Interaction Notes

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Propagation Characteristics of a Periodically Loaded Transmission Line

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Abstract

A cable passing over periodic obstacles, such as existing in an aircraft's interior, can be modeled by a periodically loaded transmission line. This report calculates the propagation characteristics of a line loaded at regular intervals with identical, symmetrical T sections. The dispersion relation determining the possible modes of wave excitation on the line is derived, from which comprehensive information on the propagation constant, the passband-stopband structures, and the phase and group velocities is obtained. The results are illustrated numerically and graphically by working out a typical example.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

SECTION I

INTRODUCTION

At sufficiently low frequencies when only the TEM mode is dominant, the propagation of voltage and current disturbances along a uniform cable can be described by the pair of transmission line equations:

$$\frac{\partial V}{\partial x} = - L \frac{\partial I}{\partial t} - RI \tag{1}$$

$$\frac{\partial I}{\partial x} = - C \frac{\partial V}{\partial t} - GV \tag{2}$$

For a time-harmonic excitation with time factor $\exp(j\omega t)$, these equations become

$$\frac{dV}{dx} = - ZI \tag{3}$$

$$\frac{dI}{dx} = - YV \tag{4}$$

where

$$Z = j\omega L + R \tag{5}$$

$$Y = j\omega C + G \tag{6}$$

are the impedance and admittance per unit length of the uniform line at angular frequency ω . Excitations with general time variation can be resolved into time-harmonic components by the method of integral transform. Consequently their treatment can also be based on equations (3) and (4).

The uniform line is of course an idealization. In practice, non-uniformities occur in the shape of cable bends, shield defects, or nearby conductors. These produce local deviations of Y and Z from their uniform values, resulting in a scattering of the waves in the cable. If the dimensions of the non-uniformities are small compared to a wavelength, the scattering effects can be described by an equivalent point loading of the cable. Fig. 1 shows a model of the non-uniformity as a symmetrical T section. The network is assumed to have zero spatial extension, and is to be inserted into the uniform line at the location of the non-uniformity. The lumped impedance and admittance elements Z_d and Y_d of the discontinuity are to be calculated from appropriate quasi-static boundary-value problems. The general situation of a transmission line of finite length, driven at one end by a harmonic voltage or current source, terminated at the other by a given load, and loaded at an arbitrary configuration of points in between, can be analyzed by the powerful transmission matrix method to be described below in Section III.

Of special interest to aircraft EMP internal coupling studies is the analysis of wave propagation along a periodically loaded transmission line. Here the loads are identical in structure, and are inserted along the line at regular intervals. Examples of such periodic loading in aircraft cables can be found in periodically applied cable clamps and periodic airframe members over which the cable runs are anchored. It is well known that a periodically loaded line acts like a bandpass filter. The line transmits waves with frequencies lying within certain discrete bands, while waves with other frequencies are effectively stopped. It can therefore restrict the range of EMP energy that can be delivered through it to the load. Hence a determination of the passband-stopband structures as a function of the loading is extremely helpful to the assessment of EMP vulnerability of aircraft systems.

The objective of this report is to calculate the propagation characteristics of an infinitely long transmission line, loaded periodically with four-terminal networks of the type shown in Fig. 1. This study will consist of deriving the so-called dispersion relation connecting the frequency with the propagation constant, showing its explicit dependence on the loading period, and on the

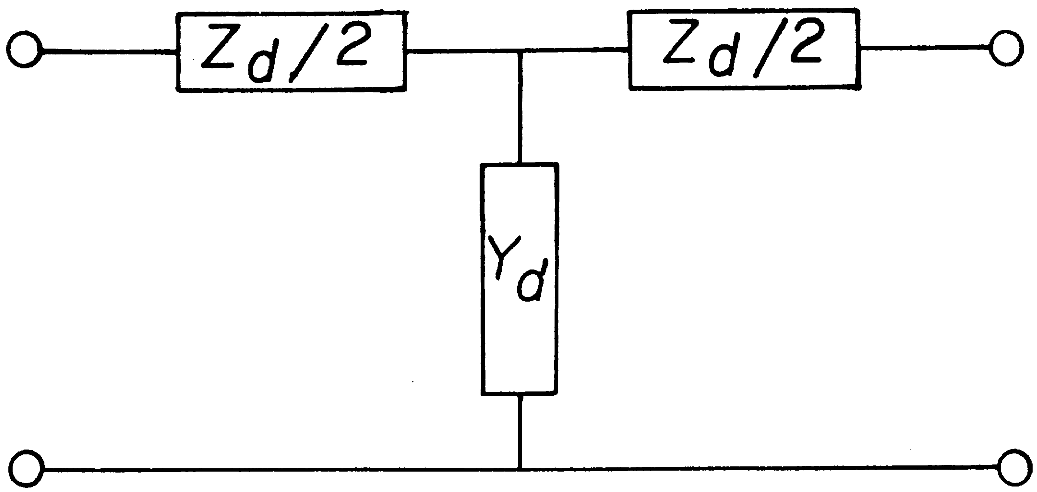

Figure 1. Model of a cable non-uniformity as a symmetrical T section. The network is to be loaded on the line at the location of the non-uniformity. The lumped impedance Z_d and admittance Y_d are to be calculated from two quasi-static boundary-value problems.

