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NUMERICAL SOLUTIONS TO THE PROBLEMS OF ELECTROMAGNETIC RADIATION AND SCATTERING BY A FINITE HOLLOW CYLINDER

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ABSTRACT

Numerical techniques for solutions to the problems of electromagnetic radiation and scattering are considered for a finite, hollow, circular cylinder of radius a. The singular-integral equations of electromagnetic scattering theory are derived along with their extensions to thin surfaces and surfaces with edges. In addition, constraints are presented which are necessary for a unique solution to the scattering problems of thin structures. The equations for a finite hollow cylinder are obtained by expanding the field quantities in Fourier series about the cylinder axis giving rise to a separate set of singular integral equations for each harmonic.

The method of moments is presented as the basic technique of digitizing the integral equations for numerical solution. It is found that the variational interpretation of the method of moments can be used as a guide for choosing the basis and testing functions. Of particular interest as basis functions are the

spline functions of finite support. The spline function properties of smoothness and best fit are also presented.

It is shown that Hallen's and Pocklington's formulations for the thin wire problem are equivalent numerically for appropriate testing functions. It is also shown that the Pocklington form is more desirable when smooth basis functions are used in conjunction with pulse or delta testing functions. In this context, the second-order sinusoidal spline is found to be an excellent current representation for both the scattering and pulse-feed problems. Slope discontinuities are easily included to approximate a delta-feed problem. Approximate operators are also considered with emphasis on the equivalence of the finite difference approximation and piecewise sinusoidal basis functions.

The problems of coupling in the higher harmonics are investigated for the first-harmonic problem. It is shown that minimum coupling of the equations is desirable in addition to the dominance of each equation over the entire structure by its respective current component. These features are obtained using a new set of equations obtained from the combination of the equations for the tangential electric field and the normal magnetic field. These equations are related to the normal derivative of the tangential magnetic field equations which are well-behaved for thick structures. The solutions of these equations are in excellent agreement with the results of other workers. Various operator approximations are also considered.

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TABLE OF CONTENTS

Page

LIST OF FIGURES

1. INTRODUCTION

Numerical techniques for electromagnetic problems are not new, but have been in existence almost as long as electromagnetic theory itself. In fact, Maxwell (1879) presented a numerical technique, known as the method of subareas, for obtaining the capacitance of a rectangular plate. With the advent of digital computers and space technology, numerical technqiues in electromagnetics began to blossom. The computer was the tool that had been needed for the analysis of the complicated structures often encountered in electromagnetic scattering theory. Until this time, other approximate techniques were classically used, including physical optics, geometrical optics, and other approximate analytical techniques such as the King-Middleton theory for the linear antenna (King, 1956).

Electromagnetic scattering problems are boundary-value problems which may generally be formulated as integral equations. Integral equation techniques have been discussed by several mathematicians including Kellog (1953), Morse and Feshbach (1953), Courant and Hilbert (1953), and Stakgold (1967). Special emphasis has been given to singular integrals and singular integral equations by Hadamard (1952), Muskhelishvili (1953), Mikhlin (1957), and Gakhov (1966). Numerical techniques for solving integral equations are discussed by Crout (1940), Hildebrand and Crout (1941), Young (1954 a and b), Kantorovich and Krylov (1964), Krylov (1962), and Vorobyev (1965) with excellent bibliographies given by Noble (1966) and Walther and Dejon (1960).

In Chapter 2, integral equations for electromagnetic scattering problems are developed. Of particular interest are the integral equations for thin, perfect electric conductors. These equations are specialized to hollow circular cylinders in Chapter 3. The equations are obtained by expanding all of the field quantities in Fourier series about the axis of the cylinder as has been done by Andreason (1965) and Kao (1969). The resultant equations are found to be harmonically decoupled. For each harmonic, additional decoupling is obtained for the axial and transverse components of currents; the decoupling is found to be advantageous for numerical solution. For thin structures, constraints are obtained which are necessary to guarantee the uniqueness of the solution to Maxwell's equations. In addition, a new method for extracting the singularities of the kernels used in the integral equations is presented using the technique of Krylov (1962).

A basic history and an introduction to numerical solution techniques in electromagnetic theory have been presented by Tanner and Andreason (1967). A straightforward presentation of the method of moments and its application to electromagnetic field problems have been given by Harrington (1968) with an extensive set of examples.

The method of moments and its variational interpretation are developed in Chapter 4. A guide for choosing the expansion functions of the method of moments is obtained from the variational interpretation. Spline functions (Ahlberg, Nilson, and Walsh, 1967 and Greville, 1969) are introduced as basis functions for the unknown currents; theorems are presented on the best approximation properties of splines.

In Chapter 5, numerical results are obtained for the linear antenna. Numerical equivalence is shown for both Hallen's and Pocklington's forms of the problem in addition to essential equivalences for various approximate operators. There is also a discussion of the choice of expansion functions for both the feed and scattering problems.

Various sets of equations and appropriate constraints for the first-harmonic problem are considered in Chapter 6. The problems of strong coupling between integral equations are observed in addition to various operator approximations. The necessity of choosing appropriate constraints for a particular set of equations is also considered.

2. ELECTROMAGNETIC FIELDS: INTEGRAL EQUATIONS

2.1 Introduction

The realm of scattering problems has led to a wide variety of solution techniques. These techniques include variation, perturbation, and integral equation methods. Common to these methods is a need for a representation of the field quantities in terms of currents, charges, and scattering bodies. If such a representation is given in an integral form, we may obtain integral equations for the induced currents and charge on scattering bodies by locating the field observation point at the scattering surfaces and enforcing appropriate boundary conditions.

Integral equations have been obtained for scattering problems by several methods. For scattering geometries which lie in coordinate planes, one can use Fourier transform methods such as in the Wiener-Hopf theory (Mittra and Lee, 1971), or as in the work of Kao on the hollow cylinder problem (Kao, 1969 and 1970). A second method used by Thiele (1973) describes the field quantities, $\bar{E}$ and $\bar{H}$, in terms of electric and magnetic vector potentials, $\bar{A}$ and $\bar{F}$ (Harrington, 1961). A third method involves the theory of Green's functions in conjunction with an integral Green's identity. The presentation of the Green's function method in this chapter is based on the work of Poggio and Miller (1973).

The resultant equations are singular integral equations* and require integral definitions if the differentiations involved are included on the kernels (or Green's functions) of the integral equations. This aspect will be discussed in Section 2.3. The integral equations will also be developed for thin structures and edges.

* Several authors refer to the equations as integro-differential equations. Since the integration and differentiation operate in a multiplicative manner rather than additive, this author prefers the singular integral terminology.

2.2 Integral Representations of the Total Electromagnetic Fields

The bases of all electromagnetic field problems are Maxwell's equations. These equations and appropriate boundary conditions may be used to formulate integral equations for the solution of electromagnetic scattering problems.

Maxwell's equations may be written in differential form as ($e^{j\omega t}$ time variation deleted)

$$\left.\begin{aligned} \nabla \times \bar{H} &= j\omega\varepsilon\bar{E} + \bar{J} \\ \nabla \times \bar{E} &= -j\omega\mu\bar{H} - \bar{K} \end{aligned}\right\} \tag{2.1a}$$

$$\left.\begin{aligned} \nabla \cdot \bar{E} &= \rho/\varepsilon \\ \nabla \cdot \bar{H} &= m/\mu \end{aligned}\right\} \tag{2.1b}$$

where $\bar{E}$ and $\bar{H}$ are the electric and magnetic field intensities, $\bar{J}$ and $\bar{K}$ are the electric and magnetic currents, and ρ and m are the electric and magnetic charge densities. The charge and current are related by the continuity equations

$$\left.\begin{aligned} -j\omega\rho &= \nabla \cdot \bar{J} \\ -j\omega m &= \nabla \cdot \bar{K} \end{aligned}\right\} \quad . \tag{2.2}$$

It has been assumed that the medium is homogeneous, linear, isotropic, and time-invariant for which ε and μ are scalar constants. Combining Maxwell's equations, one can write vector wave equations for $\bar{E}$ and $\bar{H}$ as

$$\left.\begin{aligned} \nabla \times \nabla \times \bar{E} &= k^2\bar{E} - \nabla \times \bar{K} - j\omega\mu\bar{J} \\ \nabla \times \nabla \times \bar{H} &= k^2\bar{H} + \nabla \times \bar{J} - j\omega\varepsilon\bar{K} \end{aligned}\right\} \quad . \tag{2.3}$$

To obtain integral representations for the fields $\bar{E}$ and $\bar{H}$, we make use of the vector Green's theorem

$$\int_V (\bar{Q} \cdot \nabla \times \nabla \times \bar{P} - \bar{P} \cdot \nabla \times \nabla \times \bar{Q})\, dv = \int_{\partial V} (\bar{P} \times \nabla \times \bar{Q} - \bar{Q} \times \nabla \times \bar{P}) \cdot d\bar{S} \tag{2.4}$$

where $\bar{P}$ and $\bar{Q}$ are two vector functions which have a continuous second derivative in V and on the bounding surface ∂V (Stratton, 1941). This restriction may be relaxed if the derivatives are interpreted in the sense of distributions and the resultant integrand is integrable in the sense of distributions (Arsac, 1966).

By setting $\bar{Q} = \bar{a}q$, where $\bar{a}$ is an arbitrary constant vector, we may rewrite Equation (2.4) as

$$\int_V [q\nabla \times \nabla \times \bar{P} + \bar{P}\nabla^2 q + (\nabla q)(\nabla \cdot \bar{P})]\, dv$$

$$= - \int_{\partial V} [-(\nabla q) \times (\hat{n} \times \bar{P}) + (\hat{n} \cdot \bar{P})\, \nabla q + \hat{n}q \times (\nabla \times \bar{P})]\, dS \tag{2.5}$$

where $\hat{n}$ is the inward normal on ∂V. Since q is arbitrary, we choose q to equal the free-space Green's function given by

$$q = \Phi(\bar{r},\bar{r}') = \frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r} - \bar{r}'|} \quad . \tag{2.6}$$

The function $\Phi(\bar{r},\bar{r}')$ is a well-known solution to the equation (Collin, 1960)

$$(\nabla^2 + k^2)\, \Phi(\bar{r},\bar{r}') = -\delta(\bar{r} - \bar{r}') \quad . \tag{2.7}$$

Replacing $\bar{P}$ by $\bar{E}(\bar{r}')$, one has

$$\bar{E}(\bar{r}) = -\int_V [\Phi(\nabla' \times \bar{K}) + j\omega\mu\Phi\bar{J} - \nabla'\Phi(\nabla' \cdot \bar{E})]\, dv'$$

$$+ \int_{\partial V} [(\hat{n}' \times \bar{E}) \times \nabla'\Phi + (\hat{n}' \cdot \bar{E})\, \nabla'\Phi + \Phi\hat{n}' \times (\nabla' \times \bar{E})]\, dS' \quad . \tag{2.8}$$

Using Maxwell's equations and the theorem of the rotational (Korn and Korn, 1967), Equation (2.8) becomes

$$\bar{E} = -\int_V [\bar{K} \times \nabla'\Phi + j\omega\mu\Phi\bar{J} - \nabla'\Phi\, \rho/\varepsilon]\, dv'$$

$$- \int_{\partial V} [j\omega\mu\Phi\hat{n}' \times \bar{H} - (\hat{n}' \times \bar{E}) \times \nabla'\Phi - (\hat{n}' \cdot \bar{E})\, \nabla'\Phi]\, dS' \quad . \quad (2.9)$$

It is assumed that $\bar{K}$, $\bar{J}$, and ρ are the sources that produce the incident fields for a scattering problem and that ∂V may be given as two surfaces s and s_i, where s is a surface surrounding a scattering object and s_i encloses s, s_i tending to infinity. If V contains all of the sources, then one can write

$$\bar{E}(\bar{r}) = \bar{E}^{inc}(\bar{r}) - \int_s \left[j\omega\mu\Phi\bar{J}_s + \bar{K}_s \times \nabla'\Phi + \nabla'\Phi \frac{\nabla' \cdot \bar{J}_s}{j\omega\varepsilon}\right] dS' \quad (2.10)$$

where $\bar{J}_s$, $\bar{K}_s$, and ρ_s represent the equivalent sources on the surface s (Jordan and Balmain, 1968). These sources are given by

$$\bar{J}_s = \hat{n}' \times \bar{H} ,$$

$$\bar{K}_s = -\hat{n}' \times \bar{E} ,$$

$$\text{and } \nabla' \cdot \bar{J}_s = -j\omega\varepsilon\hat{n}' \cdot \bar{E} \quad .$$

One may rewrite Expression (2.10) as

$$\bar{E}(\bar{r}) = \bar{E}^{inc}(\bar{r}) + \int_s \left[\bar{K}_s \times \nabla\Phi + \frac{1}{j\omega\varepsilon} (k^2 + \nabla\nabla\cdot)\, \bar{J}_s\Phi\right] dS' \quad (2.11a)$$

and by duality

$$\bar{H}(\bar{r}) = \bar{H}^{inc}(\bar{r}) - \int_s \left[\bar{J}_s \times \nabla\Phi - \frac{1}{j\omega\mu} (k^2 + \nabla\nabla\cdot)\, \bar{K}_s\Phi\right] dS' \quad . \quad (2.11b)$$

These expressions have been derived for $\bar{r} \notin s$, s being a closed surface. If the body is thin and $\bar{J}_s$ and $\bar{K}_s$ include currents on both sides of the surface, the expressions are still valid since $\bar{J}_s$ and $\bar{K}_s$, normal to the surface edges, which appear in the line integral resulting from the integration by parts, must vanish due to conservation of energy (Mittra and Lee, 1971).

2.3 Integral Equations for Scatterers

The integral equations obtained in this section are applicable to general scatterers including dielectric bodies. The latter are usually formulated using either an impedance boundary condition or a two-region formulation in which the fields are matched at the boundary.

In this section one deals with smooth surfaces in the neighborhood of singularities. To obtain the desired integral equations and the corresponding integral definitions, the limit shall be taken as the observation point $\bar{r}''$ approaches the point $\bar{r}$ on the surface. Although other authors use surface deformation in taking the limit, the more rigorous limiting procedure has been used here to ensure a consistent development of the Hadamard principal value (Hadamard, 1952) which is used.

Equation (2.11a) is rewritten as

$$\bar{E}(\bar{r}'') = \bar{E}^{inc}(\bar{r}'') + \int_{s-s_\Delta} \left[\bar{K}_s \times \nabla''\Phi + \frac{1}{j\omega\varepsilon}\left(k^2 + \nabla''\nabla''\cdot\right) \bar{J}_s\Phi\right] dS'$$

$$+ \int_{s_\Delta} \left[\bar{K}_s \times \nabla''\Phi + \frac{1}{j\omega\varepsilon}\left(k^2 + \nabla''\nabla''\cdot\right) \bar{J}_s\Phi\right] dS' \qquad (2.12)$$

where s_Δ is a patch on the surface surrounding the point $\bar{r}$ on the surface to which $\bar{r}''$ shall converge. Since the integral over $s - s_\Delta$ contains no kernel singularity, the limit of $\bar{r}''$ to $\bar{r}$ may be taken on this integral without special handling.

For simplicity, because the surface patch s_Δ is assumed to be sufficiently small, it may be considered as locally planar. The patch is circular of radius Δ with its center at the origin of the coordinate system as shown in Figure 2.1, where $\bar{r} = 0$. The integral over s_Δ is given by I_Δ and is written as

$$I_\Delta = \int_{s_\Delta} \left\{ \left(K_x \frac{\partial \Phi}{\partial y''} - K_y \frac{\partial \Phi}{\partial x''} \right) \hat{z} + \left[K_y \frac{\partial \Phi}{\partial z''} + \frac{J_x}{j\omega\varepsilon} \left(k^2 \Phi + \frac{\partial^2 \Phi}{\partial x''^2} \right) + \frac{J_y}{j\omega\varepsilon} \frac{\partial^2 \Phi}{\partial x'' \partial y''} \right] \hat{x} \right.$$
$$+ \left[-K_x \frac{\partial \Phi}{\partial z''} + \frac{J_y}{j\omega\varepsilon} \left(k^2 \Phi + \frac{\partial^2 \Phi}{\partial y''^2} \right) + \frac{J_x}{j\omega\varepsilon} \frac{\partial^2 \Phi}{\partial y'' \partial x''} \right] \hat{y}$$
$$\left. + \frac{1}{j\omega\varepsilon} \left(J_x \frac{\partial^2 \Phi}{\partial z'' \partial x''} + J_y \frac{\partial^2 \Phi}{\partial z'' \partial y''} \right) \hat{z} \right\} dx'\ dy' \quad . \qquad (2.13)$$

Taking the limit of $\bar{r}''$ tending to $\bar{r}$ as is done in the Appendix, one has

$$I_\Delta = \frac{1}{2} \left[\hat{n} \times \bar{K}_s(\bar{r}) - \frac{1}{j\omega\varepsilon} \nabla \cdot \bar{J}_s(\bar{r}) \right] - \frac{1}{4j\omega\varepsilon\Delta} \bar{J}_s(\bar{r}) + 0(\Delta) \ .$$

Substituting expressions in terms of $\bar{E}$ and $\bar{H}$ for $\bar{K}_s$ and $\nabla \cdot \bar{J}_s$, the integration result may be rewritten as

$$I_\Delta = \frac{1}{2} \bar{E}(\bar{r}) - \frac{1}{4j\omega\varepsilon\Delta} \bar{J}_s(\bar{r}) + 0(\Delta) \ . \qquad (2.14)$$

Equation (2.14) may also be written in terms of surface integrations as

$$I_\Delta = \frac{1}{2} \bar{E}(\bar{r}) + \overline{\left| \int_{s_\Delta} [\bar{K}_s \times \nabla\Phi + \frac{1}{j\omega\varepsilon} (k^2 + \nabla\nabla\cdot) \bar{J}_s \Phi]\ dS' \right|} \qquad (2.15)$$

where the symbol "$\overline{\mid \qquad \mid}$" refers to the Hadamard principal value

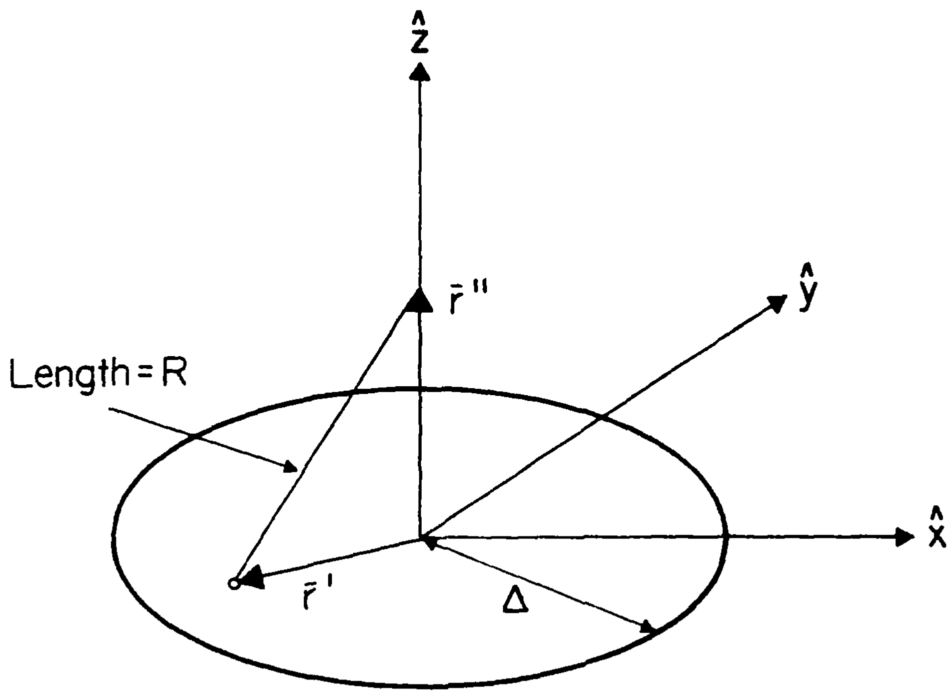

Figure 2.1. Coordinate system for the surface patch s_{Δ}.

(Hadamard, 1952) and means "the finite part of." To simplify notation, we shall use a double bar through the integral sign in lieu of this symbol to denote the Hadamard principal value.

In general, the Hadamard principal value may be evaluated in the Cauchy sense with the exception that the infinite parts thereof are discarded. In this sense the Hadamard principal value is more general than that of Cauchy and in fact includes the latter as a form of integration interpretation.

Using Equation (2.15), one writes the limiting form of Equation (2.12) on the surface as

$$\bar{E}(\bar{r}) = 2\bar{E}^{inc}(\bar{r}) + 2 \,\,{=}\!\!\!\!\!\!\int_S \left[\bar{K}_s \times \nabla\Phi + \frac{1}{j\omega\varepsilon} (k^2 + \nabla\nabla\cdot)\, \bar{J}_s\Phi\right] dS' \; . \quad (2.16)$$

By duality, $\bar{H}(\bar{r})$ is written as

$$\bar{H}(\bar{r}) = 2\bar{H}^{inc}(\bar{r}) - 2 \,\,{=}\!\!\!\!\!\!\int_S \left[\bar{J}_s \times \nabla\Phi - \frac{1}{j\omega\mu} (k^2 + \nabla\nabla\cdot)\, \bar{K}_s\Phi\right] dS' \; . \quad (2.17)$$

Although Equations (2.16) and (2.17) have been developed for a locally planar structure, the equations still apply for surfaces which are locally twice differentiable in the sense of the corresponding surface defining equations. The resultant quadratic terms in the integrand do not change the above results.

It should be noted that the Cauchy principal value or integration in the normal sense may be used if the derivatives are taken on the currents or taken outside the integral, respectively. In the latter, other vector-differential identities must be used to obtain the residue terms of the Cauchy or Hadamard forms of integration.

2.4 Integral Equations for Thin Surfaces and Surfaces with Edges

The extension of our results to thin surfaces is straightforward. A thin surface is defined as a surface that does not enclose a volume. This classification can also be used for surfaces that satisfy this assumption in only some regions.

If the surfaces of s are designated as s_+ and s_-, the $\bar{K}_s$ and $\bar{J}_s$ are written as

$$\bar{K}_s = \bar{K}_s^+ + \bar{K}_s^-$$
$$= -\hat{n}_+ \times [\bar{E}(\bar{r}_+) - \bar{E}(\bar{r}_-)] \tag{2.18a}$$

and

$$\bar{J}_s = \bar{J}_s^+ + \bar{J}_s^-$$
$$= \hat{n}_+ \times [\bar{H}(\bar{r}_+) - \bar{H}(\bar{r}_-)] \quad . \tag{2.18b}$$

The residue of the Hadamard principal value becomes $1/2[\bar{E}(\bar{r}_+) - \bar{E}(\bar{r}_-)]$ when the surface is approached on the s_+ side. Using this residue for Equation (2.12) gives

$$\bar{E}(\bar{r}_+) + \bar{E}(\bar{r}_-) = 2\bar{E}^{inc}(\bar{r}) + 2 \fint_s \left[\bar{K}_s \times \nabla\Phi + \frac{1}{j\omega\varepsilon} (k^2 + \nabla\nabla\cdot)\, \bar{J}_s\Phi\right] dS' \tag{2.19a}$$

and by duality

$$\bar{H}(\bar{r}_+) + \bar{H}(\bar{r}_-) = 2\bar{H}^{inc}(\bar{r}) - 2 \fint_s \left[\bar{J}_s \times \nabla\Phi - \frac{1}{j\omega\mu} (k^2 + \nabla\nabla\cdot)\, \bar{K}_s\Phi\right] dS' \quad . \tag{2.19b}$$

For a perfect electric conductor, Equations (2.19) may be simplified to

$$0 = \hat{n}_+ \times \bar{E}^{inc}(\bar{r}) + \frac{\hat{n}_+ \times}{j\omega\varepsilon} \fint_s (k^2 + \nabla\nabla\cdot)\, \bar{J}_s\Phi \, dS' \tag{2.20a}$$

and

$$\frac{1}{2}\left(\bar{J}_s^+ - \bar{J}_s^-\right) = \hat{n}_+ \times \bar{H}^{inc}(\bar{r}) - \hat{n}_+ \times \oint_s (\bar{J}_s \times \nabla\Phi)\, dS' \qquad (2.20b)$$

in addition to the equations for the normal components. Equation (2.20a) is a Fredholm equation of the first kind for the sum current $\left(\bar{J}_s^+ + \bar{J}_s^-\right)$ while Equation (2.20b) is an integral expression for the difference current $\left(J_s^+ - J_s^-\right)$. The latter is usually of no interest since it does not enter into the calculation of the fields away from the surface of the scatterer. Hence, for thin, perfect electric conductors only the equations due to the boundary conditions on tangential $\bar{E}$ and normal $\bar{H}$ are used.

In addition to thin structures, a scatterer may have an edge. When one approaches an edge on a scattering surface, one finds, in general, that only the fields parallel to the edge are bounded (Collin, 1960). In fact the fields parallel to the edge tend to zero as one approaches the edge. Hence, the integral equations that correspond to these fields are the only ones that exist at the edge. The additional equations needed at the edge are boundary conditions on the currents such that the currents perpendicular to the edge must be zero. These latter two conditions are also necessary for the fields parallel to the edge to be zero.

