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On the Exact Theory of a Prolate Spheroidal Receiving and Scattering Antenna

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An exact formula for the current distribution $I(\eta)$ along an unloaded (short-circuited terminals) prolate spheroidal receiving antenna of arbitrary eccentricity is derived when the incident field is a plane wave with the electric vector directed parallel to the major axis of the spheroid. A knowledge of the current $I(0)$ and the input impedance Z_0, derived earlier by Chu and Stratton, completely determines the receiving characteristics of the antenna when it is loaded by impedance Z_L at $\eta=0$. The current distribution along the antenna is obtained from Ampere's law by integrating the magnetic field at the surface of the spheroid around cross sections perpendicular to the major axis. This integration removes the azimuthal angular dependence providing sufficient simplification in the mathematics, so that the exact current distribution may be obtained from an equation involving a single infinite sum.

The plane-wave scattering problem is completely formulated. It is shown that the solution for the scattered fields can be obtained only by solving simultaneous infinite matrix equations.

Numerical results are presented that compare spheroidal antenna receiving current distributions to cylindrical antenna current distributions. Also the induced center currents are compared for a number of antenna-shape parameters. These reveal that thin cylindrical antenna theory may be extended past its theoretical limits of validity.

1. Introduction

In a recent paper (Duncan and Harrison, 1963) it is shown that external RF energy leakage into a missile containing a slot (or access door) depends upon three factors: the induced exterior surface current density, the transmitting admittance of the slot, and an eigenfunction expansion of the interior field when unit voltage is impressed across the slot. To the present time there has been no theory to accurately predict the axial current density induced by an incident plane wave on cylindrical antennas with total length/radius < 16.5. Unfortunately, this excludes most missile structures. Although a missile is approximately cylindrical in shape, it is felt that the geometry of a prolate spheroid more nearly represents a missile surface. In this paper an exact expression is obtained for the induced axial current distribution on any prolate spheroidal structure when it is illuminated by a plane wave with the electric field directed parallel to the major axis of the spheroid. This method involves the solution of a set of simultaneous infinite matrix equations.

Ordnance engineers, when supplied with the total induced axial current distribution on a missile (or prolate spheroid counterpart), can predict with reasonable accuracy, on the basis of a few tests, the current that will flow in the interior electroexplosive devices when slots and other RF leaks are introduced in the missile's surface. In the past this total induced axial current distribution on a missile was estimated from cylindrical antenna theory. However, there was no reason to expect more than qualitative results from this approximation.

Also, in this paper an analysis of a prolate spheroid as a receiving antenna is given. This is provided by the solution for the short-circuit current, i.e., the total induced axial current at the midsection of the antenna, and the input impedance obtained from the work of Chu and Stratton (1941). The receiving circuit consists of Z_0 and Z_L in series. The driving voltage $V_{oc}=I(0)Z_0$.

The complete solution of the problem of scattering from an imperfectly conducting prolate spheroidal shell is of interest, as well as a determination of the shielding qualities of such structures. From a solution of the latter problem an estimate may be obtained for the shielding characteristics of missile silos. These interesting topics are reserved for discussion in later papers.

2. Steady-State Electromagnetic Wave Scattering in Prolate Spheroidal Coordinates

A prolate spheroid is formed by rotating an ellipse about its major axis. The coordinate system along with the unit vectors is shown in figure 1. The prolate spheroidal coordinate system ξ, η, ϕ is defined as follows:

$$\left.\begin{aligned} x&=F\sqrt{(\xi^2-1)(1-\eta^2)}\cos\phi \\ y&=F\sqrt{(\xi^2-1)(1-\eta^2)}\sin\phi \\ z&=F\xi\eta \\ \rho&=\sqrt{x^2+y^2}=F\sqrt{(\xi^2-1)(1-\eta^2)} \end{aligned}\right\} \tag{1}$$

where ξ ranges from 1 to ∞, η from -1 to $+1$, and ϕ from 0 to 2π. The surface $\xi=$constant is a prolate spheroid with interfocal distance $2F$, the surface $\eta=$constant is a hyperboloid of revolution with foci at $z=\pm F$, and the surface $\phi=$constant is a plane through the z axis at an angle ϕ from the x, z plane. The family of prolate spheroids given by surfaces of constant ξ have semimajor axes of ξF and semiminor axes of $F\sqrt{\xi^2-1}$.

