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The Tapered Antenna and its Application to the Junction Problem for Thin Wires

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Abstract - When electrically thin conductors of different cross-sectional size meet, the continuity of current is assured by Kirchhoff's current law. Additional conditions must be imposed on the derivatives of the currents or the charges per unit length. The nature of the required conditions is determined from an analysis of the tapered antenna.

1. INTRODUCTION

In order to determine the total axial currents in n confluent electrically thin conductors with different radii when excited by an externally maintained electromagnetic field, n conditions must be imposed on the currents and their axial derivatives at the junction. In general, these are Kirchhoff's current law and (n - 1) conditions on the derivatives. Since the total current in a typical conductor i obeys the equation of continuity (conservation of electric charge), viz.,

$$\frac{dI_i}{ds} + j\omega q_i = 0 \tag{1}$$

where s is the variable along the axis of the conductor, a condition on dI_i/ds is equivalent to one on q_i. With rotational symmetry, $q_i = 2\pi a_i \eta_i$ where q_i is the charge per unit length and η_i is the surface density of charge. It has been assumed by some investigators [1], [2] that the appropriate condition at the junction is the continuity of the surface density of charge in the form $\eta_i = q_i/2\pi a_i$. It follows that with $\eta_1 = \eta_2 = \ldots = \eta_i = \ldots = \eta_n$ at a junction, the charge per unit length on each conductor as the junction is approached, viz., $q_i = 2\pi a_i \eta_i$, is proportional to the surface area $2\pi a_i$ of the conductor in question. The corresponding conditions on the derivatives of the currents are:

$$a_1^{-1}(\partial I_1/\partial s_1) = a_2^{-1}(\partial I_2/\partial s_2) = \ldots = a_i^{-1}(\partial I_i/\partial s_i) \ldots \tag{2a}$$

An alternative condition would require the charges per unit length (and not the surface densities) to be the same. That is, $q_1 = q_2 = \ldots = q_i = \ldots = q_n$ at the junction. This is equivalent to

$$\partial I_1/\partial s_1 = \partial I_2/\partial s_2 = \ldots = \partial I_i/\partial s_i \ldots \tag{2b}$$

Note that the two conditions (2a) and (2b) are the same only when all radii are equal.

Which of these conditions is correct? Or is perhaps neither right? An answer to these questions is sought from an investigation of the behavior of the charge per unit length along a tapered antenna when driven at an arbitrary point by a delta-function generator. How do the quantities q and η vary with the radius of the wire, r(z), when this is not a constant?

2. DIFFERENTIAL EQUATION FOR THE CURRENT IN A TAPERED ANTENNA

Consider a long tapered conductor with its axis along the z-axis of the cylindrical system (r,φ,z). A section of the conductor is shown in Fig. 1. Since rotational symmetry obtains, the tangential electric field on the surface of the conductor due to the currents and charges in it has the components E_r and E_z in the combination

$$E_{tang} = E_z \cos\theta + E_r \sin\theta \tag{3}$$

where θ is the angle between the tapered surface and the axis as shown in Fig. 1. It is related to the varying radius r(z) by the relation

$$\lim\left(\frac{\Delta r}{\Delta z}\right) = \tan\theta = \partial r(z)/\partial z \equiv r'(z) \tag{4}$$

The components of the electric field can be expressed in terms of the components of the vector potential $\vec{A}(\nabla \times \vec{A} = \vec{B})$ as follows:

$$E_z = -\frac{j\omega}{k^2}\left[\frac{1}{r}\frac{\partial}{\partial r} r\left(\frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z}\right)\right] \tag{5a}$$

$$E_r = \frac{j\omega}{k^2}\left[\frac{\partial}{\partial z}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\right] \tag{5b}$$

