + C\sin k\, z_m - \frac{j\omega\mu\varepsilon V}{2k}\left[\frac{k\Delta z}{\sin k\Delta z}\right]\sin k|z_m| ,$$

$$m=0, \pm 1, \cdots, \pm (N+1).$$

This form differs from (3a) only in the parenthetical terms of the particular solution of (10). Hence, the solution of (10) and (11) differs from the solution of Hallén's point-matched equation by a factor $(\sin k\Delta z)/k\Delta z$ for any number of subdomains (of testing set).

An approach based upon an entirely different viewpoint yields conclusions similar to those above and is even more illuminating. We define the inner product [1] (or more accurately, the reaction) between quantities g and h,

(12) $$\langle g, h\rangle \equiv \int_{-L/2}^{L/2} g(z)h(z)dz ,$$

and use the piecewise-sinusoidal functions defined in (9) to test, in the sense of the method of moments, the Pocklington equation,

(13)
$$\left[\frac{d^2}{dz^2} + k^2\right] A_z(z) = -j\omega\mu\varepsilon E_z^i(z) \quad ,$$

where E_z^i is an arbitrary impressed excitation of which (1a) is simply the specialization to a delta-gap source. Hence we wish to evaluate

(14)
$$\left\langle \left[\frac{d^2}{dz^2} + k^2\right] A_z(z),\ S_m(z) \right\rangle = -j\omega\mu\varepsilon \left\langle E_z^i(z),\ S_m(z) \right\rangle \quad .$$

Integrating by parts twice, one may write the left-hand side of (14) as

$$\frac{k}{\sin k\Delta z} \int_{-L/2}^{L/2} \left[\delta(z-z_{m+1}) - 2\cos k\Delta z\,\delta(z-z_m) + \delta(z-z_{m-1}) \right] A_z(z)dz$$

from which we easily obtain the integro-difference equation

(15)
$$A_z(z_{m+1}) - 2\cos k\Delta z\, A_z(z_m) + A_z(z_{m-1})$$
$$= -\frac{j\omega\mu\varepsilon}{k} \int_{z_{m-1}}^{z_{m+1}} E_z^i(z) \sin k(\Delta z - |z-z_m|)dz \ ,$$

$$m=0,\ \pm 1,\ \cdots,\ \pm N \quad .$$

By direct substitution, it may be verified that a solution to the difference equation (15) is

(16) $$A_z(z_m) = B \cos k z_m + C \sin k z_m - \frac{j\omega\mu\varepsilon}{k} \int_0^{z_m} E_z^i(z) \sin k(z_m - z) dz \quad ,$$

$$m=0, \pm 1, \cdots, \pm(N+1) \quad ,$$

which is precisely the point-matched Hallén's equation for arbitrary excitation. Hence, we reach the rather surprising but important conclusion that testing with piecewise sinusoids results in an integro-difference equation whose solution is identical to that of Hallén's equation with point-matching, independent of the basis set chosen to represent the current.

The fact that this result holds for arbitrary excitation is also important because Hallén's equation, as does now the integro-difference equation (15), has the advantage that the incident field appears under an integral so that the influence of a rapidly varying incident field is properly reflected in the solution. With any of the usual point-matching procedures and Pocklington's equation, often the excitation is not sampled adequately [5].

It is also of interest to note that the use of piecewise sinusoids, Eq. (9), as a basis set in (15) is precisely Richmond's so-called piecewise sinusoidal reaction matching, and one can obtain his formulas for the reaction between two sinusoidal dipoles [6] from the left-hand side of (15) and the integral

(17) $$\int_{z_1}^{z_2} \frac{e^{\pm jkz'} e^{-jk\sqrt{a^2+(z-z')^2}}}{\sqrt{a^2+(z-z')^2}} dz'$$

$$= \mp e^{\pm jkz} [\mathrm{Ci}(ku_2) + j\mathrm{Si}(ku_2) - \mathrm{Ci}(ku_1) - j\mathrm{Si}(ku_1)] ,$$

$$u_i = \mp(z-z_i) - \sqrt{a^2+(z-z_i)^2} , \quad i=1,2,$$

where Ci and Si are cosine and sine integrals, respectively. The procedure leading to (15) has the characteristics of testing with functions other than delta functions and, indeed, is Galerkin's method if piecewise sinusoids are used to represent the current, yet (15) possesses all the computational simplicity usually attributed only to point-matching.