The difficulty in treating problems of electromagnetic wave scattering in the prolate spheroidal coordinate system is compounded by two factors: the vector Helmholtz equation is not separable in these coordinates, and it is impossible to obtain a solenoidal solution of the vector Helmholtz equation, which is tangential to one of the coordinate surfaces (Morse and Feshbach, 1953). Without these solenoidal solutions that are tangential to the boundary one does not have the orthogonality conditions ordinarily used in evaluating the expansion coefficients of the general solutions for the fields.

Generally, the solving of an electromagnetic wave scattering problem in the prolate spheroidal coordinate system is a Sisyphean task. The general solution of the vector Helmholtz equation can be constructed from solutions of the scalar Helmholtz equation. This procedure is given by Schultz (1950), where he treats the problem of the scattering of a plane electromagnetic wave incident upon a perfectly conducting prolate spheroid with the direction of the propagation of the

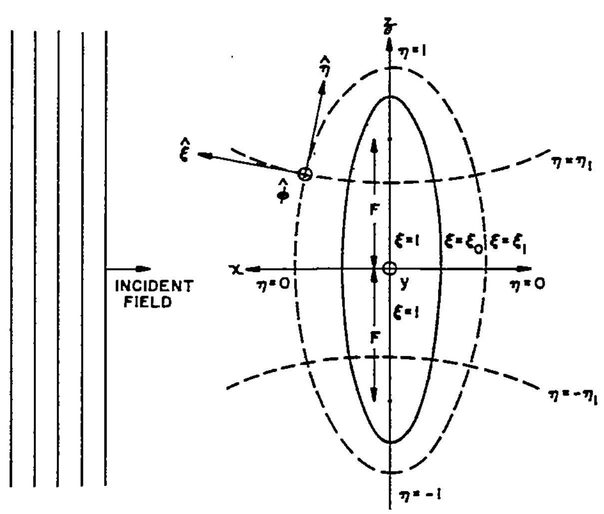

FIGURE 1. *The prolate spheroid with the incident plane-wave field showing the prolate spheroidal coordinate system.*

wave parallel to the major axis of the spheroid. This problem is considered later by Siegel et al. (1953, 1956) who discuss the more general aspects such as the convergence of the scattered field expansions and the completeness property of the general solutions of the vector Helmholtz equation. Both Schultz and Siegel et al., obtain in matching boundary conditions a pair of simultaneous infinite matrix equations for the unknown expansion coefficients of the solutions for the scattered fields.

Page (1944) treats the prolate spheroid as a receiving and scattering antenna excited by a plane wave linearly polarized with the electric field directed along the major axis of the spheroid. However, he obtains only a few approximate results for the case $\xi \approx 1$.

3. Plane-Wave Scattering from a Perfectly Conducting Prolate Spheroid

3.1. Incident and Scattered Fields

The scalar Helmholtz equation is separable in prolate spheroidal coordinates. These well-known solutions are, in Flammer's notation (1957a),

$$ {}_{\substack{e\\o}}\psi_{m\ell}^{(h)} = S_{m\ell}(\eta) R_{m\ell}^{(h)}(\xi) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\phi, \qquad h = 1, 2, 3, 4, \ldots \tag{2} $$

For a general discussion as well as the tabulation of the above prolate spheroidal wave functions the author defers to Flammer's text.[1]