where A_z depends upon the large axial current I_z, A_r on the small radial com-

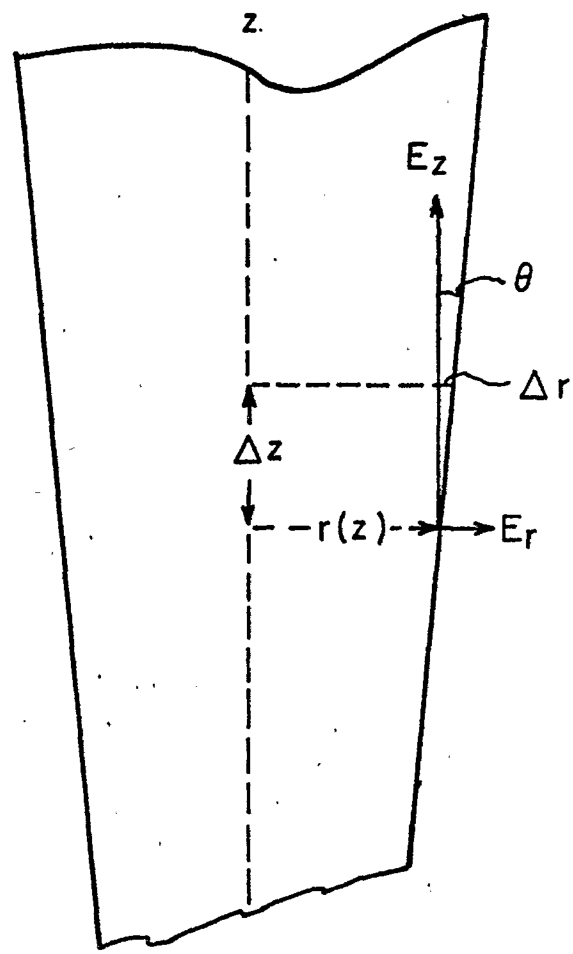

FIG. 1 SECTION OF TAPERED ANTENNA

ponent I_r. Since

$$|A_r| << |A_z| \tag{6}$$

it follows that

$$E_z \doteq -\frac{j\omega}{k^2}\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial A_z}{\partial r}\right) = \frac{j\omega}{k^2}\left(\frac{\partial^2}{\partial z^2} + k^2\right)A_z \tag{7a}$$

$$E_r \doteq -\frac{j\omega}{k^2}\frac{\partial^2 A_z}{\partial z \partial r} \tag{7b}$$

where the form on the right in (7a) is a consequence of the fact that $(\nabla^2 + k^2)A_z = 0$ at all points. With (3) through (7) it follows that

$$E_{tang} = -\frac{j\omega}{k^2} \cos\theta \left[\left(\frac{\partial^2}{\partial z^2} + k^2\right)A_z + \frac{\partial^2 A_z}{\partial z \partial r} r'(z)\right] \tag{8}$$

Except at the generator, $E_{tang} = 0$ so that

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)A_z + \frac{\partial^2 A_z}{\partial z \partial r} r'(z) = 0 \tag{9}$$

It is important to note that the partial derivative $\partial A_z/\partial z$ does not represent the total rate of change of A_z with respect to z. The operator for the total derivative is

$$d/dz = \partial/\partial z + r'(z)\, \partial/\partial r \tag{10}$$

When the conductor is electrically thin so that for all z in the range $-h_1 \leq z \leq h_2$ the following inequalities are satisfied:

$$kr(z) << 1 \quad , \quad (h_1 + h_2) >> r(z) \tag{11}$$

with (11) advantage can be taken of the special properties of the real and imaginary parts of the quantity $e^{-jkR}/kR = (\cos kR - j \sin kR)/kR$, where $R = [(z - z')^2 + r^2(z)]^{1/2}$, to simplify the z-component of the vector potential, viz., $A_z = A_{zR} + jA_{zI}$. The two parts (of which each is complex) are defined by:

$$A_{zR} = \frac{\mu_0}{4\pi} \int_{-h_1}^{h_2} I(z') \frac{\cos kR}{R} dz' \quad ; \quad A_{zI} = - \frac{\mu_0}{4\pi} \int_{-h_1}^{h_2} I(z') \frac{\sin kR}{R} dz' \tag{12}$$

In the first integral the integrand has a sharp, high peak when $z' = z$ and $(\cos kR)/R \to 1/kr(z) >> 1$. It follows that with this weighting factor it is a good approximation to set

$$A_{zR} \doteq \frac{\mu_0}{4\pi} [\Psi(z)I(z)] \tag{13}$$

where, as shown in the Appendix, $\Psi(z)$ is a proportionality factor that depends only on the radius of the wire at and near the point z. Specifically,

$$\Psi(z) = 2\{\ell n[2/kr(z)] - \gamma\} \tag{14}$$

with $\gamma = 0.5772$. The corresponding formula for a wire of constant radius with $r(z) = a$ is in the literature [3]. In the second integral in (12) the radius of the conductor can be neglected with the result

$$A_{zI} \doteq \frac{\mu_0}{4\pi} \int_{-h_1}^{h_2} I(z') \frac{\sin k(z - z')}{z - z'} dz' \tag{15}$$

which is independent of the radius of the wire.