Finally, it is observed that testing of (1a) with piecewise linear functions (8) leads directly to (6) subject only to the approximation,

$$\int_{z=z_{m-1}}^{z=z_{m+1}} A_z(z)T_m(z)dz \simeq \Delta z\, A_z(z_m)$$

which draws attention to the fundamental similarity of such testing and the difference operator approximation of (5) in Pocklington's equation. Furthermore, the above observation leads one to recognize that a solution technique in which the current is represented by piecewise linear functions in the integro-difference equation (6) is almost equivalent to Galerkin's method. Similar observations can be made relative to (13) and (15) subject to small argument approximations for the sine and cosine functions in (15). In summary, for $k\Delta z \ll 1$, the piecewise sinusoid approaches the piecewise linear function so that numerically one finds them interchangable in the various schemes discussed above and conclusions relative to one hold for the other.

GENERALIZATION TO CO-PLANAR CURRENTS

In the following, we consider the generalization of the ideas of the preceding section to co-planar currents, i.e. currents in a plane or in co-planar wires. Without loss of generality, we restrict our consideration to scattering by a rectangular conducting plate S of length 2a and width 2b. Consideration of co-planar wires is obtained merely by appropriately defining the components of the vector potential. Integral equations obtained by satisfying boundary conditions on the plate are

(18a) $$\left[\frac{\partial^2}{\partial x^2} + k^2\right]A_x(x,y) + \frac{\partial^2}{\partial x \partial y} A_y(x,y) = -j\omega\mu\varepsilon\, E_x^i(x,y) \quad ,$$

$$(x,y) \in S \,,$$

(18b) $$\left[\frac{\partial^2}{\partial y^2} + k^2\right]A_y(x,y) + \frac{\partial^2}{\partial y \partial x} A_x(x,y) = -j\omega\mu\varepsilon\, E_y^i(x,y) \quad ,$$

where

(19) $$A_p(x,y) = \frac{\mu}{4\pi} \int_{-b}^{b} \int_{-a}^{a} J_p(x',y')G(x-x', y-y')dx'dy' \quad , p=x,y \,,$$

(20) $$G(u,v) = \frac{e^{-jk\sqrt{u^2+v^2}}}{\sqrt{u^2+v^2}} \quad .$$

The terms with superscript i refer to incident field quantities which we assume to be evaluated on S. Next we define an inner product (or reaction integral)

(21) $$\langle f,g \rangle \equiv \int_{-b}^{b} \int_{-a}^{a} f(x,y)g(x,y)\, dxdy$$

and choose the set of testing functions

(22a)
$$S^{(1)}_{mn}(x,y) = \begin{cases} \dfrac{\sin k(\Delta x - |x-x_m|)}{\sin k\Delta x}\,\delta(y-y_n) & , x_{m-1} \le x \le x_{m+1} \ , \\ 0 \ , & \text{otherwise} \ , \end{cases}$$

(22b)
$$S^{(2)}_{mn}(x,y) = \begin{cases} \dfrac{\sin k(\Delta y - |y-y_n|)}{\sin k\Delta y}\,\delta(x-x_m) & , y_{n-1} \le y \le y_{n+1} \ , \\ 0 \ , & \text{otherwise} \ , \end{cases}$$

for testing (18a) and (18b), respectively. For the rectangular plate, we define the rectangular grid of points,

$$\Delta x = \frac{a}{M+1} \ , \quad \Delta y = \frac{b}{N+1} \ ,$$

$$x_m = m\Delta x \ , \quad y_n = n\Delta y \ ,$$

$$m = 0, \pm 1, \cdots, \pm(M+1) \ ,$$

$$n = 0, \pm 1, \cdots, \pm(N+1) \ .$$

We then form the inner product between (22a) , (22b), and (18a), (18b), respectively. For example, from (18a) and (22a), there results