The electromagnetic fields are considered to have the harmonic time dependence $e^{j\omega t}$ which is suppressed. The expansion of a scalar plane wave traveling in the negative x direction given in terms of prolate spheroid wave functions is according to Flammer (1957a)

$$ e^{j\beta x} = \sum_{\substack{m=0\\ \ell=0}}^{\infty} A_{m\ell}\, {}_e\psi_{m\ell}^{(1)}(\eta, \xi, \phi), \tag{3} $$

where $\beta = \dfrac{2\pi}{\lambda}$ is the propagation constant,

$$ A_{m\ell} = \frac{2j^{\ell}\epsilon_m S_{m\ell}(0)}{\displaystyle\sum_{n=0,1}^{\infty}{}' \frac{2(2m+n)!}{(2m+2n+1)n!}(d_n^{m\ell})^2} = \frac{2j^{\ell}\epsilon_m S_{m\ell}(0)}{N_{m\ell}} \tag{4} $$

$$ \left.\begin{aligned} \epsilon_m &= 1 \qquad m = 0 \\ &= 2 \qquad m > 0 \end{aligned}\right\}. \tag{5} $$

The functions ${}_e\Psi_{m\ell}^{(1)}$ are defined in (2) and normalization constants $N_{m\ell}$ are tabulated by Flammer. A very extensive tabulation of the expansion coefficients $d_n^{m\ell}$ is given by Stuckey and Layton (1964). These coefficients are also used in computing the spheroidal wave functions. The normalization of the $S_{m\ell}(\eta)$ functions is

$$ S_{m\ell}(0) = \frac{(-1)^{\frac{\ell-m}{2}}(\ell+m)!}{2^{\ell}\left(\frac{\ell-m}{2}\right)!\left(\frac{\ell+m}{2}\right)!}. \qquad (\ell - m) \text{ even} \tag{6} $$

[1] In the attempt to maintain simplicity of notation, frequency dependence of the wave functions is not explicitly shown above and in the subsequent development.

$$\frac{d}{d\eta} S_{m\ell}(0) = \frac{(-1)^{\frac{\ell-m-1}{2}}(\ell+m+1)!}{2^{\ell}\left(\frac{\ell-m-1}{2}\right)!\left(\frac{\ell+m+1}{2}\right)!} \qquad (\ell-m) \text{ odd.} \tag{7}$$

Since $P^{m}_{m+2n+1}(0)=0$, then $S_{m\ell}(0)=0$ for odd values of $(\ell-m)$. Thus

$$A_{m\ell}=0. \qquad (\ell-m) \text{ odd.} \tag{8}$$

Consider the plane-wave electromagnetic field incident upon the perfectly conducting spheroid to have the electric field polarized parallel to the z direction, i.e., along the major axis of the prolate spheroid. Then the plane-wave fields can be written

$$\left.\begin{aligned} \mathbf{E}^i &= E_0 e^{j\beta x}\hat{z} \\ \mathbf{H}^i &= \frac{E_0}{R_c} e^{j\beta x}\hat{y} \end{aligned}\right\} \tag{9}$$

where $R_c=\sqrt{\frac{\mu}{\epsilon}}$. Using (3) the magnetic field can be written

$$H^i=\frac{1}{R_c}E_0\sum_{m,\ell} A_{m\ell e}\psi^{(1)}_{m\ell}(\eta,\xi,\phi)\hat{y}\,. \tag{10}$$

Since $\mathbf{E}=-j\frac{1}{\beta}R_c\nabla\times\mathbf{H}$, the incident electric field is

$$E^i=-j\frac{1}{\beta}E_0\sum_{m,\ell} A_{m\ell}\,{}^{y}_{e}\mathbf{M}^{(1)}_{m\ell} \tag{11}$$

where

$${}^{y}_{e}\mathbf{M}^{(1)}_{m\ell}=\nabla_e\psi^{(1)}_{m\ell}\times\hat{y}. \tag{12}$$