3. INTEGRAL EQUATIONS FOR A HOLLOW CIRCULAR CYLINDER

3.1 Introduction

The structure of interest in this paper is a finite, perfectly conducting, hollow, circular cylinder, which is a member of the larger group of structures, bodies of revolution. Due to the circular symmetry, the corresponding scattering problems are often solved using a Fourier expansion of the fields and currents.

Bennett has investigated general bodies of revolution in the time domain using the tangential $\bar{H}$-field equations (Bennett, 1970). He considered only axial incidence and needed only the first-order harmonics. Harrington has solved the problem of an infinite cylinder (Harrington, 1961), and Kao has considered the finite hollow cylinder (Kao, 1969 and 1970). The work of Kao will be used for comparison in a later chapter.

In this chapter, the incident fields and surface currents are expanded in Fourier series and the corresponding integral equations obtained for each harmonic.

3.2 Incident Field and Current Expansions

Any incident field distribution and associated surface currents on a body of rotation may be expanded in Fourier series representations. For simplicity, the incident field will be expressed as a uniform plane wave with propagation direction given by $\hat{k}$ in the x-z plane as shown in Figure 3.1. The given incident plane wave is further specified to be a linear combination of E- and H-polarizations defined with total $\bar{E}$ and $\bar{H}$, respectively, contained in the x-z plane.

To develop the expressions for $\bar{E}^{inc}$ and $\bar{H}^{inc}$, the following vectors are defined as

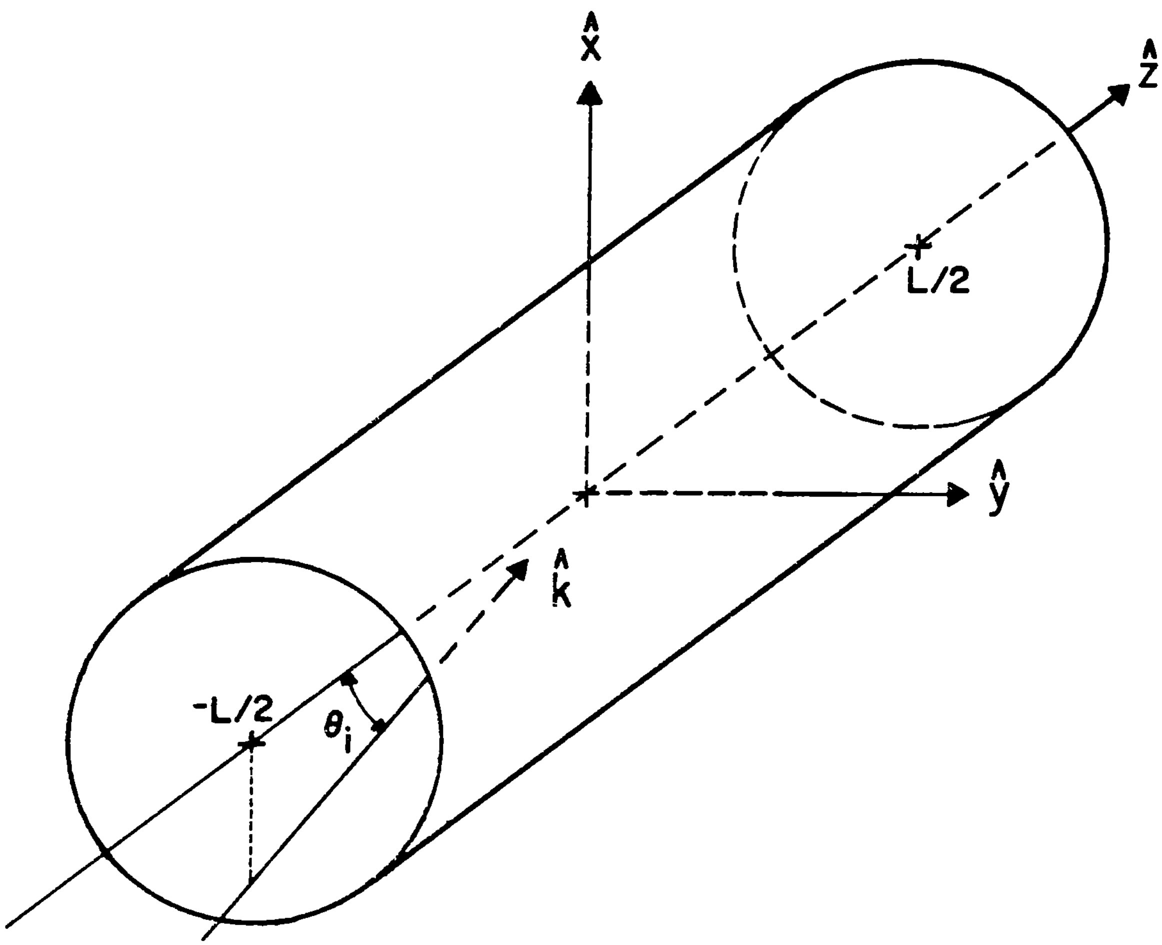

Figure 3.1. The propagation vector $\hat{k}$ with respect to the finite cylindrical structure.

$$\left.\begin{aligned}
\hat{k} &= \hat{\rho}\sin\theta_i\cos\phi - \hat{\phi}\sin\theta_i\sin\phi + \hat{z}\cos\theta_i \\
\hat{e}_e &= \hat{\rho}\cos\theta_i\cos\phi - \hat{\phi}\cos\theta_i\sin\phi - \hat{z}\sin\theta_i \\
\hat{e}_h &= -(\hat{\rho}\sin\phi + \hat{\phi}\cos\phi) \\
\hat{h}_e &= -\hat{e}_h \\
\hat{h}_h &= \hat{e}_e
\end{aligned}\right\} \quad . \tag{3.1}$$

For the electric field strengths for E- and H-polarization given as $E_e^{inc} e^{-j\bar{k}\cdot\bar{r}}$ and $E_h^{inc} e^{-j\bar{k}\cdot\bar{r}}$, respectively, one may write the incident fields as

$$\bar{E}^{inc} = \left[E_e^{inc}\hat{e}_e + E_h^{inc}\hat{e}_h\right] e^{-j\bar{k}\cdot\bar{r}}$$

and

$$\bar{H}^{inc} = \left[E_h^{inc}\hat{e}_e - E_e^{inc}\hat{e}_h\right] \frac{e^{-j\bar{k}\cdot\bar{r}}}{\eta} \tag{3.2}$$

where $\bar{k} = k\hat{k}$ and $k = 2\pi/\lambda$. Expressing $\bar{r}$ by the cylindrical coordinate triple, (ρ,ϕ,z), one has

$$e^{-j\bar{k}\cdot\bar{r}} = e^{-jk\rho\sin\theta_i\cos\phi}\, e^{-jkz\cos\theta_i} \quad . \tag{3.3}$$

The Fourier series in ϕ for the exponential term in $\cos\phi$ is well-known and is given by

$$e^{-jx\cos\phi} = \sum_{n=-\infty}^{\infty} (-j)^n J_n(x)\, e^{jn\phi} \tag{3.4a}$$

where $J_n(x)$ is the Bessel function of the first kind with order n and argument x. Two other series needed are

$$e^{-jx\cos\phi}\cos\phi = j\sum_{n=-\infty}^{\infty}(-j)^n J_n'(x)\, e^{jn\phi} \qquad (3.4b)$$

and

$$e^{-jx\cos\phi}\sin\phi = \sum_{n=-\infty}^{\infty}(-j)^n \frac{nJ_n(x)}{x}\, e^{jn\phi} \qquad (3.4c)$$

where the prime denotes derivative in x.

Substituting Expressions (3.1), (3.3), and (3.4) into Expression (3.2), the incident field expressions become

$$\bar{E}^{inc} = -e^{-jkz\cos\theta_i}\sum_{n=-\infty}^{\infty}(-j)^n \left\{\hat{\rho}\left[-jE_e^{inc}J_n'(x)\cos\theta_i + E_h^{inc}\frac{nJ_n(x)}{x}\right] + \hat{\phi}\left[E_e^{inc}\frac{nJ_n(x)}{x}\cos\theta_i + jE_h^{inc}J_n'(x)\right] + \hat{z}E_e^{inc}J_n(x)\sin\theta_i\right\} e^{jn\phi} \qquad (3.5a)$$

and

$$\bar{H}^{inc} = \frac{-e^{-jkz\cos\theta_i}}{\eta}\sum_{n=-\infty}^{\infty}(-j)^n \left\{-\hat{\rho}\left[jE_h^{inc}J_n'(x)\cos\theta_i + E_e^{inc}\frac{nJ_n(x)}{x}\right] + \hat{\phi}\left[E_h^{inc}\frac{nJ_n(x)}{x}\cos\theta_i - jE_e^{inc}J_n'(x)\right] + \hat{z}E_h^{inc}J_n(x)\sin\theta_i\right\} e^{jn\phi} \qquad (3.5b)$$

where $x = k\rho\sin\theta_i$. These expressions can be written in the form

$$\bar{E}^{inc} = e^{-jkz\cos\theta_i}\sum_{n=-\infty}^{\infty}\left[\hat{\rho}E_{\rho_n}^{inc} + \hat{\phi}E_{\phi_n}^{inc} + \hat{z}E_{z_n}^{inc}\right] e^{jn\phi}$$

and

$$\bar{H}^{inc} = e^{-jkz\cos\theta_i}\sum_{n=-\infty}^{\infty}\left[\hat{\rho}H_{\rho_n}^{inc} + \hat{\phi}H_{\phi_n}^{inc} + \hat{z}H_{z_n}^{inc}\right] e^{jn\phi} \qquad (3.6)$$

where the coefficients are functions of ρ only and the variation with respect to ϕ and z appears explicitly.

The currents on the surface of a perfectly conducting hollow cylinder of radius "a" can also be expanded in a Fourier series as

$$\bar{J}_s = \sum_{n=-\infty}^{\infty} \left[\hat{\phi} J_{\phi_n}(z) + \hat{z} J_{z_n}(z)\right] e^{jn\phi} \quad . \tag{3.7}$$

Since the surface is perfectly conducting, the magnetic surface current, $\bar{K}_s$, is set to zero. It will be shown in the next section that the currents of harmonic n are associated only with the incident fields of the same harmonic. In addition one should note that J_{z_n} and J_{ϕ_n} are even and odd, respectively, for E-polarization and odd and even, respectively, for H-polarization. Hence, one only needs solutions for the harmonic currents with non-negative n.

3.3 Integral Equations for a Hollow Cylinder

Since the scattering structure is a thin surface, the tangential electric and normal magnetic integral equations will be used as explained in Section 2.4. Equation (2.20a) and the normal H equation from Equation (2.19) are rewritten on the cylinder surface as

$$-j\omega\varepsilon\hat{\rho} \times \bar{E}^{inc} = \hat{\rho} \times \fint_s (k^2 + \nabla\nabla\cdot)\, \bar{J}_s \Phi \, dS' \tag{3.8a}$$

and

$$\hat{\rho} \cdot \bar{H}^{inc} = \hat{\rho} \cdot \fint_s (\bar{J}_s \times \nabla\Phi)\, dS' \tag{3.8b}$$

where $\Phi = e^{-jkR}/4\pi R$, $R = [(z - z')^2 + \rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi' - \phi)]^{1/2}$. Noting that $\nabla\Phi = -\nabla'\Phi$, one writes

$$-j\omega\varepsilon E_\phi^{inc}(a,\phi,z) = k^2 \int_s [J_\phi(\phi',z')\, \Phi(\bar{r},\bar{r}') \cos(\phi' - \phi)]\, a\, d\phi'\, dz'$$

$$- \fint_s \left[J_\phi(\phi',z')\, \frac{1}{a^2} \frac{\partial^2\Phi(\bar{r},\bar{r}')}{\partial\phi'\partial\phi} + J_z(\phi',z')\, \frac{1}{a} \frac{\partial^2\Phi(\bar{r},\bar{r}')}{\partial z'\partial\phi}\right] a\, d\phi'\, dz' \tag{3.9a}$$

$$-j\omega\epsilon E_z^{inc}(a,\phi,z) = k^2 \int_s [J_z(\phi',z')\ \Phi(\bar{r},\bar{r}')]\ a\ d\phi'\ dz'$$

$$- \fint_s \left[J_\phi(\phi',z')\ \frac{1}{a}\ \frac{\partial^2\Phi(\bar{r},\bar{r}')}{\partial\phi'\partial z} + J_z(\phi',z')\ \frac{\partial^2\Phi(\bar{r},\bar{r}')}{\partial z'\partial z}\right] a\ d\phi'\ dz' \qquad (3.9b)$$

and

$$H_\rho^{inc}(a,\phi,z) = \fint_s \left[J_\phi(\phi',z')\ \frac{\partial\Phi(\bar{r},\bar{r}')}{\partial z}\ \cos\ (\phi' - \phi)\right.$$

$$\left. - J_z(\phi',z')\ \frac{1}{a}\ \frac{\partial\Phi(\bar{r},\bar{r}')}{\partial\phi}\right] a\ d\phi'\ dz' \qquad (3.9c)$$

where the Cauchy principal value and standard integral interpretation have been used for the appropriate integrals.

The difference form of Φ with respect to ϕ and z enables one to simplify Equations (3.9) by replacing the differentiations in ϕ and z' by the negative of the differentiations in ϕ' and z, respectively. To complete the simplification, one may integrate by parts in ϕ' to obtain

$$-j\omega\epsilon E_\phi^{inc}(a,\phi,z) = \int_s \left\{\Phi\left[k^2 J_\phi(\alpha + \phi,z')\ \cos\ \alpha + \frac{1}{a^2}\ \frac{\partial^2}{\partial\alpha^2}\ J_\phi(\alpha + \phi,z')\right]\right\} a\ d\alpha\ dz'$$

$$+ \frac{1}{a} \fint_s \left[\frac{\partial}{\partial\alpha}\ J_z(\alpha + \phi,z')\ \frac{\partial}{\partial z}\ \Phi\right]\ a\ d\alpha\ dz' \qquad (3.10a)$$

$$-j\omega\epsilon E_z^{inc}(a,\phi,z) = \fint_s \left[J_z(\alpha + \phi,z')\ k^2\Phi + \frac{\partial^2}{\partial z^2}\ \Phi\right]\ a\ d\alpha\ dz'$$

$$+ \frac{1}{a} \fint_s \left[\frac{\partial}{\partial\alpha}\ J_\phi(\alpha + \phi,z')\ \frac{\partial}{\partial z}\ \Phi\right]\ a\ d\alpha\ dz' \qquad (3.10b)$$

and

$$H_\rho^{inc}(a,\phi,z) = \fint_s \left[J_\phi(\alpha + \phi,z')\ \frac{\partial}{\partial z}\ \Phi\ \cos\ \alpha - \Phi\ \frac{1}{a}\ \frac{\partial}{\partial\alpha}\ J_z(\alpha + \phi,z')\right]\ a\ d\alpha\ dz' \qquad (3.10c)$$

where $\alpha = \phi' - \phi$. The integration by parts is valid since the current is differentiable in ϕ and the operation is done prior to taking the limit of the principal value. All of the corresponding Hadamard principal values are thus reduced to those of a Cauchy type since the end terms on the limiting curve surrounding the singularity correspond to the infinite terms of the Hadamard principal value.

Upon substitution of Expressions (3.6) and (3.7) for the incident fields and currents on a hollow cylinder of length L, Equations (3.10) decouple for each harmonic to give

$$-j\omega\varepsilon E^{inc}_{\phi_n} e^{-jkz\cos\theta_i} = \int_{-L/2}^{L/2} \left\{ J_{\phi_n}(z') \left[k^2 G_{1n}(z - z') - \frac{n^2}{a^2} G_{0n}(z - z') \right] \right\} dz'$$

$$+ \frac{jn}{a} \fint_{-L/2}^{L/2} \left[J_{z_n}(z') \frac{\partial}{\partial z} G_{0n}(z - z') \right] dz' \qquad (3.11a)$$

$$-j\omega\varepsilon E^{inc}_{z_n} e^{-jkz\cos\theta_i} = \fint_{-L/2}^{L/2} \left[J_{z_n}(z') \left(k^2 + \frac{\partial^2}{\partial z^2} \right) G_{0n}(z - z') \right] dz'$$

$$+ \frac{jn}{a} \fint_{-L/2}^{L/2} \left[J_{\phi_n}(z') \frac{\partial}{\partial z} G_{0n}(z - z') \right] dz' \qquad (3.11b)$$

and

$$H^{inc}_{\rho_n} e^{-jkz\cos\theta_i} = \fint_{-L/2}^{L/2} \left[J_{\phi_n}(z') \frac{\partial}{\partial z} G_{1n}(z - z') - \frac{jn}{a} J_{z_n}(z') G_{0n}(z - z') \right] dz' \qquad (3.11c)$$

where the kernels, $G_{0n}(z)$ and $G_{1n}(z)$, are given by

$$G_{mn}(z) = \frac{1}{4\pi} \int_0^{2\pi} \left[\frac{e^{-jkR}}{R} \cos^m \alpha \cos n\alpha \right] a \, d\alpha \qquad (3.12)$$

with $R = \left[z^2 + 4a^2 \sin \frac{\alpha}{2} \right]^{1/2}$. Hence, the cylinder problem becomes an infinite set of one-dimensional integral equations with two coupled unknowns for each harmonic set. The distinct advantage of such an approach is the reduced complexity of the integral equations and the rapid convergence of the truncated series representation (Kao, 1970).

Equations (3.11) consist of three dependent equations of which only two are needed to obtain the currents on the surface of the cylinder. The coupling between equations is strong, being dominated by a z-derivative. To obtain further decoupling, which is advantageous for numerical solution, one combines Equations (3.11) to obtain an integral expression for $[j\omega\varepsilon\bar{E}_t + \hat{n} \times \nabla_t H_n]$ (Mittra et al., 1973), where t indicates the tangential component. The equations thus obtained for the cylinder are

$$\left[-j\omega\varepsilon E_{\phi_n}^{inc} - jkH_{\rho_n}^{inc} \cos\theta_i\right] e^{-jkz\cos\theta_i}$$
$$= \fint_{-L/2}^{L/2} \left\{ J_{\phi_n}(z') \left[\left(k^2 + \frac{\partial^2}{\partial z^2}\right) G_{1n}(z - z') - \frac{n^2}{a^2} G_{0n}(z - z') \right] \right\} dz' \tag{3.13a}$$

and

$$\left[-j\omega\varepsilon E_{z_n}^{inc} - \frac{jn}{a} H_{\rho_n}^{inc}\right] e^{-jkz\cos\theta_i}$$
$$= \fint_{-L/2}^{L/2} \left\{ J_{z_n}(z') \left[\left(k^2 - \frac{n^2}{a^2} + \frac{\partial^2}{\partial z^2}\right) G_{0n}(z - z') \right] \right\} dz'$$
$$+ \frac{jn}{a} \int_{-L/2}^{L/2} \left\{ J_{\phi_n}(z') \frac{\partial}{\partial z} \left[G_{0n}(z - z') - G_{1n}(z - z') \right] \right\} dz' \ . \tag{3.13b}$$

These equations may also be obtained by setting the ρ-derivatives of H_z and (ρH_ϕ), respectively, equal to zero.

Equations (3.11) and (3.13) are defined only for z in the interior of the interval (-L/2,L/2), except Equation (3.11a) which is defined in the closed interval [-L/2,L/2]. In certain instances, the latter may be used to insure a unique solution to Maxwell's equations, as discussed in Section 3.5.

3.4 Kernel Evaluation

The kernels of Equations (3.11) and (3.13) were expressed in (3.12) as

$$G_{mn}(z) = \frac{1}{4\pi} \int_0^{2\pi} \left[\frac{e^{-jkR}}{R} \cos^m \alpha \cos n\alpha\right] a \, d\alpha \quad . \tag{3.14}$$

Another representation may be obtained by expanding e^{-jkR}/R in a Fourier series of α and using the addition theorem for Bessel's functions. One obtains

$$G_{0n}(z) = \frac{1}{4j} \int_{-\infty}^{\infty} \left[H_n^{(2)}\left(\sqrt{k^2 - \zeta^2}\, a\right) J_n\left(\sqrt{k^2 - \zeta^2}\, a\right)\right] e^{j\zeta z} \, d\zeta \tag{3.15a}$$

and

$$G_{1n}(z) = \frac{1}{4j} \int_{-\infty}^{\infty} \left[H_n^{(2)\prime}\left(\sqrt{k^2 - \zeta^2}\, a\right) J_n'\left(\sqrt{k^2 - \zeta^2}\, a\right) + \frac{n^2}{a^2(k^2 - \zeta^2)} H_n^2\left(\sqrt{k^2 - \zeta^2}\, a\right) J_n\left(\sqrt{k^2 - \zeta^2}\, a\right)\right] e^{j\zeta z} \, d\zeta \quad . \tag{3.15b}$$

Both Expressions (3.14) and (3.15) are equally valid, where the α integration in (3.14) has been replaced by an inverse transform in z in (3.15). Kao has used the kernel representations of (3.15) in conjunction with the electric field integral equations (EFIE) given by Equations (3.11 a and b) to solve the problem for the harmonic currents (Kao, 1969 and 1970).

In this paper, the representation of (3.14) will be used in the integral equations. Since the integral of $G_{mn}(z)$ times a set of basis functions can not, in general, be done in closed form, it is desirable to expand $G_{mn}(z)$ as suggested by Krylov (1962). The expansion consists of singular and residual parts which may respectively be integrated in closed form and by numerical methods to produce an accurate scheme of integration.

A standard technique for this expansion has evolved through the years for $G_{00}(z)$ (Poggio and Mayes, 1969) where a three-term Taylor series is added and subtracted to give

$$G_{00}(z) = \frac{1}{4\pi}\int_0^{2\pi}\left[\frac{1}{R} - jk - \frac{k^2R}{2}\right] a d\alpha + \frac{1}{4\pi}\int_0^{2\pi}\left[\frac{e^{-jkR} - 1 + jkR + k^2R^2/2}{R}\right] a d\alpha \ . \tag{3.16}$$

The second integral is easily evaluated numerically while the first integral gives rise to a constant and the elliptic integrals of the first and second kinds. The latter may be expanded to extract the singularities.

A more straightforward technique for extracting the singularities involves adding and subtracting a three-term Taylor series times the Jacobian arising from the transformation of an integration in α to an integration in $y = 2a \sin\frac{\alpha}{2}$. For $G_{mn}(z)$ this becomes

$$G_{mn}(z) = \frac{a}{4\pi}\int_{-\pi}^{\pi}\left\{e^{-jkR}\cos^m\alpha\cos n\alpha - \left[1 - jkR - \frac{k^2R^2}{2} - 2\left(m + n^2 - \frac{1}{4}\right)\sin^2\frac{\alpha}{2}\right]\cos\frac{\alpha}{2}\right\}/R \ d\alpha + \frac{1}{4\pi}\int_{-2a}^{2a} \cdot \left\{\left[1 + \left(m + n^2 - \frac{1}{4}\right)\frac{z^2}{2a^2}\right]\frac{1}{R} - jk - \left[\frac{k^2a^2 + m + n^2 - \frac{1}{4}}{2a^2}\right] R\right\} dy$$

which may be written

$$G_{mn}(z) = -\frac{1}{2\pi}\ \ell n|z| + \left(k^2a^2 - m - n^2 + \frac{1}{4}\right)\frac{z^2}{8\pi a^2}\ \ell n|z| + R_{mn}(z)\ . \tag{3.17}$$

The residual, $R_{mn}(z)$, has a continuous second derivative and is expressed as

$$\begin{aligned} R_{mn}(z) = \frac{a}{4\pi}\int_{-\pi}^{\pi}\Bigg\{ & e^{-jkR}\cos^m\alpha\cos n\alpha - \left[1 - jkR - \frac{k^2R^2}{2}\right. \\ & \left. - 2\left(m + n^2 - \frac{1}{4}\right)\sin^2\frac{\alpha}{2}\right]\cos\frac{\alpha}{2}\Bigg\}/R\ \ d\alpha \\ & + \frac{1}{2\pi}\left[1 - \left(k^2a^2 - m - n^2 + \frac{1}{4}\right)\frac{z^2}{4a^2}\right]\ell n\left|2a + \sqrt{4a^2 + z^2}\right| \\ & - j\,\frac{ka}{\pi} - \left(k^2a^2 + m + n^2 - \frac{1}{4}\right)\frac{\sqrt{4a^2 + z^2}}{4\pi a}\ . \end{aligned} \tag{3.18}$$

The residual is simply evaluated numerically and the first and second derivatives may be evaluated by finite differences as long as the sample distance in z is much less than 2a.

