IN 242

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER HDL-TR-1678	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) An Integral Equation for the Interior Electromagnetic Field of a Semi-Infinite Cylindrical Waveguide in a Conducting Half-Space Excited by a Normally Incident Plane Wave		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL REPORT 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) DANIEL J. SPOHN		8. CONTRACT OR GRANT NUMBER(s) DA-1T061101A91A
9. PERFORMING ORGANIZATION NAME AND ADDRESS Harry Diamond Laboratories Washington, D. C. 20438		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Prog. Ele. 6.11.01 A
11. CONTROLLING OFFICE NAME AND ADDRESS Commander, U.S. Army Materiel Command 5001 Eisenhower Avenue Alexandria, VA 22333		12. REPORT DATE DECEMBER 1974 13. NUMBER OF PAGES 24
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

HDL Project: A102E8
AMCMS Code: 501A1184400

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Dyadic Green's functions
Integral equations
Semi-infinite cylindrical waveguide

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An integral expression for the electromagnetic field inside a semi-infinite cylindrical waveguide flanged by a perfectly conducting plane resulting from a plane wave excitation is derived using a dyadic Green's function approach. The exterior magnetic field is expressed in terms of the magnetic field which would be present in the absence of an aperture and a surface integral over the aperture which involves the aperture electric field and the dyadic Green's function. The interior magnetic field is expressed solely in terms

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) of an integral over the aperture that involves the aperture electric field and the appropriate Green's dyad for the semi-infinite cylindrical waveguide. Continuity of the tangential magnetic field in the aperture leads to an integral equation for the aperture electric field. All appropriate dyadic Green's functions have been constructed by applying the superposition theorem and boundary conditions to known dyadic Green's functions for ideal geometries.

FOREWORD

This work was funded under the HDL In-house Laboratory Independent Research (ILRI) Program. The author is grateful to Dr. John Bombardt for his encouragement and continuous guidance during the course of this work, and to Professor C-T. Tai of the University of Michigan for his consultation regarding construction of the appropriate dyadic Green's function.

TABLE OF CONTENTS

ILLUSTRATION

1. PROBLEM FORMULATION

Consider the geometry consisting of a semi-infinite, cylindrical waveguide flanged by a perfectly conducting plane. The surface of the flange is the ρ, ϕ plane at $z = 0$ and the axis of the cylindrical waveguide extends along the z-axis from $z = 0$ to $z = +\infty$; its radius is "a" (fig. 1).

For convenience of discussion, we define two regions; region I is the half-space with $z < 0$ bounded by a conducting plane at $z = 0$, and region II is the interior of the cylindrical waveguide bounded by the conducting walls at $\rho = a$. A plane wave propagates from $z = -\infty$ and is normally incident on the aperture of the waveguide. It is desired to obtain an expression for the diffracted field in region II.

The incident fields can be expressed as,

$$\begin{aligned} \overline{E}^{inc}(R) &= \overline{e} \exp\{-ikz\} \exp\{-i\omega t\} \\ \overline{H}^{inc}(R) &= \overline{h} \exp\{-ikz\} \exp\{-i\omega t\} \end{aligned} \tag{1}$$

which is assumed to be normally incident on the aperture.

The scattered field at any point is then governed by the free-space Maxwell's equations

$$\left.\begin{aligned} \nabla\cdot\overline{D} &= 0 \qquad \nabla\times\overline{E} = -\frac{\partial\overline{B}}{\partial t} = i\omega\mu_o\overline{H} \\ \nabla\cdot\overline{B} &= 0 \qquad \nabla\times\overline{H} = -i\omega\varepsilon_o\overline{E} \end{aligned}\right\} \tag{2}$$

and subject to the boundary condition

$$\hat{n} \times \overline{E}(R) = 0 \qquad R \text{ on a surface} \tag{3}$$

where $\hat{n}$ is a unit vector normal to any of the conducting surfaces.

In the formulation of the problem let the electric field in region I be expressed as

$$\overline{E}^{(1)}(R) = \overline{E}^{(0)}(R) + \overline{E}^{A}(R) \tag{4}$$

where

$\overline{E}^{(0)}(R) \equiv$ the field due to an incident field, plus a field reflected from the conducting half-space,

and

$\overline{E}^{A}(R) \equiv$ the remaining part of the scattered field due to the presence of the aperture.