(23)
$$A_x(x_{m+1},y_n) - 2\cos k\Delta x\, A_x(x_m,y_n) + A_x(x_{m-1},y_n)$$
$$+ \frac{1}{k}\left[\frac{\partial}{\partial y} \int_{x_{m-1}}^{x_{m+1}} \frac{\partial A_y}{\partial x} \sin k(\Delta x - |x-x_m|)\,dx \right]_{y=y_n}$$
$$= -\frac{j\omega\mu\varepsilon}{k} \int_{x_{m-1}}^{x_{m+1}} E^i_x(x,y_n) \sin k(\Delta x - |x-x_m|)\,dx \ ,$$

$$m=0, \pm 1, \cdots, \pm M \quad ,$$

$$n=0, \pm 1, \cdots, \pm N \quad .$$

A similar result follows from (18b). Again, the difference equation for A_x in (23) may readily be shown to be equivalent to the point-matched Hallén-like form obtainable from (18a):

(24)
$$A_x(x_m,y_n) + \frac{1}{k}\left[\frac{\partial}{\partial y}\int_0^{x_m} \frac{\partial A_y}{\partial x} \sin k(x_m - x)\, dx\right]_{y=y_n}$$

$$= B_n \cos kx_m + C_n \sin kx_m - \frac{j\omega\mu\varepsilon}{k}\int_0^{x_m} E_x^i(x,y_n) \sin k(x_m - x)\, dx,$$

$$m = 0, \pm 1, \ldots, \pm(M+1),$$

$$n = 0, \pm 1, \ldots, \pm(N+1).$$

Note that a distinct set of constants B_n, C_n is required along every constant coordinate y_n for which a solution of the difference equation (23) is formed. Eq. (24) also requires the boundary conditions

(25)
$$J_x(x_m,y_n) = 0 \quad , \quad m = \pm(M+1),\ n = 0, \pm 1, \cdots, \pm(N+1) \quad ,$$

in order to uniquely determine the solution. Finally, we note that the triplet of integrals appearing in the terms in parentheses in (23) when (19) is substituted can be reduced to a double integral with the aid of (17). The result is

(26) $$\frac{1}{k}\left[\frac{\partial}{\partial y}\int_{x_{m-1}}^{x_{m+1}} \frac{\partial A_y}{\partial x} \sin k(\Delta x-|x-x_m|)dx\right]_{y=y_n}$$

$$= D_x(x_{m+1},y_n) - 2\cos k\Delta x D_x(x_m, y_n) + D_x(x_{m-1}, y_n)$$

where

$$D_x(x, y) = -\frac{\mu}{4\pi}\int_{-b}^{b}\int_{-a}^{a} J_y(x',y')\left[\frac{x-x'}{y-y'}\right]G(x-x', y-y')dx'dy'$$

Equations similar to (23)-(26) are obtainable by testing Eq.(18b) with (22b) and proceeding as above. Equation (23) and the corresponding equation obtained from (18b) must be solved simultaneously.

Eq. (23) with (26) is much more convenient to solve than Eq. (24) since the latter necessarily incorporates additional unknowns in the form of constants of integration.

As another example, we consider an alternative formulation of the plate problem due to Mittra <u>et al</u>. [7] who begin with a set of equations originally obtained by Bouwkamp [8]. The equations used by Mittra are

(27a) $$\left[\nabla_t^2 + k^2\right]A_x(x, y) = \mu \frac{\partial H_y^i}{\partial z} ,$$

$$(x,y) \varepsilon S ,$$

(27b) $$\left[\nabla_t^2 + k^2\right]A_y(x, y) = - \mu \frac{\partial H_x^i}{\partial z} ,$$

where

$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and $\overline{A} = A_x\hat{x} + A_y\hat{y}$ is defined in Eq. (19). These equations do not uniquely determine the induced current without use of an auxiliary condition

