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An Application of Symmetrization to EMP Penetration Through Apertures

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Abstract

Upper and lower bounds for the transmission coefficient of an electrically-small planar aperture are established by an application of symmetrization.

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I. INTRODUCTION

Many of the boundary-value problems that arise in EMP shielding theory involve shield configurations that are difficult, if not impossible, to handle analytically. In principle, such boundary-value problems can be solved numerically but they cannot be solved analytically unless the configurations happen to be simple enough to permit a separation of the variables and a scalarization of the electromagnetic field. However, from this we should not infer that if a boundary-value problem cannot be solved analytically, a numerical method is the only way to obtain a solution. Actually, as a preferable alternative, one can reformulate a boundary-value problem so that what would have to be sought would be a set of upper and lower bounds on the true solution and not the true solution itself. The sandwiching of solutions between upper and lower bounds is an old device which has yielded important results in geometry and mathematical physics [1] and has recently started to play a promising role in EMP shielding theory.

As a paradigm, we shall consider in this note the problem of determining the shielding effectiveness of a plane perfectly conducting thin wall with a small aperture of arbitrary shape. That is, we shall determine how much of the electromagnetic energy that exists on one side of the wall will leak through the aperture to the other side. From the works of Bouwkamp [2] and of Meixner and Andrejewski [3] we know what the fields and the leakage would be if the opening in the wall were a circular aperture. Using their analytical solutions as a point of departure we shall delimit the shielding effectiveness of the wall as the circular aperture is topologically transformed into differently shaped apertures.

II. SYMMETRIZATION

A transformation of a geometric figure is topological when adjacencies are not destroyed and no new adjacencies are created. In other words, under a topological transformation the parts of the figure that are in contact remain in contact and the parts that are not in contact remain apart. The distinguishing feature of a topological transformation is that neither breaks nor fusions can arise.

Symmetrization is a topological transformation, and of the several kinds of symmetrization that have been invented we shall restrict our attention to the simplest, i.e., symmetrization of a plane figure with respect to a straight line.

To symmetrize a plane figure with respect to a straight line L, we suppose the figure to consist of line segments that are parallel to each other and perpendicular to L (see Fig. 1). Then we shift each line segment along its own line until the line segment is bisected by L. The shifted line segments compose the symmetrized figure. For example, a semicircle of radius R, when symmetrized with respect to its bounding diameter, changes into an ellipse with semiaxes R and $R/2$.

Symmetrization leaves the area A unchanged and decreases, or, more accurately, never increases the perimeter P. Moreover, symmetrization of a plane conducting plate decreases (never increases) the electrostatic capacity of the plate.

A plane figure symmetrized infinitely many times yields a circle and, consequently, of all conducting plates of a given area the circular plate has the minimum capacity.

Accordingly, if C_o denotes the electrostatic capacity of a plane conducting plate and if C_{sym} denotes the electrostatic capacity of the circular plate that is obtained by completely symmetrizing the original plate, we get the inequality

