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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The problem of an infinite wire above a plane infinite ground has been investigated, with a view toward studying the types of waves that this structure can support with no incident radiation. We find that, although a propagation constant can be meaningfully defined, the concepts of impedance and admittance per unit length cannot be simply generalized from transmission-line theory. The difficulty associated with the solutions of the two-dimensional

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Helmholtz equation is discussed, and the outgoing waves that increase in amplitude exponentially with the distance from the source are reasonably interpreted.

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1. INTRODUCTION

A very important set of problems in electromagnetic theory centers on phenomena involving the generation and propagation of current waves in a single long wire parallel to the earth. In practice, the concern is with specific terminations and a specific set of excitation conditions, via a series or shunt generator, or via an incident wave (which, in EMP problems, is a broadband pulse). Since this is a very general and difficult problem, this study is limited to a simple related case.

Assume that the wire is infinitely long and very thin (its diameter is small compared with other linear dimensions in the problem). Simple harmonic fields are considered that can propagate along this structure, with no incident radiation. This has been done previously by Wait,[1] Kikuchi,[2] dos Santos,[3] and Carson.[4] The main concern here lies in the determination of the propagation constant and the attempts to define transmission-line constants, such as the impedance and admittance per unit length. Also considered is an approach to the puzzling choice between outgoing waves that increase exponentially in amplitude with distance from the wire and incoming waves that decrease exponentially.

The fields of the wire above ground are determined in section 2 by matching boundary conditions for a perfectly conducting wire[1] and for a wire of finite conductivity.[2] These boundary conditions lead to an equation for the propagation constant Γ in terms of the frequency ω. Section 3 reviews the concepts of impedance and admittance for transmission lines and decribes why they are not generally applicable to the present problem. This is not in agreement with Wait[1] and Kikuchi.[2] It is concluded, however, that Carson's work[4] is valid but limited to low frequencies. Section 4 presents some limiting cases from Wait,[1] which could be of interest for applications. Electric and magnetic Hertz vectors are discussed in appendix A. A derivation of the Green function for the two-dimensional Helmholtz operator is given in appendix B, with special emphasis on the behavior of incoming and outgoing waves far from the source; some of the problems associated with the definition of potential differences and currents for high-frequency fields are discussed in appendix C.

Rationalized MKSA units are used and the time-dependence of harmonic fields is chosen to be $\exp(i\omega t)$.

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[1] *J. R. Wait, Radio Science 7, 675 (1972).*
[2] *H. Kikuchi, Electrotech. J. Japan 2, 73 (1956).*
[3] *A. F. dos Santos, Proc. IEEE 119, 1103 (1972).*
[4] *J. R. Carson, Bell Sys. Tech. J. 5, 539 (1926).*

2. FIELDS OF A WIRE ABOVE GROUND

2.1 Fields of a Line Current

First, the fields generated by a current in an infinite wire of radius a located at a height h above a semi-infinite ground with a plane boundary have to be determined. The coordinate system was chosen here in such a way that the boundary is the yz-plane and the wire axis--running in the z-direction--is located at the coordinates $x = h$, $y = 0$.

Considering only waves propagating in the z-direction, the symmetry of the configuration is such that the z-dependence of all fields can be separated as a factor $\exp(-\Gamma z)$, where

$$\Gamma = i\beta \tag{1}$$

is a complex constant. If the positive z-axis is chosen in the direction of the wave propagation, the factor $\exp[-i(\beta z - \omega t)]$ shows that the real part of β has to be positive, whereas the existence of losses implies that its imaginary part has to be negative.

The solution is based on the fields of a line current in an infinite homogeneous medium. The magnetic field does not have a component along the wire, so that the only need is an electric Hertz vector of the form

$$\vec{\Pi}(\vec{x}) = \Pi^{p}(\vec{x})\hat{e}_3 = \exp(-i\beta z) f(x,y)\hat{e}_3, \tag{2}$$

where Π^{p} satisfies equation (A-18); consequently, f obeys

$$\begin{aligned}(\partial^2/\partial x^2 + \partial^2/\partial y^2 + k^2 - \beta^2) f(x,y) \\ = (i/\varepsilon\mu) I \delta(y)\delta(x-h).\end{aligned} \tag{3}$$

This is essentially the Green function for the two-dimensional Helmholtz equation, discussed in appendix B. Assume that

$$\mathrm{Re}(k^2 - \beta^2) > 0; \tag{4}$$

otherwise, we would be dealing with an evanescent cylindrical wave. Being concerned with the response of the system to a current in the wire, a solution was sought that corresponds to outgoing waves, which is given by equation (B-11). The fact that this solution increases exponentially with large R, as discussed in appendix B and dos Santos,[3] limits its validity to a region up to a distance from the wire large enough to be in the radiation zone but not so large that this exponential factor changes the nature of the solution. The resulting potential is

> Note: The italicized numerals parenthesized on the left side of mathematical formulas in this report represent reference and equation numbers--that is, numbers "(1-3)" denote that our equation (5) is basically the same as equation (3) in reference 1 by Wait.[1] The italicized numerals marked with an asterisk (as in our eq 11) denote a disagreement between the referenced equation and our equation.

(1-3)
$$\Pi^p(\vec{x}) = -(1/4\varepsilon\omega)I\exp(-i\beta z)H_0^{(2)}[\sqrt{k^2-\beta^2}\sqrt{(x-h)^2+y^2}], \tag{5}$$

which, according to equation (B-15), can be rewritten as

(1-4)
$$k^2\Pi^p(\vec{x}) = -(i\mu\omega I/4\pi)\exp(-\Gamma z) \times \int_{C_-} d\lambda u^{-1}\exp(-u|x-h|)\exp(-i\lambda y), \tag{6}$$

where

(1-4')
$$u = \sqrt{\lambda^2+\beta^2-k^2} \tag{7}$$

The field Π^p has the correct source for the problem of the wire above ground, and the boundary conditions can be satisfied by means of solutions of the homogeneous equation of the same general form.

2.2 Perfectly Conducting Wire

Wait[1] addresses the problem of a perfectly conducting wire in a medium of (possibly complex) constants ε_1, μ_1 separated by a plane from another medium characterized by ε_2, μ_2. The corresponding quantities are then distinguished by indices 1 or 2, so that

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[1]*J. R. Wait, Radio Science* 7, *675 (1972).*
[3]*A. F. dos Santos, Proc. IEEE* 119, *1103 (1972).*

$$k_n^2 = \varepsilon_n \mu_n \omega^2 , \qquad n = 1,\ 2 , \tag{8}$$

$$u_n = \sqrt{\lambda^2+\beta^2-k_n^2} , \tag{9}$$

$$\kappa_n^2 = k_n^2 - \beta^2 \tag{10}$$

and so on. It is assumed that the Hertz potentials described in appendix A have the form

(1-5)*

$$\Pi_1 = A \int_{C_-} d\lambda u_1^{-1} \{\exp(-u_1 |x-h|) + R(\lambda) \exp[-u_1(x+h)]\} \exp(-i\lambda y) , \tag{11}$$

(1-6)*

$$\Pi_1' = A \int_{C_-} d\lambda u_1^{-1} M(\lambda) \exp[-u_1(x+h)] \exp(-i\lambda y) \tag{12}$$

(1-7)*

$$\Pi_2 = A \int_{C_-} d\lambda u_1^{-1} T(\lambda) \exp(-u_1 h+u_2 x) \exp(-i\lambda y) \tag{13}$$

(1-8)*

$$\Pi_2' = A \int_{C_-} d\lambda u_1^{-1} N(\lambda) \exp(-u_1 h+u_2 x) \exp(-i\lambda y) \tag{14}$$

where

$$A = - i(\mu_1 \omega I/4\pi k_1^2) \exp(-\Gamma z) . \tag{15}$$

The functions R, M, T, and N are determined by the boundary conditions, and the path C_- is chosen so that outgoing waves are obtained in both media as shown in figure 1. The first term in Π_1 corresponds to a source at x = h, y = 0; whereas the terms containing R, M, T, and N are solutions of the homogeneous Helmholtz equation. The ohmic currents in the ground are taken into account through the complex ε_2 in the usual manner. The factors $u_1^{-1} \exp(-u_1 h)$ could have been included in the functions, since they have no special significance as far as image locations are concerned.