By defining

$${}^{y}_{e}\mathbf{N}^{(1)}_{m\ell}=\frac{1}{\beta}\nabla\times{}^{y}_{e}\mathbf{M}^{(1)}_{m\ell} \tag{13}$$

the incident magnetic field can be written

$$H^i=\frac{1}{\beta}\frac{1}{R_c}E_0\sum_{m,\ell} A_{m\ell}\,{}^{y}_{e}\mathbf{N}^{(1)}_{m\ell}. \tag{14}$$

Each of the two sets of functions, $({}^{x}_{\substack{e\\0}}\mathbf{M}^{(h)}_{m\ell},\ {}^{y}_{\substack{e\\0}}\mathbf{M}^{(h)}_{m\ell},\ {}^{z}_{\substack{e\\0}}\mathbf{M}^{(h)}_{m\ell})$ and $({}^{x}_{\substack{e\\0}}\mathbf{N}^{(h)}_{m\ell},\ {}^{y}_{\substack{e\\0}}\mathbf{N}^{(h)}_{m\ell},\ {}^{z}_{\substack{e\\0}}\mathbf{N}^{(h)}_{m\ell})$,

forms a complete set of solenoidal vector wave functions, i.e., they can be used to represent any solenoidal solution of the vector Helmholtz equation (Siegel et al., 1958), where

$$\left.\begin{aligned} {}^{u}_{\substack{e\\0}}\mathbf{M}^{(h)}_{m\ell} &= \nabla_{\substack{e\\0}}\psi^{(h)}_{m\ell}\times\hat{u} \qquad & u=x,y,z \\ {}^{u}_{\substack{e\\0}}\mathbf{N}^{(h)}_{m\ell} &= \frac{1}{\beta}\nabla\times{}^{u}_{\substack{e\\0}}\mathbf{M}^{(h)}_{m\ell}, \qquad & u=x,y,z \end{aligned}\right\}. \tag{15}$$

The scattered fields are required to satisfy the Sommerfeld radiation condition. Since only the radial function $R_{m\ell}^{(h=4)}(\xi)$ has the proper radial asymptotic behavior,

$$R_{m\ell}^{(4)}(\xi)\overrightarrow{F\beta\xi\to\infty}\ \frac{1}{F\beta\xi}e^{-j\left(F\beta\xi-\frac{m+1}{2}\pi\right)} \tag{16}$$

only these radial functions must be utilized in the scattered field expansion. The functional dependence of the scattered field components on the variable ϕ must be odd or even corresponding to the dependence of the incident field components. Since there are two boundary conditions to be satisfied it is to be expected that the scattered electric field may be represented by two series of vector functions each with undetermined coefficients (Siegel et al., 1956),

$$\mathbf{E}^s=\sum_{m,\ell}\{\alpha_{m\ell}\,{}_0^x\mathbf{M}_{m\ell}^{(4)}+\beta_{m\ell}\,{}_e^y\mathbf{M}_{m\ell}^{(4)}\}. \tag{17}$$

The corresponding magnetic field is

$$\mathbf{H}^s=j\frac{1}{R_c}\sum_{m,\ell}\{\alpha_{m\ell}\,{}_0^x\mathbf{N}_{m\ell}^{(4)}+\beta_{m\ell}\,{}_e^y\mathbf{N}_{m\ell}^{(4)}\}. \tag{18}$$

Delineative expressions for the vector wave functions are found in Flammer's text (1957a).