It is shown in the Appendix that with (10)

$$\frac{4\pi}{\mu_0} [\frac{\partial^2}{\partial z^2} + k^2]A_{zR} = [I''(z) + k^2I(z)]\Psi(z) \tag{16}$$

$$\frac{4\pi}{\mu_0}\frac{\partial^2 A_{zR}}{\partial z\,\partial r} = -2I'(z)/r(z) \tag{17}$$

$$\frac{4\pi}{\mu_0}\left[\frac{\partial^2}{\partial z^2} + k^2\right]A_{zI} = k^2 g(z) \tag{18}$$

where

$$g(z) = 2\int_{-k(h_1+z)}^{k(h_2-z)} I(u,z)\left[\frac{u\cos u - \sin u}{u^3}\right]du \tag{19}$$

for an antenna that extends from $-h_1$ to h_2.

When these quantities are substituted in (9), the resulting equation is:

$$[I''(z) + k^2 I(z)]\Psi(z) - 2[r'(z)/r(z)]I'(z) + jk^2 g(z) = 0 \tag{20}$$

However, with (14), $d\Psi/dz \equiv \Psi'(z) = -2r'(z)/r(z)$ so that

$$I''(z) + [\Psi'(z)/\Psi(z)]I'(z) + k^2 I(z) = -jk^2 g(z)/\Psi(z) \tag{21}$$

This is the differential equation for the zero-order current in the tapered wire.

3. THE CHARGE PER UNIT LENGTH AND ITS SCALE FACTOR

The charge per unit length is related to the current through the equation of continuity (1). Thus, with $I'(z) = -j\omega q(z)$, (21) becomes

$$q'(z)\Psi(z) + \Psi'(z)q(z) = (jk^2/\omega)[I(z)\Psi(z) - jq(z)] \tag{22}$$

In order to determine the local behavior of the zero-order charge per unit length near any point z, the near-zone or quasi-stationary approximation is adequate. This is obtained from (22) by neglecting the terms with $k^2 = \omega^2/c^2$ as a coefficient. The result is the simple differential equation

$$q'(z)\Psi(z) + \Psi'(z)q(z) \doteq 0 \tag{23}$$

or

$$\frac{d}{dz}[q(z)\Psi(z)] \doteq 0 \tag{24}$$

This has the solution

$$q(z)\Psi(z) = \text{constant.} \tag{25}$$

This is an important result. It indicates that the radius-dependent scale factor for the charge per unit length along a tapered conductor is $\Psi^{-1}(z)$ where $\Psi(z) = 2\{\ell n[2/kr(z)] - \gamma\}$. That is, $q(z) \sim \text{constant}/\Psi(z)$.

An informative application of the constancy of the quantity $q(z)\Psi(z)$ is to a conductor that has the radius a_1 from $z = -h_1$ to $z = 0$ and the radius a_2 from $z = 0$ to $z = h_2$. Over the range $-h_1 \leq z \leq -5a_1$, the radius-dependent scale factor for the charge per unit length is Ψ_1^{-1} where $\Psi_1 = 2[\ell n(2ka_1) - \gamma]$; over the range $5a_2 \leq z \leq h_2$, the factor is Ψ_2^{-1} where $\Psi_2 = 2[\ell n(2/ka_2) - \gamma]$. In the electrically very short range $-5a_1 \leq z \leq 5a_2$, the scale factor changes continuously from Ψ_1^{-1} to Ψ_2^{-1} with most of the change occurring quite near $z = 0$. It follows that no significant error is introduced in a zero-order approximation if the scale factor Ψ_1^{-1} is used in the range $-h_1 \leq z \leq 0$ and the factor Ψ_2^{-1} in the range $0 \leq z \leq h_2$ with a discontinuous change at $z = 0$ instead of a rapid by continuous one. Thus, the condition (25) can be expressed as follows for the junction between two sections of conductor with radii a_1 and a_2 and a step at $z = 0$:

$$q_1\Psi_1 = q_2\Psi_2 \tag{26}$$

where $q_1 = q_1(-5a_1) \doteq q_1(0)$; $q_2 = q_2(5a_2) \doteq q_2(0)$; and

$$\Psi_1 = 2[\ell n(2/ka_1) - \gamma] \quad , \quad \Psi_2 = 2[\ell n(2/ka_2) - \gamma] \tag{27}$$