The boundary conditions at x = 0 have to be satisfied for all y and z; they give four independent equations for M, N, R, and T. These conditions represent the continuity of the tangential components of $\vec{E}$ and $\vec{H}$,

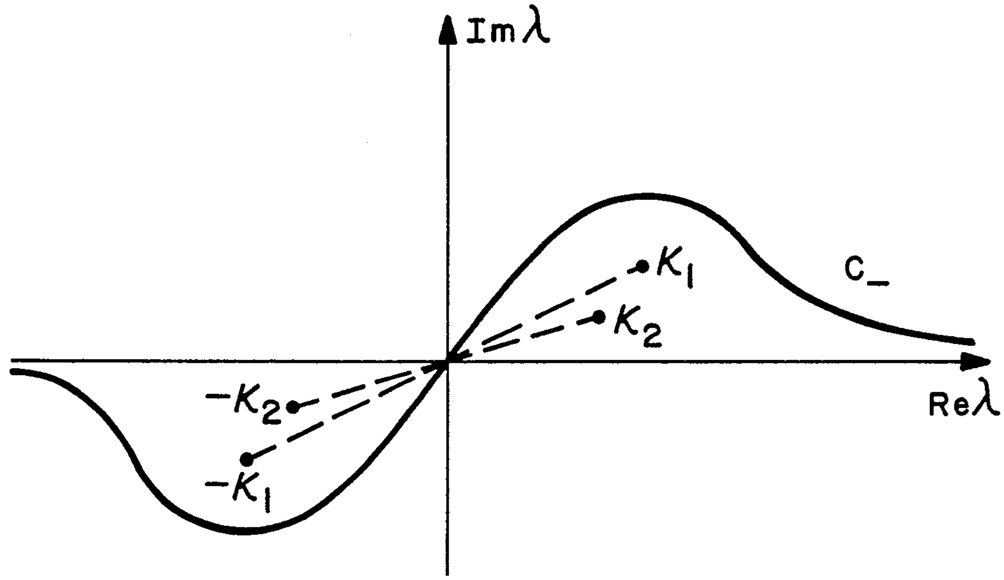

Figure 1. Contour C_ for integrals in equations (11) through (15). Each set of branch points corresponds to one medium; increasing the conductivity of that medium moves the branch points closer to the real axis, and they can eventually cross it for large enough conductivity.

(1-2") $$E_{1y} = E_{2y} \quad \text{at } x = 0, \tag{16}$$

(1-2") $$E_{1z} = E_{2z} \quad \text{at } x = 0, \tag{17}$$

(1-2") $$H_{1y} = H_{2y} \quad \text{at } x = 0, \tag{18}$$

(1-2") $$H_{1z} = H_{2z} \quad \text{at } x = 0. \tag{19}$$

Equations (16) and (17) together with Maxwell's equation for $\nabla \times \vec{E}$ imply that the normal component of $\vec{B}$ is continuous; the discontinuity in the normal component of $\vec{D}$ gives the surface charge. Equations (16) through (19) lead to

(1-11) $$i\lambda\beta(1 + R) - \mu_1 \omega u_1 M = i\lambda\beta T + \mu_2 \omega u_2 N, \tag{20}$$

(1-9) $$\kappa_1^2 (1 + R) = \kappa_2^2 T, \tag{21}$$

(1-12) $$i\lambda\beta M - \varepsilon_1 \omega u_1 \; (1 - R) = i\lambda\beta N - \varepsilon_2 \omega u_2 \; T, \qquad (22)$$

(1-10) $$\kappa_1^2 \; M = \kappa_2^2 \; N \qquad (23)$$

respectively, where equations (A-29), (A-30), (A-32), and (A-33) are used to relate the potentials to the fields. To obtain E_{1z}, we solve for R and find

(1-13) $$R = \frac{\lambda^2\beta^2(1-K)^2+(\varepsilon_1\omega u_1-\varepsilon_2\omega u_2 K)(\mu_1\omega u_1+\mu_2\omega u_2 K)}{-\lambda^2\beta^2(1-K)^2+(\varepsilon_1\omega u_1+\varepsilon_2\omega u_2 K)(\mu_1\omega u_1+\mu_2\omega u_2 K)} \qquad (24)$$

where

(1-13') $$K = \kappa_1^2/\kappa_2^2 \qquad (25)$$

equation (A-31) then gives

(1-13") $$E_{1z} = \kappa_1^2 \Pi_1 \; . \qquad (26)$$

In the case where

(1-13'") $$\mu_1 = \mu_2 = \mu \; , \qquad (27)$$

the expression (24) reduces to

(1-14) $$R(\lambda) = -1 + \frac{2k_1^2}{\kappa_1^2} \times \frac{\lambda^2 - u_1 u_2}{k_1^2 u_2 + k_2^2 u_1} \; , \qquad (28)$$

whence,

(1-15*) $$E_{1z} = -i \; (\mu\omega I/2\pi)\exp(-i\beta z)B(\beta). \qquad (29)$$

The factor $B(\beta)$ can be written in terms of the integral representation of $H_0^{(2)}$ and, since the integral vanishes for an odd function of λ (the path of integration can be chosen symmetric about the origin), it becomes

(1-16)*

$$B(\beta) = (-i\pi/2)\ (1 - \beta^2/k_1^2)\left\{H_0^{(2)}\left[\kappa_1 \sqrt{(x-h)^2+y^2}\right] - H_0^{(2)}\left[\kappa_1 \sqrt{(x+h)^2+y^2}\right]\right\} + \int_{C_-} d\lambda[(\lambda^2-u_1u_2)/(k_1^2u_2+k_2^2u_1)] \times \exp[-(x+h)u_1] \cos \lambda y. \quad (30)$$

The boundary conditions at the wire require that

(1-16')

$$E_{1z} = 0 \quad \text{for} \quad (x-h)^2 + y^2 = a^2. \quad (31)$$

The first term in $B(\beta)$ is constant at this surface; the other terms vary slowly. Consequently, if the radius of the wire is small enough, a good approximation is obtained by setting the field equal to zero at any particular point on the surface. Wait[1] chooses a point at $x = h$, $y = a$, which leads to the modal equation

(1-17)

$$D_1(\beta) = 0\ , \quad (32)$$

where

(1-17)*

$$D_1(\beta) = (-i\pi/2)(1 - \beta^2/k_1^2)\left[H_0^{(2)}(\kappa_1 a) - H_0^{(2)}(\kappa_1 \sqrt{4h^2+a^2})\right] + \int_{C_-} d\lambda[(\lambda^2-u_1u_2)/(k_1^2u_2+k_2^2u_1)]\exp(-2u_1h)\cos\lambda a \quad (33)$$

This equation gives the value of the propagation constant β for a given frequency ω that corresponds to the mode under consideration.

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[1] *J. R. Wait, Radio Science* <u>*7*</u>*, 675 (1972).*

The validity of the above assumption about the boundary condition at the wire can be expressed in terms of the rate at which that part of the field not from the line charge changes over the wire. This is a good approximation if

$$2a \left[\frac{1}{\alpha} \frac{\partial \alpha}{\partial x} \right]_{x=h,\ y=o} << 1 \quad , \tag{34}$$

where

$$\alpha(x,y) = \int_{C_-} d\lambda u_1^{-1} R(\lambda) \exp[-u_1(x+h)] \exp(-i\lambda y) . \tag{35}$$

Alternatively, the locations where the field E_{1z} vanishes can be determined, which would correspond to a slightly deformed wire.

There is no contribution to the other tangential component of the electric field from the line source, as can be seen from equations (5) and (A-16). Contributions from the other terms have to be small as does the normal component of $\vec{H}$, which also has to vanish on the surface of a perfect conductor.