Equation (3.17) and the derivatives may be written in final form as

$$G_{mn}(z) = -\frac{1}{2\pi}\ \ell n|z| + R_{mn}^{(0)}(z)\quad , \tag{3.19a}$$

$$\frac{d}{dz}G_{mn}(z) = \frac{-1}{2\pi z} + R_{mn}^{(1)}(z)\quad , \tag{3.19b}$$

and

$$\frac{d^2}{dz^2}G_{mn}(z) = \frac{1}{2\pi z^2} + \frac{\left(k^2a^2 - m - n^2 + \frac{1}{4}\right)}{4\pi a^2}\ \ell n|z| + R_{mn}^{(2)}(z) \tag{3.19c}$$

where the residuals are given by

$$R_{mn}^{(0)}(z) = \frac{\left(k^2a^2 - m - n^2 + \frac{1}{4}\right)}{8\pi a^2} z^2 \ln|z| + R_{mn}(z) \quad ,$$

$$R_{mn}^{(1)}(z) = \frac{\left(k^2a^2 - m - n^2 + \frac{1}{4}\right)}{8\pi a^2} (z + 2z\ln|z|) + \frac{d}{dz} R_{mn}(z) \quad ,$$

and

$$R_{mn}^{(2)}(z) = \frac{3\left(k^2a^2 - m - n^2 + \frac{1}{4}\right)}{8\pi a^2} + \frac{d^2}{dz^2} R_{mn}(z) \quad .$$

Figures 3.2 and 3.3 are plots of the real parts for the components of $G_{m1}(z)$, $\frac{d}{dz} G_{m1}(z)$, and $\frac{d^2}{dz^2} G_{m1}(z)$ for m equal to 0 and 1, respectively, with ka = 1. The residuals have been approximated by piecewise linear functions and the logarithmic terms have been normalized to the sample distance in z.

It should be noted that the singularities dominate only for $|z| < 2a$ and cancel with terms in the residuals for $|z| > 2a$. This property enables one to approximate $G_{00}(z)$ for 2a small as

$$G_{00}(z) \simeq \frac{a}{2} \frac{e^{-jk\tilde{R}}}{\tilde{R}}$$

where $\tilde{R} = [z^2 + a^2]^{1/2}$. In the numerical problem, 2a small corresponds to 2a much less than the sampling distance in z.

3.5 Additional Constraints Required for a Unique Solution

Equations (3.11) and (3.13) consist of five integral equations derived from Maxwell's equations for the harmonic currents on a hollow cylinder. Several authors have shown for a variety of structures with edges that Maxwell's equations are not complete (Mittra and Lee, 1971;

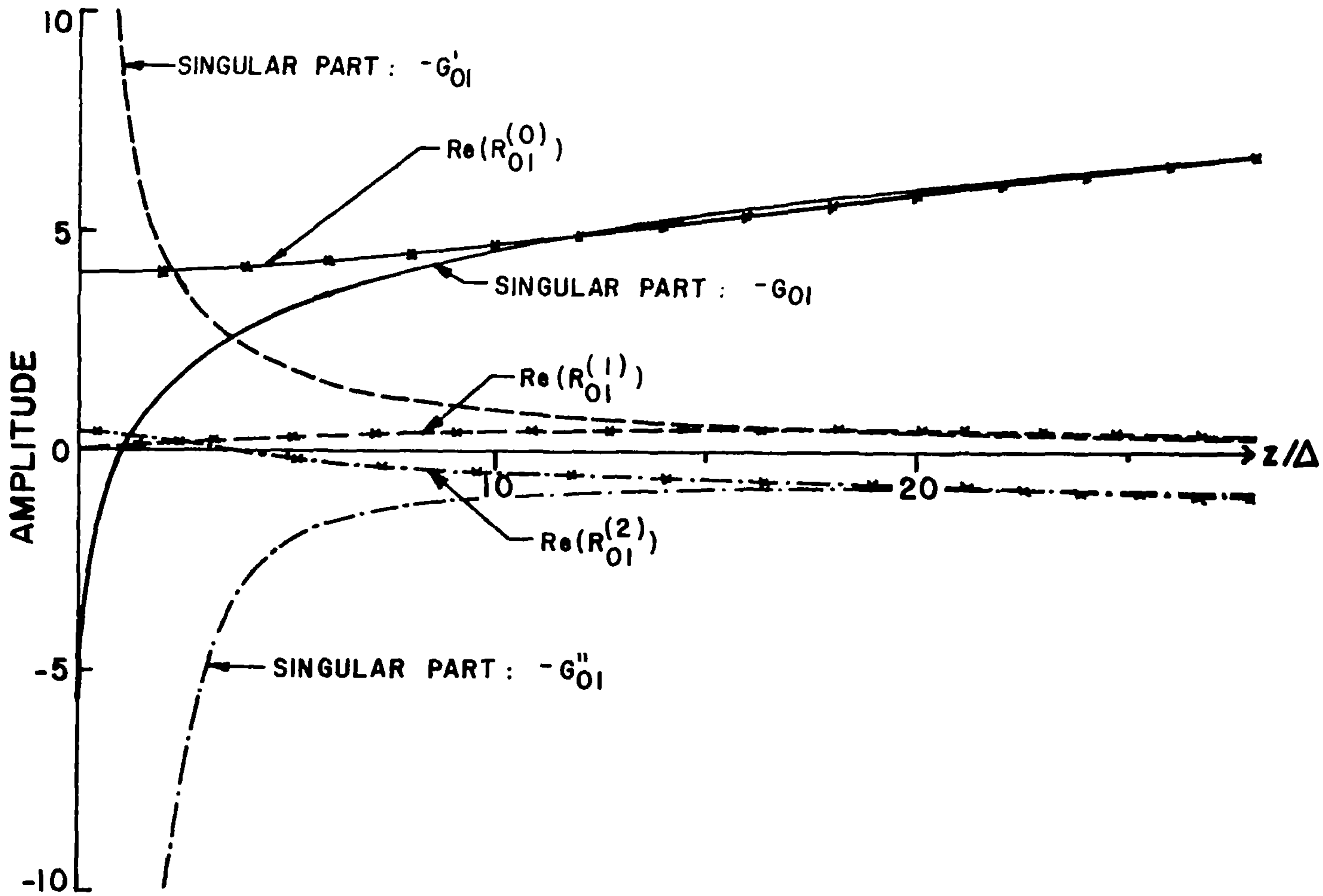

Figure 3.2. The real parts of the singular and residual portions of $G_{01}(z)$ and its first and second derivatives. The derivatives are denoted by primes. The sample spacing is given by $\Delta = \lambda/29$.

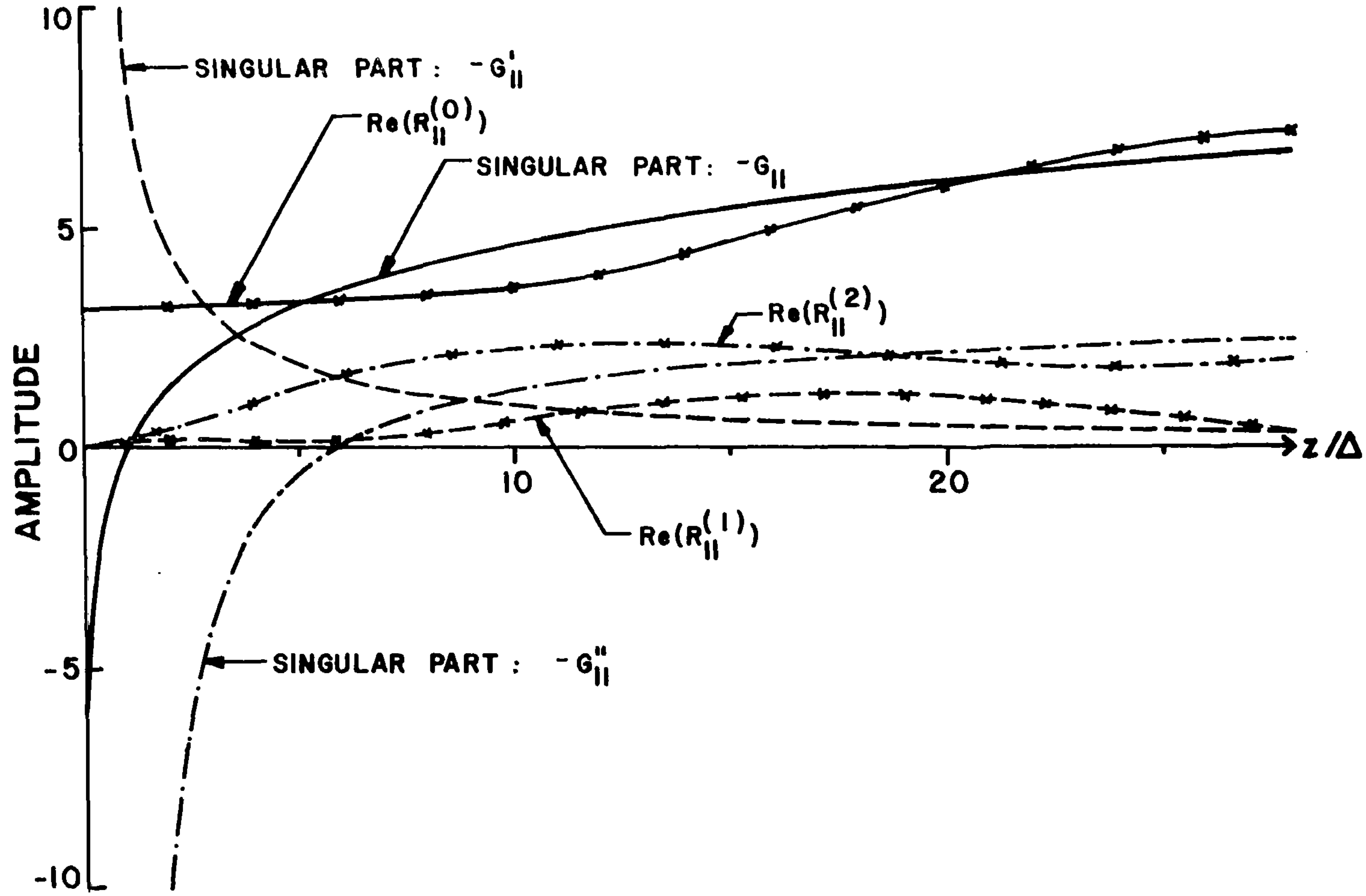

Figure 3.3. The real parts of the singular and residual portions of $G_{11}(z)$ and its first and second derivatives. The derivatives are denoted by primes. The sample spacing is given by $\Delta = \lambda/29$.

Collin, 1960; Jones, 1964). In order to obtain a unique solution, a condition for finite energy must be imposed resulting in specific asymptotic behaviors of the fields and currents, requiring that the fields parallel to the edge asymptotically approach zero.

To complete the statement of the hollow cylinder problem, E_{ϕ_n} and H_{ϕ_n} must be set to zero at the ends of the cylinder. H_{ϕ_n} equal to zero directly implies that J_{z_n} must be zero, while E_{ϕ_n} equal to zero provides two additional constraints necessary to constrain the two degrees of freedom that occur for J_{ϕ_n}. It should be noted that the latter condition requires the asymptotic behavior $J_{\phi_n} = 0\left(1/\sqrt{L/2 - |z|}\right)$ with respect to an edge at $z = \pm L/2$. With the correct asymptotic behavior for J_{z_n} and J_{ϕ_n}, one finds that $\lim\limits_{z \to \pm L/2} H_{\rho_n}(z) = 0$ is consistent with $E_{\phi_n}(\pm L/2) = 0$. Both the constraint on the behavior of E_{ϕ_n}, and $J_{z_n}(\pm L/2) = 0$ will henceforth be referred to as the constraints for the cylinder problem.

Any two of the five integral equations in Section 3.3 might conceivably be used to solve for the harmonic currents. The only requirement is that the equations in conjunction with their constraints be independent. For $n = 0$, it is clear that either Equation (3.11b) or (3.13b) be used in conjunction with one of the other three equations. For $n \neq 0$, any of the ten combinations may be used.

In Chapter 6, four particular combinations will be considered for $n = 1$ that give an insight into the numerical difficulties that arise for various sets of equations and constraints. The sets to be considered are Equations (3.11 a and b), Equations (3.11b) and (3.13a), Equations (3.11c) and (3.13a), and Equations (3.13 a and b). Equations (3.11 a and b), the electric field integral equations (EFIE), and also Equations (3.11c) and (3.13a), $n \neq 0$, are complete if

$J_{z_n}(\pm L/2) = 0$ and $E_{\phi_n}(\pm L/2) = 0$ are enforced. The other two sets of equations require different constraints.

Consider Equations (3.11b) and (3.13a). For $z \in (-L/2, L/2)$, the equations have homogeneous solutions given by $H_{\rho_n} = A \cos kz + B \sin kz$ and $E_{\phi_n} = j\eta(A \sin kz - B \cos kz)$. If we require $J_{z_n}(\pm L/2) = 0$ and $A = B = 0$, then homogeneous solutions do not exist and the solution is unique since the constraints are also satisfied. To assure that A and B are zero, $E_{\phi_n}(\pm L/2) = 0$ may be used if the structure length is not a multiple of $\lambda/2$. To avoid the half-wavelength anomaly, one may instead require H_{ρ_n} and E_{ϕ_n} to be zero at the same point in the open interval $(-L/2, L/2)$.

Equations (3.13 a and b) have homogeneous solutions such that H_{ρ_n}, E_{ϕ_n}, and E_{z_n} each satisfy the equation $\left(k^2 - \frac{n^2}{a^2} + d_z^2\right) \psi = 0$. The homogeneous solutions are zero and the original constraints are satisfied if H_{ρ_n} is set equal to zero at two distinct points z_1 and z_2 in $(-L/2, L/2)$ where $\sqrt{k^2 - \frac{n^2}{a^2}}\ |z_2 - z_1| \neq p\pi,\ p \neq 0$. One can also satisfy the above conditions by setting E_{ϕ_n} and H_{ρ_n} to zero at the same point in the interval $(-L/2, L/2)$. This latter form of constraining the problem does not have anomalies as do the conditions on H_{ρ_n}.

To emphasize the need for choosing constraints consistent with the equation set to be solved, consider the set of Equations (3.13 a and b) for $n = ka = 1$ and normal incidence of an E-polarized plane wave. In this instance, Equations (3.13 a and b) have forcing functions identical to zero. If E_{ϕ_n} is set to zero, one obtains no information on the currents, since $E_{\phi_n}^{inc} = 0$, and H_{ρ_n} equal to a constant becomes a homogeneous solution of the problem. Hence, one must choose a consistent set of equations and constraints to assure a unique solution to the hollow cylinder scattering problem.

4. THE METHOD OF MOMENTS FOR THE SOLUTION OF SCATTERING PROBLEMS

4.1 Introduction

The integral equations of Chapter 3 are linear integral equations. A formalism has evolved that encompasses most of the techniques for solving linear integral operators in addition to linear differential operators. This formalism, called the method of moments, has been discussed by Kantorovich (1964) in the context of degenerate or separable kernels and has been the subject of lengthy treatises by Vorobyev (1965) and Harrington (1968). Vorobyev based his treatment of the subject on the theory of degenerate kernels, including a section on singular integrals, while Harrington presented a brief discussion of the method with a wealth of examples in electromagnetics. There are also excellent discussions of the topic by Thiele (1973) and Poggio and Miller (1973).

The presentation in this chapter will be based on the work of Harrington with an extensive discussion of the variational interpretation, particularly with regard to the choices of the basis and testing functions to be defined. Equivalence of various combinations of basis and testing functions will be discussed in addition to the implications of using approximate operators. The final topic, of special importance with its best approximation property and smoothness features, is an introduction to the spline theory of approximation.

4.2 The Method of Moments

Consider a linear operator L operating on a response function f defined in the domain (a,b) and equated to a source function g defined in the range equal to the domain of f. This linear operator equation is written as

$$Lf = g \tag{4.1}$$

where usually g is a given function and f is the unknown of interest.

If the solution for f in Equation (4.1) for a specific g exists and is unique, then we may write

$$f = L^{-1}g \tag{4.2}$$

where L^{-1} is the inverse operator. In many problems, an explicit form of L^{-1} can not be obtained and it becomes necessary to solve the problem in an approximate manner. Both the approximate and exact solutions may be obtained in the context of the moment method.

Define the inner product of the functions f and g as

$$\langle f,g\rangle = \int_a^b f(x)\ g(x)\ dx \quad . \tag{4.3}$$

Associated with the inner product is the adjoint operator L^a of the operator L satisfying the relation

$$\langle Lf,h\rangle = \langle f,L^a h\rangle \quad . \tag{4.4}$$

It is clear the inner product of the source of the equation Lf = g with the function h corresponds to $L^{-1}g$ if $L^a h$ is the Dirac delta function. The inability to obtain an explicit form for L^{-1} or a closed form for h is consistent.

To apply the method of moments to Equation (4.1), one expands the unknown function f in the complete series

$$f = \sum_n \alpha_n f_n \tag{4.5}$$

where the f_n are called the basis functions. Equation (4.1) may be written

$$\sum_n \alpha_n L f_n = g \quad . \tag{4.6}$$

In most numerical problems, one truncates the summation, in which case, Equation (4.6) may be written as

$$\sum_{n=1}^{N} \alpha_n L f_n = g + e \tag{4.7}$$

where e represents the error in the truncation.

To solve for the α_n, one takes the inner product of Equation (4.7) with the weighting or testing functions w_m to obtain

$$\sum_{n=1}^{N} \alpha_n \langle w_m, Lf_n \rangle = \langle w_m, g \rangle + \langle w_m, e \rangle \quad , \quad m = 1, \ldots, M \, . \tag{4.8}$$

If the scalars, $\langle w_m, e \rangle$, are set to zero for each m, Equation (4.8) may be written as

$$\sum_{n=1}^{N} \alpha_n \langle w_m, Lf_n \rangle = \langle w_m, g \rangle \tag{4.9}$$

or in matrix form as

$$\begin{bmatrix} \ell_{11} & \ell_{12} & \cdot & \cdot & \cdot \\ \ell_{21} & \ell_{22} & \cdot & \cdot & \cdot \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \tag{4.10}$$

where $\ell_{mn} = \langle w_m, Lf_n \rangle$ and $g_m = \langle w_m, g \rangle$. If the matrix $[\ell_{mn}]$ is nonsingular, then the α_n are given by

$$[\alpha_n] = [\ell_{mn}]^{-1}[g_m] \quad . \tag{4.11}$$

In Equation (4.11), it was assumed that $M = N$. For $M \neq N$, one may calculate the α_n by taking the pseudo-inverse which consists of multiplying Equation (4.10) by the conjugate transpose of $[\ell_{mn}]$ before inverting. The underdetermined system, $M < N$, is often found to be poorly behaved or ill-conditioned and is rarely used in this form. The overdetermined system, $M > N$, is usually well-behaved and represents a least-squares solution to Equation (4.10). A thorough discussion of this topic has been presented by Rao and Mitra (1971). This paper deals with the case $M = N$, for which Equation (4.10) is used directly.

4.3 Variational Principles

The moment method is equivalent to Galerkin's method as presented by Stakgold (1967). Historically, Galerkin's method has been restricted to the case $w_m = f_m$ for which equivalence to the Rayleigh-Ritz variational method for linear operators is well known (Kantorovich, 1964). It will be shown that the moment method is identical to the Rayleigh-Ritz variational method.

Given the operator equation $Lf = g$, it is desired to calculate the functional

$$\rho = \langle f, h \rangle \quad . \tag{4.12}$$

One shall need to introduce the adjoint problem

$$L^a w = h \quad . \tag{4.13}$$

In the context of the calculus of variations, a variational form of ρ, which is stationary when f and w are respective solutions to $Lf = g$ and $L^a w = h$, is given as

$$\rho = \langle f,h \rangle + \langle g,w \rangle - \langle Lf,w \rangle \tag{4.14}$$

or in the scale-independent form (Schwinger-Levine, 1948) as

$$\rho = \frac{\langle f,h \rangle \ \langle g,w \rangle}{\langle Lf,w \rangle} \quad . \tag{4.15}$$

If f and w are expanded as

$$f = \sum_{n=1}^{N} \alpha_n f_n \quad \text{and} \quad w = \sum_{n=1}^{M} \beta_m w_m \quad , \tag{4.16}$$

then one obtains an approximate expression for ρ. Applying the Rayleigh-Ritz conditions to ρ, one has

$$\text{and} \qquad \left.\begin{aligned} \frac{\partial \rho}{\partial \alpha_i} &= 0 \ , \quad i = 1, \ . \ . \ ., N \\ \frac{\partial \rho}{\partial \beta_i} &= 0 \ , \quad i = 1, \ . \ . \ ., M \end{aligned}\right\} \quad . \tag{4.17}$$

These conditions for the ρ of Equation (4.15) are implied by Equation (4.9) for the problem $Lf = g$ and a similar equation for $L^a w = h$. The conditions are identical to Equation (4.9) and the adjoint equation when ρ is defined as in (4.14). Hence the method of moments and Rayleigh-Ritz variational method are equivalent for linear operators.

This variational equivalence can be useful as a guide for choosing the testing functions w_m. Harrington (1968) suggests that the w_m be chosen to closely represent the adjoint response w. If ρ represents the response f to the original problem, then w corresponds to a Green's function. He concludes that since a Green's function is usually poorly behaved, one should expect slower convergence of the solution than for computation of a continuous linear functional.

If the operator problem Lf = g has been solved by the moment method, then Expressions (4.14) and (4.15) reduce to the original functional which may also be written

$$\rho = \langle w,g \rangle \quad . \tag{4.18}$$

This form is useful since the knowledge of the adjoint operator is not needed and the associated adjoint problem does not have to be solved.

Equation (4.18) leads to a straightforward procedure to aid one in choosing both w_m and f_n. Typically, w_m is obtained by shifting a function w about the point x_m such that

$$w_m(x) = w(x - x_m) \quad . \tag{4.19}$$

The function $w(x - y)$ should be chosen such that $\rho(y)$ is a continuous, or at least bounded, function of y for a given source g. If this is not done, one may obtain a substantial error in the solution as the x_m approach the discontinuity or singularity. For an isolated discontinuity, one may want to use smoother testing functions in the region of the discontinuity than used elsewhere.

In a similar manner, we define the functional

$$\rho_f = \sum_{n=1}^{N} \alpha_n \langle w,Lf_n \rangle \quad . \tag{4.20}$$

Expressing w in terms of (x - y), the f_n should be chosen such that the behavior of ρ_f approximates the behavior of ρ. In other words, one should choose the f_n such that $\left(\sum_{n=1}^{N} \alpha_n Lf_n\right)$ is a reasonable representation of g. This criterion for choosing the basis functions is in general more stringent than only requiring a reasonable representation for f, but is a more consistent choice and should give faster convergence.

For example, a series of pulse functions may nicely represent a function f, but has no meaning when operated on by a second order differential operator L and then set equal to a continuous function g at N points. The equation does have meaning if the weight functions are continuous, but one could just as easily solve the problem using piecewise quadratic functions with a continuous first derivative. In the latter, Lf is represented by a series of pulse functions which may adequately represent a continuous g.

Hence, the variational interpretation of the method of moments is an important guide for choosing the basis and testing functions to be used in the moment method.

4.4 Basis and Testing Functions with Comments on Approximate Operators

Both basis and testing functions are described in terms of either entire-domain bases or subsectional bases. Entire-domain basis functions are defined in the entire domain of the operator L. Representative bases are Fourier, Chebyshev, Maclaurin, and Legendre series consisting of the corresponding trigonometric and polynomial functions. Such bases provide fast convergence if they reasonably represent the response function, f, but have a major drawback in the computer time required to calculate the matrix elements due to the entire domain integration.

The matrix elements for subsectional bases are efficient to compute in contrast to the entire-domain bases. In addition, subsectional bases may be used for problems in which entire-domain bases are not suitable. The basic form of subsectional bases is defined by

$$f_n(x) = \begin{cases} p_n(x) \ , & x \text{ in } \Delta_n \\ 0 \ , & x \text{ not in } \Delta_n \end{cases} \tag{4.21}$$

where the various types of subsectional bases are thus defined by the functions $p_n(x)$. It is sometimes convenient to let the subsections, Δ_n, overlap as for the B-splines to be discussed in the next section.

Examples of subsectional bases presented by Thiele (1973) are

Piecewise uniform (pulse function):

$$f(x) = P(x;\Delta) = \begin{cases} 1, & |x| < \Delta/2 \\ 0, & |x| > \Delta/2 \end{cases} \tag{4.22}$$

Piecewise linear (triangle function):

$$f(x) = T(x;\Delta) = \begin{cases} 1 - |x|/\Delta, & |x| \leq \Delta \\ 0 \ , & |x| > \Delta \end{cases} \tag{4.23}$$

Piecewise sinusoidal:

$$f(x) = \begin{cases} \sin k\,(\Delta - |x|)/\sin k\Delta, & |x| \leq \Delta \\ 0 \ , & |x| > \Delta \end{cases} \tag{4.24}$$

Quadratic interpolation:

$$f(x) = \begin{cases} A + Bx + Cx^2, & |x| < \Delta/2 \\ 0 \ , & |x| > \Delta/2 \end{cases} \tag{4.25}$$

Sinusoidal interpolation:

$$f(x) = \begin{cases} A + B \sin kx + C \cos kx, & |x| < \Delta/2 \\ 0 \ , & |x| > \Delta/2 \end{cases} \ . \tag{4.26}$$

These are currently the most commonly used bases in numerical methods for electromagnetic field problems. The first three examples are B-splines, while the last two examples are B-splines only if continuity of the function and its derivative are enforced.

The spline form of sinusoidal interpolation is particularly advantageous for the wire problem (zero-order harmonic) if the $\left(k^2 + \frac{\partial^2}{\partial z^2}\right)$ operation is transferred to the current [see Equation (3.11b)]. The resultant matrix elements become an integration of the kernel over a subsection plus the end terms of the integration by parts. The use of subsectional bases is referred to as subsectional collocation whereas the use of the Dirac delta is referred to as point-matching.