(28) $$H_z^i + \frac{1}{\mu}\frac{\partial A_y}{\partial x} - \frac{1}{\mu}\frac{\partial A_x}{\partial y} = 0 \quad , (x, y)\varepsilon C \quad ,$$

where C denotes the boundary of S. In their work, Mittra et al. obtain from Eqs. (27a) and (27b), Hallen-like equations which contain two arbitrary functions in the homogeneous part of the solution to the partial differential equation. Application of the condition (28) determines a relationship between the two functions, and the boundary condition constraining the current normal to the edge of the plate to be zero then uniquely determines these two functions. Point-matching the Hallen-like equation in the interior of S enables one to determine the current when a suitable set of basis functions is chosen.

The principal advantages in numerical solutions of the equations resulting from this formulation over those of the E-field formulation, Eq. (18), are (a) the decoupling of the two components of current appearing in Eq.(27) and (b) the elimination of any partial derivatives of vector potentials except through the auxiliary condition (28).

Again, in order to obtain the advantages of the Hallén-like form of Mittra *et al.* without the additional unknown functions to determine, we consider approximating the partial derivatives (27) by finite differences. In the simplest approximation, we write, for example,

$$\left.\frac{\partial^2 A_x(x, y)}{\partial x^2}\right|_{\substack{x=x_m \\ y=y_n}} \simeq \frac{A_x(x_{m+1},y_n) - 2A_x(x_m,y_n) + A_x(x_{m-1},y_n)}{\Delta x^2}$$

and similarly for $\partial^2/\partial y^2$.

Hence Eq.(27a) becomes

$$\begin{aligned} &\left[A_x(x_{m+1},y_n) - 2A_x(x_m, y_n) + A_x(x_{m-1}, y_n)\right]/\Delta x^2 \\ (29) \qquad + &\left[A_x(x_m,y_{n+1}) - 2A_x(x_m, y_n) + A_x(x_m, y_{n-1})\right]/\Delta y^2 \\ &+ k^2 A_x(x_m, y_n) = \mu \frac{\partial}{\partial z} H_y^i(x_m,y_n) \end{aligned}$$

and a similar equation is obtainable for A_y. Eq.(28) is then enforced by approximating the first partial derivatives appearing therein by the appropriate difference approximations calculated on the bounding contour C.

While a direct comparison of solutions obtained using (29) versus the Hallén form used by Mittra *et al.* is not possible, we can make an indirect comparison. This is done by comparing homogeneous solutions of (27) with those of (29). Such a comparison is inclusive and complete because equality of homogeneous solutions implies equality of particular solutions since the latter can, in principle, always be synthesized from the former. We find below, however, that equality between homogeneous solutions of the differential equations (27) and the difference equations (29) obtains only in the limit of decreasing subdomain size. The relationship between the resulting solutions is the two dimensional analog of the relationship between Eqs.(3) and (6).

A spectral component of homogeneous solutions of (27), when point-matched, has the form

(30) $$e^{-jk_x m\Delta x} e^{-jk_y n\Delta y}, \quad k_x^2 + k_y^2 = k^2 \quad ,$$

and a general homogeneous solution may be synthesized by means of a weighted integral of such components. Homogeneous solutions of the partial difference equation (29), on the other hand, have as spectral components

(31) $$e^{-j\alpha m\Delta x} e^{-j\beta n\Delta y} \quad ,$$

where the constants α and β are related by the condition

(32) $$\frac{\cos\alpha\Delta x-1}{\Delta x^2} + \frac{\cos\beta\Delta y-1}{\Delta y^2} + \frac{k^2}{2} = 0$$

as may be verified by direct substitution of (31) into the homogeneous form of (29). For that portion of the spectrum such that $\alpha\Delta x \ll 1$ and $\beta\Delta y \ll 1$, the condition (32) becomes approximately