2.3 Wire of Finite Conductivity

Kikuchi[2] addresses the problem of a wire of finite conductivity. In this case, the tangential component of $\vec{E}$ is continuous at the surface of the wire. He matches the z-component outside the wire with that inside, the latter being obtained from the current, under the assumption that the field is not significantly affected by the presence of the ground; it is

$$E_z = \frac{\kappa_c I}{2\pi a \sigma_c} \frac{J_0(\kappa_c a)}{J_1(\kappa_c a)} \exp(-\Gamma z) \quad , \tag{36}$$

where σ_c is the conductivity of the wire, and

$$\kappa_c^2 = (\varepsilon_c - i\sigma_c/\omega) \mu \omega^2 + \Gamma^2 \quad , \tag{37}$$

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[2] *H. Kikuchi, Electrotech. J. Japan* 2*, 73 (1965).*

ε_c being the dielectric constant of the conductor. Equation (36) can be obtained from Sommerfeld's equations (20) and (20a).[5] In the case of a good conductor, the asymptotic limits of the Bessel functions can be used to obtain

$$\lim_{\sigma_c \to \infty} \frac{J_o(\kappa_c a)}{J_1(\kappa_c a)} = i \ ,$$

since the magnitude of κ_c tends to infinity and its argument tends to $-\pi/4$.

Kikuchi's scalar and vector potentials[2] become

(2-12)* $$A_{1z} = -(i\mu_1 I/4)\exp(-\Gamma z)\{H_o^{(2)}[\kappa_1\sqrt{(x-h)^2+y^2}\,] - H_o^{(2)}(\kappa_1\sqrt{(x+h)^2+y^2}\,] + (2i/\pi)\int_{C_-}[\exp(-u_1x-u_1h) \times \exp(-i\lambda y)/(u_1+u_2)]\,d\lambda\} \quad (39)$$

(2-12) $$A_{2z} = (\mu_1 I/2\pi)\exp(-\Gamma z)\int_{C_-}\{\exp(u_2x - u_1h)\exp(-i\lambda y)/(u_1+u_2)\}\,d\lambda \quad (40)$$

(2-12)* $$\Phi_1 = -(\Gamma I/4\omega\varepsilon_1)\exp(-\Gamma z)\{H_o^{(2)}[\kappa_1\sqrt{(x-l)^2+y^2}\,] - H_o^{(2)}[\kappa_1\sqrt{(x+h)^2+y^2}\,] + (2i\,k_1^2/\pi)\int_{C_-}[\exp(-u_1x - u_1h) \times \exp(-i\lambda y)/(k_1^2\,u_2+k_2^2u_1)]\,d\lambda\} \quad (41)$$

(2-12)* $$\Phi_2 = -i(k_1^2\,\Gamma I/2\pi\omega\varepsilon_1)\exp(-\Gamma z) \times \int_{C_-}[\exp(u_2x - u_1h)\exp(-i\lambda y)/(k_1^2\,u_2+k_2^2u_1)]\,d\lambda \quad (42)$$

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[2] *H. Kikuchi, Electrotech. J. Japan 2, 73 (1965).*
[5] *A. Sommerfeld, Lectures on Theoretical Physics, Vol. III, Electrodynamics, Academic Press, New York, 1952.*

The same field component (equation 29) can be obtained from

(2-5)
$$E_{1z} = \Gamma\Phi_1 - i\omega A_{1z} \tag{43}$$

Matching boundary conditions at the surface of the wire at the point $x = h - a$, $y = 0$, the modal equation takes the form

(2-6)
$$- i\mu\omega\, D_2(\beta) = \left(\kappa_c/a\sigma_c\right) J_0\left(\kappa_c a\right)/J_1\left(\kappa_c a\right) \approx i\kappa_c/a\sigma_c\ , \tag{44}$$

where

$$D_2(\beta) = (-i\pi/2)(1 - \beta^2/k_1^2)\left\{H_0^{(2)}(\kappa_1 a) - H_0^{(2)}[\kappa_1(2h-a)]\right\}$$

$$+ \int_{C_-} d\lambda\ [(\lambda^2 - u_1u_2)/(k_1^2u_2 + k_2^2u_1)]$$

$$\times \exp\ [-(2h - a)\ u_1]. \tag{45}$$

For a perfect conductor, equation (44) reduces to

$$D_2(\beta) = 0\ , \tag{46}$$

which is equivalent to equation (32).

The discussion about the approximations in section 2.2 applies here, with the added assumption about the field distribution inside the wire being unaffected by the presence of the ground.

3. TRANSMISSION-LINE PARAMETERS

A transmission line is characterized by such distributed parameters as a series impedance per unit length Z_1 and a shunt admittance per unit length Y_1 as shown in figure 2.

The differential equations for voltage and current as functions of the distance along the line are

(2-8)
$$\partial V/\partial z = - Z_1 I\ , \tag{47}$$

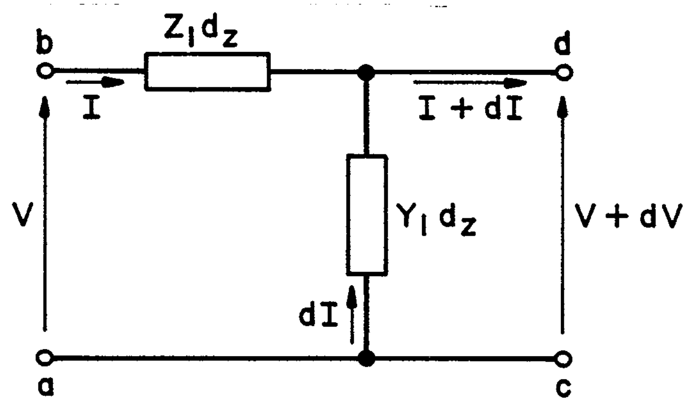

Figure 2. Variables and parameters that characterize an infinitesimal section of a transmission line.

(2-8)

$$\partial I/\partial z = - Y_1 V \ . \qquad (48)$$

For an ideal transmission line without any losses,

$$Z_1 = i\omega L_1 \ , \qquad (49)$$

$$Y_1 = i\omega C_1 \ , \qquad (50)$$

where L_1 is the series inductance per unit length and C_1 is the shunt capacitance per unit length. Both of these geometric properties of the circuit are defined for static fields; their usefulness is questionable, however, when retardation effects are important.

If losses are taken into account, they are reflected in a series resistance per unit length R_1 and a shunt conductance per unit length G_1; thus, equations (49) and (50) change to

$$Z_1 = R_1 + i\omega L_1 \ , \qquad (51)$$

and

$$Y_1 = G_1 + i\omega C_1 \ . \qquad (52)$$

If the parameters are independent of Z, equations (47) and (48) reduce to

$$\partial^2 V/\partial z^2 - Z_1 Y_1 V = 0 \ , \tag{53}$$

and the solution is

$$V = A\, e^{-\Gamma z} + B\, e^{\Gamma z} \ , \tag{54}$$

where

$$\Gamma^2 = Z_1 Y_1 \ . \tag{55}$$

The second term in equation (54) corresponds to an increasing exponential contrary to our assumptions, and we set B equal to zero, so that

$$V = V_o\, e^{-\Gamma z} \ . \tag{56}$$

Similarly, from equation (75),

$$I = (\Gamma/Z_1)\, V_o\, e^{-\Gamma z} = I_o\, e^{-\Gamma z} \ ; \tag{57}$$

the constant ratio,

$$Z_o = V/I = Z_1/\Gamma = \sqrt{Z_1/Y_1} \ , \tag{58}$$

is called the characteristic impedance of the line. For the ideal case,

$$\Gamma = i\omega \sqrt{L_1 C_1} \ , \tag{59}$$

$$Z_o = \sqrt{L_1/C_1} \ . \tag{60}$$

Equation (C-5) can be written in the form

$$\oint_{(C)} \vec{E} \cdot d\vec{r} + \int_{(S)} \partial \vec{B}/\partial t \cdot d\vec{S} = 0 \ , \tag{61}$$

and the potential V is defined, conventionally, as

$$V = - \int_a^b \vec{E} \cdot d\vec{r} \ . \tag{62}$$

Considering the circuit abdc of figure 2, the change in potential dV comes in part from the line integral along the conductors related to R_1, and in part from the magnetic field through the second term in equation (61), represented by L_1. Similarly, the current dI between the conductors comes partly from an actual conduction current described by the conductance G_1, and partly from a displacement current due to the motion of charges on the conductors and represented by the capacitance C_1; this is also discussed in appendix C.