Although a guide has been given in the last section for choosing the basis and testing functions, it is helpful to consider equivalences of various combinations of the subsectional basis functions. For this purpose, the discussion is restricted to convolution or difference operators typical of electromagnetic scattering problems which are given by

$$\begin{aligned} Lf(x) &= \int_{-\infty}^{\infty} K(x - x')\, f(x')\, dx' \\ &= K \circledast f \end{aligned} \tag{4.27}$$

where the basis functions, testing functions, and source are assumed to be identical to zero outside of the domain (a,b) of the original operator, L. The method of moments is thus expressed as

$$\sum_{n=1}^{N} \alpha_n [w_m \cdot (K \circledast f_n)] = w_m \cdot g, \; m = 1, \ldots, M \; . \tag{4.28}$$

Changing the degree of smoothness of w_m makes only minor changes in the source vector of Equation (4.10) as long as the continuity is retained as discussed in Section 4.3. One defines

$$\left.\begin{aligned} f_n(x) &= \tilde{f}(x - x_n) \\ \text{and} \qquad w_m(x) &= \tilde{w}(x - x_m) \end{aligned}\right\} \tag{4.29}$$

where the support of both $f_n(x)$ and $w_m(x)$ is contained in (a,b) for all m and n. The matrix elements are written as

$$\begin{aligned} \ell_{mn} &= w_m \cdot (K \circledast f_n) \\ &= \tilde{w} \circledast K \circledast \tilde{f}\Big|_{x=x_m-x_n} \quad . \end{aligned} \tag{4.30}$$

It is clear that the matrix will not distinguish the basis and testing functions from the use of $(\tilde{w} \circledast \tilde{f})$ as a basis set in conjunction with point-matching of the source.

As a specific example, consider the thin-wire scattering problem for which Equation (3.11b) is rewritten as

$$-j\omega\varepsilon E^{inc}_{z_0} = \fint_{-L/2}^{L/2} \left[\left(k^2 + d_z^2\right) G(z - z')\ I(z')\right] dz' \tag{4.31}$$

where

$$G(z) = \frac{1}{8\pi^2} \int_0^{2\pi} e^{-jkR}/R \ d\alpha \quad .$$

Equation (4.31) is commonly referred to as Pocklington's integral equation. The basis and testing functions are given as

$$f(z) = \begin{cases} \sin k\,(\Delta - |z|)/\Delta \sin k\Delta, & |z| \le \Delta \\ 0 & , |z| > \Delta \end{cases} \tag{4.32a}$$

and

$$w(z) = P(z;\Delta)/\Delta \quad . \tag{4.32b}$$

The current I(z) is expanded in terms of f as

$$I(z) = \sum_{n=-N}^{N} I_n f(z - n\Delta) \tag{4.33}$$

where $\Delta = 1/2\ L/(N + 1)$. The boundary condition $I(\pm L/2) = 0$ is satisfied by this expansion. The testing functions are similarly given by $w_m(z) = w(z - m\Delta)$ for $m = -N, \ldots, N$. The matrix elements are given by

$$\ell_{mn} = [\psi_{m,n+1} + \psi_{m,n-1} - 2\psi_{m,n} \cos k\Delta] \frac{k}{\Delta^2 \sin k\Delta} \tag{4.34}$$

where

$$\psi_{m,n} = \int_{(m-1/2)\Delta}^{(m+1/2)\Delta} G(z - n\Delta)\, dz \quad .$$

The same matrix is obtained when the definitions of w and f are interchanged and also for point-matching with basis functions given as the convolution of the previous f and w. In the latter case, f and w are given by

$$f(z) = \begin{cases} \dfrac{\cos k(z + \Delta/2) + \cos k(z - \Delta/2) - 2 \cos k\Delta}{k\Delta \sin k\Delta}, & |z| \le \dfrac{\Delta}{2} \\ \dfrac{1 - \cos k(|z| - 3\Delta/2)}{k\Delta \sin k\Delta} & , \dfrac{\Delta}{2} < |z| \le \dfrac{3\Delta}{2} \\ 0 & , \dfrac{3\Delta}{2} < |z| \end{cases} \tag{4.35a}$$

$$w(z) = \delta(z) \quad . \tag{4.35b}$$

For normal incidence on this scatterer, the source term is a constant. Hence the source vector obtained using Expressions (4.32b) and (4.35b) is the same. The expression obtained using (4.32a) as a testing function form differs by the constant multiplier $[1 + O(k^2\Delta^2)]$ where "O" is defined in Chapter 2 and is read "the order of." Since it is usually desirable for $k\Delta$ to be much less than 1, the three methods are essentially equivalent.

In many problems of interest, the matrix elements ℓ_{mn} can not be written in closed form or require an unreasonable amount of computer time for calculation. This problem is often circumvented by approximating the operator. Harrington comments that any such approximation has a corresponding moment method solution using approximation of the response f. A classic form of such an approximation is the finite difference method used for differential operators.

Consider Equation (4.31) with $\left(k^2 + \frac{\partial^2}{\partial z^2}\right)$ described by finite differences and f and w given by

$$\begin{aligned} f(z) &= \delta(z) \\ w(z) &= P(z)/\Delta \ . \end{aligned} \tag{4.36}$$

The matrix elements are given by

$$\ell_{mn} = (\psi_{m,n+1} + \psi_{m,n-1} - 2\psi_{mn})/\Delta^3 \ . \tag{4.37}$$

Expressions (4.34) and (4.37) differ by a constant multiplier $[1 + O(k^2\Delta^2)]$ and are essentially equivalent for $k\Delta$ small. One can obtain exact equivalence if piecewise sinusoidal basis functions are used over the entire domain $(-\infty,\infty)$. In this case, the response f would be approximated by piecewise sinusoids in the domain of the original operator, and by simple sinusoids outside of this range. The resultant operation of $\left(k^2 + \frac{\partial^2}{\partial z^2}\right)$ on the approximation to f gives a series of Dirac delta functions in the interval of the domain of the original operator and zero elsewhere.

Hence one finds not only an equivalence for various combinations of basis and testing functions, but also an equivalence of basis functions and approximate operators. The knowledge of such equivalences along with an appropriate choice of either entire-domain or subsectional bases leads to a tractable problem for computer solution.

4.5 An Introduction to Spline Approximation

Currently the trend of the method of moments for electromagnetic scattering problems is toward the spline form of approximation. To date, this trend has not been expressed in the context of spline theory. An introduction is provided here so that the use of splines in later chapters will be clear in addition to giving one an understanding of the features of such an approximation and how it might be implemented.

Spline theory in its current form was introduced by Schoenberg (1946). This original work was based on the cubic spline, which is a mathematical model of the draftsman's spline, hence the name. The basic principle of the latter is to constrain the spline to go through several points in a plane. The curve is mathematically approximated by a cubic between constraints and has a continuous second derivative at the constraints.

The basic principle of splines has been extended in two basic directions. The first extension was to several dimensions, while the second extension was to more complex forms than polynomials, referred to as generalized splines. Rigorous treatments of this topic are found in the books by Ahlberg, Nilson, and Walsh (1967) and by Greville (1969).

The treatment presented here is for generalized splines; the reader is referred to the above texts for the proofs of the theorems.

Define a linear differential operator L by

$$L = a_n(x)\ D^n + a_{n-1}(x)\ D^{n-1} + . . . + a_0(x) \tag{4.38}$$

where $D^n = \dfrac{d^n}{dx^n}$, each $a_j(x)$ is in $C^n[a,b]^*$, and $a_n(x)$ does not vanish in [a,b]. The formal adjoint of L is written as

$$L^a = (-1)^n\ D^n\ \{a_n(x)\cdot\} + (-1)^{n-1}\ D^{n-1}\ \{a_{n-1}(x)\cdot\} + . . . + a_0(x)\quad . \tag{4.39}$$

If the constraining points are defined as a mesh Δ:
$a = x_0 < x_1 < ... < x_N = b$, then the generalized spline $S_\Delta(x)$ in $K^{2n-1}(a,b)$, the class of functions with an absolutely continuous (2n - 2)th derivatives on [a,b] and with the (2n - 1)th derivative in $L^2(a,b)$ normed space, satisfies the equation

$$L^a L S_\Delta = 0 \tag{4.40}$$

between constraint points. This is a spline of odd order (2n - 1) with continuity through the (2n - 2)th derivative. An even order spline may also be defined. The even order spline $S_\Delta(x)$ in $K^{2n}(a,b)$ satisfies the equation

$$D L^a L S_\Delta = 0 \tag{4.41}$$

between constraint points.

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* denotes space of nth differentiable real functions on [a,b].

The two most attractive features of splines are the minimum norm and best approximation properties. For the odd order spline the minimum norm property is stated in the following theorem.

Theorem 4.1: Let Δ and $Y = \{y_i | \ i = 0,1,\ldots,N\}$ be given. Then of all functions $f(x)$ in $K^n(a,b)$ such that $f(x_i) = y_i$ $(i = 0,1,\ldots,N)$, the generalized spline $S_\Delta(Y;x)$ in $K^{2n-1}(a,b)$, when it exists, minimizes

$$\int_a^b [Lf(x)]^2 \, dx \quad . \tag{4.42}$$

If $g(x)$ also minimizes (4.42), then $g(x)$ and $S_\Delta(Y;x)$ differ by a homogeneous solution of $Lf = 0$.

This theorem implies that splines would not, in general, have the oscillatory problems characteristic of entire-domain bases.

The property of best approximation is stated in a similar manner.

Theorem 4.2: Let Δ and $f(x)$ in $K^n(a,b)$ be given. Then, of generalized splines $S_\Delta(x)$ on Δ, $S_\Delta(f;x)$ with $S_\Delta(f;x_i) = f(x_i)$ $(i = 0,1,\ldots,N)$, when it exists, minimizes

$$\int_a^b [Lf(x) - LS_\Delta(x)]^2 \, dx \quad . \tag{4.43}$$

If $S_\Delta(x)$ also minimizes (4.43), then $S_\Delta(x)$ and $S_\Delta(f;x)$ differ by a homogeneous solution of $Lf = 0$.

Theorems 4.1 and 4.2 also hold for the even order spline $S_\Delta(x)$ in $K^{2n}(a,b)$ if one equates integrals of the given function and the spline between the constraints rather than equating the function and the spline at the constraints.

The simplest way to incorporate splines as basis functions is to represent the basis splines as the B-splines discussed by Greville (1969). These are simply splines of minimal and usually finite support. The piecewise sinusoidal bases in Equation (4.24) and its second-order form of Equation (4.35a) represent generalized splines of the first and second order, respectively, where

$$L = k + j \frac{d}{dz} \quad . \tag{4.44}$$

4.6 Conclusions

The method of moments has been developed in this chapter, with an extensive discussion of closely related topics. It was found that the moment method may be interpreted in the variational context of Rayleigh-Ritz. The resultant stationary functional provides the worker with insight for choosing the basis and testing functions of the moment method. Several bases were discussed along with comments of equivalences of various basis and testing combinations in addition to approximate operators and smoother bases.

The final topic considered was an introduction to spline theory. It is found that most of the subsectional bases currently in use are splines, but have not previously been presented in the context of the extensive theory of splines.

5. ZERO HARMONIC PROBLEM

5.1 Introduction

The zero harmonic problem gives rise to two classic problems, the loop antenna and the linear antenna. These problems are dominated respectively by the ϕ- and z-components of the current that are decoupled for this harmonic. The linear antenna and the associated z-component of the current will be of interest in this paper.

The equation for the linear antenna was originally presented by Pocklington (1897); a variation of Pocklington's equation was developed by Hallen (1938). Hallen's method solves Pocklington's equation in two steps. The first step is to obtain the solution of the differential operator equation in terms of the source and two homogeneous solutions. This solution has the form of a Fredholm integral equation of the first kind having a weak kernel singularity with the conditions that the current be zero at the structure ends. The latter is necessary to determine the amplitudes of the differential operator homogeneous solutions. The solution of the resultant integral equation is the second step.

An extensive treatise on approximate analytical techniques for solving the integral equations of linear antennas has been presented by King (1956). In addition, numerical techniques have been considered by Mei (1965), Harrington (1968), Poggio and Mayes (1969), and Thiele (1973) for linear antennas.

In this chapter, the equivalence of the numerical formulations for Hallen's and Pocklington's equations will be presented along with a discussion of the thin-wire kernel as defined in Section 3.4. Numerical results will be presented as well as a discussion of other equivalences and operator approximations.

5.2 Hallen's Formulation

For the zero harmonic, Equations (3.11b) and (3.13b) are identical and may be written as

$$-j\omega\varepsilon E_{z_0}^{inc}(z) = \left(k^2 + \frac{d^2}{dz^2}\right) \int_{-L/2}^{L/2} [I(z')\ K(z - z')]\ dz' \qquad (5.1)$$

where $I(z) = 2\pi a J_{z_0}(z)$ and the kernel K is given by

$$K(z - z') = \frac{1}{2\pi a} G_{00}(z - z')$$

$$= \frac{1}{8\pi^2} \int_0^{2\pi} \left[\frac{e^{-jkR}}{R}\right] d\alpha \quad . \qquad (5.2)$$

Equation (5.1) is referred to as Pocklington's integral equation.

One defines the z-component of the vector potential as

$$A_z(z) = \int_{-L/2}^{L/2} [I(z')\ K(z - z')]\ dz' \quad . \qquad (5.3)$$

Hence, Equation (5.1) may be written as a second-order differential equation for A_z with the solution given by

$$A_z = \frac{j}{2\eta} \int_{-L/2}^{L/2} \left[E_{z_0}^{inc}(z') \sin k\left|z - z'\right|\right] dz'$$

$$+ B \sin kz + C \cos kz \qquad (5.4)$$

where $\eta = \sqrt{\mu/\varepsilon}$ and the last two terms of the expression are homogeneous solutions of the differential equation. It should be noted that the Green's function $(\sin k|z|/2k)$, denoted by $g(z)$, for the differential equation is not unique since boundary conditions for A_z are not given.

Substituting Expression (5.3) into Equation (5.4) gives

$$\int_{-L/2}^{L/2} [I(z')\ K(z - z')]\ dz' = -\frac{j}{2\eta} \int_{-L/2}^{L/2} \left[E_{z_0}^{inc}(z')\ \sin k|z - z'|\right] dz'$$

$$+ B \sin kz + C \cos kz \quad . \tag{5.5}$$

This is a Fredholm equation of the first kind for the unknown I(z) commonly referred to as Hallen's integral equation. The constants B and C are determined indirectly by enforcing $I(\pm L/2) = 0$.

5.3 The Equivalence of Pocklington's and Hallen's Formulations

The analytical equivalence of Equations (5.1) and (5.5) is clear since each can be derived directly from the other. The numerical equivalence in the context of the method of moments is not as straightforward. In fact, Mei concluded that they were not equivalent and that Hallen's formulation is preferred (Mei, 1965). However, his conclusion was based on a particular choice of testing and basis functions; these were Dirac delta testing and pulse basis functions in both cases.

Numerical equivalence does exist when the same basis functions are used and the Pocklington testing functions are obtained from those of the Hallen's form by convolution with piecewise sinusoids. For example, consider pulse bases in conjunction with Dirac delta and piecewise sinusoidal testing functions for the Hallen and Pocklington forms, respectively. The Green's function of the differential operator may be represented as

$$g(z - m\Delta) = \frac{\sin k|z - m\Delta|}{2k}$$

$$= \sum_{-\infty}^{\infty} \beta_{n-m} S_{\Delta}(z - n\Delta) \tag{5.6}$$

where $\beta_n = \sin(k|n|\Delta)/2k$ and $S_\Delta(z)$ is the piecewise sinusoid given by

$$S_\Delta(z) = \begin{cases} \dfrac{\sin k(\Delta - |z|)}{\sin k\Delta} , & |z| \leq \Delta \\ \\ 0 , & |z| > \Delta \end{cases} . \tag{5.7}$$

Hence, linear combinations of the moment method equations for the Pocklington formulation may be equated to the point samples of the Hallen formulation. In the latter, the homogeneous terms may be written as

$$B \sin kz + C \cos kz = \left[g(z - z') \frac{d}{dz'} A_z(z') - A_z(z') \frac{d}{dz'} g(z - z') \right]_{z' = -L/2}^{L/2} . \tag{5.8}$$

With the homogeneous terms of Hallen's equation defined as in Expression (5.8), the currents at the ends become unknowns rather than boundary conditions which is consistent with the Pocklington formulation.

In general, one would impose the boundary conditions on the current and solve Hallen's or Pocklington's equations only on the interior of the structure. This is necessary since the matrix elements corresponding to a non-zero current at the structure ends are infinite. In addition, the homogeneous terms of Equation (5.5) are generally used for the solution of Hallen's equation in lieu of obtaining the derivative of A_z at $\pm L/2$.

For a thin linear antenna consisting of thirty-five sections in addition to two half-sections at the ends, the coefficients of the current for both formulations were identical to eight significant figures for structure lengths of $\lambda/2$, λ, and 2λ. These results are plotted in Figure 5.1 along with the results of the Hallen formulation using sixty-one sections (Poggio and Mayes, 1969) and the King-Middleton second-order theory (King, 1956) for a center-fed half-wavelength dipole.

An additional equivalence occurs for the Dirac delta source in the Pocklington formulation. In this instance, the forcing vector of the matrix equation is identical for both piecewise sinusoid and pulse testing functions. Since the matrix due to a pulse expansion and piecewise sinusoid testing is the same as for the piecewise sinusoid expansion and pulse testing, the latter is equivalent to Hallen's formulation for determining the coefficients. It should be noted that although the coefficients are identical, the current has a more realistic behavior for the piecewise sinusoid expansion.

5.4 Operator Approximations

Various forms of operator approximation are generally implemented either to enable one to evaluate terms that do not exist in closed form or to simplify the problem for easier handling. The evaluation of terms not existing in closed form was introduced in Section 3.4 by approximating the kernels of the integral equations as the sum of singular and residual terms, the latter to be treated numerically. This form of approximation will not be discussed here. Operator approximations that are intended to simplify the problem at hand will be considered in this section.

The most common approximation for the thin linear antenna problem, which is used in this chapter, is the use of the thin wire kernel given in Section 3.4 as

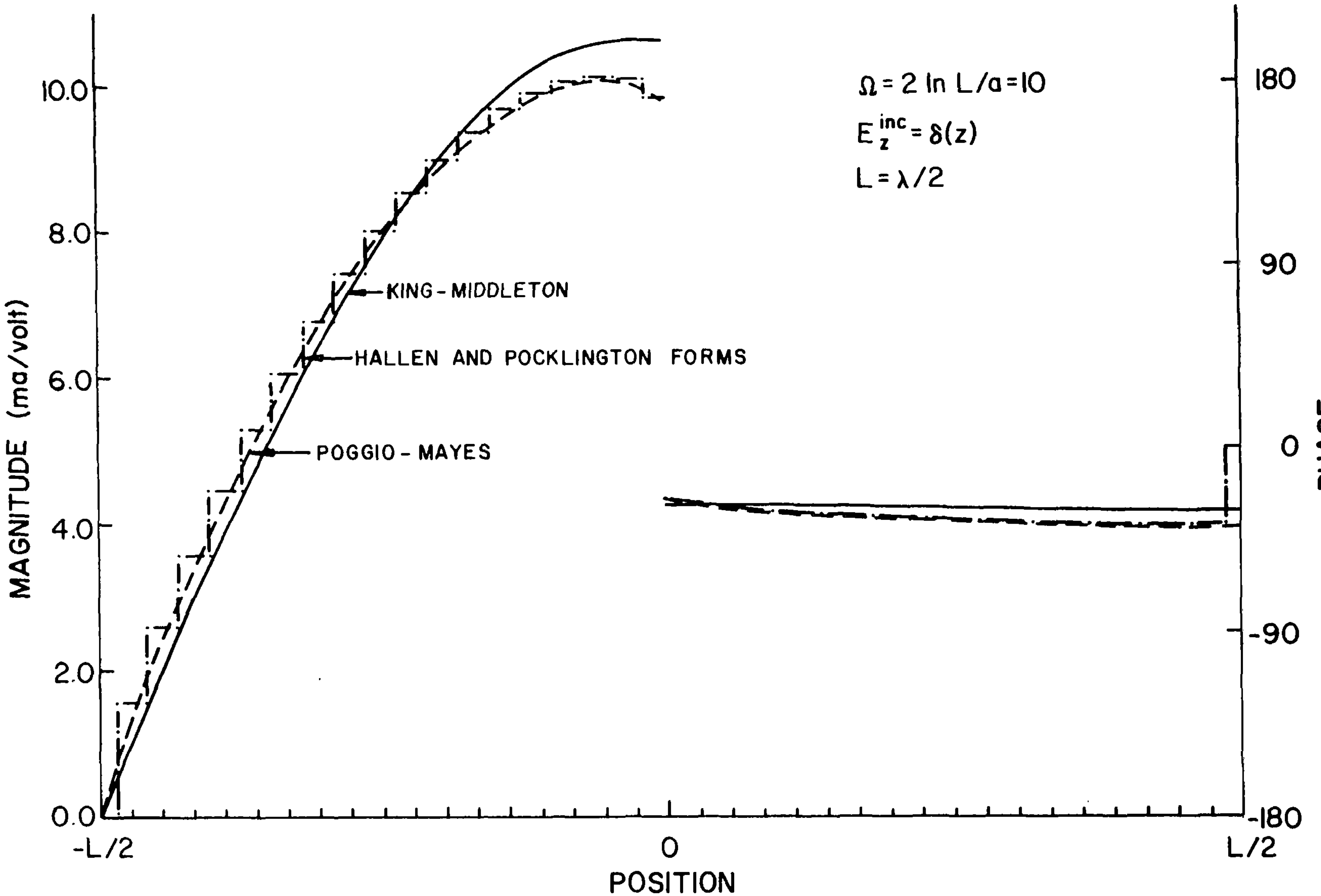


Figure 5.1. Solutions for the current on a center-fed, half-wavelength dipole using the Hallen formulation of Poggio and Mayes, the King-Middleton theory, and the forms presented in this paper. The current is symmetrical and has been plotted with the phase on the right.

$$K(z - z') \approx \frac{1}{4\pi} \frac{e^{-jk\tilde{R}}}{\tilde{R}} \tag{5.9}$$

where $\tilde{R} = [(z - z')^2 + a^2]^{1/2}$. Expression (5.9) is the exact kernel for the extended boundary condition of problems for which the currents on end caps are included (Al-Badwaihy and Yen, 1974). King (1969) has also commented that this kernel yields essentially the same results for the tubular antenna as does the exact kernel.

Consider the diagonal matrix elements of the Hallen formulation in Section 5.3 given by

$$\ell_{nn} = \frac{1}{8\pi^2} \int_{-\Delta/2}^{\Delta/2} \int_{0}^{2\pi} \left[\frac{e^{-jkR}}{R}\right] d\alpha \, dz \quad . \tag{5.10}$$

Expanding the exponential, one obtains

$$\ell_{nn} = \frac{1}{8\pi^2} \int_{-\Delta/2}^{\Delta/2} \int_{0}^{2\pi} \frac{d\alpha \, dz}{R} - \frac{jk\Delta}{4\pi} + O(k^2\Delta^2) \quad . \tag{5.11}$$

The first term may be written as

$$\frac{1}{4\pi^2} \int_{0}^{2\pi} \ell n \left[\frac{\Delta + \sqrt{\Delta^2 + 16a^2 \sin^2 \frac{\alpha}{2}}}{|4a \sin \frac{\alpha}{2}|}\right] d\alpha \approx \frac{\ell n(\Delta/a)}{2\pi} \, , \quad 4a \ll \Delta \tag{5.12}$$

where the only approximation is the omission of the sine term under the radical. The same result is obtained using the approximate thin-wire kernel, expanding, and neglecting similar second-order terms. Hence, the integrations of the thin wire and exact kernels differ by $O[(4a/\Delta)^2]$.

Another type of operator approximation often used in the Pocklington formulation is the finite difference approximation to derivatives. This approximation with delta expansion functions was shown in Section 4.4 to differ from a piecewise sinusoid in the Pocklington formulation by $O(k^2\Delta^2)$. The comparison of the finite difference approach and the use of piecewise sinusoids is shown in Figure 5.2 for both half-wavelength and two wavelength wires having thirty-five unknowns in each case. The maximum percentage error in the current coefficients for both lengths was less than $k^2\Delta^2$.

In using finite differences, it is common practice to take forward or backward differences adjacent to the ends rather than central differences used in the previous approximation. This approach defines the derivatives in terms of the function entirely in the domain of interest. For the center-feed problem, the source is zero near the ends and one has numerically

$$A_z(\pm L/2) = A_z[\pm(L/2 - \Delta)] \quad . \tag{5.13}$$

King (1969) argues that since the kernel is so highly peaked about $z = z'$, $A_z(z)$ varies in much the same way as does $I(z)$. Hence

$$\begin{aligned} A_z(z) - A_z(L/2) &= \psi(z)[I(z) - I(L/2)] \\ &\approx \Psi[I(z) - I(L/2)] \end{aligned} \tag{5.14}$$

where Ψ is the approximately constant value of $\psi(z)$. From this argument one finds

$$I(\pm L/2) \approx I[\pm(L/2 - \Delta)] \quad . \tag{5.15}$$

This is found to approximate the numerical results as shown in Figure 5.3 where comparison is made to the solution obtained by using central differences over the entire structure.