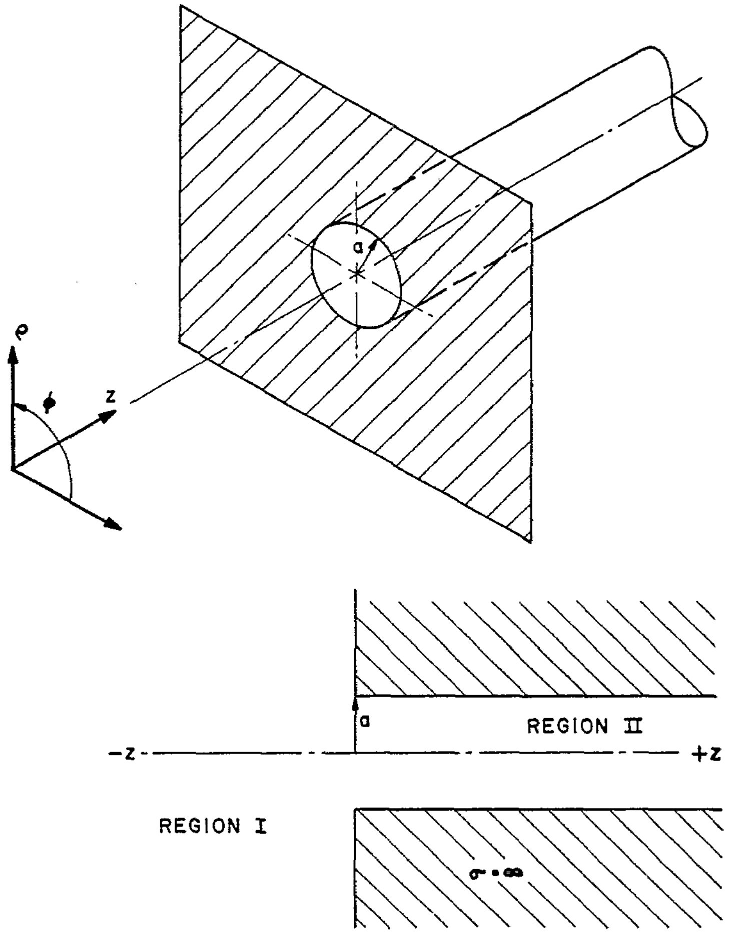

Figure 1. Semi-infinite waveguide flanged by a perfectly conducting plane.

In region II the diffracted electric field is expressed as $\overline{E}^{(2)}(R)$. In addition, the scattered field at any location must satisfy the homogeneous vector wave equations.

$$\left.\begin{aligned} \nabla \times \nabla \times \overline{E}(R) - k^2 \overline{E}(R) &= 0 \\ \text{and} \qquad & \\ \nabla \times \nabla \times \overline{H}(R) - k^2 \overline{H}(R) &= 0 \end{aligned}\right\} \qquad (5)$$

The approach employed consists of deriving appropriate dyadic Green's functions for regions I and II and then obtaining integral expressions for the electric field in the regions by applying the vector form of Green's theorem. Then, by applying Maxwell's equations, expressions for the magnetic field in both regions can be obtained. Observing that the tangential components of the magnetic field are continuous through the aperture results in an integral equation for the aperture electric field. The substitution of the solution to the integral equation into the integral expression for the electric field in region II and solution of the integral results in an expression for the electric field inside the cylinder. The corresponding magnetic field can then be found by application of Maxwell's equations.

2. DERIVATION OF INTEGRAL EQUATIONS

The integral equations are formulated using the general approach described by Tai.[1] Essentially, this approach involves integrating the vector wave equations by employing Green's theorem.

Prior to the solution of the equations, the properties of the Green functions that are employed must be discussed. Two types of dyadic Green's functions are used in the solution of electromagnetic boundary value problems; $\bar{\bar{G}}_e$ is an electric-type function and $\bar{\bar{G}}_m$ is a magnetic type. These functions satisfy the following equations.

$$\nabla \times \bar{\bar{G}}_e = \bar{\bar{G}}_m \tag{6}$$

and

$$\nabla \times \bar{\bar{G}}_m - \bar{\bar{I}}\delta(R - R') + k^2\bar{\bar{G}}_e \tag{7}$$

where $\bar{\bar{I}}$ = a unit dyad (Idemfactor)

and $k^2 = \omega^2\mu_o\varepsilon_o$.