distributed and lumped impedances and admittances of the uniform line and its lumped loads. The dispersion relation governs the propagation of the eigenmodes of the infinite line. The reason for the interest in these infinite-line modes is that the voltage and current excitations on the finite line encountered in practice are composed of these eigenmodes. The composition is determined by taking the applied source and the load termination as boundary conditions at the two ends.

The contents of this report are distributed as follows. Section II summarizes the salient features of waves on a periodic line. The transmission matrix method for calculating wave propagation along transmission lines is introduced in Section III. In Sections IV, V and VI the dispersion relation, the passband-stopband structures and the phase and group velocities of a transmission line loaded periodically with symmetrical T sections are explicitly derived. The report concludes with an illustrative example worked out in detail in Section VII.

SECTION II

GENERAL PROPERTIES OF WAVES ON A PERIODIC LINE

The phenomenon of wave propagation in periodic structures occurs in many areas of physics and engineering. Over the years its principal features have been delineated and the general procedures for its study worked out. Before embarking on the specific calculations for the periodically loaded transmission line, it is advantageous to anticipate the results by summarizing below the general properties of waves in a one-dimensional periodic structure.

1. At a given frequency ω , the waves on a line of period a have the general form

$$\psi(x) = e^{-jkx} u_k(x) \qquad (7)$$

where k and $u_k(x)$ are determined by the detailed structure of the line, and $u_k(x)$ is a periodic function with period a . Thus the excitations are "plane waves" modulated by the periodicity of the line. This result is known in mathematics as Floquet's theorem.

2. Upon analyzing the periodic function $u_k(x)$ into a Fourier series

$$u_k(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(-\frac{j2n\pi x}{a}\right) \qquad (8)$$

one concludes from (7) that the excitations consist of an infinite sequence of spatial harmonics with propagation constants

$$k_n = k + \frac{2n\pi}{a}, \quad n = 0, \pm 1, \pm 2, \ldots \qquad (9)$$

Hence at each frequency, there are an infinite number of eigenmodes.

3. The frequency ω is an even periodic function of k. In fact, the dispersion relation is of the general form

$$\cos ka = F(\omega) \tag{10}$$

4. When there are no resistive elements in the line, there exist an infinite series of frequency bands in which k is alternately real and purely imaginary. The line is a bandpass filter.

These general results will be verified and illustrated by the calculations in the following sections.

SECTION III

TRANSMISSION MATRICES

Consider a length of uniform transmission line containing a number of point loads, as shown in Fig. 2. Each load is assumed to have the structure shown in Fig. 1. The variation of the voltage and current along the uniform sections is described by the transmission line equations (3) and (4). The variation in the loads is governed by Kirchhoff's laws.

At any reference plane along the line, such as plane A in Fig. 2, the voltage and current can be combined into a two-component vector:

$$\begin{pmatrix} V(A) \\ I(A) \end{pmatrix}$$

Then the voltage and current at any other reference plane in the same uniform section of the line, such as plane B in Fig. 2, are related to those at plane A through a transmission matrix T_u:

$$\begin{pmatrix} V(B) \\ I(B) \end{pmatrix} = T_u \begin{pmatrix} V(A) \\ I(A) \end{pmatrix} \tag{11}$$

This matrix is determined by solving equations (3) and (4). It is given by

$$T_u = \begin{pmatrix} \cosh \gamma a & -Z_o \sinh \gamma a \\ -\frac{1}{Z_o} \sinh \gamma a & \cosh \gamma a \end{pmatrix} \tag{12}$$