3.2. Boundary Equations

The boundary condition on the electric field at the surface of a perfectly conducting prolate spheroid is

$$\hat{\xi}\times[\mathbf{E}^i+\mathbf{E}^s]_{\xi=\xi_0}=0. \tag{19}$$

This boundary condition gives two equations,

$$j\frac{1}{\beta}E_0\sum_{m,\ell}A_{m\ell}\,[m\xi_0S_{m\ell}(\eta)R_{m\ell}^{(1)}(\xi_0)\sin\phi\sin m\phi+(\xi_0^2-1)S_{m\ell}(\eta)R_{m\ell}^{(1)\prime}(\xi_0)\cos\phi\cos m\phi]$$

$$\begin{aligned}=\sum_{m,\ell}\{\alpha_{m\ell}\,&[-m\xi_0S_{m\ell}(\eta)R_{m\ell}^{(4)}(\xi_0)\cos\phi\cos m\phi-(\xi_0^2-1)S_{m\ell}(\eta)R_{m\ell}^{(4)\prime}(\xi_0)\sin\phi\sin m\phi]\\&+\beta_{m\ell}\,[m\xi_0S_{m\ell}(\eta)R_{m\ell}^{(4)}(\xi_0)\sin\phi\sin m\phi+(\xi_0^2-1)S_{m\ell}(\eta)R_{m\ell}^{(4)\prime}(\xi_0)\cos\phi\cos m\phi]\}.\end{aligned} \tag{20}$$

$$j\frac{1}{\beta}E_0\sum_{m,\ell}A_{m\ell}[\eta(\xi_0^2-1)S_{m\ell}(\eta)R_{m\ell}^{(1)\prime}(\xi_0)\sin\phi\cos m\phi+\xi_0(1-\eta^2)S'_{m\ell}(\eta)R_{m\ell}^{(1)}(\xi_0)\sin\phi\cos m\phi]$$

$$\begin{aligned}=\sum_{m,\ell}\{\alpha_{m\ell}&[\eta(\xi_0^2-1)S_{m\ell}(\eta)R_{m\ell}^{(4)\prime}(\xi_0)\cos\phi\sin m\phi+\xi_0(1-\eta^2)S'_{m\ell}(\eta)R_{m\ell}^{(4)}(\xi_0)\cos\phi\sin m\phi]\\+\beta_{m,\ell}&[\eta(\xi_0^2-1)S_{m\ell}(\eta)R_{m\ell}^{(4)\prime}(\xi_0)\sin\phi\cos m\phi+\xi_0(1-\eta^2)S'_{m\ell}(\eta)R_{m\ell}^{(4)}(\xi_0)\sin\phi\cos m\phi]\}.\end{aligned} \tag{21}$$

The following notation was used in (20) and (21):

$$S'_{m\ell}(\eta)=\frac{d}{d\eta}S_{m\ell}(\eta) \tag{22}$$

and
$$R'_{m\ell}(\xi)=\frac{d}{d\xi}R_{m\ell}(\xi). \tag{23}$$

From these equations it is found that a solution for the scattered field involves solving simultaneous infinite matrix equations. An approximate solution to the infinite matrix equations can be obtained by truncating the infinite matrices at a large order. That was the technique used by Schultz (1950) and Siegel et al. (1956).

Some of the expansion coefficients of (17) and (18) can be obtained without having to solve infinite matrix equations. From (20) using the orthogonality properties,

$$\left.\begin{aligned}\int_0^{\pi}\begin{matrix}\sin\\ \cos\end{matrix}\ \ell\phi\ \begin{matrix}\sin\\ \cos\end{matrix}\ m\phi d\phi&=\frac{\pi}{2}\delta_{m\ell}\\ \int_0^{\pi}\cos m\phi d\phi&=\pi\delta_{m0}\\ \int_{-1}^{+1}S_{1\ell}(\eta)S_{1p}(\eta)d\eta&=\delta_{\ell p}N_{1\ell}\end{aligned}\right\} \tag{24}$$

the following relation is obtained

$$(\alpha_{1\ell}-\beta_{1\ell})=-j\frac{1}{\beta}E_0\frac{[\xi_0R^{(1)}_{1\ell}(\xi_0)+(\xi_0^2-1)R^{(1)\prime}_{1\ell}(\xi_0)]}{[\xi_0R^{(4)}_{1\ell}(\xi_0)+(\xi_0^2-1)R^{(4)\prime}_{1\ell}(\xi_0)]}A_{1\ell}. \tag{25}$$