Considering the wire above ground, the main problem in further generalizing these concepts lies in the absence of a well defined circuit to which equation (61) could be applied.

Kikuchi[2] defines the series impedance and shunt admittance in terms of a scalar potential in a particular Lorentz gauge. From equation (C-7), which is valid in an arbitrary gauge,

(2-5)

$$\partial V/\partial z = - i\omega A_z - E_z \ , \tag{63}$$

and, evaluating the potential at the surface of the wire at a point closest to the ground, equation (47) shows that

(2-7)

$$Z_1 = \frac{i\omega \ A_{1z} \ (h-a,0,z) + E_{1z} \ (h-a,0,z)}{I \ \exp(-\Gamma z)} \ . \tag{64}$$

The shunt admittance is then simply obtained from equation (48), and

[2] *H. Kikuchi, Electrotech. J. Japan* <u>*2*</u>*, 73 (1956).*

(2-7) $$Y_1 = \frac{\Gamma\ I\ \exp(-\Gamma z)}{V_1(h-a,0,z)}\ . \tag{65}$$

Clearly, the z-dependence of numerators and denominators cancel. It is also possible to rewrite equation (64) in the form

$$Z_1 = \frac{\Gamma\ V_1(h-a,0,z)}{I\ \exp(-\Gamma z)}\ , \tag{66}$$

which, together with equation (65), yields the expression (55) for Γ^2 regardless of the choice of scalar potential. It is this physically meaningful propagation constant that is determined from the modal equations (32) or (44).

Wait[1] rewrites equation (32) in the form

$$\beta^2 = k_1^2\ [\Lambda + 2(Q - iP)][\Lambda + 2(N - iM)]^{-1}, \tag{67}$$

where

(1-27)* $$Q - iP = \tfrac{1}{2} \int_{C_-} [\exp(-2u_1 h)/(u_1+u_2)]\exp(-i\lambda a)\,d\lambda, \tag{68}$$

(1-28)* $$N - iM = \tfrac{1}{2} \int_{C_-} [\exp(-2u_1 h)/(u_2+u_1 k_2^2/k_1^2)]\exp(-i\lambda a)\,d\lambda. \tag{69}$$

According to equation (55) and following the decomposition used by Kikuchi,[2] Wait[1] defines

(1-24) $$Z_1 = (i\mu\omega/2\pi)[\Lambda + 2(Q-iP)]\ , \tag{70}$$

(1-25) $$Y_1 = (2\pi i\varepsilon_1\omega)[\Lambda + 2(N-iM)]^{-1}, \tag{71}$$

but there is no really firm motivation for this choice.

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[1] *J. R. Wait, Radio Science 7, 675 (1972).*
[2] *H. Kikuchi, Electrotech. J. Japan 2, 73 (1956).*

Alternatively, it is possible to define a gauge independent, path-dependent potential at the wire, using equation (62) and integrating along the straight vertical line shown in figure 3. Equation (66) would then give the impedance per unit length as

$$Z_1 = (\Gamma/I)\ \exp(\Gamma z)\left[-\int_{-\infty}^{0} E_{2x}\ (x,0,z)\ dx\right.$$

$$\left. - \int_{0}^{h-a} E_{1x}\ (x,0,z)\ dx\right]\ , \tag{72}$$

and the admittance per unit length would be

$$Y_1 = \Gamma^2/Z_1\ . \tag{73}$$

The ground may be chosen as a reference for this potential; the difference between the two definitions can be expressed in terms of

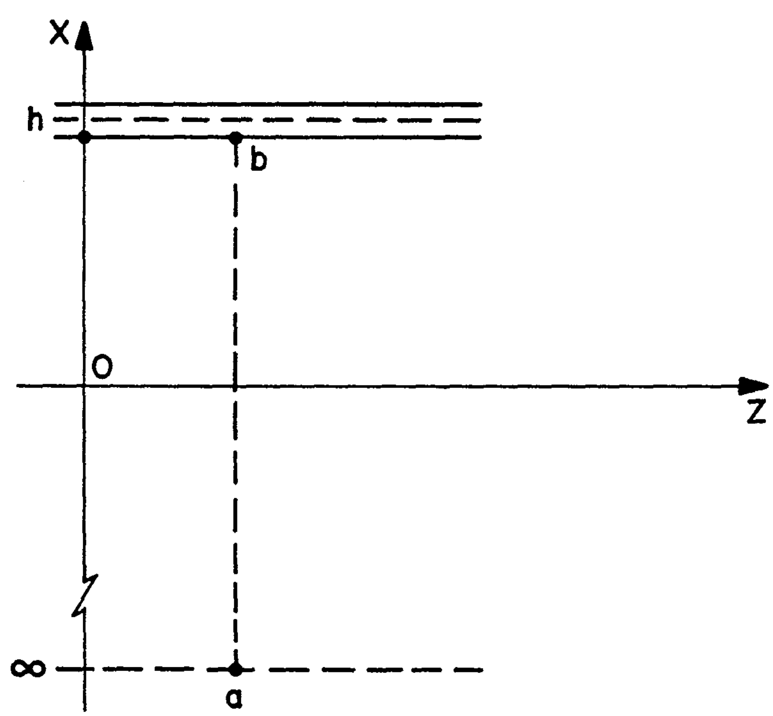

Figure 3. Path of integration for the definition of potential difference.

$$\int_{-\infty}^{0} E_{2x}(x,0,z)\, dx = i\beta K \exp(-\Gamma z)$$

$$\times \int_{C_-} [\exp(-u_1 h)/u_1][1 - \lambda^2(1-K)u_2^{-1}(u_1+u_2K)^{-1}]$$

$$\times (1 + R)\, d\lambda. \tag{74}$$

The ground is not an equipotential surface, and the losses contribute to the series resistance.

Ultimately, the definition and usefulness of concepts such as a voltage and an impedance must be closely related to the type of measurement to be performed.

4. SPECIAL CASES

Some of the special cases discussed by Wait[1] are presented here to conclude our analysis. Kikuchi[2] also refers to some limiting cases.

From equation (29), a solution can be obtained to the basic equations by superposition of those solutions for a fixed ω but arbitrary β, where the boundary condition at the wire (which determines β as a function of ω through the modal equation) is replaced by one that represents in some sense a voltage, V, applied to a gap of length 2b in the wire. In that case, the tangential field at the wire vanishes everywhere except at the gap, where it is

$$E_{1z}(a) = V/(2b) \ . \tag{75}$$

The discontinuous integral of Dirichlet,

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin\beta b}{\beta b} \exp(-i\beta z)\, d\beta = \begin{cases} 1 \text{ for } |z| < \beta \\ 0 \text{ for } |z| > \beta \end{cases} \tag{76}$$

is used to show that equation (77),

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[1] *J. R. Wait, Radio Science* 7, *675 (1972).*
[2] *H. Kikuchi, Electrotech. J. Japan* 2, *73 (1956).*

(1-18) $$E_{1z} = -\frac{V}{2\pi}\int_{-\infty}^{\infty}\frac{\sin\beta b}{\beta b}\,\frac{B(\beta)}{D(\beta)}\,\exp(-i\beta z)\,d\beta\,, \qquad (77)$$

obeys the new boundary conditions; since $D(\beta)$ in equation 33 is equal to $B(\beta)$ in equation (30) at the point on the wire surface, $x = h$, $y = a$, and approximately equal to it on the whole surface of the wire.

The current along the wire is linearly related to the field through

$$I(z,\beta) = [i(2\pi/\mu\omega)/B(\beta)]E_{1z}(z,\beta) \qquad (78)$$

from equation (29) for each component β; thus, the current is

(1-19) $$I(z) \approx \frac{V}{i\mu\omega}\int_{-\infty}^{\infty}\frac{\sin\beta b}{\beta b}\,\frac{1}{D(\beta)}\,\exp(-i\beta z)\,d\beta\,. \qquad (79)$$

The function $D(\beta)$ vanishes for the solution β_o of the modal equation (32), leading to a pole in the integrand in equation (79). This pole lies in the fourth quadrant, as pointed out at the beginning of section 2, and gives a contribution when the integral is evaluated by closing the contour through the lower half of the complex β-plane. The residue gives the first term in

(1-19) $$I(z) \approx -\frac{2\pi V}{\mu\omega}\;\frac{\sin\beta_o b}{\beta_o b}\left[\frac{\partial D(\beta)}{\partial\beta}\right]_{\beta=\beta_o} + I_r, \qquad (80)$$

and the second term I_r comes from other singularities in this half-plane such as branch-line integrations.