Also, by vector manipulation

$$\nabla \times \nabla \times \bar{\bar{G}}_e - k^2\bar{\bar{G}}_e = \bar{\bar{I}}\delta(\bar{R} - \bar{R}') \tag{8}$$

and

$$\nabla \times \nabla \times \bar{\bar{G}}_m - k^2\bar{\bar{G}}_m = \nabla \times [\bar{\bar{I}} (\bar{R} - \bar{R}')] \tag{9}$$

It can also be shown that

$$\nabla \cdot \bar{\bar{G}}_m = 0 \tag{10}$$

and

[1]Tai, C-T., Dyadic Green's Functions in Electromagnetic Theory, Intext Educational Publishers, Scranton, Pa., 1971, Section 17.

$$\nabla\cdot\overline{\overline{G}}_e = -\frac{1}{k^2}\nabla\cdot\overline{\overline{I}}\delta(\overline{R}-\overline{R}') \quad . \tag{11}$$

Dyadic Green's function can be classified further as to the boundary conditions that they satisfy. Functions satisfying Dirichlet boundary conditions denoted with the subscript 1, satisfy

$$\left.\begin{aligned} \hat{n}\times\overline{\overline{G}}_{e1} &= 0 \\ \hat{n}\times\overline{\overline{G}}_{m1} &= 0 \end{aligned}\right\} \tag{12}$$

evaluated on the boundary. Functions satisfying Neumann boundary conditions, denoted by the subscript 2, satisfy

$$\left.\begin{aligned} \hat{n}\times\nabla\times\overline{\overline{G}}_{e2} &= 0 \\ \hat{n}\times\nabla\times\overline{\overline{G}}_{m2} &= 0 \end{aligned}\right\} . \tag{13}$$

From the above equations, the following relationships can be generated:

$$\left.\begin{aligned} \nabla\times\overline{\overline{G}}_{e1} &= \overline{\overline{G}}_{e2} \\ \nabla\times\overline{\overline{G}}_{m2} &= \overline{\overline{I}}\delta(\overline{R}-\overline{R}') + k^2\overline{\overline{G}}_{e1} \\ \nabla\times\overline{\overline{G}}_{e2} &= \overline{\overline{G}}_{m1} \\ \nabla\times\overline{\overline{G}}_{m1} &= \overline{\overline{I}}\delta(\overline{R}-\overline{R}') + k^2\overline{\overline{G}}_{e2} \end{aligned}\right\} . \tag{14}$$

In formulating the integral expressions, we start with the statement of Green's theorem as given by Stratton[2]

$$\iiint_V [\overline{P}\cdot\nabla\times\nabla\times\overline{Q} - (\nabla\times\nabla\times\overline{P})\cdot\overline{Q}]\, dV = -\oint\!\!\oint_S \hat{n}\cdot[\overline{P}\times\nabla\times\overline{Q} + (\nabla\times\overline{P})\times\overline{Q}]\, dS. \tag{15}$$

[2]Stratton, J. A., Electromagnetic Theory, McGraw Hill, New York, 1941, p. 250.

By letting $\overline{Q}$ be equal to any of the three vector functions, $\overline{Q}_x$, $\overline{Q}_y$, $\overline{Q}_z$, and by defining the dyadic function

$$\overline{\overline{Q}} = \overline{Q}_x\hat{x} + \overline{Q}_y\hat{y} + \overline{Q}_z\hat{z}$$

an appropriate dyadic Green's theorem can be obtained. Then by letting $\overline{P} = \overline{E}^{(1)}$ and $\overline{\overline{Q}} = \overline{\overline{G}}_{e1}^{(1)}$, we have

$$\iiint_{V_1}\left[\overline{E}^{(1)}\cdot\nabla\times\nabla\times\overline{\overline{G}}_{e1}^{(1)} - (\nabla\times\nabla\times\overline{E}^{(1)})\cdot\overline{\overline{G}}_{e1}^{(1)}\right] dV = \oint\!\!\oint_S \hat{n}\cdot\left[\overline{E}^{(1)}\times\nabla\times\overline{\overline{G}}_{e1}^{(1)}\right.$$

$$\left. + (\nabla\times\overline{E}^{(1)})\times\overline{\overline{G}}_{e1}^{(1)}\right] dS \,. \qquad (16)$$