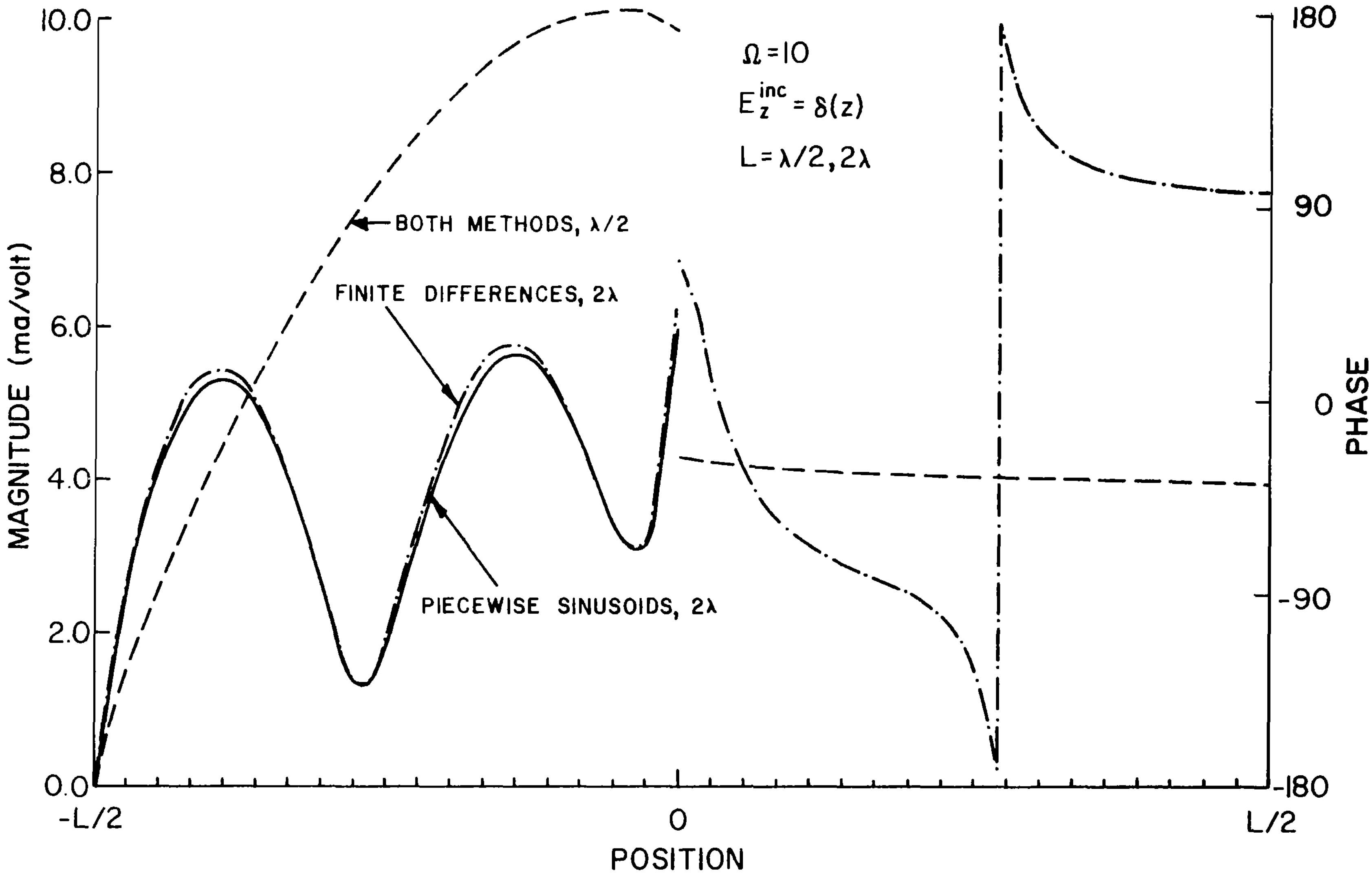


Figure 5.2. Comparison of the finite difference approach to the use of piecewise sinusoids. The phase is plotted on the right with the phase difference of the two methods being negligible for both lengths considered. The current for $L = 2\lambda$ is plotted for twice the actual amplitude.

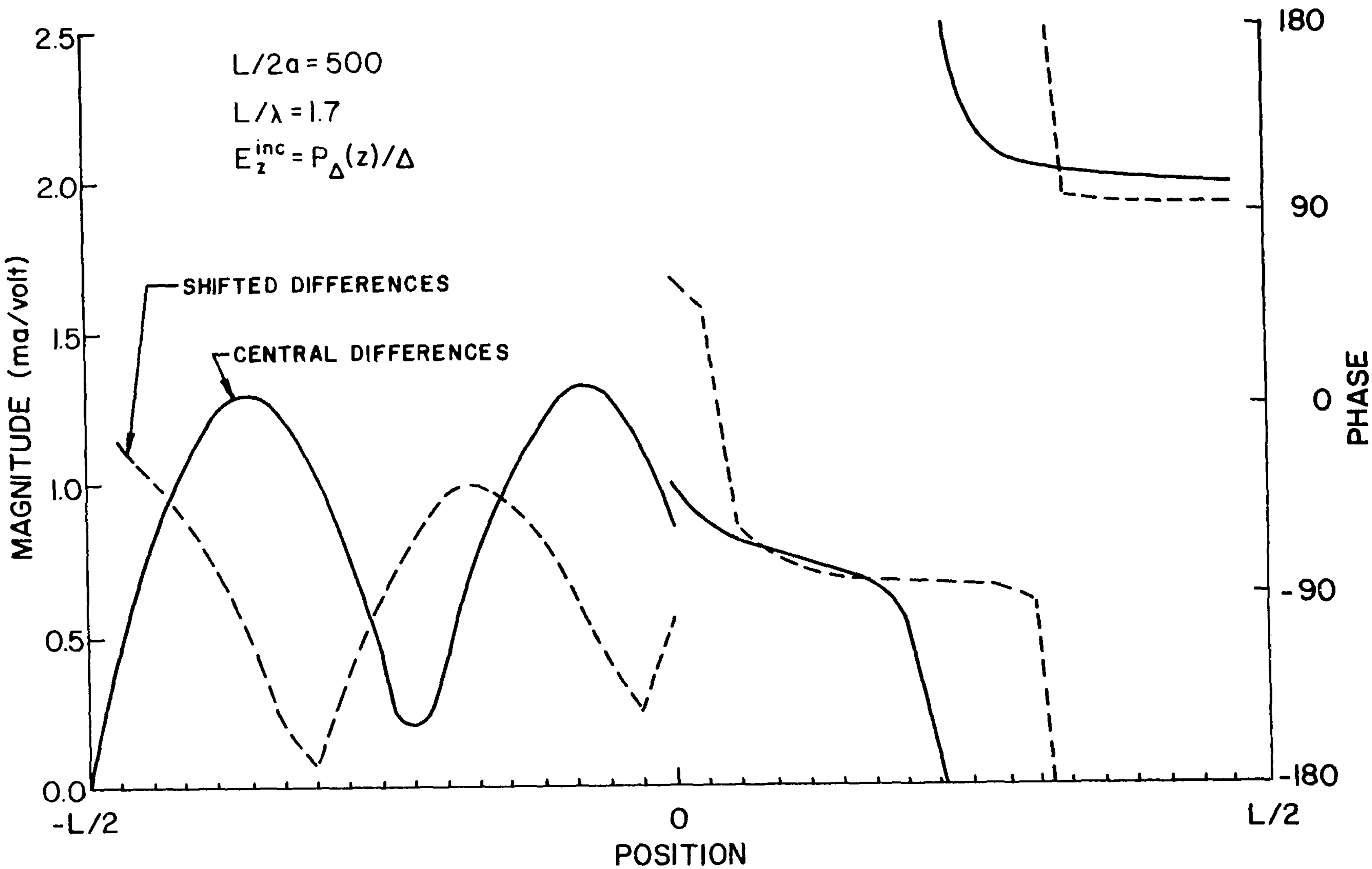


Figure 5.3. Comparison of the currents obtained using both central and forward (or backward) differences at the ends in the finite difference approach. The phase is plotted on the right.

5.5 Basis and Testing Functions: Choice and Interpretation

Equation (5.1) has the form Lf = g where f represents the current I and g represents the incident electric field. The operator L is a linear singular integral operator which may be separated into a second-order differential operator and a singular integral having an essentially logarithmic kernel. A guide for choosing the basis and testing functions for the moment method solution of Lf = g was given in Section 4.3 in the context of variational properties.

The three choices to be considered are pulse, piecewise sinusoid, and second-order sinusoidal spline basis functions in conjunction with piecewise sinusoid, pulse, and delta testing functions, respectively. Since the operator is essentially a second-order differentiation, the testing and basis functions were chosen such that convolution would give a function with a bounded second derivative in the domain of the operator. In terms of the variational interpretation, these choices give stationary forms for the vector potential, voltage, and electric field, respectively.

The first two choices have identical coefficients for a delta source, as indicated in Section 5.3, while no solution exists for the third choice of expansion functions. Though the coefficients are identical, the second choice would seem to be a more accurate representation due to the current continuity as shown in Figure 5.4. For a pulse source, the solution for the second choice is the same as for a delta source with new solutions for the other two choices as shown in Figure 5.5. The third choice would seem to give the most accurate representation for the pulse source since it gives the only bounded approximation to the source.

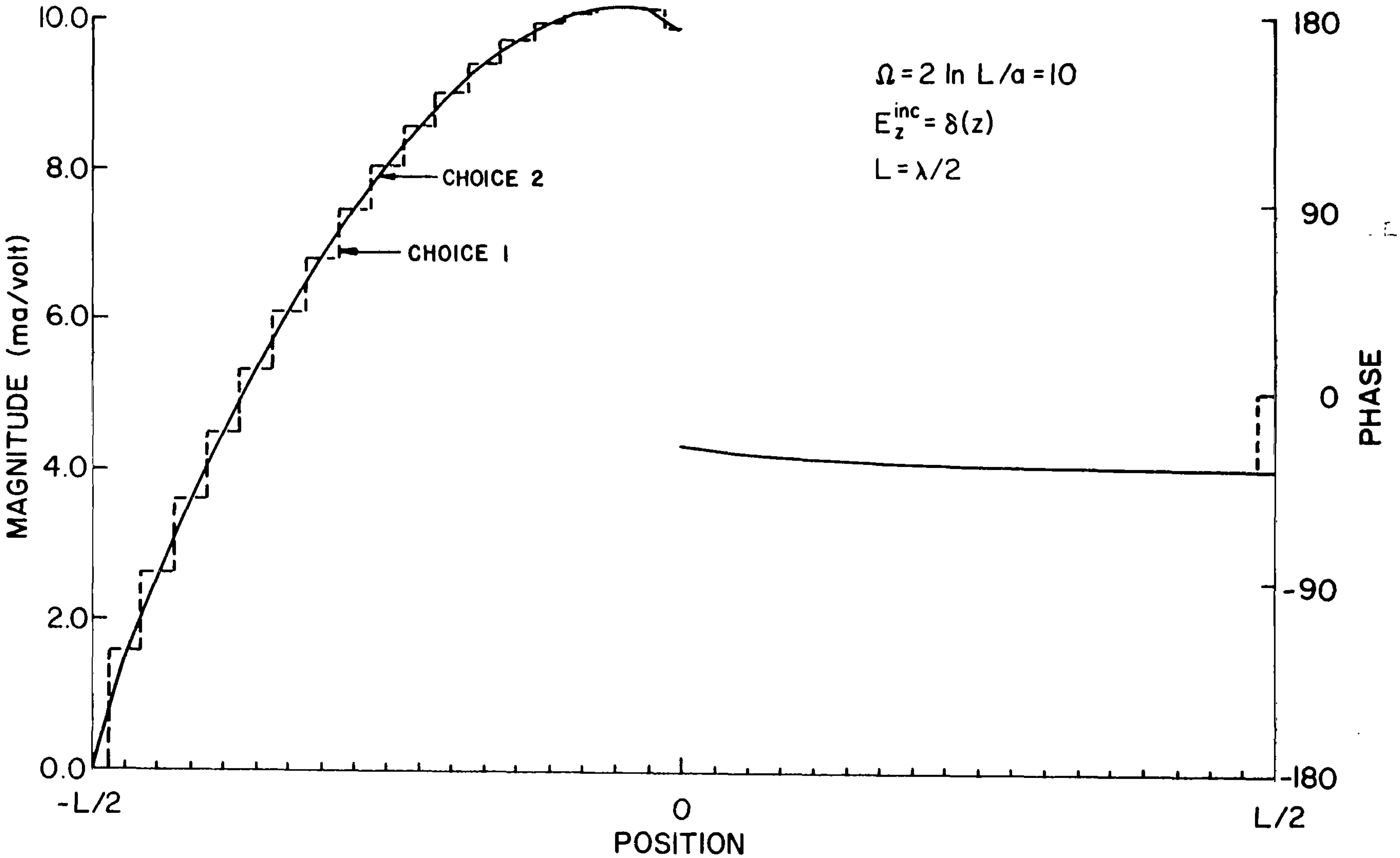


Figure 5.4. Comparison of the current solutions using the first and second choices of expansion functions. The phase is plotted on the right.

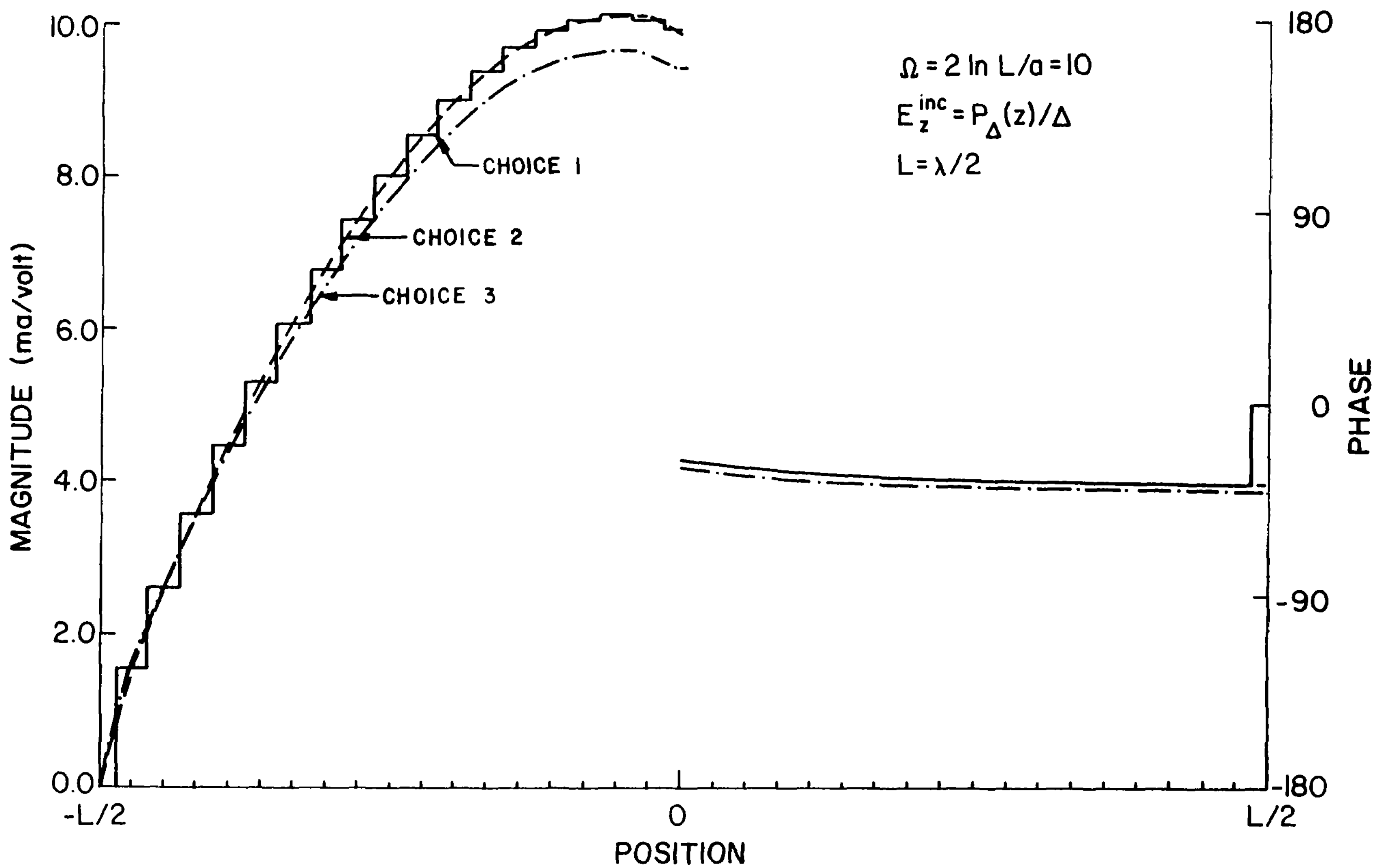


Figure 5.5. Current solutions for a pulse-fed half-wavelength dipole using all three choices of expansion functions. The phase is plotted on the right.

It is desirable to obtain some additional insight into the problem for various types of feeds in order to estimate which solutions are the best representations of the actual current and associated source. In terms of the continuity of weight and source convolution, the first choice will offer the most stable current coefficients with respect to the source description, but the pulse representation of the average current will give the worst representation for g.

Since the choice of piecewise sinusoids with pulse testing functions has the same solution for both pulse and delta sources, it is desirable to determine which of these sources is best represented by the piecewise sinusoidal expansion of the current. For a thin wire with sample distance much greater than the radius, Lf becomes a series of highly peaked functions which have the appearance of a delta type feed. If one considers $w \circledast Lf$, one obtains approximately a pulse function as would result from convolving the pulse testing and delta source functions. Hence, the second choice is the best representation for delta sources with the above restriction on radius.

The Lf obtained from the third choice approximates a series of pulse functions and is thus well-suited to pulse feed and scattering problems. The smoothness of the current expansion makes it the most desirable for all but the delta feed problem. A slope discontinuity may be included along with a pulse sample at the feed point to make this expansion also consistent with a delta feed. In addition, this expansion gives a reasonable representation of the near fields which are often the desired quantities. As shown in Figure 5.6, the point-wise values of current are essentially identical for the scattering problem for all three choices, but with the second-order sinusoidal spline having the desired smoothness.

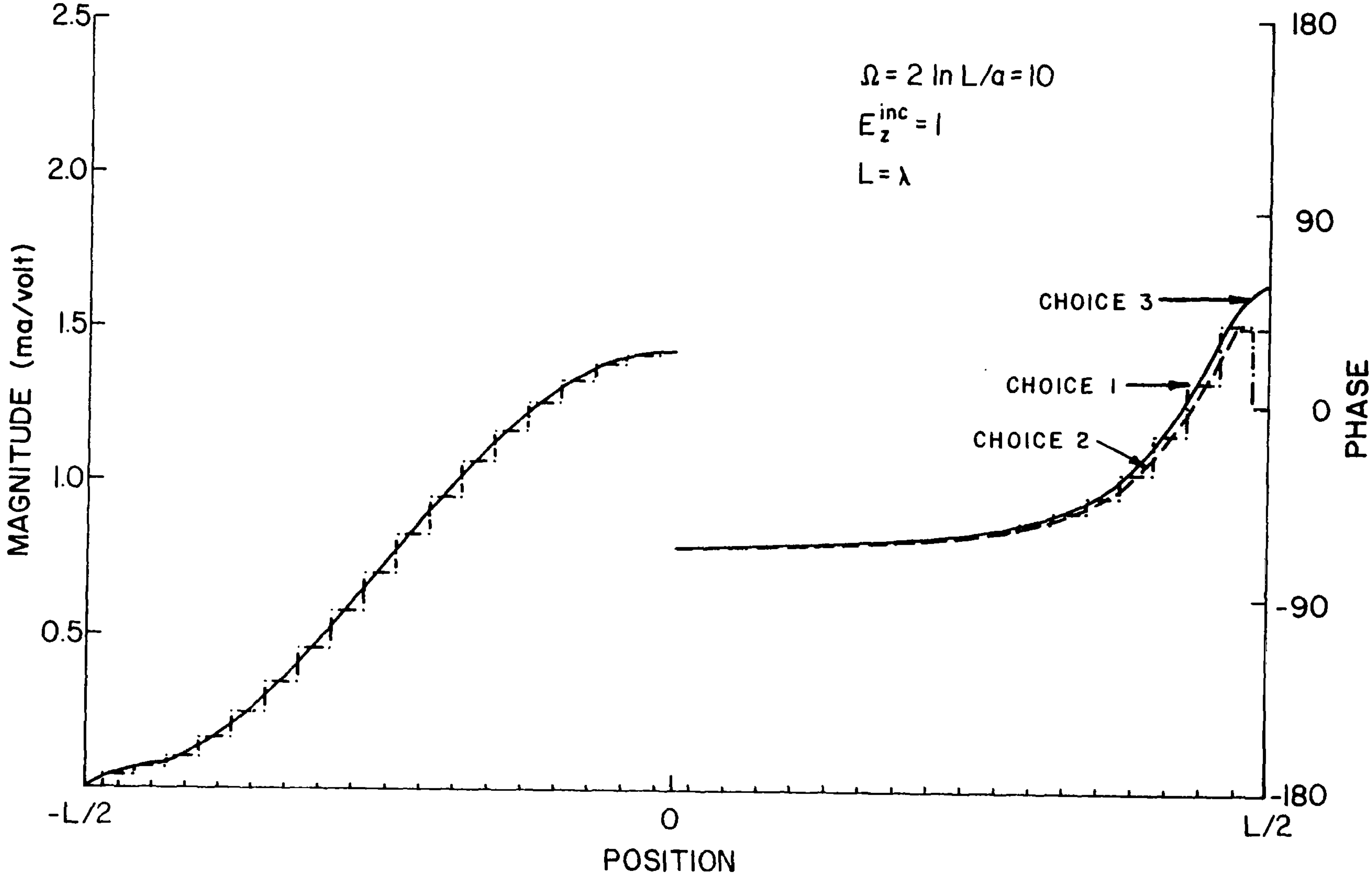


Figure 5.6. Comparison of the induced currents calculated for normal incidence scattering by a one-wavelength wire using all three choices of expansion functions. The magnitude difference of the last two choices is negligible. The phase is plotted on the right.

The currents for pulse feed and scattering problems of λ and 2λ wires are shown in Figure 5.7 using the second-order sinusoidal spline.

5.6 Conclusions

In this chapter, the Hallen and Pocklington formulations have been shown to be numerically equivalent for appropriate choices of testing and basis functions. Equivalences that result from operator approximations have also been discussed.

In comparing various combinations of basis and testing functions, it was found that a delta testing function in conjunction with a second-order sinusoidal spline offers the best representation of both the current and electric field for all of the sources considered except a Dirac delta feed. The latter can also be included in this expansion scheme if a slope discontinuity in the basis functions is incorporated at the feed point along with a pulse testing function.