Another limit of certain interest is that of the wire on the ground--that is, $h \to 0$. In that case, the modal equation (32) reduces to

(1-20*) $$\int_{C_-} d\lambda[(\lambda^2 - u_1u_2)/(k_1^2u_2 + k_2^2u_1)]\;, \qquad (81)$$

because a<<h implies that a also has to tend to zero for this solution to apply. The integrand behaves like $1/\lambda$ for large λ, since

(1-21)*

$$\frac{\lambda^2 - u_1 u_2}{k_1^2 u_2 + k_2^2 u_1} \approx \frac{1}{\lambda} \frac{k_1^2 + k_2^2 - 2\beta^2}{k_1^2 + k_2^2} , \tag{82}$$

so that the numerator has to vanish for the integral to converge, giving the solution

(1-21)

$$\beta_o = [(k_1^2 + k_2^2)/2]^{\frac{1}{2}} , \tag{83}$$

It is not clear, however, to what extent this applies to a real wire on the ground, due to the restriction to a vanishing diameter from the condition $a \ll h$.

5. SUMMARY AND CONCLUSIONS

This report presents a solution to the problem of waves supported by a single infinite wire above a (imperfectly) conducting ground.

The modal equation, obtained using some simplifying assumptions, determines the propagation constant for a given frequency. The separation of the square of this propagation constant into factors representing the series impedance and shunt admittance of an equivalent transmission line is highly ambiguous at best. Thus, these concepts, as well as that of a voltage or potential difference to which they are related, are ill defined for time-varying fields when the frequencies involved are sufficiently high. In particular, the scalar potential that--together with the vector potential--allows the determination of the fields can be changed in an arbitrary manner by gauge transformations. On the other hand, a voltage defined as the line integral of the electric field is path-dependent for time-varying fields. Care has to be exercised in interpreting the measurements of "voltmeters" and, to some extent, "ammeters," in cases where the high-frequency content of a signal is significant.

An important aspect of the problem, overlooked by Kikuchi[2] and Wait,[1] is the specification of the path of integration for the integral representation of the Hankel functions for complex arguments. As shown by dos Santos,[3] an integral taken along the real axis corresponds to incoming waves, and the radiation of energy by this system requires the selection of a path deformed around the branch points, as shown in

[1]*J. R. Wait, Radio Science 7, 675 (1972).*
[2]*H. Kikuchi, Electrotech. J. Japan 2, 73 (1956).*
[3]A. F. dos Santos, Proc. IEEE 119, 1103 (1972).

figure 1. These solutions have the property that they lead to fields that *increase* exponentially with distance from the source. This is, of course, unphysical for the problem under consideration. We attribute this behavior to the simplification involved in reducing it to a two-dimensional configuration. A study of the Green function shows that there is a region of space where the exponential increase is not noticeable but far enough from the wire to correspond to the radiation zone of the fields; in this region the fields can be associated with physically meaningful outgoing waves. The flow of energy at larger distances comes mainly from the source at the "beginning" of the infinite wire, and does not have a physical interpretation.

An area for further study and of great practical interest is the behavior of this system under an incident wave--a problem that has been studied with different approaches involving other approximations.

APPENDIX A. ELECTRIC AND MAGNETIC HERTZ VECTORS

The scalar potential Φ and the vector potential $\vec{A}$ are not independent in a Lorentz gauge, but they are related by the Lorentz condition

$$\nabla \cdot \vec{A} + \varepsilon\mu \partial\Phi/\partial t = 0 \; . \tag{A-1}$$

Consequently, they can be expressed in terms of a single three-vector field $\vec{\Pi}$, the Hertz vector, through the equations

$$\Phi = - \nabla \cdot \vec{\Pi} \; , \tag{A-2}$$

$$\vec{A} = \varepsilon\mu \partial\vec{\Pi}/\partial t . \tag{A-3}$$

Maxwell's equations for the electric and magnetic fields reduce to the wave equation for the potentials, and $\vec{\Pi}$ satisfies

$$(\partial^2/\partial t^2 - \nabla^2)\, \vec{\Pi} = \vec{p}/\varepsilon \tag{A-4}$$

where the polarization vector $\vec{p}$ is related to the sources, the charge density ρ and current density $\vec{j}$, through

$$\rho = - \nabla \cdot \vec{p} \; , \tag{A-5}$$

$$\vec{j} = \partial\vec{p}/\partial t \; . \tag{A-6}$$

These last two equations imply that

$$\partial\rho/\partial t + \nabla \cdot \vec{j} = 0 \; , \tag{A-7}$$

which is an expression of the conservation of charge, implicit in Maxwell's equations. The electric and magnetic fields are related to the Hertz vector through

APPENDIX A

$$\vec{E} = -\varepsilon\mu\partial^2\vec{\Pi}/\partial t^2 + \nabla\nabla\cdot\vec{\Pi} \ , \qquad \text{(A-8)}$$

$$\vec{B} = \varepsilon\mu\nabla\times(\partial\vec{\Pi}/\partial t) \ . \qquad \text{(A-9)}$$

For monochromatic waves, all fields have a time dependence exp(iωt); for instance,

$$\vec{E}(\vec{x},t) = \vec{E}(\vec{x}) \exp(i\omega t) \ , \qquad \text{(A-10)}$$

where $\vec{E}(\vec{x})$ can be complex and it is understood that $\vec{E}(\vec{x},t)$ is equal to the real part of the right-hand side. Equations (A-8) and (A-9) reduce to

$$\vec{E} = k^2\vec{\Pi} + \nabla\nabla\cdot\vec{\Pi} \ , \qquad \text{(A-11)}$$

$$\vec{B} = i\varepsilon\mu\omega\nabla\times\vec{\Pi} \ , \qquad \text{(A-12)}$$

where

Note: The italicized numerals parenthesized on the left side of mathematical formulas in this report represent reference and equation numbers--that is, numerals "(1-2)" denote that our equation (A-13) is basically the same as equation (3) in reference 1 by Wait.[1]

(1-2') $$k^2 = \varepsilon\mu\omega^2 \qquad \text{(A-13)}$$

and equation (A-4) becomes

$$(\nabla^2 + k^2)\vec{\Pi} = (i/\varepsilon\omega)\vec{J} \ . \qquad \text{(A-14)}$$

[1]*J. R. Wait, Radio Science 7, 675 (1972).*

In a case when $\vec{\Pi}$ has only one component, the z-axis can be chosen in that direction and consequently

$$\vec{\Pi} = \Pi \; \hat{e}_3 \; . \qquad \text{(A-15)}$$

Then, equations (A-11) and (A-12) reduce to

$$\vec{E} = k^2 \Pi \hat{e}_3 + \nabla \; \partial\Pi/\partial z \; , \qquad \text{(A-16)}$$

$$\vec{B} = i\varepsilon\mu\omega(\hat{e}_1 \partial\Pi/\partial y - \hat{e}_2 \partial\Pi/\partial x) \; . \qquad \text{(A-17)}$$

As can be seen from equation (A-14), $\vec{j}$ must also be in the z-direction, and Π obeys the scalar equation

$$(\nabla^2 + k^2)\Pi = (i/\varepsilon\omega) j \; . \qquad \text{(A-18)}$$

The magnetic field derived from such a Hertz vector has no component in the z-direction, as shown by equation (A-17). Thus, even when $\vec{j}$ has a fixed direction, the most general solution for the fields requires a Hertz vector with more than one component. Alternatively, another field, the magnetic Hertz vector $\vec{\Pi}'$, can be defined by taking into account the symmetry of the equations for a source-free field. It satisfies

$$(\partial^2/\partial t^2 - \nabla^2) \; \vec{\Pi}' = 0 \; , \qquad \text{(A-19)}$$

and the corresponding electromagnetic fields are

$$\vec{H} = - \nabla^2 \vec{\Pi}' + \nabla\nabla \cdot \vec{\Pi}' \; , \qquad \text{(A-20)}$$

$$\vec{E} = - \mu \nabla \times \partial \vec{\Pi}'/\partial t. \qquad \text{(A-21)}$$