3.3. Induced Current Distribution

The incident plane-wave field induces on the prolate spheroid a surface current density given by

$$\mathbf{J}=\hat{\xi}\times[\mathbf{H}^i+\mathbf{H}^s]_{\xi=\xi_0}. \tag{26}$$

To obtain this surface current density would involve the solution of the set of simultaneous infinite matrix equations discussed in the previous section. If the induced current distribution on the prolate spheroid is defined as in the case of linear antennas, i.e., as the total current through a cross section perpendicular to the antenna being a function of the position of the cross section, the current distribution can be obtained exactly.

Using Ampere's law it is easily shown that the current density $I(\eta)$ is

$$I(\eta)=\rho(\eta)\int_0^{2\pi}[H^i_\phi+H^s_\phi]_{\xi=\xi_0}d\phi, \tag{27}$$

where
$$\rho(\eta)=F\sqrt{(\xi_0^2-1)(1-\eta^2)}. \tag{28}$$

From the substitution of (14) and (20) into (27), the following is obtained after the indicated integration:

$$I(\eta)=\pi\rho(\eta)\frac{1}{R_c}\sum_{\ell=0}^{\infty}\{A_{1\ell}[E_0S_{1\ell}(\eta)R^{(1)}_{1\ell}(\xi_0)]-j\beta(\alpha_{1\ell}-\beta_{1\ell})[S_{1\ell}(\eta)R^{(4)}_{1\ell}(\xi_0)]\}. \tag{29}$$

To reduce (29) to this simple form the defining equations for the wave functions were used.

The substitution of (25) into the previous equations yields

$$I(\eta)=\pi\frac{1}{R_c}E_0\rho(\eta)\sum_{\ell=0}^{\infty}A_{1\ell}S_{1\ell}(\eta)\left[R_{1\ell}^{(1)}(\xi_0)-R_{1\ell}^{(4)}(\xi_0)\frac{\xi_0R_{1\ell}^{(1)}(\xi_0)+(\xi_0^2-1)R_{1\ell}^{(1)\prime}(\xi_0)}{\xi_0R_{1\ell}^{(4)}(\xi_0)+(\xi_0^2-1)R_{1\ell}^{(4)\prime}(\xi_0)}\right]. \tag{30}$$

Equation (30) can be simplified by use of the Wronskian relation (Flammer, 1957a)

$$R_{1\ell}^{(1)}(\xi_0)R_{1\ell}^{(2)\prime}(\xi_0)-R_{1\ell}^{(1)\prime}(\xi_0)R_{1\ell}^{(2)}(\xi_0)=\frac{1}{\beta F(\xi_0^2-1)}, \tag{31}$$

to yield

$$I(\eta)=-j\pi\frac{E_0(\xi_0^2-1)^{1/2}(1-\eta^2)^{1/2}}{\beta R_c}\sum_{\ell=1,3,5,\ldots}^{\infty}\frac{A_{1\ell}S_{1\ell}(\eta)}{[\xi_0R_{1\ell}^{(4)}(\xi_0)+(\xi_0^2-1)R_{1\ell}^{(4)\prime}(\xi_0)]} \tag{32}$$

for the axial current distribution. Since $S_{1\ell}(+1)=0$, the current vanishes at the ends of the prolate spheroid.