(33) $$\alpha^2 + \beta^2 = k^2 \quad .$$

Hence, for small Δx and Δy, homogeneous solutions of (27) and (29) are very nearly equal over a large portion of the spectrum. We point out that (29) is also obtainable by testing (27a) with the two-dimensional piecewise linear function

(34) $$T_{mn}(x,y) = \begin{cases} \left[\dfrac{\Delta x - |x-x_m|}{\Delta x}\right]\left[\dfrac{\Delta y - |y-y_n|}{\Delta y}\right], & \begin{array}{l} x_{m-1} \le x \le x_{m+1} \\ y_{n-1} \le y \le y_{n+1} \end{array} \\ 0 \quad , \text{otherwise} \end{cases} ,$$

when one takes note of the approximations of the form,

(35) $$\int_{y_{n-1}}^{y_{n+1}} A_x(x_m, y)\left[\frac{\Delta y - |y-y_n|}{\Delta y}\right] dy \simeq \Delta y \, A_x(x_m, y_n) ,$$

together with the corresponding approximations for integration over the interval $[x_{m-1}, x_{m+1}]$.

Similar to the observations made in the one-dimensional case, one sees that there exists a close relationship between the finite difference approximation of the (partial) differential-integral equation and a corresponding Hallén-like equation with point-matching. Again, a close relationship between the original partial differential equation tested with either piecewise sinusoids or piecewise linear testing functions and the corresponding Hallén-like equation is established. Furthermore, the resulting integro-difference equation enjoys the rapid convergence and incorporates the relatively simple kernel of Hallén-like equations without the added complications due to the attendant unknown constants of integration.

CONCLUSIONS

It is shown above that testing Pocklington's integral equation with piecewise sinusoids results in a system of linear, integro-difference equations whose numerical solution is identical to the collocation solution of Hallen's equation for any choice of basis functions. This formulation in terms of integro-difference equations enjoys the advantages normally associated with collocation solutions of Hallen's equation, which are listed below.

(1) The method exhibits the same rapid convergence rate associated with solutions of Hallen's equation.

(2) Only well-behaved kernels (exact) need be calculated numerically.

(3) The method admits the use with equal ease of any form of excitation, e.g., delta-gap voltage source and incident field illumination, and assures that the forcing function is adequately sensed [5].

(4) The simplicity of collocation is retained.

On the other hand, the method does not suffer the major disadvantage of Hallén-type equations; specifically, in the technique, no complicating arbitrary constants of integration are introduced--the system of difference equations retains the boundary conditions of the problem. We also mention that, when one uses piecewise sinusoids for basis functions as well as for testing, the method suggested above readily specializes to Richmond's piecewise sinusoidal reaction matching technique.

One draws an equivalence between piecewise linear testing of Pocklington's equation and the approximation of its derivative operator by the corresponding difference operator. Furthermore, piecewise linear and piecewise sinusoidal testing yield systems of integro-difference equations which approach a common limit as the number of testing functions is increased. Hence, in this limit, observations pertinent to one hold for the other testing set.

Conclusions similar to those above hold for two-dimensional integral equations discussed under Generalization to Co-Planar Currents. However, in view of the large matrices encountered in handling planar structures, it is even more important to have a solution technique which is simple, which converges rapidly, and yet which does not introduce additional unknowns.

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Table 1. Illustration of the convergence of the values of Θ to kΔz with decreasing subdomain size.

$k\Delta z$	$\Theta = \cos^{-1}\left[1 - \frac{k^2 \Delta z^2}{2}\right]$
1.5000	1.6961
1.3000	1.4152
1.0000	1.0472
0.8000	0.8230
0.6000	0.6094
0.5000	0.5054
0.4000	0.4027
0.2500	0.2507
0.1250	0.1251
0.6000×10^{-1}	0.6001×10^{-1}
0.6000×10^{-2}	0.6000×10^{-2}