The intermediate potentials, still in a Lorentz gauge, are

$$\Phi = 0, \qquad \text{(A-22)}$$

$$\vec{A} = \mu \nabla \times \vec{\Pi}' \ . \tag{A-23}$$

For monochromatic waves, equations (A-19) through (A-21) become

$$(\nabla^2 + k^2)\vec{\Pi}' = 0 \ , \tag{A-24}$$

$$\vec{H} = k^2 \vec{\Pi}' + \nabla\nabla \cdot \vec{\Pi}' \ , \tag{A-25}$$

$$\vec{E} = - i\mu\omega \nabla \times \vec{\Pi}' \ . \tag{A-26}$$

Furthermore, if $\vec{\Pi}'$ is also in a fixed direction along the *z*-axis,

$$\vec{\Pi}' = \Pi' \hat{e}_3 , \tag{A-27}$$

$$(\nabla^2 + k^2)\Pi' = 0 \ . \tag{A-28}$$

We use both Hertz potentials to obtain

$$E_x = \partial^2\Pi/\partial x \partial z - i\mu\omega \partial \Pi'/\partial y \ , \tag{A-29}$$

(1-1)
$$E_y = \partial^2\Pi/\partial y \partial z + i\mu\omega \partial \Pi'/\partial x , \tag{A-30}$$

(1-1)
$$E_z = (k^2 + \partial^2/\partial z^2)\Pi \ , \tag{A-31}$$

(1-2)
$$H_x = \partial^2\Pi'/\partial x \partial z + i\varepsilon\omega \partial \Pi/\partial y \ , \tag{A-32}$$

(1-2)
$$H_y = \partial^2\Pi'/\partial y \partial z - i\varepsilon\omega \partial \Pi/\partial x \ , \tag{A-33}$$

(1-2)
$$H_z = (k^2 + \partial^2/\partial z^2)\Pi' \ . \tag{A-34}$$

The one-component field Π is determined by the current density, which has to be in the *z*-direction and the boundary conditions, whereas the one-component field Π' is determined by the boundary conditions alone.

APPENDIX B. GREEN FUNCTION FOR THE TWO-DIMENSIONAL HELMHOLTZ EQUATION

Since the Green function depends only on the difference of the coordinates of the field point and the source point, a source point can be chosen at the origin and equation to be solved is

$$(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \kappa^2)G(x,y) = -\delta(x)\delta(y). \qquad \text{(B-1)}$$

The Fourier transform of the Green function is determined in the usual manner; thus,

$$G(x,y) = \frac{1}{(2\pi)^2}\iint \frac{\exp[-i(k_x x+k_y y)]dk_x dk_y}{k_x^2 + k_y^2 - \kappa^2} . \qquad \text{(B-2)}$$

The k_x-integration is done first. For real (positive) κ, a purely real path of integration would encounter two poles, at

$$k_x = \pm\sqrt{\kappa^2 - k_y^2} \qquad \text{(B-3)}$$

for $|k_y| < \kappa$; these poles become purely imaginary for $|k_y| > \kappa$. To specify the path of integration around these poles, note that κ^2 can be given either a positive or negative imaginary part; consequently, there are two integrals

$$G_\pm(x,y) = \frac{1}{(2\pi)^2}\lim_{\varepsilon\to 0+}\iint_{-\infty}^{\infty}\frac{\exp[-i(k_x x+k_y y)]dk_x dk_y}{k_x^2 + k_y^2 - \kappa^2 \mp i\varepsilon} . \qquad \text{(B-4)}$$

The positions of the poles in the complex k_x-plane for different values of k_y are shown in figures B-1 and B-2.

For positive x, the contour around the lower half-plane can be closed without adding a contribution from the infinite arc, while this has to be done around the upper half-plane for negative x. Equation (B-4) may be expressed in the form

$$G_\pm(x,y) = \frac{1}{(2\pi)^2}\lim_{\varepsilon\to 0+}\iint_{-\infty}^{\infty}\frac{\exp[-i(k_x x+k_y y)]dk_x dk_y}{(k_x+i\sqrt{k_y^2-\kappa^2\mp i\varepsilon})(k_x-i\sqrt{k_y^2-\kappa^2\mp i\varepsilon})}, \qquad \text{(B-5)}$$

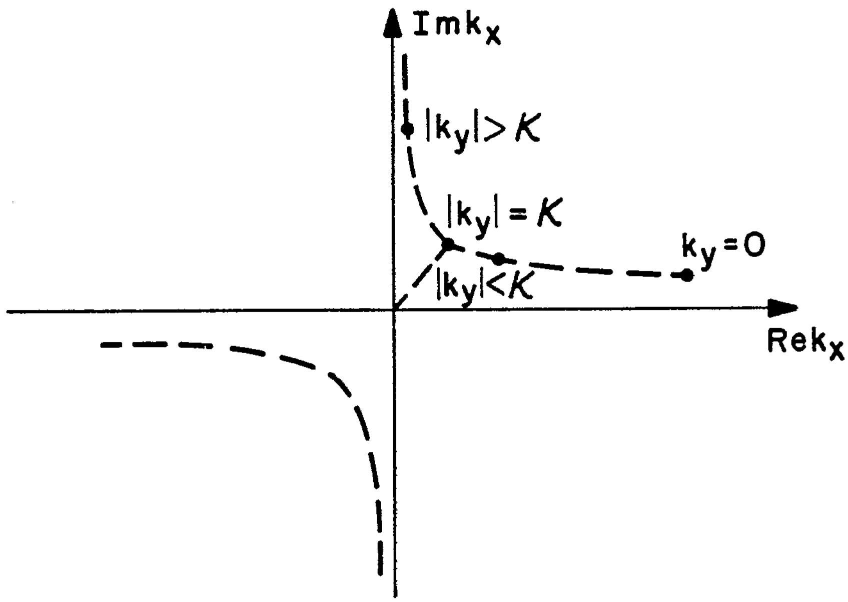

Figure B-1. Location of the poles of the integrand of G_+ in equation (B-4) as a function of k_x.

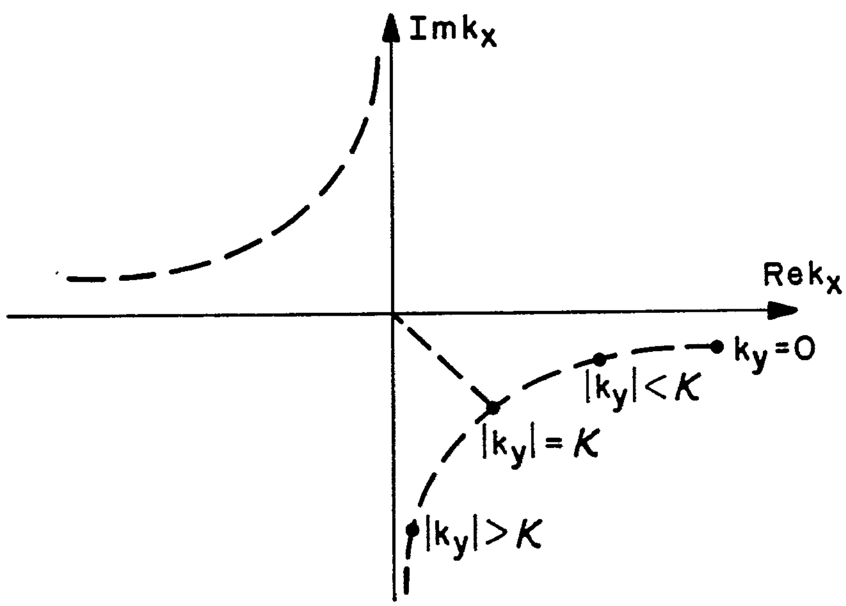

Figure B-2. Location of the poles of the integrand of G_- in equation (B-4) as a function of k_x.