$$C_o > C_{sym} . \qquad (1)$$

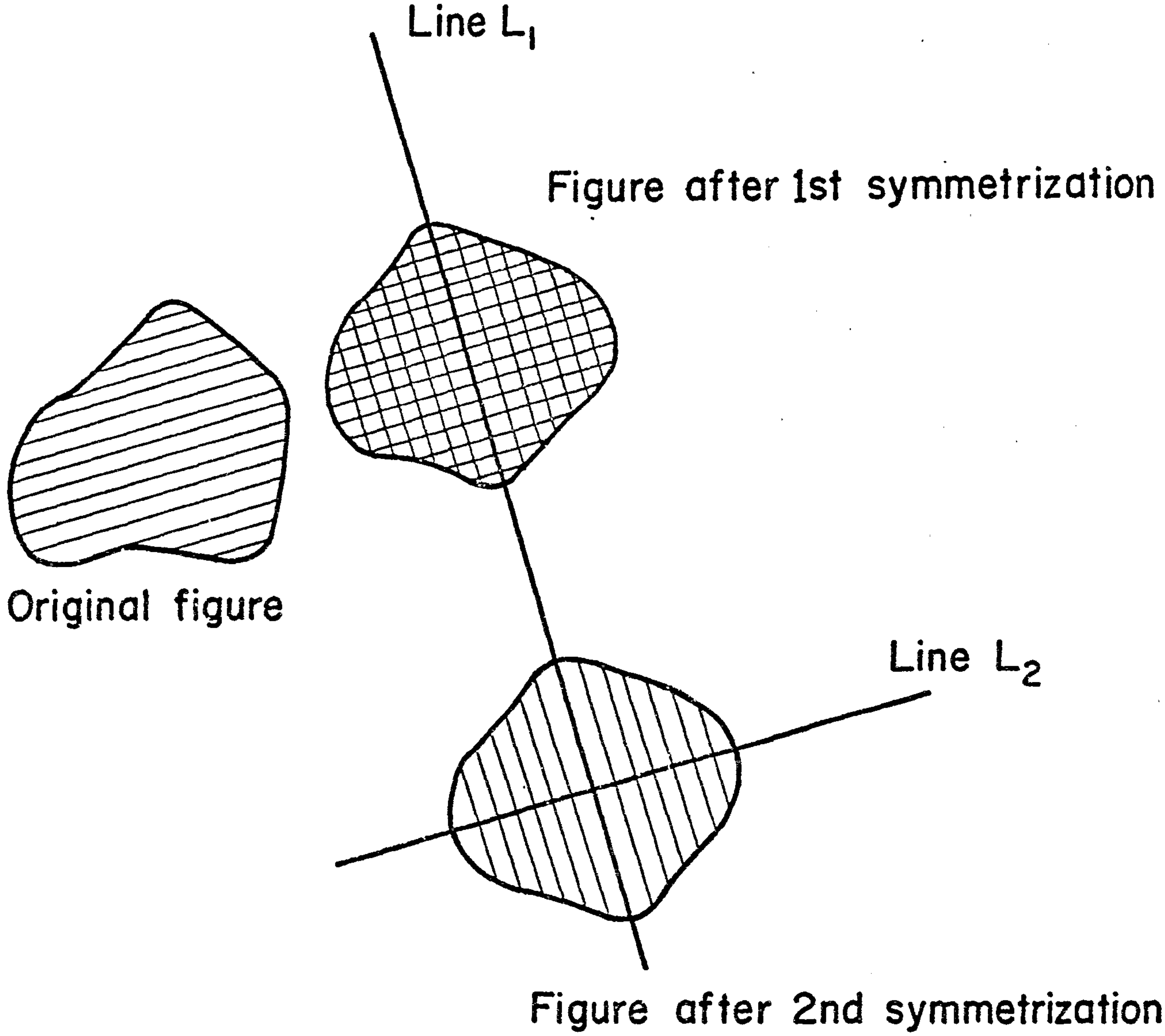

Figure 1. Symmetrization of a plane figure.

This places a lower bound on C_o. To obtain an upper bound we invoke the conjecture that of all plates with a given outer radius r_{out} the circular plate has the maximum capacity. Thus we have the inequality

$$C_{out} > C_o \tag{2}$$

where C_{out} is the electrostatic capacity of a circular plate whose radius is equal to the outer radius of the original plate. From (1) and (2) it follows that C_o is sandwiched between C_{out} and C_{sym} or

$$C_{out} > C_o > C_{sym} . \tag{3}$$

If A denotes the area of the original plate and if P denotes its perimeter, then C_{sym} is the electrostatic capacity of a circular plate whose radius is given by

$$r_{sym} = \sqrt{A/\pi} \tag{4}$$

and C_{out} is the electrostatic capacity of a circular plate whose radius r_{out} lies within the following bounds [4]

$$\sqrt{A/\pi} \le r_{out} \le P/2\pi \tag{5}$$

Since the electrostatic capacity of a circular plate (disk) in MKS units is given by

$$C = 8\varepsilon_o r \tag{6}$$

where r is the radius of the plate and ε_o is the dielectric constant of free space, it follows from (3), (4), and (5) that the capacitance C_o of a plate of area A and perimeter P satisfies

$$8\varepsilon_o P/2\pi > C_o > 8\varepsilon_o \sqrt{A/\pi} \qquad (7)$$

where $\varepsilon_o = 8.854 \times 10^{-12}$ farads per meter.

Let us now apply (7) to the case of an elliptic plate with semimajor axis a and semiminor axis b . For an ellipse, we know that

$$A = \pi ab \qquad (8)$$

and

$$P = 2\pi a \left(1 - \frac{1}{2^2} e^2 - \frac{3^2}{2^2 \cdot 4^2} \frac{e^4}{3} - \frac{3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \frac{e^6}{5} + \ldots\right) \qquad (9)$$

where the eccentricity e is given by

$$e = [(a^2 - b^2)/a^2]^{\frac{1}{2}} . \qquad (10)$$

Assuming that the eccentricity is small, we find from (8), (9), and (10) that

$$P = 2\pi \sqrt{A/\pi} \left(1 + \frac{3}{64} e^4 - \frac{2}{64} e^6 + \ldots\right) . \qquad (11)$$

It then follows from (7) and (11) that

$$8\varepsilon_o \sqrt{A/\pi} \left(1 + \frac{3}{64} e^4 - \frac{2}{64} e^6\right) > C_o > 8\varepsilon_o \sqrt{A/\pi} . \qquad (12)$$

Professor Smythe, using a different approach, has shown that [5]

$$C_o = 8\varepsilon_o \sqrt{A/\pi} \left(1 + \frac{e^4}{64} + \frac{e^6}{64}\right) \text{for } e << 1 . \qquad (13)$$

We see that our result (12) compares favorably with his result (13) when e is small.