4. Prolate Spheroidal Antenna

It is of interest to see how the spheroidal coordinates relate to the antenna dimensions:

$$\left.\begin{aligned}
&h=\frac{1}{2}\text{ antenna length}=F\xi_0\\
&a=\text{antenna midsection radius}=F(\xi_0^2-1)^{1/2}\\
&\text{and}\quad \frac{h}{a}=\frac{\xi_0}{(\xi_0^2-1)^{1/2}};\qquad \xi_0=\frac{\dfrac{h}{a}}{\left[\left(\dfrac{h}{a}\right)^2-1\right]^{1/2}}
\end{aligned}\right\}. \tag{33}$$

The structural shape parameter Ω common to cylindrical antenna theory is

$$\Omega=2\,\ell n\,\frac{2h}{a}$$

$$\Omega=2\left[\ell n\,(2\xi_0)-\frac{1}{2}\,\ell n\left(\xi_0^2-1\right)\right]. \tag{34}$$

The prolate spheroid was treated by Chu and Stratton (1941) as a driven antenna obtaining its input impedance and the driven current distribution. The antenna is considered to be ϕ-symmetrically driven by a delta gap voltage V. The driven current distribution obtained is

$$I_d(\eta)=-j\frac{\pi}{2}\frac{V\beta F}{R_c}(\xi_0^2-1)\,(1-\eta^2)^{1/2}\sum_{\ell=0}^{\infty}\frac{A_{1\ell}\,j^{-\ell}R_{1\ell}^{(4)}(\xi_0)S_{1\ell}(\eta)}{\xi_0R_{1\ell}^{(4)}(\xi_0)+(\xi_0^2-1)R_{1\ell}^{(4)\prime}(\xi_0)}, \tag{35}$$

and the input impedance obtained is

$$Z_0=\frac{V}{I_d(0)}=j\frac{2}{\pi}\frac{R_c}{\beta F(\xi_0^2-1)}\left\{\sum_{\ell=0}^{\infty}\frac{A_{1\ell}\,j^{-\ell}R_{1\ell}^{(4)}(\xi_0)S_{1\ell}(0)}{[\xi_0R_{1\ell}^{(4)}(\xi_0)+(\xi_0^2-1)R_{1\ell}^{(4)\prime}(\xi_0)]}\right\}^{-1}. \tag{36}$$

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Chu and Stratton point out that the series in (35) for $\eta=0$, and the series in (36), do not converge for any ξ_0 because of the assumption that the antenna is driven by a delta gap voltage. They carefully state that the introduction of a finite driving gap considerably decreases the higher order terms of the series and fortunately produces no appreciable change for the lower order terms. This is shown to be true by the results of Flammer (1957b) who obtains, using a variational method, the input admittance of a prolate spheroidal monopole antenna driven by a coaxial line. His results show good agreement with the results of Chu and Stratton when the series in (36) is truncated after four terms. However, for large values of βF and $\Omega < 9.9$, Flammer's input admittance in both real and imaginary parts is less than that obtained by Chu and Stratton. Because of the simplicity of the expressions Chu and Stratton give, only their work is presented in this paper.

A knowledge of the current at the center of the receiving antenna, $I(0)$, and the input impedance Z_0 of the transmitting antenna completely determines the receiving characteristics of the antenna when it is loaded by an impedance, Z_L at $\eta=0$. The equivalent circuit of the receiving antenna is the load impedance Z_L in series with Z_0 and the ideal voltage source $V_{oc}=I(0)Z_0$ accounting for the excitation of the antenna.

It is of interest to obtain simplifying special forms of (32). To that end suppose that $\beta F = n\pi/2$ where n is some odd integer, then, according to Flammer (1957a),

$$S_{1\ell}(\eta)=S_{1\ell}(0)\frac{\cos\frac{\eta\pi}{2}}{(1-\eta^2)^{1/2}}. \tag{37}$$

Substituting (37) into (32) yields

$$I(\eta)=I(0)\cos\frac{\eta\pi}{2}, \tag{38}$$

or

$$I(z)=I(0)\cos\frac{\pi}{2}\frac{z}{h}. \tag{39}$$

Note that (39) is the predicted induced current distribution occurring on a very thin cylindrical antenna that is illuminated as the spheroid.