From equations (5) and (8) we have

$$\nabla\times\nabla\times\overline{E}(R) = k^2\overline{E}(R)$$

and

$$\nabla\times\nabla\times\overline{\overline{G}}(R) = k^2\overline{\overline{G}}(R) + \overline{\overline{I}}\delta(R - R') \,.$$

So, by employing these relationships and integrating over the volume, we obtain

$$\overline{E}(R) = \oint\!\!\oint_{S'} \hat{n}\times\overline{E}(R')\cdot\left[\nabla\times\overline{\overline{G}}_{e1}(R'|R)\right] dS' \,. \qquad (17)$$

Further vector manipulation and application of the transposition property of dyadic Green's functions,

$$\widetilde{\overline{\overline{G}}_{e1}(R'|R)} = \overline{\overline{G}}_{e1}(R|R')$$

and

$$\widetilde{\nabla'\times\overline{\overline{G}}_{e1}(R'|R)} = \nabla\times\overline{\overline{G}}_{e2}(R|R')$$

yields

$$\overline{E}(R) = -\oint\!\!\oint_{S'}\left[\nabla\times\overline{\overline{G}}_{e2}(R|R')\right]\cdot[\hat{n}\times\overline{E}(R')]\, dS' \qquad (18)$$

as a general expression for the field in a source-free region. However, for region I, the assumed incident and reflected plane waves must be added to the right-hand side of equation (15). This procedure is consistent with the method employed by Levine and Schwinger[3] in solving a similar problem. So, for the field in region I, we have

$$E^{(1)}(R) = E^{(o)}(R) - \oint\!\!\oint_{S'} \left[\nabla \times \overline{\overline{G}}_{e2}(R|R')\right] \cdot [\hat{n}' \times \overline{E}(R')]\, dS .$$

Note that the surface over which the integration is to be carried out consists of the ρ, ϕ plane at $z = 0$ where $n \times \overline{E}(R) = 0$ everywhere except in the aperture and over the hemisphere at $R = \infty$, where there is no contribution because of the asymptotic properties of the Green function and the radiation condition. Consequently, the only contribution to the surface integral is over the aperture. Therefore, for region I,

$$\overline{E}^{(1)}(R) = \overline{E}^{(o)}(R) - \oint\!\!\oint_{A'} \left[\nabla \times \overline{\overline{G}}_{e2}(R|R')\right] \cdot [\hat{n}' \times \overline{E}(R')]\, dA' \qquad (19)$$

where A' is the aperture in the conducting plane.

By taking the curl of equation (19) and applying Maxwell's equations, we obtain the corresponding expression for the magnetic field in region I,

$$\overline{H}^{(1)}(R) = \frac{1}{i\omega\mu_o} \Big\{ \nabla \times \overline{E}^{(o)}(R) - \oint\!\!\oint_{A'} \overline{I}\delta(R - R') \cdot \hat{n}' \times \overline{E}(R')\, dA' - \iint_{A'} k^2 \overline{\overline{G}}^{(1)}_{e2}(R|R') \cdot \hat{n}' \times \overline{E}(R')\, dA' \Big\} . \qquad (20)$$

Observing that there is no current source in region II and applying the boundary condition and radiation condition to equation (18), we obtain

$$\overline{E}^{(2)}(R) = - \oint\!\!\oint_{A'} \left(\nabla \times \overline{\overline{G}}^{(2)}_{e2}\right) \cdot \hat{n} \times \overline{E}^{(2)}(R')\, dA' \qquad (21)$$

with

$$\overline{H}^{(2)}(R) = - \frac{1}{i\omega\mu_o} \iint_{A'} \overline{\overline{I}}\delta(R - R')\hat{n}' \times \overline{E}(R')\, dA' + k^2 \oint\!\!\oint_{A'} G^{(2)}_{e2}(R|R') \cdot \hat{n}' \times \overline{E}(R')\, dA' . \qquad (22)$$

Since the tangential components of the magnetic field are continuous through the aperture, we have