For example, when $ka_1 = .01$, $ka_2 = .02$,

$$\Psi_1 = 2[\ln 200 - .5772] = 2[5.298 - .577] = 2 \times 4.721 = 9.442$$

$$\Psi_2 = 2[\ln 100 - .5772] = 2[4.605 - .577] = 2 \times 4.028 = 8.056$$

Evidently,

$$q_1/q_2 = \Psi_2/\Psi_1 = 0.85 \qquad (28)$$

4. GENERALIZATION AND CONCLUSION

The behavior of the charge per unit length along a tapered cylinder indicates that the product of the charge per unit length and the expansion parameter Ψ is the quantity that is constant in a change of radius. Since for electrically thin conductors the surface area of the junction itself is negligible and the transverse distribution of charge is unimportant, it may be concluded that in addition to Kirchhoff's current law the following conditions must be satisfied at a junction of n conductors with different radii:

$$q_1\Psi_1 = q_2\Psi_2 = \dots = q_i\Psi_i = \dots = q_n\Psi_n \qquad (29)$$

In (29) q_i is the charge per unit length on a conductor with radius a_i and $\Psi_i = 2[\ln(2/ka_i) - \gamma]$. When the conductors are all electrically very thin, all of the Ψ_i's may be so large that the relations (29) become approximately:

$$q_1 \doteq q_2 \doteq \dots \doteq q_i \doteq \dots \doteq q_n \qquad (30)$$

There appears to be no justification for the conditions (2a) or their equivalent:

$$\frac{q_1}{a_1} \doteq \frac{q_2}{a_2} \doteq \dots \doteq \frac{q_i}{a_i} \doteq \dots \doteq \frac{q_n}{a_n} \qquad (31)$$

5. REFERENCES

[1] J. L. Lin, W. L. Curtis, and M. C. Vincent, "Radar cross sections of a rectangular conducting plate by wire-mesh modeling," IEEE Trans. Antennas Propagat., vol. AP-22, pp. 718-720, Sept. 1974.

[2] F. M. Tesche, "Numerical considerations for the calculation of currents induced on intersecting wires using the Pocklington integro-differential equation," Interaction Notes, Note 150, January 1974.

[3] R. W. P. King and T. T. Wu, "Analysis of crossed wires in a plane-wave field," Interaction Notes, Note 216, July 1974.

6. APPENDIX

1. Evaluation of $\Psi(z)$

The function $\Psi(z)$ is defined as follows:

$$\Psi(z) = \int_{-h_1}^{h_2} \frac{\cos kR}{R} dz' = \int_{-k(h_1+z)}^{k(h_2-z)} \frac{\cos[U^2 + A^2(z)]^{1/2}}{[U^2 + A^2(z)]^{1/2}} dU \qquad \text{(A-1)}$$

where $A(z) = kr(z)$. The integral on the right can be separated into two parts as follows :

$$\Psi(z) = \int_{-k(h_1+z)}^{k(h_2-z)} \frac{dU}{[U^2 + A^2(z)]^{1/2}} - \int_{-k(h_1+z)}^{k(h_2-z)} \frac{1 - \cos[U^2 + A^2(z)]^{1/2}}{[U^2 + A^2(z)]^{1/2}} dU \qquad \text{(A-2)}$$

Since $A^2(z)$ is negligible everywhere except in the range $-5kr(z) \leq U \leq 5kr(z)$ near $U = 0$ in the first integral, (A-2) may be approximated as follows:

$$\Psi(z) = \int_{-5kr(z)}^{5kr(z)} dU/[U^2 + A^2(z)]^{1/2} + \left(\int_{-k(h_1+z)}^{-5kr(z)} + \int_{5kr(z)}^{k(h_2-z)} \right) dU/U - \mathrm{Cin}\, k(h_2 - z) - \mathrm{Cin}\, k(h_1 + z) \qquad \text{(A-3)}$$

where $\mathrm{Cin}\, x = {}_0\int^x [(1 - \cos U)/U]dU$. Since $r(z)$ is assumed to vary slowly over distances of the order of $5r(z)$, it is satisfactory to treat $A(z)$ as approximately constant over this small range in the first integral at the value where $U = 0$ and its contribution is greatest.