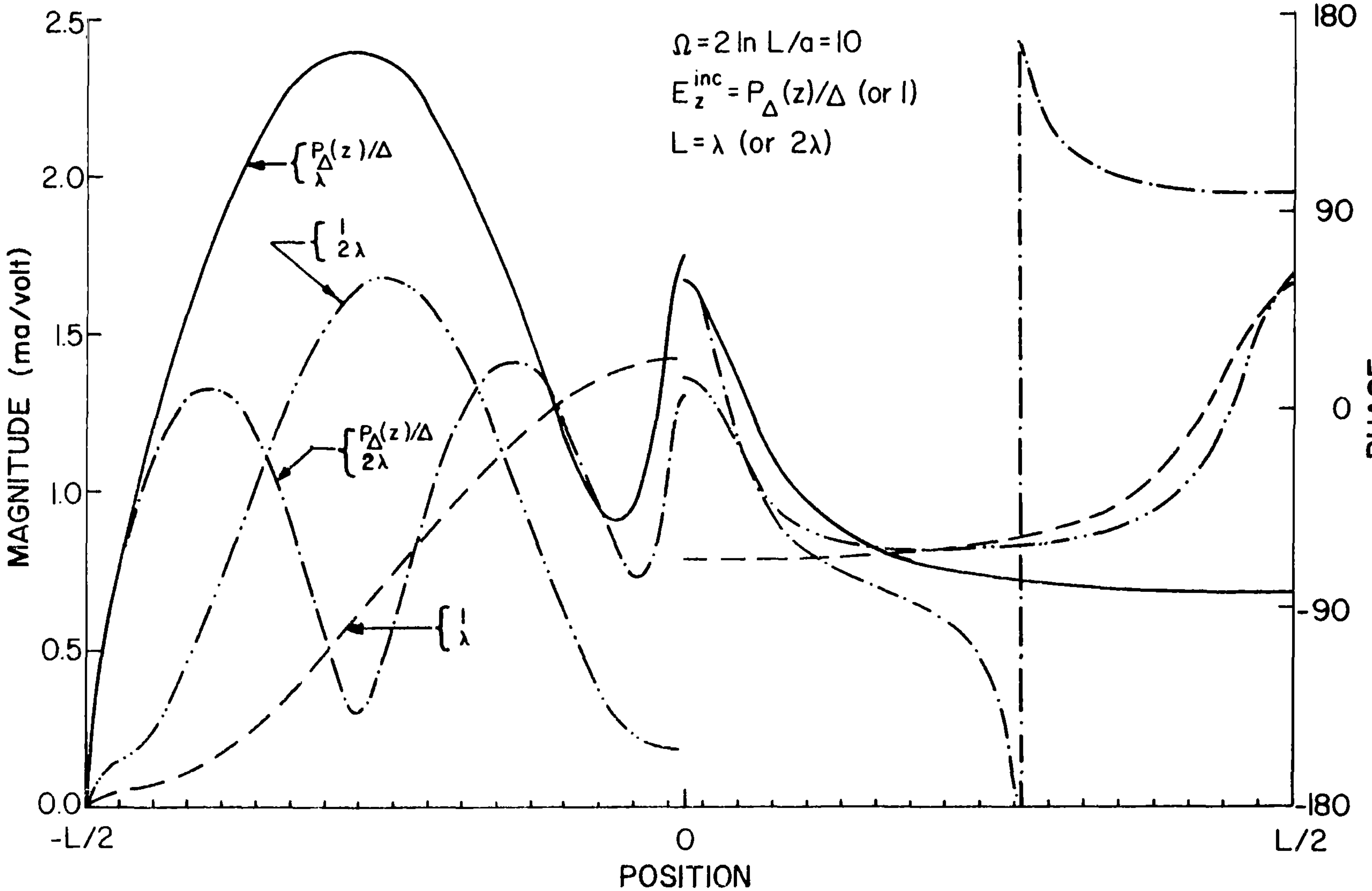


Figure 5.7. Currents for both the pulse-fed and normal incidence scattering problems calculated for one- and two-wavelength structures using second-order sinusoidal splines. The phase is plotted on the right. The pulse-fed two-wavelength dipole is plotted for half the actual magnitude.

6. FIRST HARMONIC PROBLEM

6.1 Introduction

In this chapter, the first harmonic problem is investigated to obtain insight into the nature of coupled integral equations. Particular interest will be given to the comparison of the various sets of integral equations. The currents on an infinite cylinder in addition to the results of Kao (1970) will be used to verify the numerical results.

Other considerations of importance are the uniqueness constraints, the description of the edge behavior, and finite difference methods.

6.2 The Infinite Cylinder

The currents on an infinite cylinder are derived for both E- and H-polarized incident plane waves as done by Harrington (1961). It is assumed that no fields exist inside the cylinder and that all field quantities have the variation $e^{-jkz\cos\theta_i}$ that is suppressed.

For E-polarization, the z-component of the electric field is given by

$$E_z^{inc} = E_0 \sin\theta_i \, e^{-jk_\rho \rho\cos\phi} \tag{6.1}$$

where $k_\rho = k \sin\theta_i$. Using Equation (3.4a) one has

$$E_z^{inc} = E_0 \sin\theta_i \sum_{n=-\infty}^{\infty} (-j)^n \, J_n(k_\rho \rho) \, e^{jn\phi} \tag{6.2}$$

where $J_n(k_\rho \rho)$ is the Bessel function of the first kind. The scattered field may be represented by outward-traveling waves as

$$E_z^s = E_0 \sin\theta_i \sum_{n=-\infty}^{\infty} (-j)^n \, \alpha_n H_n^{(2)}(k_\rho \rho) \, e^{jn\phi} \tag{6.3}$$

where $H_n^{(2)}$ is the Hankel function of the second kind. The sums of the scattered and incident field components comprise the total fields. Since the total tangential electric field is zero at the surface of the cylinder, the α_n are given by

$$\alpha_n = \frac{-J_n(k_\rho a)}{H_n^{(2)}(k_\rho a)} \quad . \tag{6.4}$$

The nth harmonic component of the surface current is given by

$$J_{z_n} = H_{\phi_n}(a^+) = \frac{1}{j\omega\mu} \frac{\partial E_{z_n}}{\partial\rho}(a^+)$$

which becomes upon substitution

$$J_{z_n} = \frac{(-j)^n \; 2E_0 \sin\theta_i}{\pi\eta k a H_n^{(2)}(k_\rho a)} \quad . \tag{6.5}$$

In a similar manner, one may expand H_z in a Fourier series and set the ρ-derivative of the total magnetic field to zero at $\rho = a$ to obtain

$$J_{\phi_n} = \frac{(-j)^{n-1} \; 2H_0}{\pi k a H_n^{(2)'}(k_\rho a)} \quad . \tag{6.6}$$

It is convenient to normalize the harmonic current components to an incident electric field of $\eta/4$ for the same harmonic. In the case of normal incidence, one has for E-polarization

$$J_{z_n} = \frac{1}{2\pi k a J_n(ka) \; H_n^{(2)}(ka)} \tag{6.7a}$$

and for H-polarization

$$J_{\phi_n} = \frac{1}{2\pi ka J_n'(ka)\, H_n^{(2)'}(ka)} \quad . \tag{6.7b}$$

6.3 The Finite Cylinder

Five integral equations were obtained in Section 3.3 for the currents on a hollow cylinder. The constraints for four sets of these equations were considered in Section 3.5. These sets of equations are designated: I, Equations (3.11 a and b); II, Equations (3.11c) and (3.13a); III, Equations (3.11b) and (3.13a); and IV, Equations (3.13 a and b). It can be shown that Kao (1969, 1970) solved equation set II using the Hallen's formulation for the operator $\left(k^2 + \frac{d^2}{dz^2}\right)$ and the Bessel's function product form of the Green's function given by Expression (3.15). The resultant matrix equation was obtained using point-matching with piecewise quadratic basis functions, which are continuous functions, but do not have continuous derivatives.

The numerical results obtained by this author have been determined by point-matching with a quadratic spline basis set. The basis set is modified in the end sections in the same manner as Kao (1970) by using a quadratic times $1/\sqrt{L/2 - |z|}$. The currents are normalized to the incident E-field and expressed in series form as

$$J_{\binom{\phi}{z}_m}(z) = \sum_{-(N+1)}^{N+1} J_{\binom{\phi}{z}_{mn}} f_n\left(\frac{z}{\Delta}\right) \tag{6.8}$$

with the basis functions given by

$$
f_n(x) = \begin{cases}
Q(x-n) , & \text{for } n \in [-(N-2),(N-2)] \\[6pt]
\left.\dfrac{(y-1/2)^2}{\sqrt{y+1/2}}\, P_1(y)\right|_{y=L/2\Delta-1/2-|x|} , & n = \pm(N+1) \\[6pt]
\left.\left[\left(\dfrac{9}{8}-\dfrac{5y}{4}\right)\sqrt{y+1/2}\, P_1(y) + \dfrac{(y-3/2)^2}{2} P_1(y-1)\right]\right|_{y=L/2\Delta-1/2-|x|} , & n = \pm N \\[6pt]
\left.\left[Q(y-1)\,\theta\left(y-\dfrac{1}{2}\right) + \left(\dfrac{1}{8}+\dfrac{3y}{4}\right)\sqrt{y+\dfrac{1}{2}}\;\; P_1(y)\right]\right|_{y=L/2\Delta-1/2-|x|} , & n = \pm(N-1)
\end{cases}
$$

where θ is the Heaviside function, P_1 is the pulse function of unity width, and the function Q is the quadratic B-spline given by

$$
Q(x) = \begin{cases}
\dfrac{3}{4} - x^2 , & |x| \le \dfrac{1}{2} \\[6pt]
\left(|x| - \dfrac{3}{2}\right)^2/2 , & \dfrac{1}{2} < |x| \le \dfrac{3}{2} \\[6pt]
0 , & \dfrac{3}{2} < |x|
\end{cases} .
$$

For clarity, the $f_n(x)$ are plotted in Figure 6.1 for $n \in [-(N+1),-(N-2)]$. The kernels of the integral equations are expressed in the forms of Expressions (3.19) with the residuals approximated by piecewise linear functions.

Equation set I is the well-known set of electric field integral equations (EFIE) which is solved in conjunction with the constraints $J_{z_n}(\pm L/2) = E_{\phi_n}(\pm L/2) = 0$. For $ka = |n|$, the kernel on J_{ϕ_n} in Equation (3.11a) is bounded and both Equation (3.11a) and Equation (3.11b)

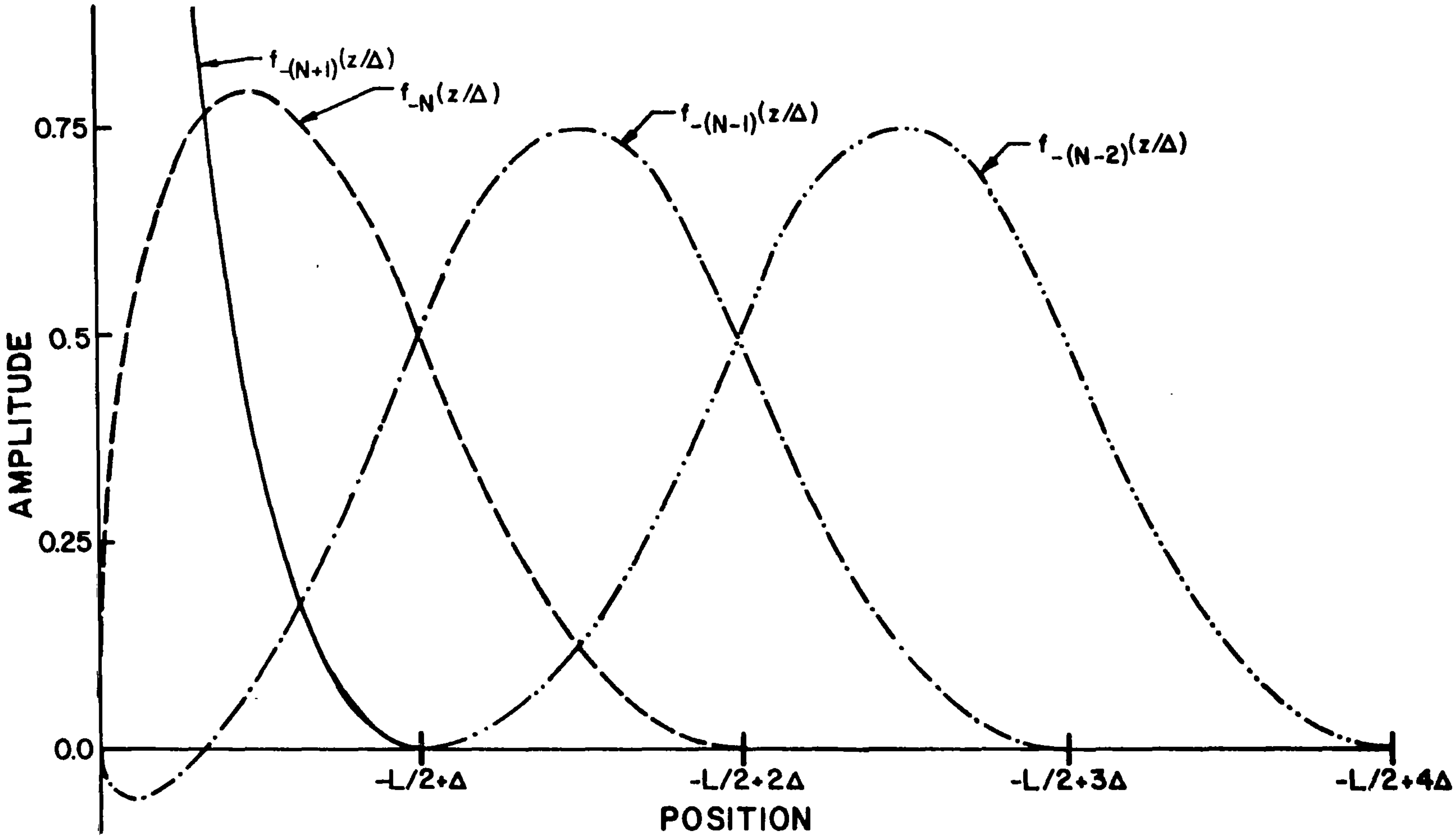
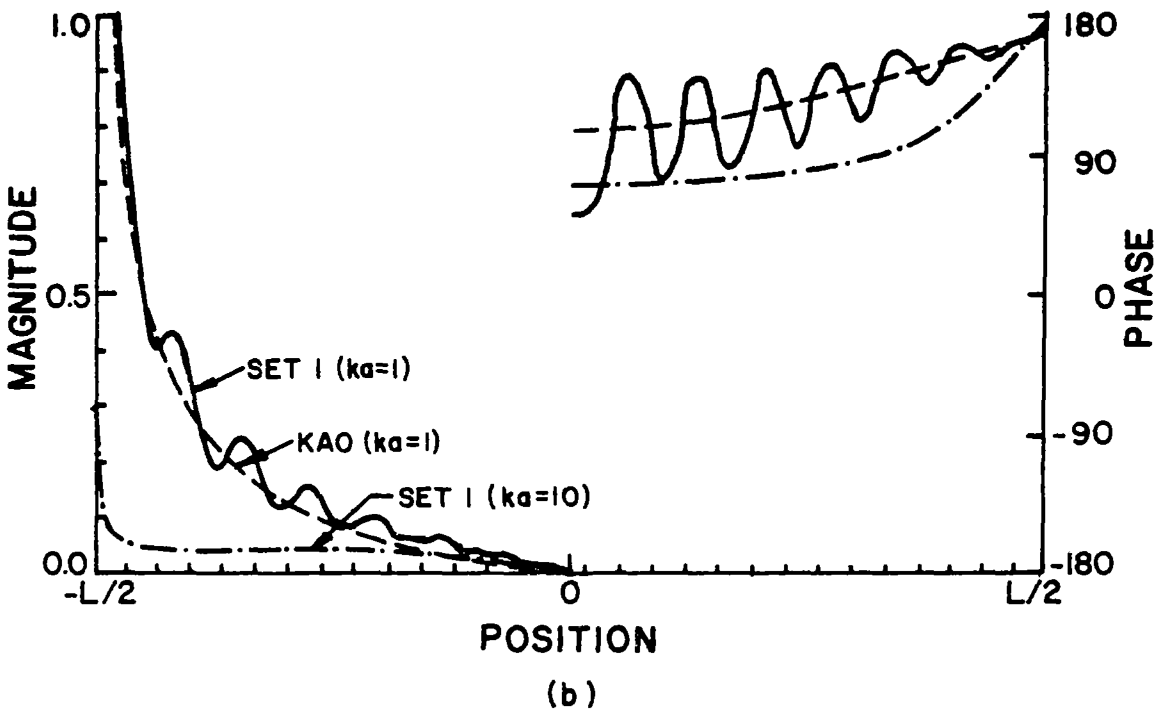


Figure 6.1. The four basis functions used about the point -L/2.

are dominated by the behavior of J_{z_n}. Hence the evaluation of J_{ϕ_n} may become ill-conditioned as shown in Figure 6.2 for the first harmonic with ka = 1 as compared to ka = 10. The conditioning becomes even worse for axial incidence as shown in Figure 6.3. Though J_{ϕ_1} shows poor behavior, J_{z_1} appears to be well-behaved and compares well to Kao's results.

By eliminating J_{z_n} from Equation (3.11a), one obtains Equation (3.13a) that should be well-conditioned for evaluation of J_{ϕ_n} as long as the second derivative is adequately described for numerical calculation. Solving this equation in conjunction with Equation (3.11c), the normal-H equation, and the same constraints as used for set I, one obtains a different type of conditioning problem. Since the J_{ϕ_n} and J_{z_n} components are unbounded and bounded, respectively, at the ends of the structure, J_{ϕ_n} dominates Equation (3.11c) near the ends. Hence, the evaluation of J_{z_n} near the ends becomes ill-conditioned. This behavior appears as a rise in the J_{z_n} current near the ends as shown for the first harmonic in Figure 6.4 for ka = 1 and 10. In comparison with the results of equation set I, the behavior is worse for ka = 10 as would be expected since the contribution of J_{z_n} in Equation (3.11c) varies as (1/a).

To avoid the ill-conditioning of equation sets I and II, consider set III with the constraints $J_{z_n}(\pm L/2) = 0$ and E_ϕ and H_ρ near one end equal to zero. Equation (3.13a) was found to be well-conditioned for the J_{ϕ_n} computation in set II. Equation (3.11b) is dominated by J_{z_n} due to the second derivative term. Since neither current component is dominated by the other component in both of these equations, one might expect good behavior for both components. This is indeed the case for the first harmonic as is shown in Figure 6.5 for a length of λ.



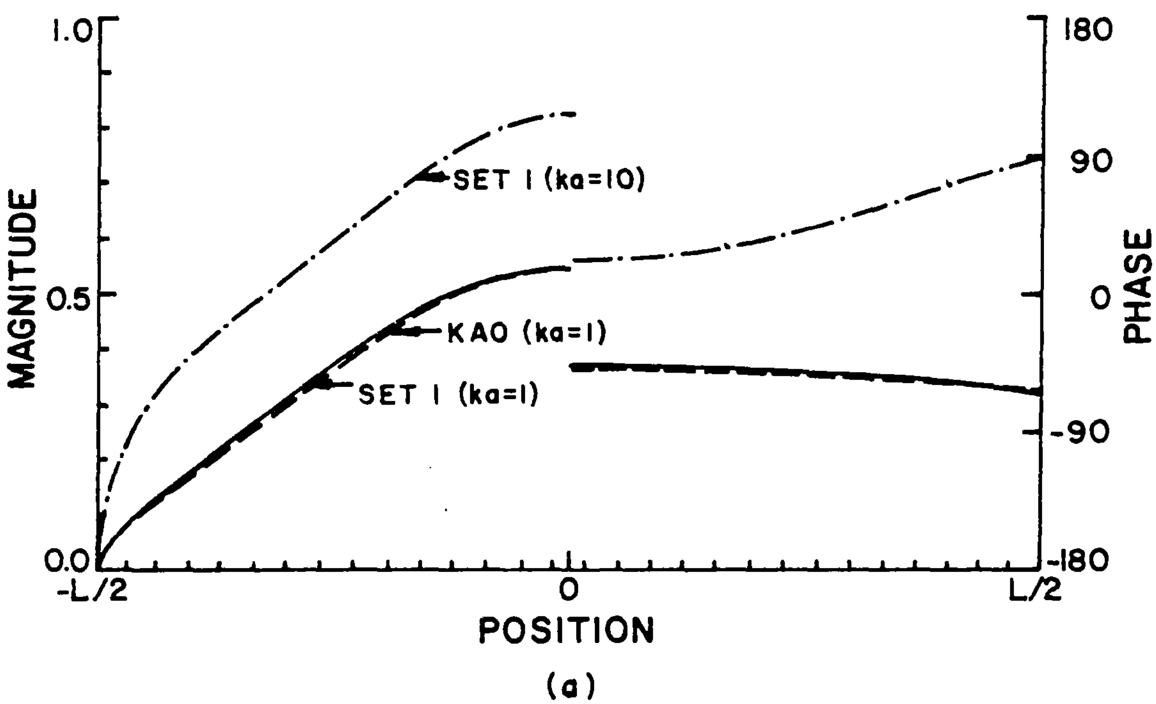
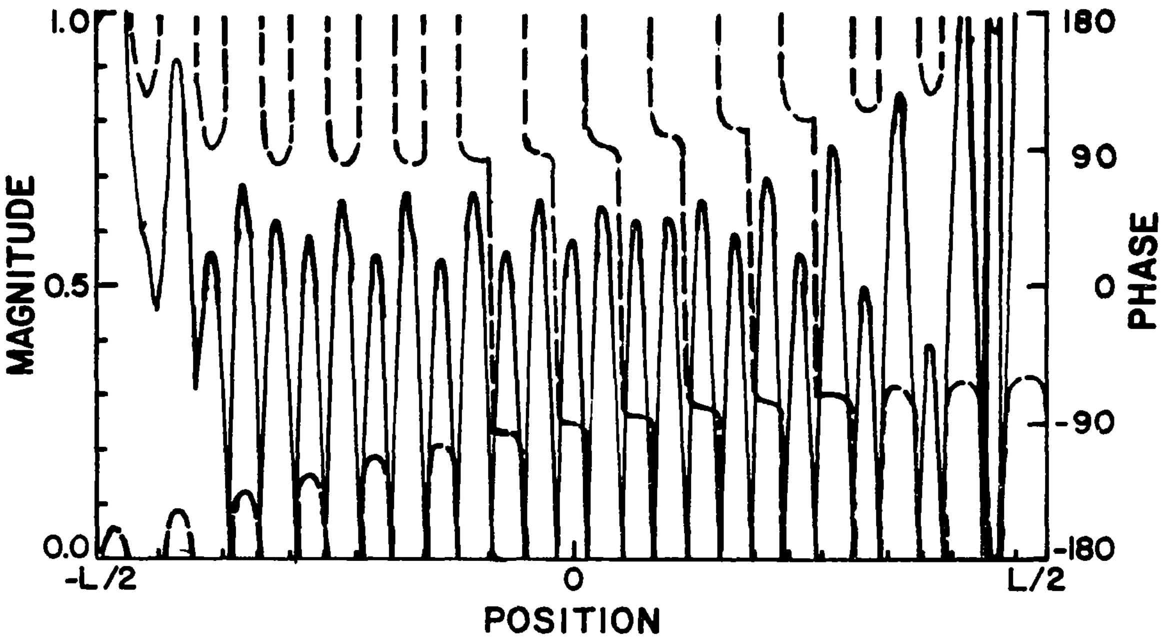


Figure 6.2. Calculated first-harmonic currents on a one-wavelength hollow cylinder for ka = 1 and 10 using the EFIE. The phase is plotted on the right with figures a and b representing J_{z_1} and J_{ϕ_1}, respectively. The incident field is E-polarized and normally incident.



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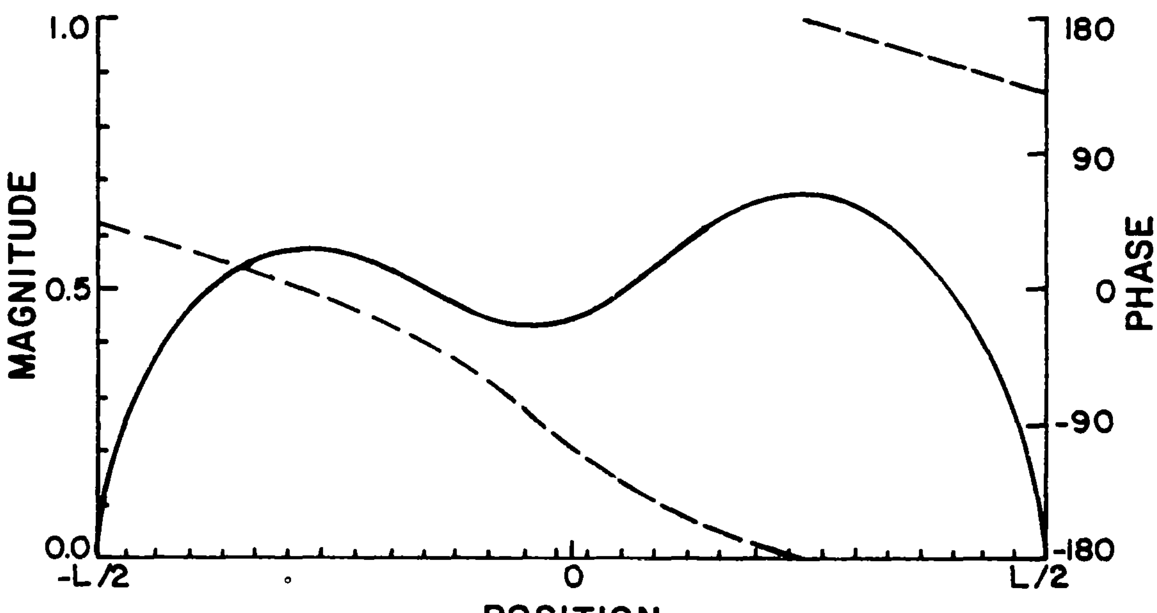
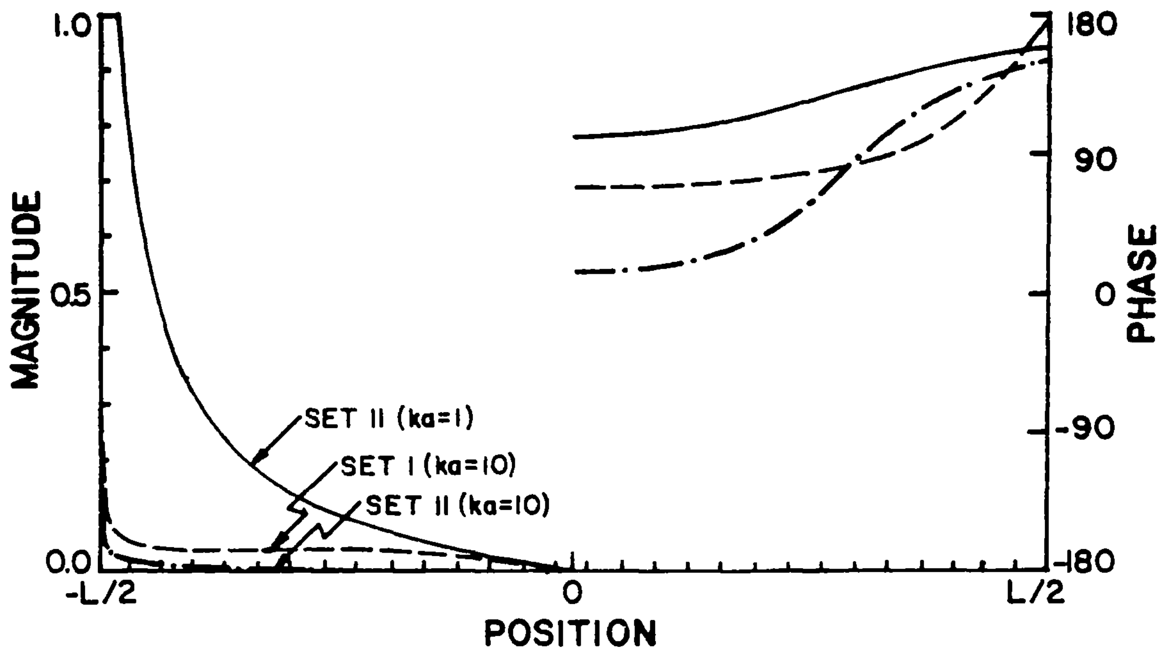


(a)

Figure 6.3. First-harmonic currents obtained using the EFIE for axial incidence on a one-wavelength hollow cylinder. The magnitude and phase are plotted as solid and dashed curves, respectively, with figures a and b representing J_{z_1} and J_{ϕ_1}, respectively.