[3]Levine, H. and J. Schwinger, On the Theory of Electromagnetic Wave Diffraction by an Aperture in an Infinite Plane Conducting Screen, Comm. Pure and Applied Mathematics III, 1950.

$$\hat{n} \times \overline{H}^{(1)}(R)|_{z=0} = \hat{n} \times \overline{H}^{(2)}(R)|_{z=0} \ . \tag{23}$$

By taking the cross product of the normal with equations (20) and (22) and equating the expressions we obtain

$$\hat{n} \times \left(\nabla \times \overline{E}^{o}(R)\right)|_{z=0} = \oint_{A} \hat{n} \times \left[\overline{\overline{G}}^{(1)}_{e2}(R|R') - \overline{\overline{G}}^{(2)}_{e2}(R|R')\right] \cdot \hat{n}' \times E(R') \, dA' \tag{24}$$

defined within the aperture.

However, if

$$\overline{K}(R) \equiv \hat{n} \times \nabla \times \overline{E}^{o}(R)$$

is defined to be the apparent surface current on the aperture, there is

$$\overline{K}(R) = k^2 \oint_{A'} \hat{n} \times \left[\overline{\overline{G}}^{(1)}_{e2}(R|R') - \overline{\overline{G}}^{(2)}_{e2}(R|R')\right] \cdot \hat{n}' \times \overline{E}(R') \, dA' . \tag{25}$$

By solving equation (25) for the aperture electric field $E(R')$ and substituting the result into equation (21), an expression for the electric field in the cylindrical waveguide is obtained. Then by applying Maxwell's equation (22), the accompanying magnetic field can be found.

3. CONSTRUCTION OF DYADIC GREEN'S FUNCTIONS

In the formulation of the integral equation, several dyadic Green's functions were employed without being expressed explicitly. Since only two of the functions actually appear in the final integral equation, only those need be derived explicitly. Those to be derived are $\overline{\overline{G}}^{(1)}_{e2}(\overline{R}|\overline{R}')$ and $\overline{\overline{G}}^{(2)}_{e2}(\overline{R}|\overline{R}')$.

An approach that may be used is an eigenfunction expansion technique involving vector functions for the solution of the equation

$$\nabla \times \nabla \times \overline{\overline{G}} - k^2 \overline{\overline{G}} = \overline{\overline{I}}\delta(R - R') \quad . \tag{26}$$

This method, known as the Ohm-Rayleigh technique is explained by Tai, and is used extensively by him in deriving dyadic Green's function for many idealized geometries.[1,4,5]

---

[1]Tai, C-T., Dyadic Green's Functions in Electromagnetic Theory, Intext Educational Publishers, Scranton, Pa., 1971.

[4]Tai, C-T., On the Eigenfunction Expansion of Dyadic Green's Functions, Proc. of IEEE, Volumn 61, No. 4, April 1973, pg. 480-481.

[5]Tai, C-T., On the Eigenfunction Expansion of Dyadic Green's Functions, Report 011136-1-T, Radiation Laboratory, University of Michigan, April 1973.

Instead of rigorously applying the Ohm-Rayleigh technique, the dyadic Green's functions of interest can be constructed by applying the principle of superposition, employing those functions already derived by Tai.

In the case of region I, the free space dyadic Green's function can be used.[5]

$$\bar{\bar{G}}_{eo}(R|R') = \frac{i}{4\pi}\left\{\int_0^\infty d\lambda \sum_n \frac{(2-\delta_0)}{\lambda k_\lambda}\left(\bar{M}_{\substack{e\\o}n\lambda}(\pm k_\lambda)\bar{M}'_{\substack{e\\o}n\lambda}(\mp k_\lambda) + \bar{N}_{\substack{e\\o}n\lambda}(\pm k_\lambda)\bar{N}'_{\substack{e\\o}n\lambda}(\mp k_\lambda)\right)\right\} - \frac{\hat{z}\hat{z}\delta(R-R')}{k^2} . \tag{27}$$

The upper sign corresponds to the situation when $z<z'$ and the lower sign when $z>z'$.