With these considerations the integrals reduce to

$$\Psi(z) = \ln[4(h_1 + z)(h_2 - z)/r^2(z)] - \mathrm{Cin}\, k(h_2 - z) - \mathrm{Cin}\, k(h_1 + z) \qquad \text{(A-4)}$$

If it is now assumed that the conductors are sufficiently long so that over most of their extension except close to the ends the conditions

$$k(h_1 + z) \geq \pi/2 \quad , \quad k(h_2 - z) \geq \pi/2 \qquad \text{(A-5)}$$

obtain, the following approximations can be made:

$$\text{Cin } k(h_2 - z) \doteq \gamma + \ln k(h_2 - z) \tag{A-6}$$

$$\text{Cin } k(h_1 + z) \doteq \gamma + \ln k(h_1 + z) \tag{A-7}$$

When these are substituted in (A-4), the final result is

$$\Psi(z) = 2\{\ln[2/kr(z)] - \gamma\} \tag{A-8}$$

which is independent of the length of the conductor.

2. Evaluation of derivatives

With (10), the following derivatives are readily evaluated:

$$\frac{4\pi}{\mu_0}\frac{\partial A_{zR}}{\partial z} = \frac{4\pi}{\mu_0}\left[\frac{d}{dz} - r'(z)\frac{\partial}{\partial r}\right]A_{zR} = \left[\frac{d}{dz} - r'(z)\frac{\partial}{\partial r}\right][I(z)\Psi(z)]$$

$$= I(z)\Psi'(z) + I'(z)\Psi(z) - r'(z)\frac{\partial\Psi(z)}{\partial r}$$

$$= -2I(z)[r'(z)/r(z)] + I'(z)\Psi(z) + 2I(z)[r'(z)/r(z)] = I'(z)\Psi(z) \tag{A-9}$$

Similarly,

$$\frac{4\pi}{\mu_0}\frac{\partial^2 A_{zR}}{\partial z^2} = \left[\frac{d}{dz} - r'(z)\frac{\partial}{\partial r}\right]I'(z)\Psi(z) = I''(z)\Psi(z) \tag{A-10}$$

Also,

$$\frac{4\pi}{\mu_0}\frac{\partial A_{zR}}{\partial r} = I(z)\frac{\partial\Psi(z)}{\partial r} = -\frac{2I(z)}{r(z)} \tag{A-11}$$

$$\frac{4\pi}{\mu_0}\frac{\partial^2 A_{zR}}{\partial z\,\partial r} = \left[\frac{d}{dz} - r'(z)\frac{\partial}{\partial r}\right]\left[-2\frac{I(z)}{r(z)}\right] = -\frac{2I'(z)}{r(z)} \tag{A-12}$$

It follows that

$$\frac{4\pi}{\mu_0}\left(\frac{\partial^2}{\partial z^2} + k^2\right) A_{zR} = \left[I''(z) + k^2 I(z)\right]\Psi(z) \tag{A-13}$$

Since A_{zI} is independent of r, $\partial A_{zI}/\partial r = 0$ and (15) gives:

$$\frac{4\pi}{\mu_0}\frac{\partial A_{zI}}{\partial z} = -\int_{-h_1}^{h_2} I(z')\left[\frac{k \cos k(z - z')}{z - z'} - \frac{\sin k(z - z')}{(z - z')^2}\right]dz' \tag{A-14}$$

$$\frac{4\pi}{\mu_0}\frac{\partial^2 A_{zI}}{\partial z^2} = -\int_{-h_1}^{h_2} I(z')\left[-\frac{k^2 \sin k(z - z')}{z - z'} - \frac{2 \cos k(z - z')}{(z - z')^2} + \frac{2 \sin k(z - z')}{(z - z')^3}\right] dz' \tag{A-15}$$

It follows that

$$\frac{4\pi}{\mu_0}\left(\frac{\partial^2}{\partial z^2} + k^2\right) A_{zI} = 2\int_{-h_1}^{h_2} I(z')\left[\frac{k \cos k(z - z')}{(z - z')^2} - \frac{\sin k(z - z')}{(z - z')^3}\right] dz' \tag{A-16}$$

With the substitution $u = k(z - z')$, this becomes

$$\frac{4\pi}{\mu_0}\left(\frac{\partial^2}{\partial z^2} + k^2\right) A_{zI} = k^2 g(z) \tag{A-17}$$

where

$$g(z) = 2\int_{-k(h_1+z)}^{k(h_2-z)} I(u)\left[\frac{u \cos u - \sin u}{u^3}\right] du \tag{A-18}$$