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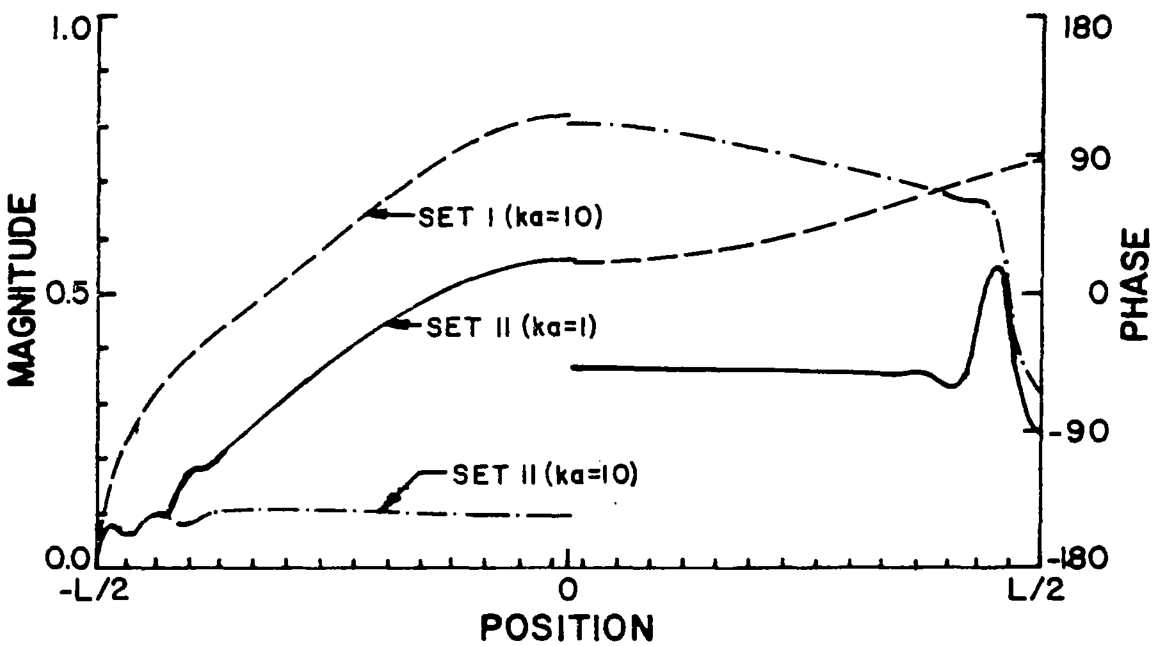
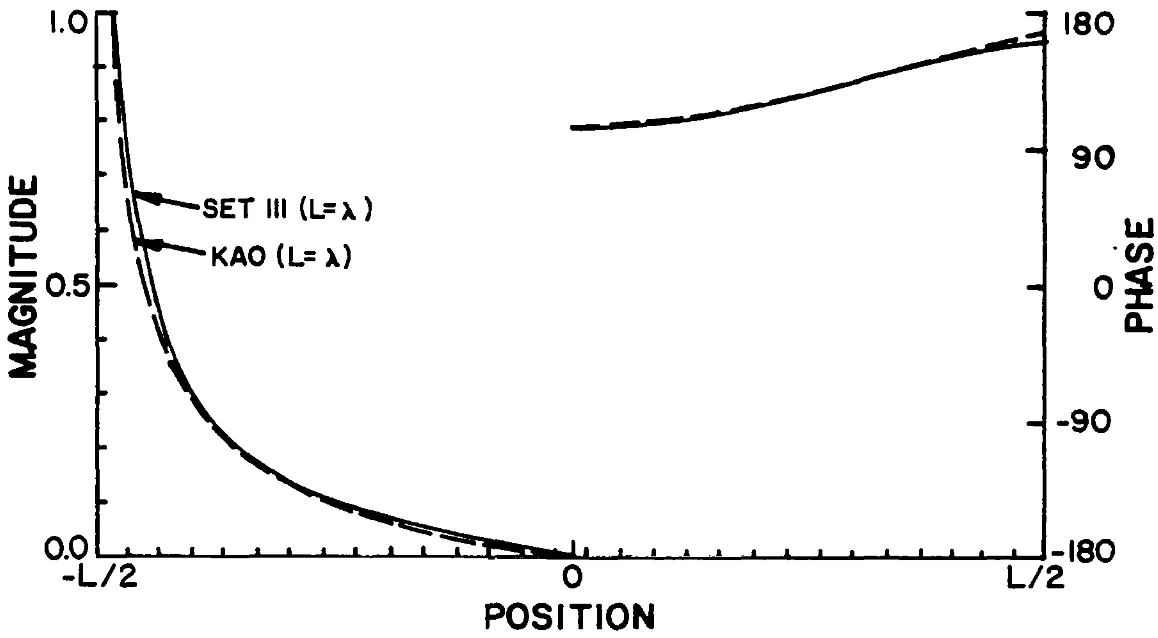


(a)

Figure 6.4. First-harmonic currents on a one-wavelength cylinder obtained from equation set II for ka = 1 and 10. The phase is plotted on the right with figures a and b representing J_{z_1} and J_{ϕ_1}, respectively. The incident field is E-polarized and normally incident.



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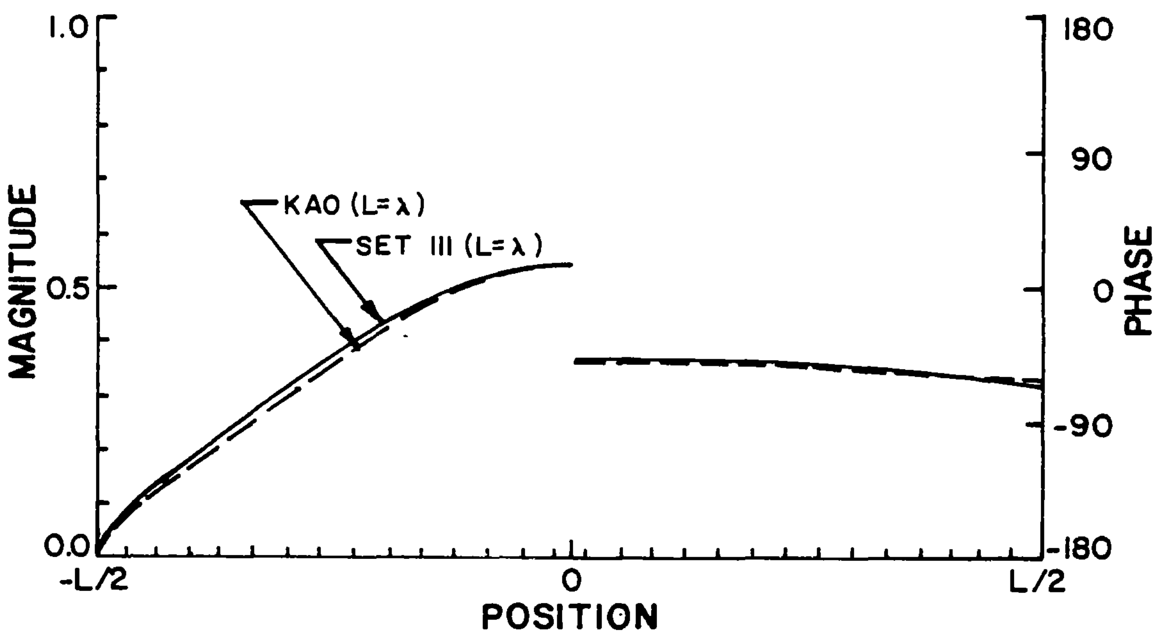


(a)

Figure 6.5. First-harmonic currents on a one-wavelength cylinder using equation set III for ka = 1. The phase is plotted on the right with figures a and b representing J_{z_1} and J_{ϕ_1}, respectively. The incident field is E-polarized and normally incident.

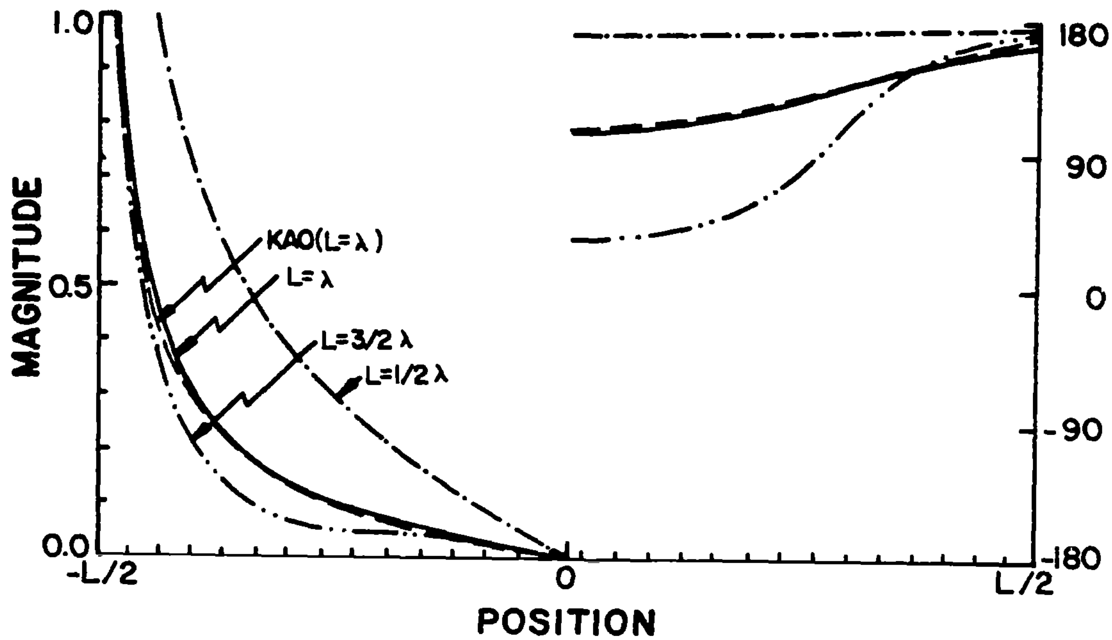
Equation (3.13b) has the same features as Equation (3.11b) for the J_{z_n} component, but lesser coupling from the J_{ϕ_n} component due to the continuity of the coupling term kernel. The solution to this set of equations, set IV, in conjunction with the constraints $J_{z_n}(\pm L/2) = H_{\rho_n}[\pm(L - \Delta)/2] = 0$ is plotted in Figures 6.6 through 6.8 for E- and H-polarization of normal incidence and for axial incidence, respectively.

It has been shown that equation sets I and II are not well-suited in general for determining the currents on a hollow cylinder. Sets III and IV produce well-behaved results that are nearly identical with additional decoupling being a desirable feature of set IV.

6.4 Additional Observations

In Section 3.5, it was pointed out that homogeneous solutions of the form $H_{\rho_n} = A \cos kz + B \sin kz$ and $E_{\phi_n} = j\eta(A \sin kz - B \cos kz)$ may exist for equation set III. Hence, the constraints must be chosen to eliminate such solutions. In the previous section, it was found that the boundary conditions on J_{z_n} and the zero constraints of H_{ρ_n} and E_{ϕ_n} in the same section gave well-behaved solutions. If the latter two constraints are replaced by $E_{\phi_n}(\pm\lambda/2) = 0$, which permits a homogeneous solution, then the solution obtained by matrix methods may not have the proper behavior. This is indeed the case as is shown in Figure 6.9 for a length of λ. Hence, one must choose the constraints on the problem in a manner to not only satisfy $H_{\phi_n}(\pm L/2)$ and $E_{\phi_n}(\pm L/2)$ equal to zero, but to force all homogeneous solutions to also be identically zero.

Up to this point, it has been assumed that the correct asymptotic behavior will be used in the current expansion. It would computationally be desirable if the square root behavior were replaced by a simple polynomial. Quadratic splines have been used in place of the quadratic



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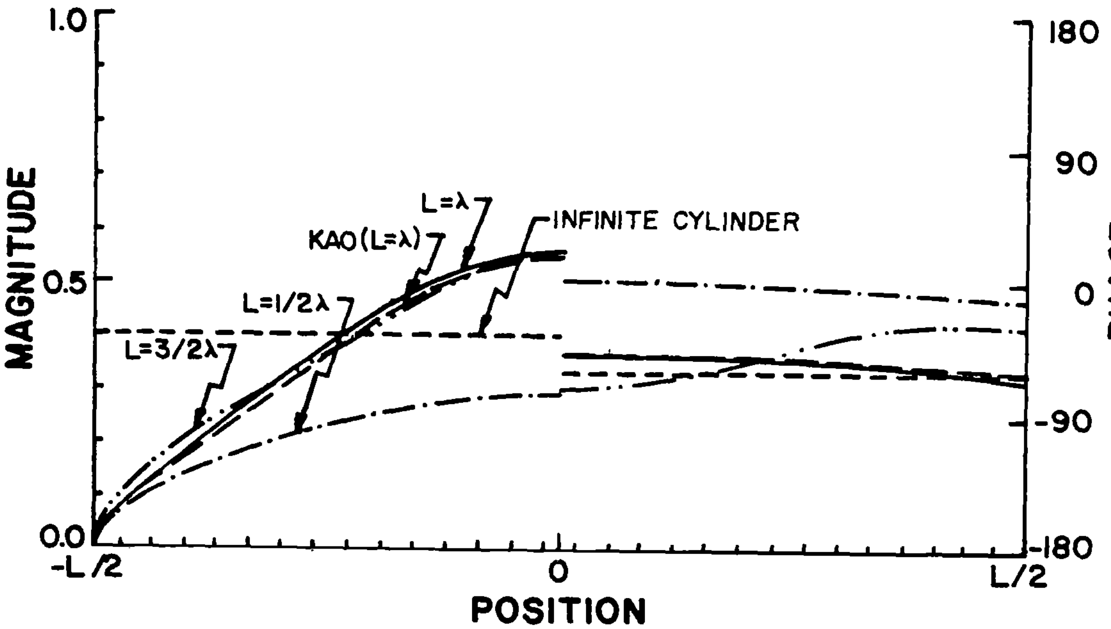
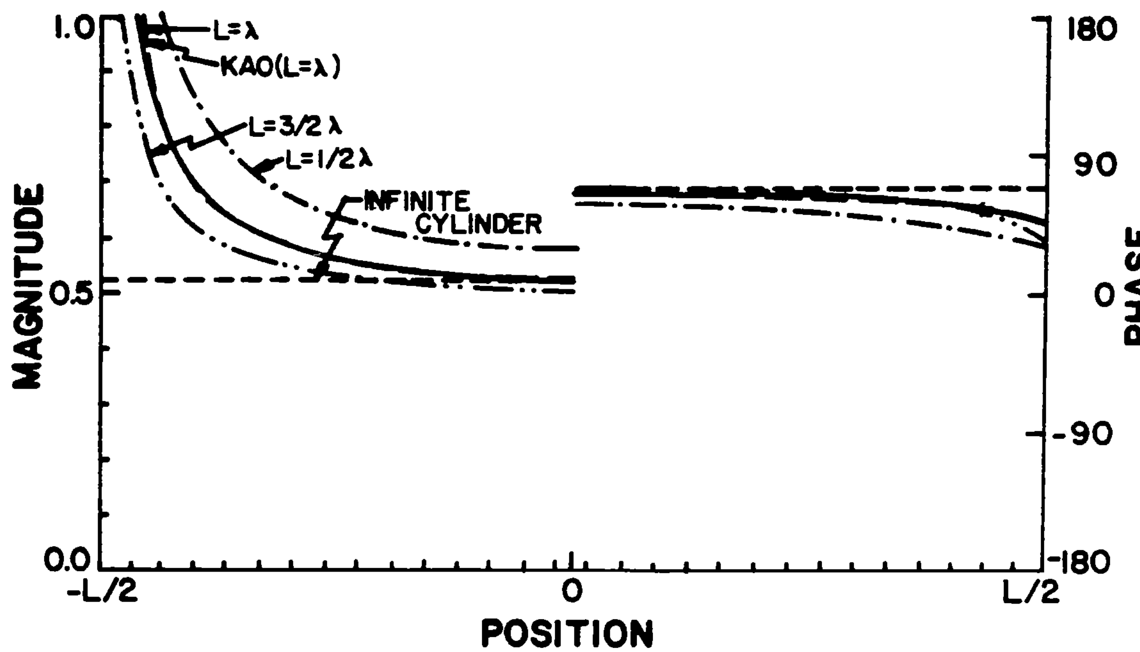


(a)

Figure 6.6. First harmonic currents using equation set IV for ka = 1. The phase is plotted on the right with figures a and b representing J_{z_1} and J_{ϕ_1}, respectively. The incident field is E-polarized and normally incident.



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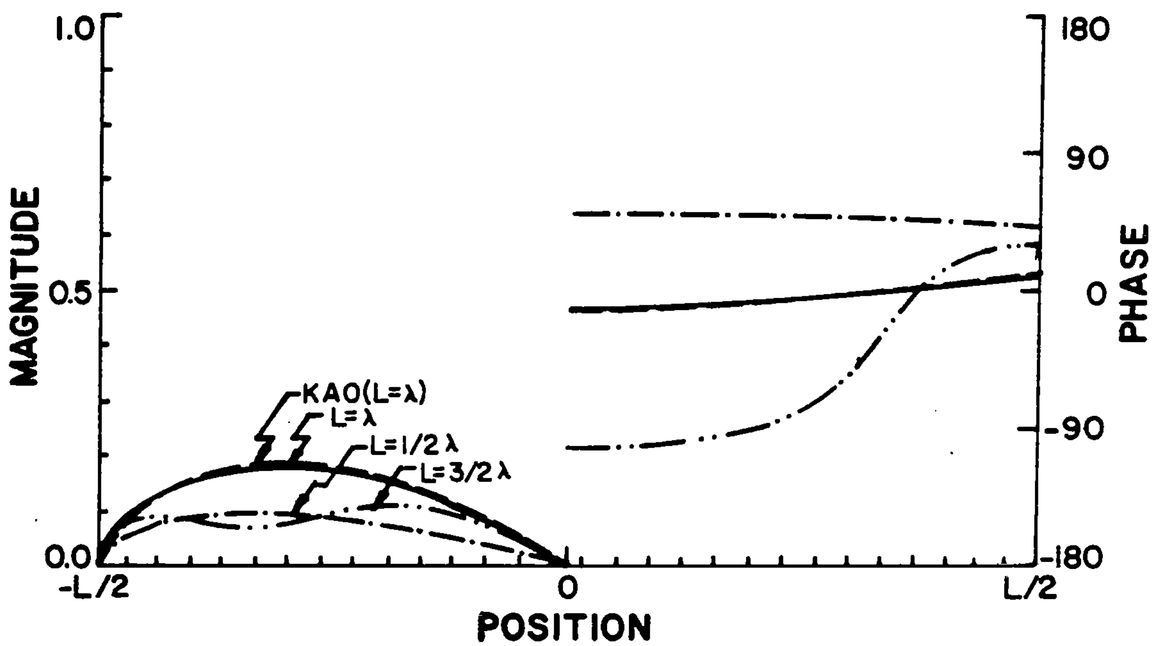
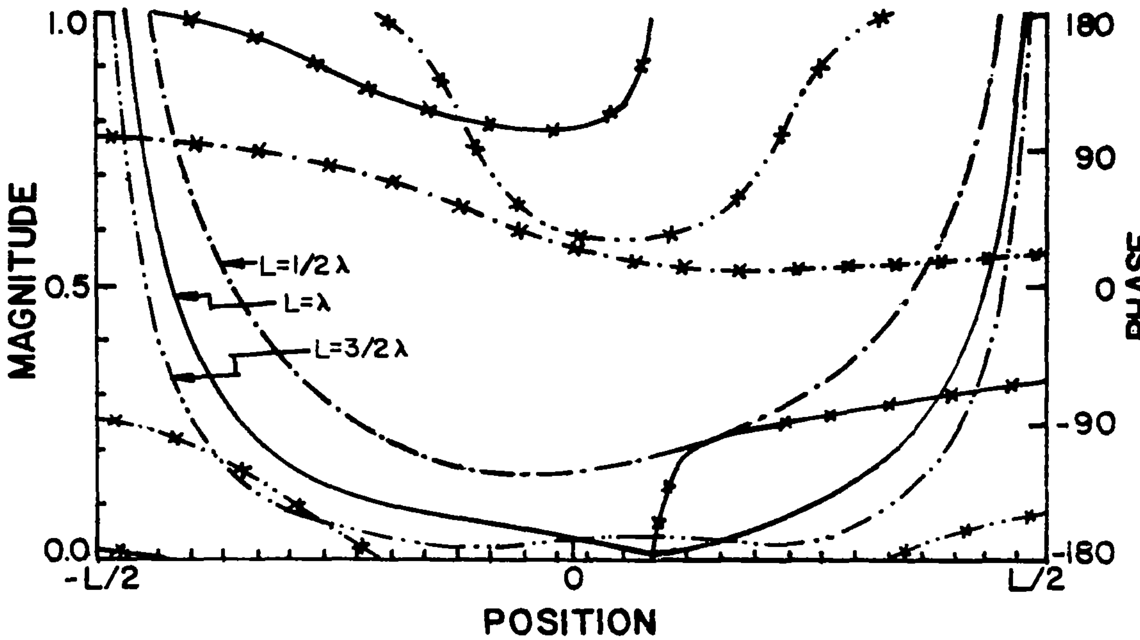


(a)

Figure 6.7. First harmonic currents using equation set IV for ka = 1. The phase is plotted on the right with figures a and b representing J_{z_1} and J_{ϕ_1}, respectively. The incident field is H-polarized and normally incident.



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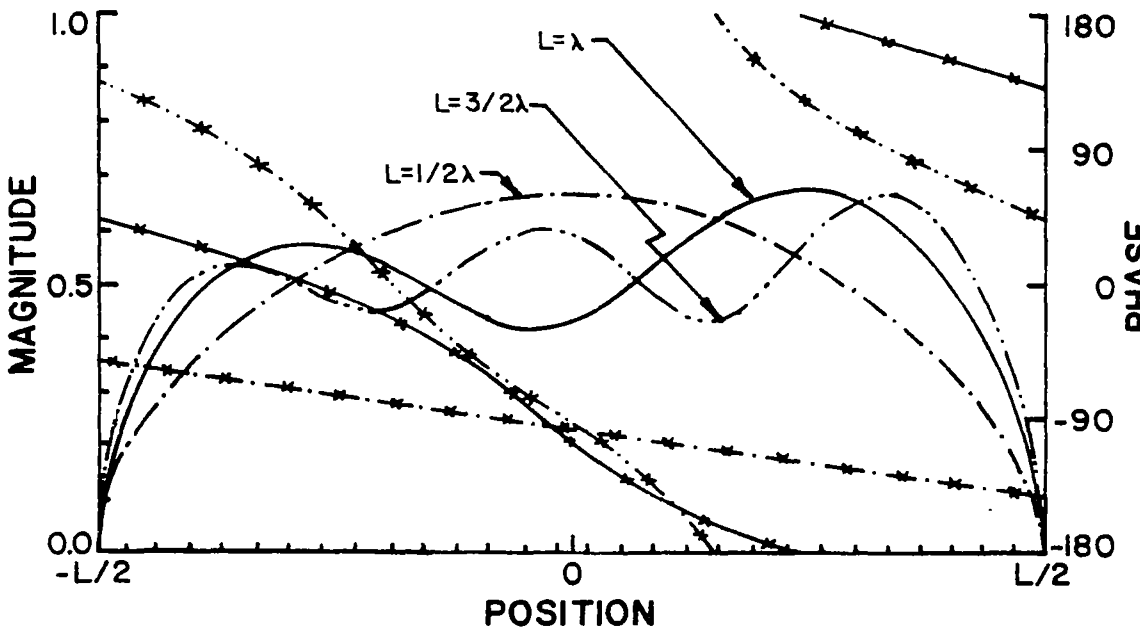

(a)

Figure 6.8. First-harmonic currents using equation set IV for ka = 1. The phase is denoted by x's with figures a and b representing J_{z_1} and J_{ϕ_1}, respectively. The incident field is axially incident.