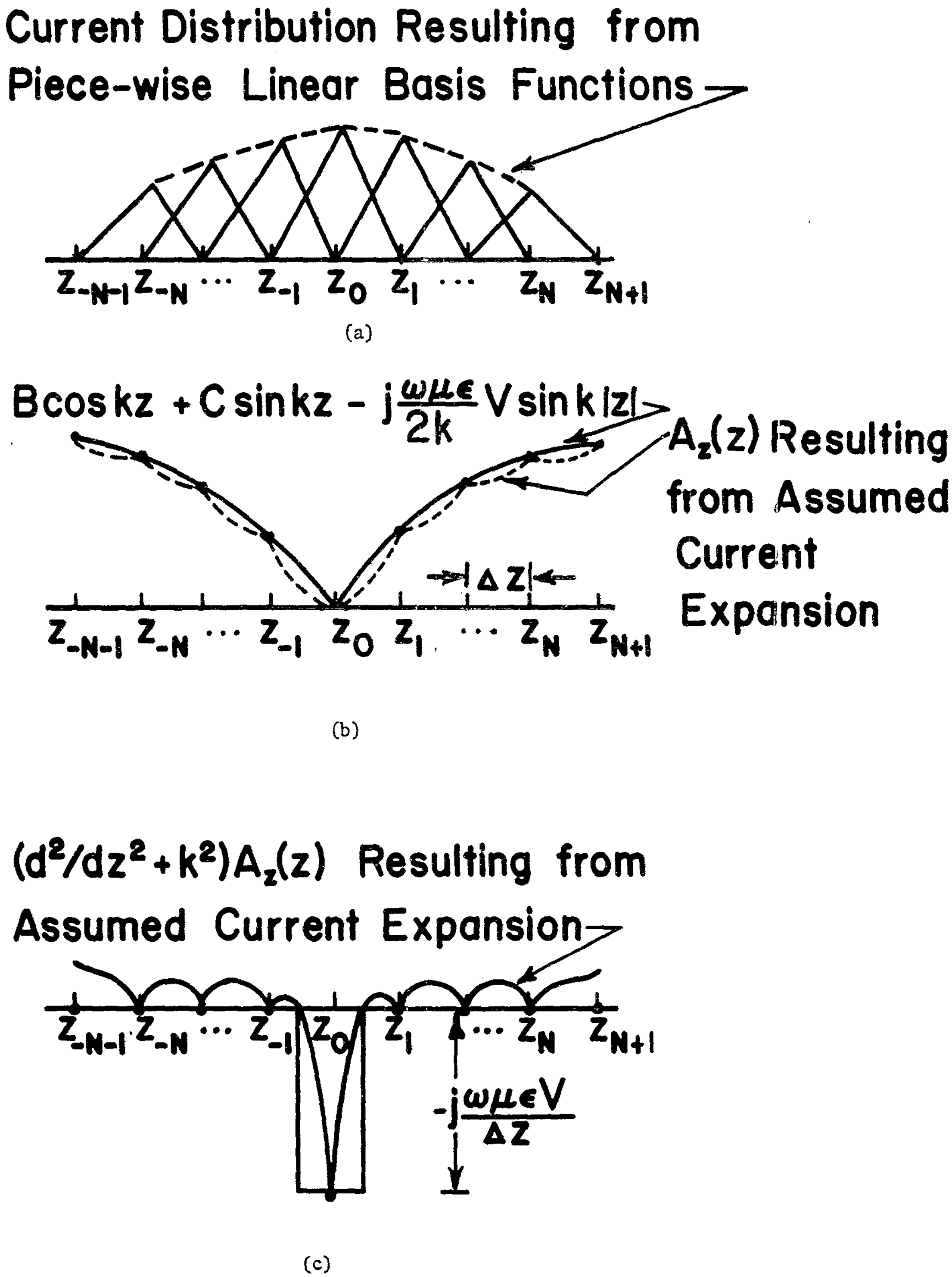

Figure 1. Illustration of (a) piecewise linear current representation, (b) a point-matching solution of Hallen's equation, and (c) a point-matching solution of Pocklington's equation.