III. TRANSMISSION THROUGH APERTURE IN ELECTRIC WALL

Suppose we have an electric wall with a circular aperture. Orienting a cartesian coordinate system (x,y,z) so that the center of the aperture is at (0,0,0) and the wall lies along the plane $z = 0$, we consider the question of how much power is transmitted through the aperture into the half-space $z \geq 0$ when a linearly polarized, time-harmonic, plane electromagnetic wave is incident on the aperture.

For normal incidence it is known that the transmission coefficient t is given by [2,3]

$$(t)_{\text{circle}} = \frac{64}{27\pi^2} (ka)^4 \left[1 + \frac{22}{25} (ka)^2 + \frac{7312}{18375} (ka)^4 + \ldots \right] \tag{14}$$

where a = radius of aperture , $k = \omega/c$, and $ka << 1$.

If now the circular aperture is deformed into an ellipse with semimajor axis a and semiminor axis b , we see from (5), (8), and (9) that

$$P = 2\pi \sqrt{A/\pi} \left(1 + \frac{3}{64} e^4 \right) \quad \text{for} \quad e << 1$$

and hence

$$\sqrt{A/\pi} \leq r_{out} \leq \sqrt{A/\pi} \left(1 + \frac{3}{64} e^4 \right) . \tag{15}$$

Substituting r_{out} for a in (14) we get

$$\frac{64}{27\pi^2} k^4 (A/\pi)^2 \leq (t)_{\text{ellipse}} \leq \frac{64}{27\pi^2} k^4 (A/\pi)^2 \left(1 + \frac{3}{16} e^4 \right) . \tag{16}$$

The question now arises as to whether or not $(t)_{\text{ellipse}}$ is sensitive to changes in polarization. We know that if natural light is incident on a very narrow slit, the transmitted field is polarized perpendicular to the slit. From this we infer that $(t)_{ellipse}$ for polarization parallel to the minor axis is greater than $(t)_{ellipse}$ for polarization parallel to the major axis. This suggests that the upper bound in (16) applies to the case of polarization parallel to the minor axis.

If the circular aperture is deformed into a square whose area is given by $A = 4a^2$, we see from (5) that for the square

$$\sqrt{A/\pi} \le r_{out} \le (2/\sqrt{\pi})\sqrt{A/\pi} \ . \tag{17}$$

Substituting r_{out} for a in (14) we get

$$\frac{64}{27\pi^2} k^4 (A/\pi)^2 \le (t)_{square} \le \frac{64}{27\pi^2} k^4 (A/\pi)^2 (1.622) \ . \tag{18}$$

Similarly, for an equilateral triangle of side a, we see that

$$\sqrt{A/\pi} \le r_{out} \le \frac{3^{3/4}}{\sqrt{\pi}} \sqrt{A/\pi} = 1.286 \sqrt{A/\pi}$$

and hence

$$\frac{64}{27\pi^2} k^4 (A/\pi)^2 \le (t)_{triangle} \le \frac{64}{27\pi^2} k^4 (A/\pi)^2 (2.736) \tag{19}$$

These examples support the assertion that the transmission through a small aperture $(k^2 A \ll 1)$ decreases as the shape of the aperture is changed from that of an equilateral triangle to that of a square, an ellipse, and finally a circle. That is, the transmission decreases as the shape of the aperture approaches a circle.

IV. FINAL REMARKS

The above calculations are based on an exact solution (the solution for a circular aperture) and on certain global inequalities.

To generalize the applicability of this method of estimation we must direct our efforts toward the development of new inequalities and the construction of new exact solutions.

Such a generalization of the method would be most useful in the design of EMP shields and accordingly deserves serious attention.

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[4] G. Pólya and G. Szegö, Aufgaben und Lehrsätze aus der Analysis, Berlin, Vol. 2, 1925, p.21.

[5] W.R. Smythe, Static and Dynamic Electricity, Third edition, McGraw-Hill, New York, 1968, p.228, problem 48C.