where the square root is assumed to be real and positive for $k_y > \kappa$ and $\varepsilon = 0$. That is, the first factor corresponds to the pole in the lower half-plane. Evaluating the residues yields

$$G_{\pm}(x,y) = \frac{1}{4\pi} \lim_{\varepsilon \to 0+} \int_{-\infty}^{\infty} \frac{\exp(-ik_y y - |x| \sqrt{k_y^2 - \kappa^2 \mp i\varepsilon})\, dk_y}{\sqrt{k_y^2 - \kappa^2 \mp i\varepsilon}} \quad . \qquad \text{(B-6)}$$

The integral is separated into Fourier sine and cosine transforms, giving

$$G_{\pm}(x,y) = \frac{1}{2\pi} \lim_{\varepsilon \to 0+} \int_{0}^{\infty} \frac{\cos(k_y y) \exp(-|x| \sqrt{k_y^2 - \kappa^2 \pm i\varepsilon})\, dk_y}{\sqrt{k_y^2 - \kappa^2 \mp i\varepsilon}} \quad . \qquad \text{(B-7)}$$

The sine transform vanishes due to the integrand being an odd function of k_y. Using equation (1.4.27)[1] yields

$$G_{\pm}(x,y) = (2\pi)^{-1} \lim_{\varepsilon' \to 0+} K_o[(\mp i\kappa + \varepsilon')\sqrt{x^2+y^2}] \quad , \qquad \text{(B-8)}$$

where K_o is a modified Bessel function, related to the Hankel functions by

$$K_o(\zeta) = \frac{1}{2} i\pi\, H_o^{(1)}(i\zeta) = -\frac{1}{2} i\pi\, H_o^{(2)}(-i\zeta) \quad . \qquad \text{(B-9)}$$

Equation (B-9) in this appendix is obtained from equation (7.2.15).[2] We finally can write

$$G_{+}(x,y) = (i/4) H_o^{(1)}(\kappa R) \quad , \qquad \text{(B-10)}$$

$$G_{-}(x,y) = -(i/4)\, H_o^{(2)}(\kappa R) \quad , \qquad \text{(B-11)}$$

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[1] A. Erdélyi, Editor, Bateman Manuscript Project, Tables of Integral Transforms, Vol. I, McGraw-Hill Book Company, Inc., New York, 1954.
[2] A. Erdélyi, Bateman Manuscript Project, Higher Transcendental Functions, Vol. II, McGraw-Hill Book Company, Inc., New York, 1953.

where

$$R = \sqrt{x^2+y^2} \quad . \tag{B-12}$$

From equations (7.13.1) and (7.13.2),[2] the asymptotic limits of the Hankel functions are,

$$H_o^{(1)}(\zeta) \sim (2/\pi\zeta)^{\frac{1}{2}} \exp[i(\zeta-\pi/4)] \quad , \tag{B-13}$$

$$H_o^{(2)}(\zeta) \sim (2/\pi\zeta)^{\frac{1}{2}} \exp[-i(\zeta-\pi/4)] \quad , \tag{B-14}$$

which shows that G_- is the Green function that corresponds to outgoing waves. In Morse and Feshbach,[3] the time dependence is assumed to be $\exp(-i\omega t)$ and G_+ is chosen for the Green function (eq 7.2.18).

It is, of course, possible to rewrite equation (B-6) in the form

$$G_{\pm}(x,y) = \frac{1}{4\pi} \int_{(C_{\pm})} \frac{\exp(-ik_y y-|x|\sqrt{k_y^2-\kappa^2})\,dk_y}{\sqrt{k_y^2-\kappa^2}} \quad , \tag{B-15}$$

where the paths of integration are shown in figures B-3 and B-4. The path chosen around the branch points at $\pm\kappa$ specifies the branch of the square-root function at each point of the contour.

We are interested in values of κ^2 which are obtained from

$$\kappa^2 = k^2 - \beta^2 \quad , \tag{B-16}$$

where k^2 is given by equation (A-13) of appendix A. For real k^2, equate the imaginary parts in equation (B-16) and obtain

$$\kappa_r \kappa_i = -\beta_r \beta_i > 0 \quad , \tag{B-17}$$

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[2] *Erdélyi, Editor, Bateman Manuscript Project, Higher Transcendental Functions, Vol. II, McGraw-Hill Book Company, Inc., New York, 1953.*

[3] *P. M. Morse and H. Feshbach, Methods of Theoretical Physics, McGraw-Hill Book Company, Inc., New York, 1953. (Also see pp. 822-825).*

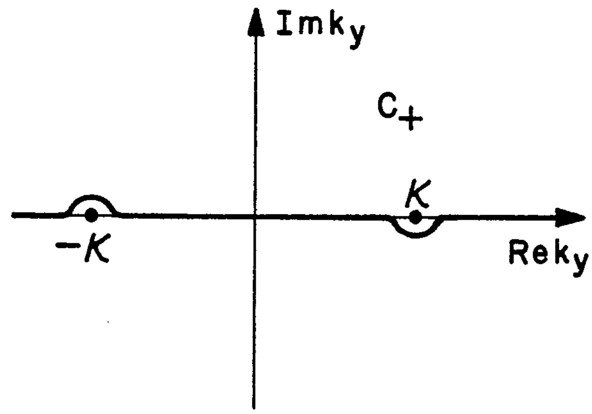

Figure B-3. Path of integration C_+ in the k_y-plane for G_+ in the case of real κ. It shows the deformation of the path around the branch points at $\pm\kappa$.

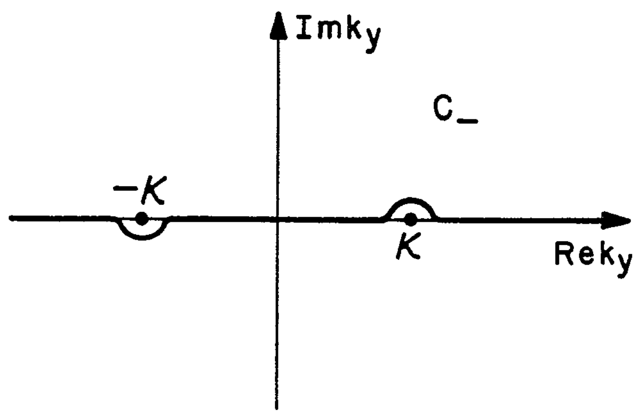

Figure B-4. Path of integration C_- in the k_y-plane for G_- in the case of real κ. It shows the deformation of the path around the branch points at $\pm\kappa$.

which shows that the branch points in figures B-3 and B-4 move into the first and third quadrants, and the paths of integration have to be changed as shown in figures B-5 and B-6; this agrees with the contours chosen by dos Santos.[4]

Furthermore, the asymptotic formulas (B-13) and (B-14) show that when the exponent is no longer imaginary but complex, the behavior of the Green function at large distances is determined by the exponential factor. Thus, for real κ, G_+ corresponds to incoming waves and G_- to outgoing ones; both decrease like $1/\sqrt{R}$ at large distances. On the other hand, when κ has a positive imaginary part, G_+ corresponds to incoming

[4]*A. F. dos Santos, Proc. IEEE* <u>*119*</u>*, 1103 (1972).*

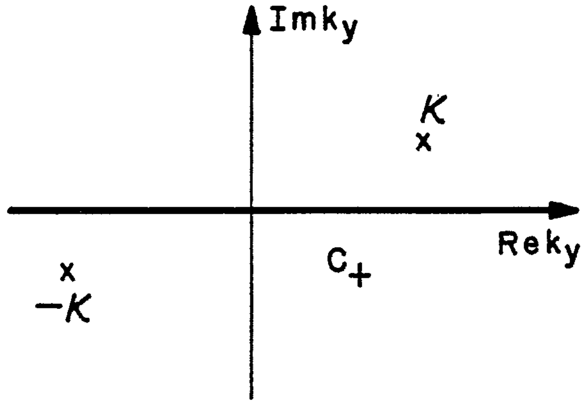

Figure B-5. The contour C_+, for complex values of κ.