Where

$$k_\lambda = (k^2 - \lambda^2)^{1/2}$$

$$\bar{M}_{\substack{e\\o}n}(k_\lambda) = \left[\mp \frac{nJ_n(\lambda\rho)}{\rho} \begin{matrix}\sin\\ \cos\end{matrix} n\phi\hat{\rho} - \frac{\partial}{\partial\rho} J_n(\lambda\rho) \begin{matrix}\cos\\ \sin\end{matrix}\phi n\ \hat{\phi}\right] e^{ik_\lambda z}$$

and

$$\bar{N}_{\substack{e\\o}n\lambda}(k_\lambda) = \frac{1}{k_\lambda}\left[ik_\lambda \frac{\partial}{\partial\rho} J_n(\lambda\rho) \begin{matrix}\cos\\ \sin\end{matrix} n\phi\hat{\rho} \mp \frac{ik_\lambda n}{\rho} J_n(\lambda\rho) \begin{matrix}\sin\\ \cos\end{matrix} n\phi\ \hat{\phi} + \lambda^2 J_n(\lambda\rho) \begin{matrix}\cos\\ \sin\end{matrix} n\phi\ \hat{z}\right] e^{ik_\lambda z} \tag{28}$$

with $\lambda = q_{mn/a}$ and $J_n(q_{mn}) = 0$

and

$$\kappa_\lambda = \left(\lambda^2 + k_\lambda^2\right)^{1/2} .$$

---

[5]Tai, C-T., On the Eigenfunction Expansion of Dyadic Green's Functions, Report 011136-1-T, Radiation Laboratory, University of Michigan, April 1973, pg 15.

The Green function for the half-space of region I can be constructed by assuming that it is composed of two components, a free-space Green function and a scattered Green function

$$\bar{\bar{G}}_{e1}^{(1)} = \bar{\bar{G}}_{eo} + \bar{\bar{G}}_{es} \quad . \tag{29}$$

By applying the boundary condition that at $z = 0$, $\hat{z} \times \bar{\bar{G}}_{e1}^{(1)} = 0$, the scattered Green function, $\bar{\bar{G}}_{es}$, is found to be

$$\bar{\bar{G}}_{es}^{(1)} = \frac{i}{4\pi} \int_0^\infty d\lambda \sum_n \frac{(2 - \delta_o)}{\lambda k_\lambda} \Big\{ - \bar{M}_{\substack{e\\o}n\lambda}(k_\lambda)\, \bar{M}'_{\substack{e\\o}n\lambda}(k_\lambda)$$

$$+ \bar{N}_{\substack{e\\o}n\lambda}(k_\lambda)\, \bar{N}_{\substack{e\\o}n\lambda}(k_\lambda) \Big\} \quad . \tag{30}$$

So

$$\bar{\bar{G}}_{e1}^{(1)}(R|R') = \frac{i}{4\pi} \int_0^\infty d\lambda \sum_n \frac{(2 - \delta_o)}{\lambda k_\lambda} \Big\{ \bar{M}_{\substack{e\\o}n\lambda}(\pm k_\lambda)\, \bar{M}_{\substack{e\\o}n\lambda}(\mp k_\lambda)$$

$$+ \bar{N}_{\substack{e\\o}n\lambda}(\pm k_\lambda)\, \bar{N}'_{\substack{e\\o}n\lambda}(\mp k_\lambda) - \bar{M}_{\substack{e\\o}n\lambda}(k_\lambda)\, \bar{M}'_{\substack{e\\o}n\lambda}(k_\lambda)$$

$$+ \bar{N}_{\substack{e\\o}n\lambda}(k_\lambda)\, \bar{N}'_{\substack{e\\o}n}(k_\lambda) \Big\} - \frac{\hat{z}\hat{z}\, \delta(R - R')}{k^2} \quad , \tag{31}$$

with the upper sign corresponding to the situation with $z<z'$ and the lower sign with $z>z'$.