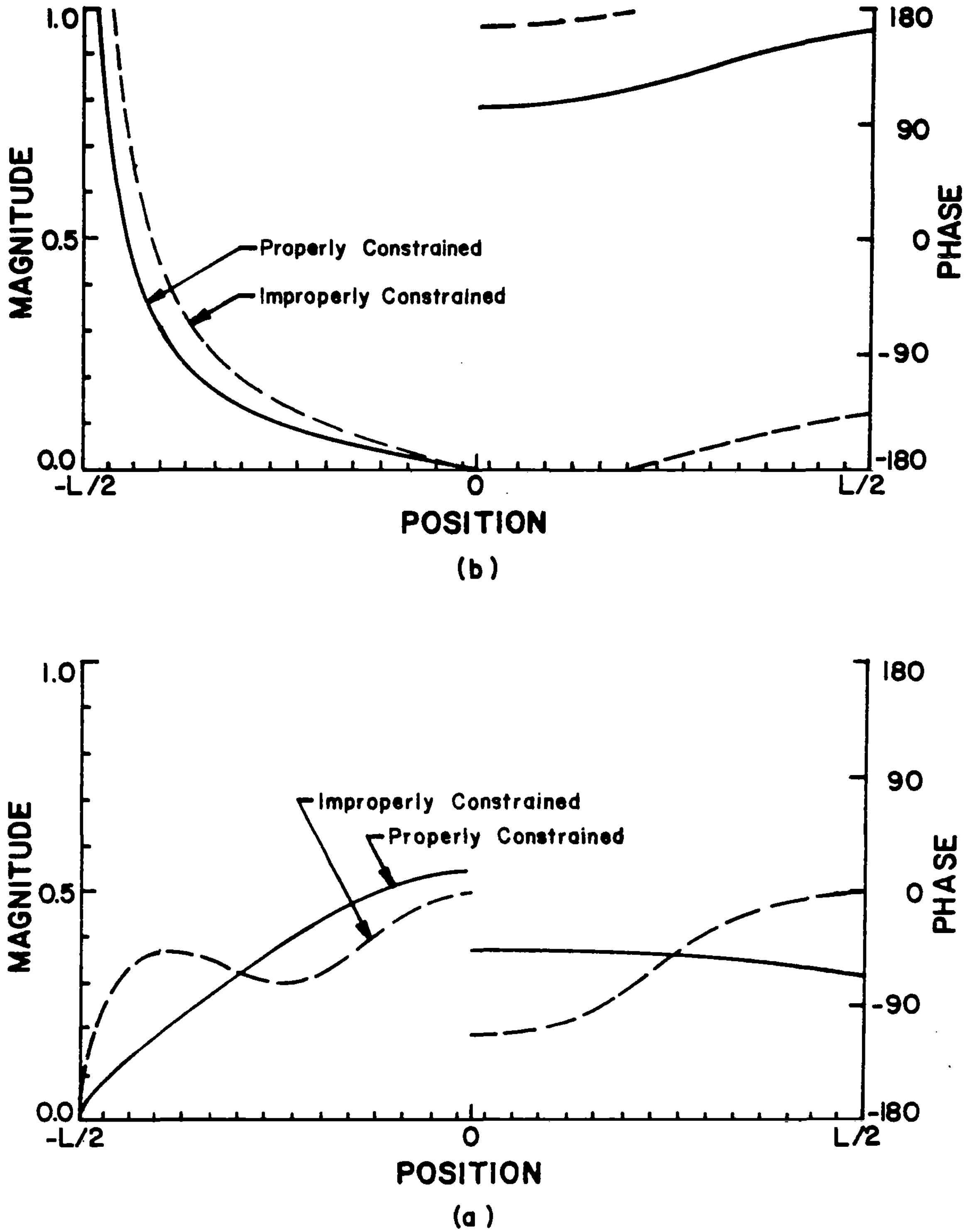

Figure 6.9. Comparison of properly and improperly constrained first-harmonic solutions for a one-wavelength cylinder using equation set III with ka = 1. The phase is plotted on the right with figures a and b representing J_{z_1} and J_{ϕ_1}, respectively. The incident field is E-polarized and normally incident.

functions times inverse square root functions for equation set IV. The results are not encouraging as shown in Figure 6.10. In most of the cases considered, the polynomial current amplitudes were substantially less than the currents with the correct asymptotic behavior, but had a similar phase behavior.

Due to the complications of the numerical development, it is often desirable to approximate the derivatives with finite differences. The use of finite differences is reasonable for observation points internal to the surface edges where the fields are analytic along the surface. Finite differences were applied to equation set II, for ka = 1. Delta testing and pulse expansion functions were used in conjunction with two section central differences for both the first and second derivatives on the interior of the structure. Equation (3.13a) was not enforced in the end sections in lieu of setting J_{z_n} in the same sections to zero. Equation (3.11c) was enforced in the last sections using both central and shifted differences. The shifted difference, which was required to obtain a reasonable solution as shown in Figure 6.11, uses only samples on the structure. A method of employing the difference routine in a manner more consistent with the other investigations would be to use half-sections at the ends and enforce $E_\phi(\pm L/2) = 0$ or other appropriate constraints where derivatives do not even occur at the ends of the cylinder.

6.5 Conclusions

In this chapter, the first harmonic problem has been investigated to understand the characteristics of coupled singular integral equations. The solution of the infinite cylinder problem was developed for

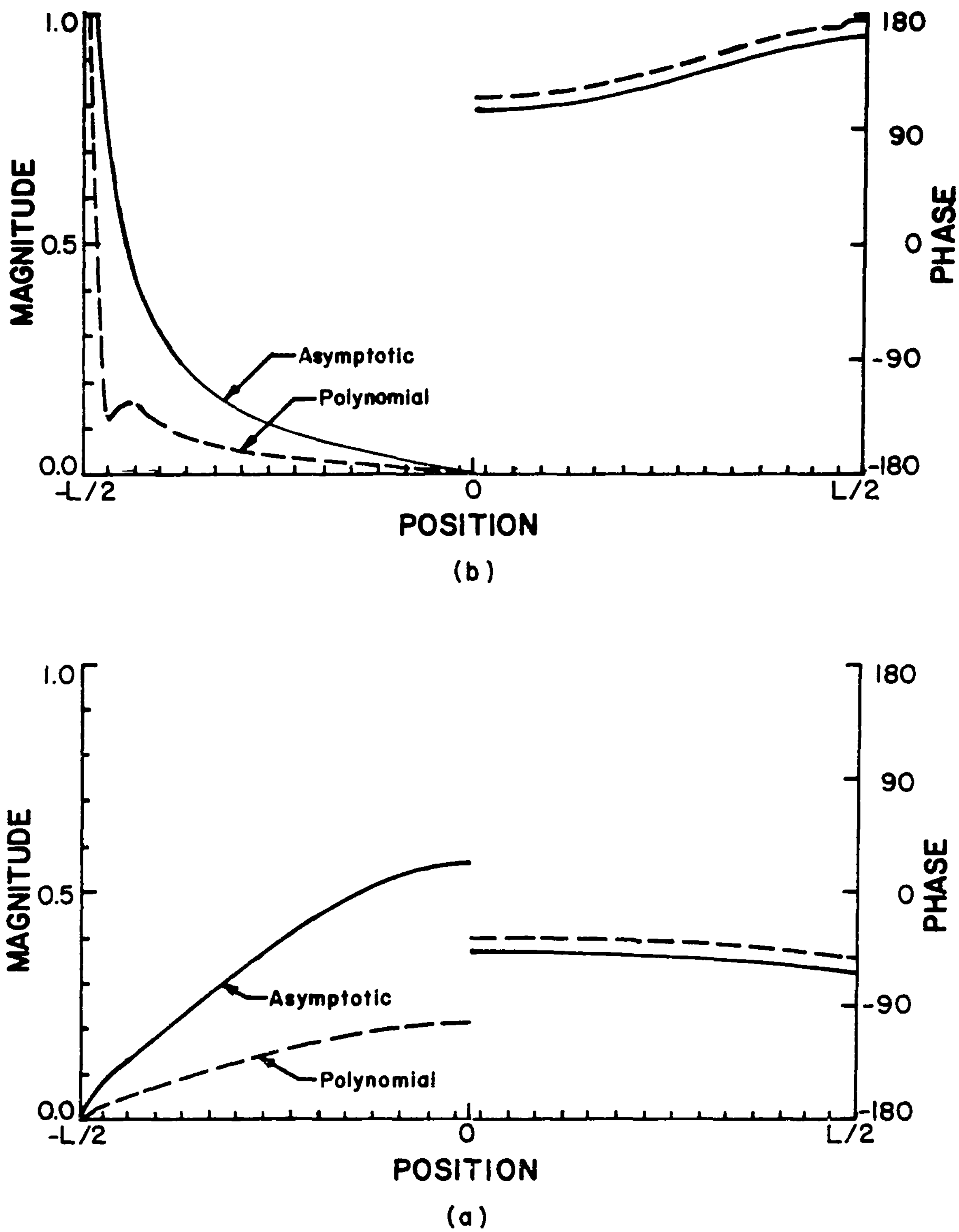

Figure 6.10. Comparison of the first-harmonic solutions for a one-wavelength cylinder with ka = 1 using both the correct asymptotic current behavior at the ends and a polynomial behavior. The phase is plotted on the right with figures a and b representing J_{z_1} and J_{ϕ_1}, respectively. The incident field is E-polarized and normally incident.

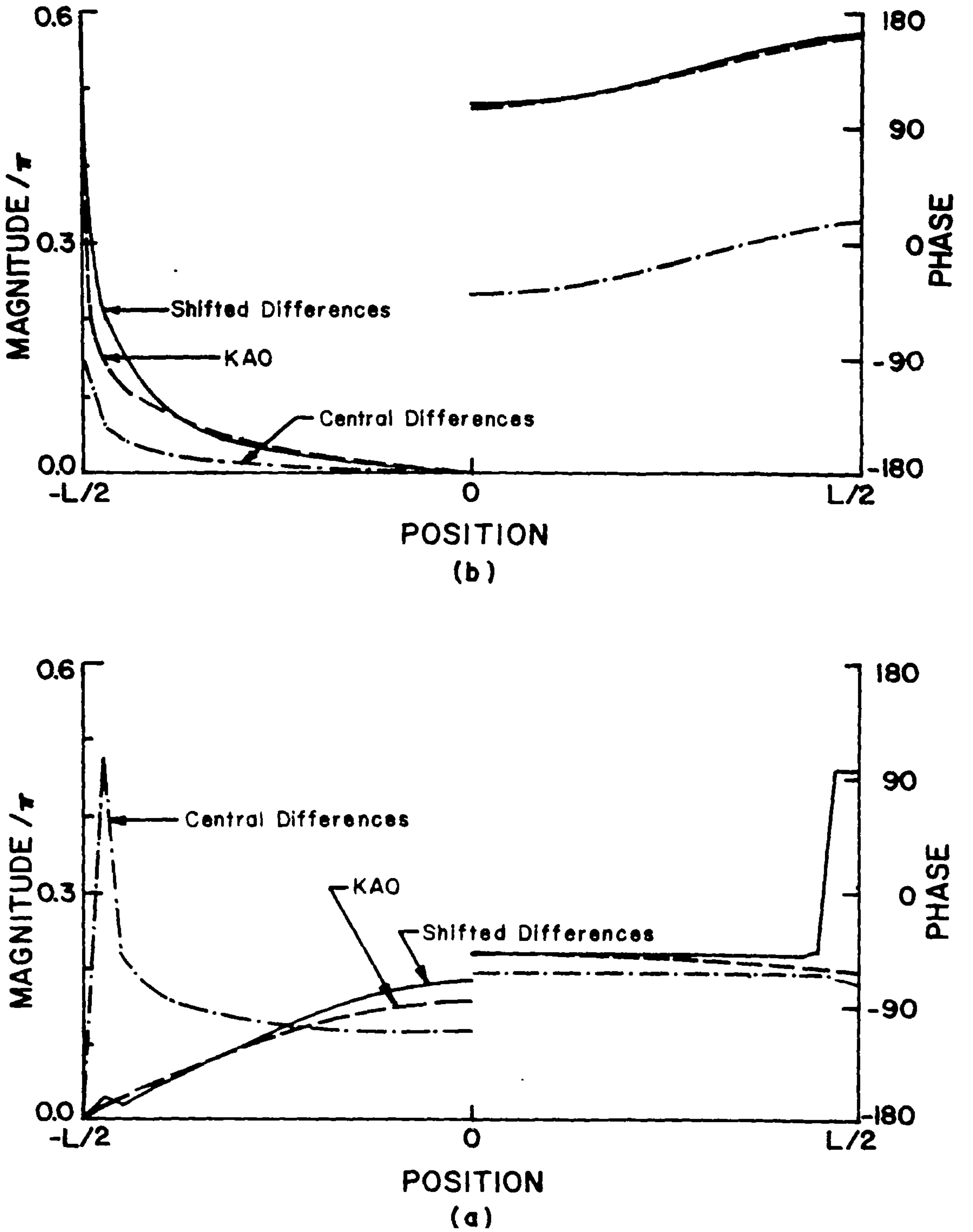

Figure 6.11. Comparison of the first-harmonic currents using both central and forward (or backward) differences at the ends of the cylinder for solution by finite differences. The structure has $L = \lambda$ and $ka = 1$. The phase is plotted on the right with figures a and b representing J_{z_1} and J_{ϕ_1}, respectively. The incident field is E-polarized and normally incident.

comparison with the solutions of the finite cylinder problem. Several sets of integral equations were investigated, with the conclusion that those arising from the ρ-derivative of both H_z and (ρH_ϕ) had the minimum coupling and hence the best numerical conditioning.

Additional conclusions are that the constraints must be chosen in the proper manner for uniqueness, the asymptotic behavior of currents must be used, and finite difference schemes may be used in a judicious manner.

7. CONCLUSIONS AND RECOMMENDATIONS

In this paper, the integral equations for a hollow circular cylinder and the numerical techniques involved in the approximate solution of such equations have been investigated. The integral equations for three-dimensional scattering structures were derived by evaluating the fields in a source-free region not including the scatterer and by letting the observation point converge to the surface of the scatterer. The integral equations thus obtained make use of the Hadamard principal value form of integration (Hadamard, 1952) which allows more highly singular integrands than the Cauchy principal value. The integral equations were also obtained for thin scatterers with edges.

All of the field quantities for the hollow circular cylinder were expanded in Fourier series and the corresponding integral equations were obtained for each harmonic. A new efficient procedure was presented for extracting the singularities and residuals of the kernels in the integral equations which may be integrated respectively in closed form and by numerical methods. Constraints on the integral equations were also presented and found to be necessary for obtaining a unique solution to Maxwell's equations.

The method of moments was used to obtain numerical solutions to the integral equations. The fundamentals of this method were presented along with the variational properties of the method. It was found that the variational interpretation could be used to gain insight into the choice of both the basis and testing functions. Several common expansion functions were defined in addition to a brief introduction to spline functions. The spline functions have desirable continuity built-in, with the B-splines being useful as subsectional basis functions.

The linear antenna (or zero harmonic) problem was investigated to obtain insight into the choice of expansion functions and operator approximations. It was found that the Hallen and Pocklington formulations for the linear antenna are both analytically and numerically equivalent for appropriate combinations of weight and basis functions. The finite difference approximation for the derivatives of the Pocklington formulation was found to be equivalent to a piecewise sinusoidal basis set on the order of $(k^2\Delta^2)$. It was concluded that for most problems of interest the most useful and easily obtained solution to the linear antenna problem is obtained by point-matching Pocklington's equation with second-order sinusoidal spline basis functions.

The first harmonic problem is representative of the higher harmonics and has been solved using point-matching with quadratic spline basis functions, modified by the asymptotic behavior in the end sections. Of particular interest were the problems of strong coupling between equations. Such coupling was found for the electric field integral equations when $ka = |n|$, where n is the harmonic number, and for a set of equations involving the normal H-field. The strong coupling leads to poor behavior of the numerical solution. Maximum decoupling was obtained with a set of equations obtained by setting the expression $[j\omega\varepsilon\bar{E}_t + \hat{n} \times \nabla_t H_n]$ equal to zero. The weak coupling resulted in a well-behaved solution. Additional facets of the first harmonic problem that were considered showed that the constraints on the integral equations must be chosen in an appropriate manner to suppress homogeneous solutions of the problem, finite differences may be used to describe the derivatives if end behavior is taken into account, and the replacement of the asymptotic current behavior by polynomials is not well-suited to the first harmonic problem.

The feed modeling and near-field calculations for the linear antenna with the second-order sinusoidal spline basis expansion are recommended for further investigation. For the first harmonic problem, the finite difference operator approximation and the necessity of the analytic asymptotic behavior in end sections are suggested areas for future study.

APPENDIX: LIMITING PROCEDURE FOR INTEGRALS

In this appendix, the limits of the various types of integrals involved in I_Δ, Section 2.3, are derived. These limits lead to three types of integral interpretations, the Hadamard interpretation being the most general.

To take the limits, various degrees of differentiability are required. These requirements enable one to make use of a remainder theorem (Davis, 1963).

Theorem: Let f(x) be n + 1 times differentiable at $x = x_0$. Then,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$+ \frac{(x - x_0)^{n+1}}{(n+1)!}\left[f^{(n+1)}(x_0) + \varepsilon(x)\right] \qquad \text{(A.1)}$$

where

$$\lim_{x \to x_0} \varepsilon(x) = 0 \quad .$$

The integral I_Δ is given by

$$I_\Delta = \int_{s_\Delta} \left\{ \left(K_x \frac{\partial \Phi}{\partial y''} - K_y \frac{\partial \Phi}{\partial x''} + \frac{J_x}{j\omega\varepsilon} \frac{\partial^2 \Phi}{\partial z'' \partial x''} + \frac{J_y}{j\omega\varepsilon} \frac{\partial^2 \Phi}{\partial z'' \partial y''} \right) \hat{z} \right.$$

$$+ \left[K_y \frac{\partial \Phi}{\partial z''} + \frac{J_x}{j\omega\varepsilon} \left(k^2 \Phi + \frac{\partial^2 \Phi}{\partial x''^2} \right) + \frac{J_y}{j\omega\varepsilon} \frac{\partial^2 \Phi}{\partial x'' \partial y''} \right] \hat{x}$$

$$\left. + \left[-K_x \frac{\partial \Phi}{\partial z''} + \frac{J_x}{j\omega\varepsilon} \frac{\partial^2 \Phi}{\partial y'' \partial x''} + \frac{J_y}{j\omega\varepsilon} \left(k^2 \Phi + \frac{\partial^2 \Phi}{\partial y''^2} \right) \right] \hat{y} \right\} dx'\, dy' \qquad \text{(A.2)}$$

where one is interested in the limit of I_Δ as δ tends to zero. I_Δ consists of six basic types of integrals for which the limits as δ tends to zero shall be derived.

The six basic types of integrals comprising I_Δ shall be considered individually using $f(x',y')$ to represent the current. The first type is written as

$$I_1(\delta) = \int_{s_\Delta} f(x',y')\ \Phi(\bar{r}'',\bar{r}')\ ds' = \int_0^{2\pi} \int_0^{\Delta} f(\rho',\phi')\ \frac{e^{-jkR}}{4\pi R}\ \rho'\ d\rho'\ d\phi' \tag{A.3}$$

where $\delta = |z'' - z|$. If f is finite in the Δ region, then*

$$I_1(\delta) = \frac{j}{4\pi k}\left(e^{-jk\sqrt{\Delta^2+\delta^2}} - e^{-jk\delta}\right) + 0(\Delta) \quad .$$

Hence, one obtains

$$I_1(0) = 0(\Delta) \quad . \tag{A.4}$$

It should be noted that this integral may be evaluated directly on the surface since the 1/R singularity is integrable.

The second type may be written as

$$\begin{aligned} I_2(\delta) &= \int_{s_\Delta} f(x',y')\ \frac{\partial \Phi}{\partial y''}\ dx'\ dy' \\ &= \int_0^{2\pi} \int_0^{\Delta} f(\rho',\phi')\left(\rho'\ \sin\phi'\ \frac{1 + jkR}{4\pi R^3}\right) e^{-jkR}\ \rho'\ d\rho'\ d\phi' \quad . \end{aligned} \tag{A.5}$$

For f differentiable in the Δ region, one can write

$$I_2(\delta) = 0(\Delta) \quad . \tag{A.6}$$

Unlike type 1, this integral must be done in the Cauchy sense on the surface.

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* $f(x) = 0[g(x)]$ as $x \to x_0$ implies that $\lim\limits_{x\to x_0} |f(x)/g(x)| = A$.

Type three integrals are given by

$$I_3(\delta) = \int_{s_\Delta} f(x',y') \frac{\partial\Phi}{\partial z''} dx' \, dy'$$

$$= \int_0^{2\pi} \int_0^{\Delta} f(\rho',\phi') \left(-\delta \frac{1 + jkR}{4\pi R^3}\right) e^{-jkR} \rho' \, d\rho' \, d\phi' \quad . \tag{A.7}$$

For f differentiable in the Δ region, $I_3(\delta)$ becomes

$$I_3(\delta) = \frac{f(0,0)\delta}{2} \left[e^{-jk\sqrt{\delta^2+\Delta^2}}/\sqrt{\delta^2 + \Delta^2} - e^{-jk\delta}/\delta\right] + 0(\Delta)$$

$$= - \frac{f(0,0)}{2} + 0(\Delta) \qquad \text{as} \qquad \delta \to 0 \quad . \tag{A.8}$$

As a result, the type three integral must be evaluated as a Cauchy principal value plus the residue $[-f(0)/2]$ for $\bar{r}$ on the surface.

The remaining three types of integrals involve the second derivatives of Φ and will be derived for f twice differentiable in the Δ region. The type four integral is given by

$$I_4(\delta) = \int_{s_\Delta} f(x',y') \left(x'y' \frac{3 + j3kR - k^2R^2}{4\pi R^5}\right) e^{-jkR} dx' \, dy' \quad . \tag{A.9}$$

To obtain a non-zero integration, f must be odd in both x' and y'. The lowest order terms in Δ would be due to a product x'y' term in f. Hence, one obtains

$$I_4(\delta) = 0(\Delta) \quad . \tag{A.10}$$

The expression for I_5 is

$$I_5(\delta) = \int_{s_\Delta} f(x',y') \left(-x'\delta \frac{3 + j3kR - k^2R^2}{4\pi R^5}\right) e^{-jkR} dx' \, dy' \quad . \tag{A.11}$$

Constant and quadratic terms in addition to the y' linear term integrate to zero over s_Δ. The remaining terms are the $\frac{\partial f}{\partial x'} \cdot x'$ term and the higher order terms which integrate to $O(\Delta)$. Hence, one can write Equation (A.11) as

$$I_5(\delta) = \left[\frac{\partial f}{\partial x}(0)\right](-\delta) \int_0^\Delta \rho^3 \frac{3 + j3kR - k^2R^2}{4R^5} e^{-jkR} d\rho + O(\Delta)$$

$$= -\frac{1}{2}\frac{\partial f}{\partial x}(0) + O(\Delta) \text{ , as } \delta \to 0 \text{ .} \tag{A.12}$$

In this case one obtains a residue in a manner similar to the Cauchy residue of the type three integral.

The final integral type may be written as

$$I_6(\delta) = \int_{s_\Delta} f(x',y') \left(-\frac{1 + jkR}{4\pi R^3} + y'^2 \frac{3 + j3kR - k^2R^2}{4\pi R^5}\right) e^{-jkR} dx' \, dy' \text{ .} \tag{A.13}$$

The linear terms of f integrate to zero in s_Δ and the quadratic and higher order terms give $O(\Delta)$. Thus, Equation (A.13) may be written as

$$I_6(\delta) = \frac{f(0)}{4} \int_0^\Delta \left(-2 \frac{1 + jkR}{R^3} + \rho^2 \frac{3 + j3kR - k^2R^2}{R^5}\right) e^{-jkR} \rho \, d\rho + O(\Delta)$$

$$= -f(0) \frac{1 + jk\Delta}{4\Delta} e^{-jk\Delta} + O(\Delta) \text{ , as } \delta \to 0 \text{ .}$$

This may be rewritten as

$$I_6(\delta) = -\frac{f(0)}{4\Delta} + O(\Delta) \text{ .} \tag{A.14}$$

Substituting these limits into Equation (A.2) for I_Δ, one has

$$I_\Delta = \left[\hat{n} \times \bar{K}_s(0) - \frac{1}{j\omega\varepsilon} \nabla \cdot \bar{J}_s(0)\right] /2 - \bar{J}_s(0)/4j\omega\varepsilon\Delta + 0(\Delta) \quad . \qquad (A.15)$$

The first two terms are the residues of the surface integration, the last term is dropped as Δ tends to zero, and the third term cancels with the remaining surface integral evaluation over $s - s_\Delta$ at the curve bounding s_Δ.

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