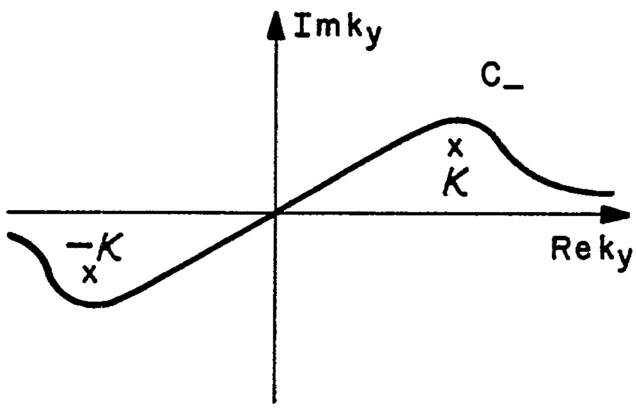

Figure B-6. The contour C_-, which has to be deformed as shown when the branch points move into the first and third quadrants.

waves and decreases exponentially with distance, while G_- represents outgoing waves that increase exponentially with distance. Neither of these solutions is completely satisfactory in terms of a physical interpretation.

When the medium is conductive, equation (A-13) is changed to

$$k^2 = \varepsilon\mu\omega^2(1 - i\sigma/\varepsilon\omega) , \tag{B-18}$$

and equation (B-17) is changed to

$$\kappa_r\kappa_i = - \beta_r\beta_i - \mu\sigma\omega . \tag{B-19}$$

Thus, $\kappa_r \kappa_i$ is positive for

$$\sigma < -\beta_r \beta_i / \mu\omega ; \qquad \text{(B-20)}$$

the above conclusions are unchanged, whereas for

$$\sigma > -\beta_r \beta_i / \mu\omega , \qquad \text{(B-21)}$$

the behavior at large distances of G_+ and G_- are interchanged.

The energy flow for a nonconductive medium and an outgoing wave can be represented by the arrows in figure B-7, which show that the increase in the energy flux in the radial direction comes from a decrease of the flux in the axial direction. This unphysical behavior can be related to the assumption of a source at $z = -\infty$ which supplies energy in amounts increasing exponentially with distance from the axis. If σ is large enough, the energy dissipated in the volume reverses these conclusions when the inequality (B-21) is satisfied.

If $\sigma = 0$ and $-\beta_i << \beta_r$, then

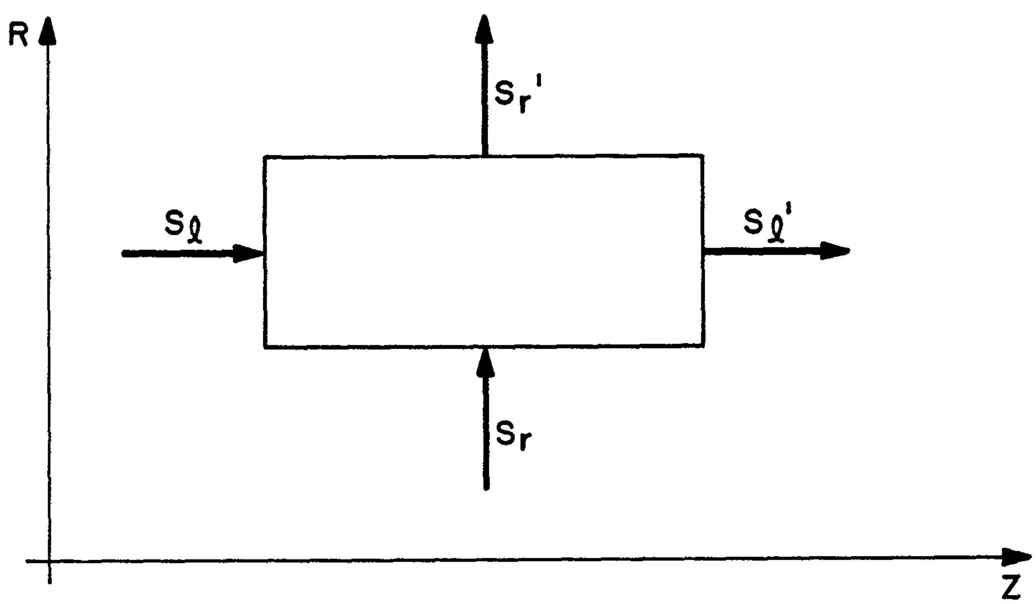

Figure B-7. Diagram of the flow of energy at large distances from the axis.

$$\kappa_r \approx \sqrt{k^2-\beta_r^2} \,, \tag{B-22}$$

$$\kappa_i \approx -\beta_i\beta_r/\kappa_r \,, \tag{B-23}$$

and κ_i is small compared with κ_r as long as k is not close to β_r. Hence, the factor $1/\sqrt{\kappa R}$ becomes important when $R \approx 1/\kappa_r$, whereas the exponential increase only starts to matter when $R \approx 1/\kappa_i$, and the fields would have the expected behavior of outgoing radiation in the region

$$\kappa_i \approx -\beta_i\beta_r/\kappa_r \,. \tag{B-24}$$

The solutions for larger values of R would have to be ignored.

When κ becomes imaginary, which corresponds to evanescent cylindrical waves and no radiation, the exponentially increasing solution becomes meaningless and the other solution, the one that is connected to incoming waves, has to be chosen.

APPENDIX C. POTENTIAL DIFFERENCES AND CURRENTS

The scalar potential, Φ, is originally defined in electrostatics in terms of the field $\vec{E}$ by

$$\Phi_a - \Phi_b = \int_a^b \vec{E} \cdot d\vec{r} \quad , \tag{C-1}$$

which is independent of the path chosen between the points a and b because

$$\nabla \times \vec{E} = 0 \ . \tag{C-2}$$

The inverse of equation (C-1) is

$$\vec{E} = -\nabla\Phi \ . \tag{C-3}$$

For a time-independent distribution of currents, this translates into Kirchoff's second law, which states that the potential drop around a closed path is zero.

When the fields are time dependent, equation (C-2) changes to

$$\nabla \times \vec{E} = -\partial\vec{B}/\partial t \ , \tag{C-4}$$

whence

$$\oint_{(C)} \vec{E} \cdot d\vec{r} = -dF/dt, \tag{C-5}$$

where

$$F = \int_{(S)} \vec{B} \cdot \overrightarrow{dS} \ , \tag{C-6}$$

and (S) is a surface bounded by the fixed contour (C). That is Kirchoff's second law is recovered if the right-hand side of equation (C-5) is written as a potential drop (Faraday's law). When most of the flux is localized in a certain part or parts of the circuit, an inductance that corresponds to a definite potential drop can be defined; otherwise, it is no longer possible to specify the potential (with respect to some reference point) at a given point of the circuit.

The scalar potential Φ becomes a part of a set, together with the vector potential $\vec{A}$, and is related to the fields by

$$\vec{E} = -\nabla\Phi - \partial\vec{A}/\partial t \qquad \text{(C-7)}$$

$$\vec{B} = \nabla\times\vec{A} \ . \qquad \text{(C-8)}$$

These potentials can be changed through a gauge transformation to

$$\vec{A}' = \vec{A} + \nabla\Lambda \qquad \text{(C-9)}$$

$$\Phi' = \Phi - \partial\Lambda/\partial t \ , \qquad \text{(C-10)}$$

where Λ is an arbitrary function of $\vec{x}$ and t. This implies that the scalar potential is no longer determined by the fields and, in particular, it could be set equal to zero.

Another definition of potential difference can be based on equation (C-1), but the integral is now path dependent. The differences for different paths are given by equations (C-5) and (C-6). Precisely what is measured by a voltmeter has to be determined by a detailed study of the instrument.

Another measured quantity that has to be clearly defined is the current. Maxwell's equation

$$\nabla\times\vec{H} = \vec{j} + \partial\vec{D}/\partial t \qquad \text{(C-11)}$$

gives

$$\oint_{(C)} \vec{H} \cdot d\vec{r} = I + \int_{(S)} \partial\vec{D}/\partial t \cdot \overrightarrow{dS} \,, \qquad \text{(C-12)}$$

where

$$I = \int_{(S)} \vec{j} \cdot \overrightarrow{dS} \qquad \text{(C-13)}$$

is the conduction current and the second term of the right-hand side is called the displacement current. When a current is measured by placing a loop around a wire, what is really determined is the left-hand side of equation (C-12), which is often called the total current.