Now, to determine $\bar{\bar{G}}_{e2}^{(1)}(R|R')$ we apply the transposition property

$$\widetilde{\nabla \times \bar{\bar{G}}_{e1}(R|R')} = \nabla' \times \bar{\bar{G}}_{e2}(R'|R) \quad . \tag{32}$$

By noting that the vector functions are related by

$$\nabla \times \bar{N}_{\substack{e\\o}n\lambda}(k_\lambda) = \kappa_\lambda\, \bar{M}_{\substack{e\\o}n\lambda}(k_\lambda) \tag{33}$$

and

$$\nabla \times \overline{M}_{\substack{e\\o}n\lambda}\left(k_\lambda\right) = \kappa_\lambda \overline{N}_{\substack{e\\o}n\lambda}\left(k_\lambda\right) ,$$

and performing the appropriate vector manipulation we have

$$\overline{\overline{G}}^{(1)}_{e_2}(R|R') = \frac{i}{4\pi}\int_0^\infty d\lambda \sum_n \frac{\left(2-\delta_{on}\right)}{\lambda k_\lambda}\left[\overline{N}_{\substack{e\\o}n\lambda}\left(\pm k_\lambda\right)\overline{N}'_{\substack{e\\o}n\lambda}\left(\mp k_\lambda\right)\right.$$

$$+ \overline{M}_{\substack{e\\o}n\lambda}\left(\pm k_\lambda\right)\overline{M}'_{\substack{e\\o}n\lambda}\left(\mp k_\lambda\right) - \overline{N}_{\substack{e\\o}n\lambda}\left(k_\lambda\right)\overline{N}'_{\substack{e\\o}n\lambda}\left(k_\lambda\right)$$

$$\left. + \overline{M}_{\substack{e\\o}n\lambda}\left(k_\lambda\right)\overline{M}'_{\substack{e\\o}n\lambda}\left(k_\lambda\right)\right] - \frac{\hat{z}\hat{z}\ \delta(R-R')}{k^2} , \qquad (35)$$

but with the upper sign corresponding to the situation with $z>z'$ and the lower sign with $z<z'$.

In region II, the semi-infinite, cylindrical waveguide, a similar superposition scheme can be employed, this time using the interior dyadic Green's function for an infinite cylinder.[5]

$$\overline{\overline{G}}_{e_1 i}(R|R') = \frac{i}{4\pi}\sum_{m=1}^{\infty}\sum_{n=o}^{\infty}\left(2-\delta_{on}\right)\left[\frac{1}{\lambda^2 k_\lambda I_\lambda}\overline{N}_{\substack{e\\o}n\lambda}\left(\pm k_\lambda\right)\right.$$

$$\left. + \frac{1}{\mu^2 k_\mu I_\mu}\overline{M}_{\substack{e\\o}n\mu}\left(\pm k_\mu\right)\overline{M}'_{\substack{e\\o}n\mu}\left(\mp k_\mu\right)\right] - \frac{\hat{z}\hat{z}\ \delta(R-R')}{k^2} \qquad (36)$$

with the $\binom{\text{upper}}{\text{lower}}$ sign corresponding to the situation with $\binom{z>z'}{z<z'}$,

where

$$k_\lambda = (k^2-\lambda^2)^{1/2}$$

$$k_\mu = (k^2-\mu^2)^{1/2} \qquad k = \frac{\omega}{c}$$

$$I_\lambda = \frac{a^2}{2} J'_n\left(\lambda_a\right)$$

$$I_\mu = \frac{a^2}{2\mu^2}\left(\mu^2 - \frac{\mu^2}{a^2}\right)J_n^2(\mu a)$$

[5]Tai, C-T., On the Eigenfunction Expansion of Dyadic Green's Functions, Report 011136-1-T, Radiation Laboratory, University of Michigan, April 1973, pg 13.

$$\lambda = q_{mn/a} \text{ with } J_n\left(q_{mn}\right) = 0$$

$$\mu = p_{mn/a} \text{ with } J_n'\left(p_{mn}\right) = 0$$

As before, we can assume that the dyadic Green's function can be written in terms of a known dyadic function and a scattered term. So, in region II, the dyadic Green function can be written as

$$\overline{\overline{G}}_{e1}^{(2)} = \overline{\overline{G}}_{e1i} + \overline{\overline{G}}_{e1s} \tag{37}$$