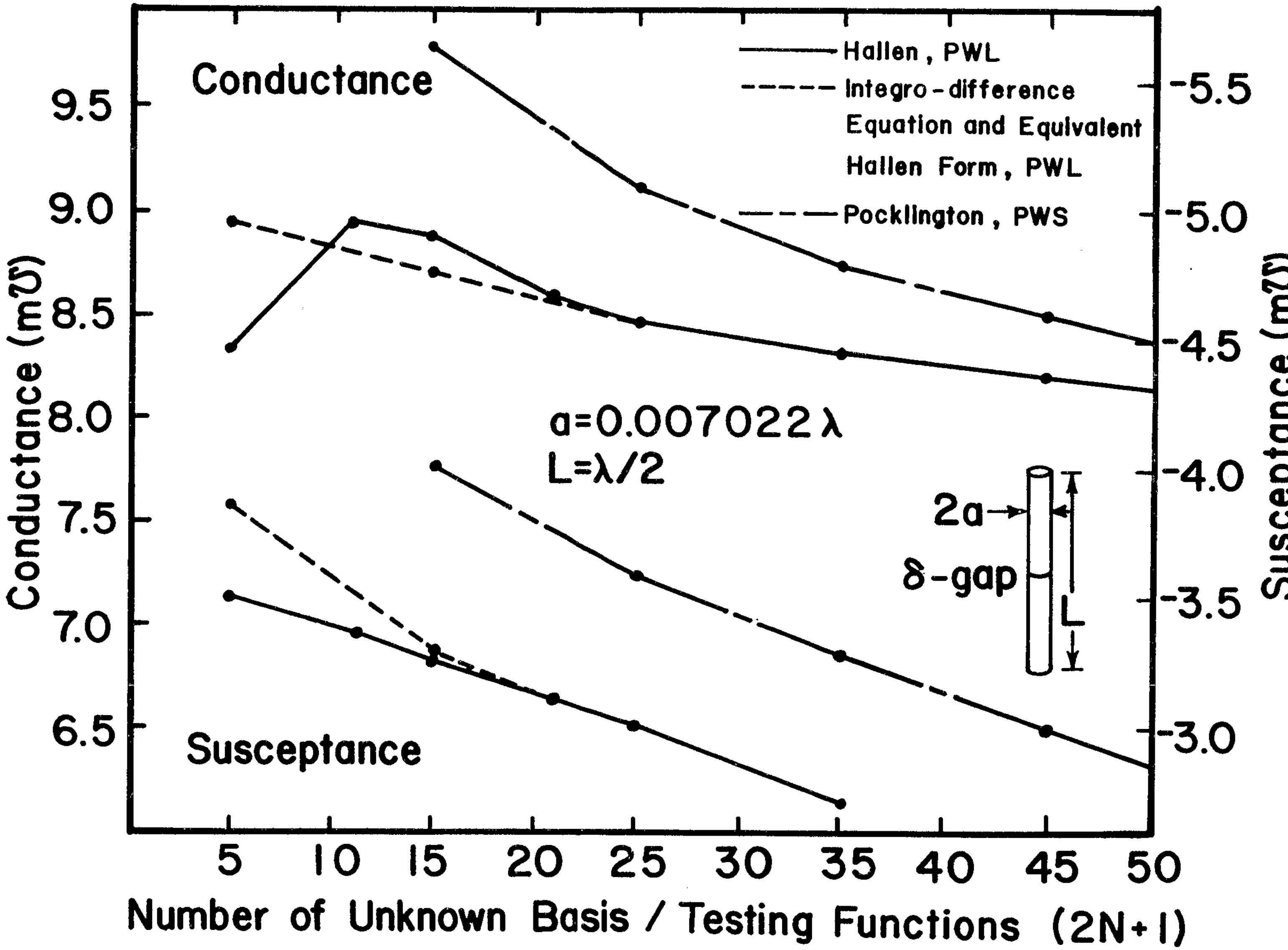

Figure 2. Comparison of the convergence rates of dipole admittance calculated from Eqs. (3), (6), and (7) with piecewise linear current representation and Eq. (2) with piecewise sinusoidal current representation.