5. Numerical Results

Fortunately, the series solution for the current distribution (32) is highly convergent. In fact, using only the first four terms of the series will yield results that are accurate to four significant figures. By using the wave functions tabulated by Flammer (1957a) a few center currents were computed and presented in table 1. For a direct comparison the corresponding currents in a cylindrical tube are also presented in table 1. The data for the cylindrical tube were obtained using the theory of Harrison et al. (1967). It must be expected that the cylinder theory does not hold for the smaller Ω because it considers (as does all existing cylindrical receiving antenna theory) the cylinder to be sufficiently thin so that the incident radiation is azimuthally symmetric. Also it considers that the cylinder is an infinitesimally thin, perfectly conducting cylindrical tube. However, this is also the model taken in all other cylindrical receiving antenna theory. Evidently when these considerations are no longer valid it will be necessary to use the prolate spheroid theory.

The current distributions induced on cylinders and corresponding prolate spheroids are shown in figure 2. Although extended past its theoretical limits of validity, the cylindrical antenna theory is shown to yield reasonably accurate results for the magnitudes of the currents. However, there is considerable error in the computed phases.

TABLE 1. *Short circuit current in milliamperes for* $E_0 = \beta$

βh	Prolate spheroid I(0)	Cylindrical antenna I(0)
	$\Omega = 9.90$	$\Omega = 9.90$
1.200	1.662 + j7.306	3.852 + j10.45
1.571	27.37 − j0.04285	17.96 − j12.20
2.000	7.200 − j10.86	5.431 − j9.754
2.356	4.633 − j9.001	4.023 − j8.470
2.500	4.232 − j8.640	3.791 − j8.255
2.800	3.787 − j8.288	3.543 − j8.143
3.000	3.640 − j8.278	3.469 − j8.279
3.142	3.573 − j8.365	3.436 − j8.478
3.200	3.551 − j8.425	3.424 − j8.590
	$\Omega = 5.31$	$\Omega = 5.31$
1.212	8.136 + j14.39	27.72 + j2.128
1.586	27.37 − j0.2501	16.60 − j10.95
2.020	18.23 − j10.08	12.82 − j10.91
2.380	15.19 − j10.55	12.34 − j11.13
2.525	14.65 − j10.60	12.36 − j11.44
2.828	14.11 − j10.85	12.43 − j12.63
3.030	13.99 − j11.22	12.23 − j14.01
3.173	13.94 − j11.64	11.69 − j15.35
3.232	13.91 − j11.85	11.30 − j15.99

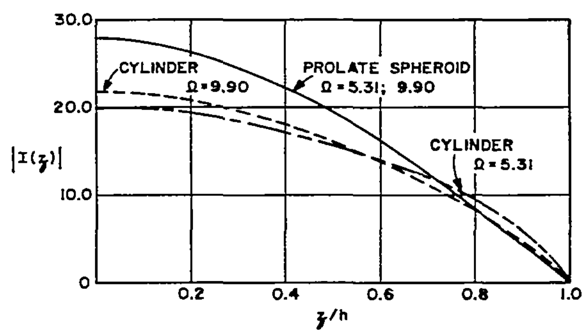

FIGURE 2. *Comparing the current distributions induced on cylinders and spheroids.*

$E_0 = \beta$, and the ordinate is in units of milliamperes per volt. When $\Omega = 9.90$, $\beta h = 1.571$ and $\Omega = 5.31$ $\beta h = 1.586$.

6. Conclusion

It has been shown that the axial current distribution induced on a prolate spheroid by a plane-wave field can be obtained in exact form. The resulting expression is in terms of the complicated spheroidal wave functions, but these have been tabulated to a large extent.

The main applications of the work presented here are in receiving antenna theory and missile vulnerability studies. This theory is of particular interest since there exists no cylindrical antenna theory that holds for thick structures, i.e., $h/a \leq 16.5$. To aid in the application of the results, an analytically simple limiting form of the induced axial current distribution is presented.

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