By applying the boundary conditions that

$$\hat{z} \times \overline{\overline{G}}_{e1}^{(2)}\Big|_{z = 0} = 0 ,$$

it is found that

$$\overline{\overline{G}}_{e1s} = \sum_m \sum_n C_\lambda \overline{N}_{\substack{e\\o}n\lambda}\left(-k_\lambda\right) \overline{N}'_{\substack{e\\o}n\lambda}\left(-k_\lambda\right)$$

$$- C_\mu \overline{M}_{\substack{e\\o}n\mu}\left(-k_\mu\right) \overline{M}_{\substack{e\\o}n\mu}\left(-k_\mu\right) \tag{38}$$

where

$$C_\lambda \equiv \frac{i\left(2 - \delta_{on}\right)}{4\pi\mu^2 k_\lambda I_\lambda}$$

and

$$C_\mu \equiv \frac{i\left(2 - \delta_{on}\right)}{4\pi\mu^2 k_\mu I_\mu}$$

So the dyadic Green's function for region II can be written

$$\overline{\overline{G}}_{e1}^{(2)} = \sum_m \sum_n \Big\{ C_\lambda \Big[\overline{N}_{\substack{e\\o}n\lambda}(\pm k_\lambda)\, \overline{N}'_{\substack{e\\o}n\lambda}(\mp k_\lambda)$$

$$+ \overline{N}_{\substack{e\\o}n\lambda}(-k_\lambda)\, \overline{N}'_{\substack{e\\o}n\lambda}(-k_\lambda) \Big]$$

$$+ C_\mu \Big[\overline{M}_{\substack{e\\o}n\mu}(\pm k_\mu)\, \overline{M}_{\substack{e\\o}n\mu}(\mp k_\mu)$$

$$- \overline{M}_{\substack{e\\o}n\mu}(-k_\mu)\, \overline{M}_{\substack{e\\o}n\mu}(-k_\mu) \Big] \Big\}$$

$$- \frac{\hat{z}\hat{z}\ \delta(R - R')}{k^2} \qquad (39)$$

where again $\binom{\text{upper}}{\text{lower}}$ sign corresponds to the condition $\binom{z>z'}{z<z'}$.

Now recall from before that dyadic Green's function have the property

$$\nabla' \times \overline{\overline{G}}_{e1}(R'|R) = \nabla \times \overline{\overline{G}}_{e2}(R|R') \quad . \qquad (40)$$

By using this property and the vector relations between the vector functions $\overline{M}_{\substack{e\\o}n\mu}(h)$ and $\overline{N}_{\substack{e\\o}n\mu}(h)$, we obtain

$$\overline{\overline{G}}_{e2}^{(2)} = \sum_m \sum_n \Big\{ C_\lambda \Big[\overline{M}_{\substack{e\\o}n\lambda}(\pm k_\lambda)\, \overline{M}'_{\substack{e\\o}n\lambda}(\mp k_\lambda)$$

$$+ \overline{M}_{\substack{e\\o}n\lambda}(k_\lambda)\, \overline{M}'_{\substack{e\\o}n\lambda}(-k_\lambda) \Big]$$

$$+ C_\mu \Big[\overline{N}_{\substack{e\\o}n\mu}(\pm k_\mu)\, \overline{N}'_{\substack{e\\o}n\mu}(\mp k_\mu)$$

$$- \overline{N}_{\substack{e\\o}n\mu}(-k_\mu)\, \overline{N}'_{\substack{e\\o}n\mu}(-k_\mu) \Big] \Big\}$$

$$- \frac{\hat{z}\hat{z}\ \delta(R - R')}{k^2} \qquad (41)$$

where the $\begin{pmatrix} \text{upper} \\ \text{lower} \end{pmatrix}$ sign correspond to the situation where $\begin{pmatrix} z>z' \\ z<z' \end{pmatrix}$.

Equations (39) and (41) express the two dyadic Green's function required for the integrand of equation (25).

LITERATURE CITED

(1) Tai, C-T., Dyadic Green's Functions in Electromagnetic Theory, Intext Educational Publishers, Scranton, Pa, 1971.

(2) Stratton, J. A., Electromagnetic Theory, McGraw-Hill, New York, 1941, pg. 250.

(3) Levine, H. and J. Schwinger, On the Theory of Electromagnetic Wave Diffraction by an Aperture in an Infinite Plane Conducting Screen, Comm. Pure and Applied Mathematics III, 1950.

(4) Tai, C-T., On the Eigenfunction Expansion of Dyadic Green's Functions, Proc. of IEEE, Volume 61, No. 4, April 1973, pg. 480-481.

(5) Tai, C-T., On the Eigenfunction Expansion of Dyadic Green's Functions, Report 011136-1-T, Radiation Laboratory, University of Michigan, April 1973.