where a is the separation between planes A and B , and

$$Z_o = \sqrt{\frac{Z}{Y}} \quad , \quad \gamma = \sqrt{YZ} \tag{13}$$

are respectively the characteristic impedance and the propagation constant of

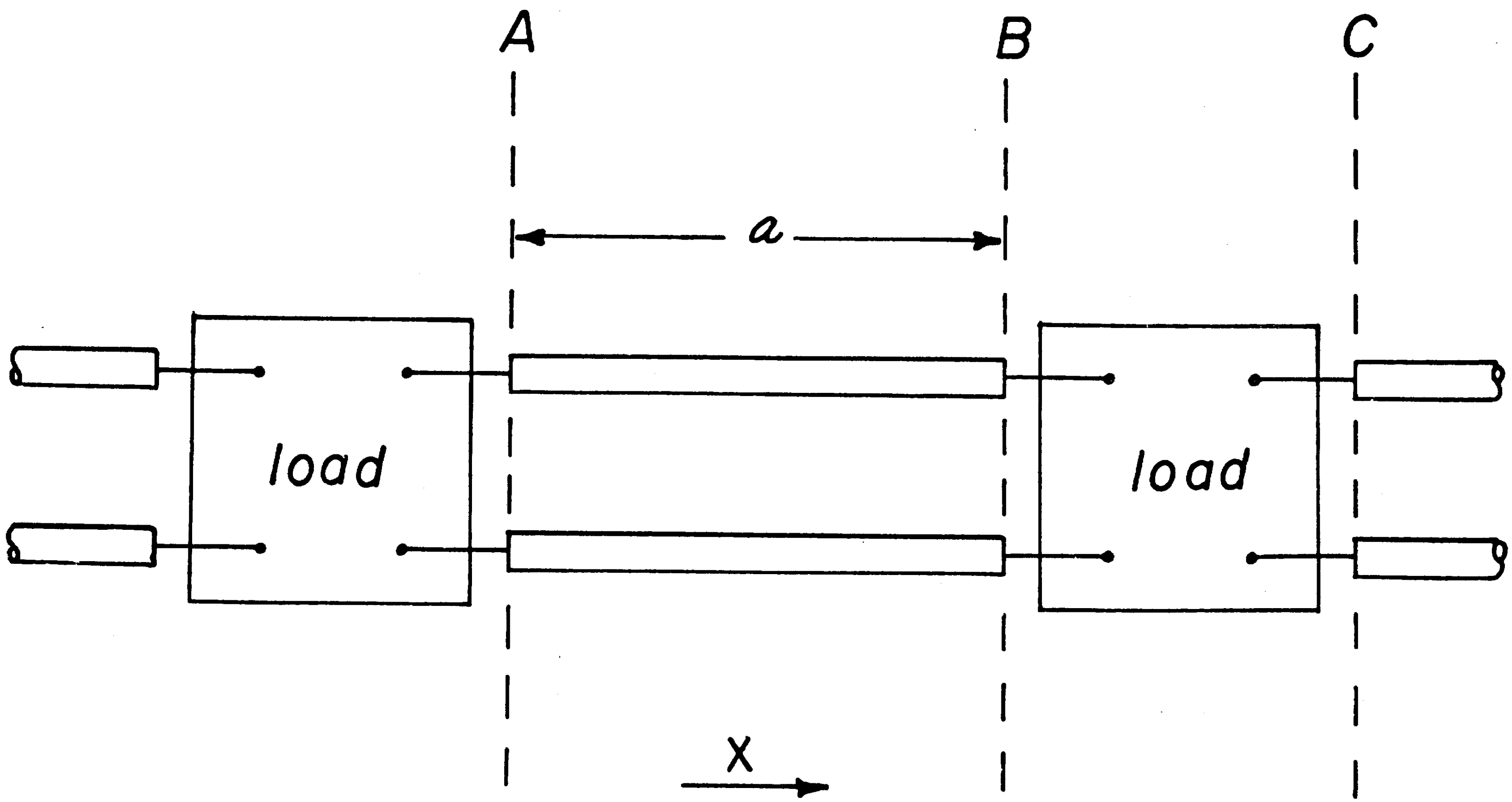

Figure 2. Section of a uniform transmission line loaded at discrete points with four-terminal networks.

the uniform section.

Similarly, the variation of the voltage and current across a four-terminal network, such as between reference planes B and C in Fig. 2, can be described by a transmission matrix T_d :

$$\begin{pmatrix} V(C) \\ I(C) \end{pmatrix} = T_d \begin{pmatrix} V(B) \\ I(B) \end{pmatrix} \tag{14}$$

Upon applying Kirchhoff's laws to Fig. 1, one finds that

$$T_d = \begin{pmatrix} 1 + \frac{1}{2}Y_dZ_d & -Z_d - \frac{1}{4}Y_dZ_d^2 \\ -Y_d & 1 + \frac{1}{2}Y_dZ_d \end{pmatrix} \tag{15}$$

It is to be noted that both T_u and T_d have determinant 1:

$$\det T_u = 1 \quad , \quad \det T_d = 1 \tag{16}$$

Matrices with this property are said to be unimodular.

The relation between the voltage and current at reference plane C and those at plane A can be obtained by matrix multiplication. Thus

$$\begin{pmatrix} V(C) \\ I(C) \end{pmatrix} = T \begin{pmatrix} V(A) \\ I(A) \end{pmatrix} \tag{17}$$

where

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = T_dT_u \tag{18}$$

with

$$T_{11} = (1 + \tfrac{1}{4}Y_dZ_d)\cosh\gamma a + \frac{Z_d}{Z_o}(1 + \tfrac{1}{4}Y_dZ_d)\sinh\gamma a$$

$$T_{22} = (1 + \tfrac{1}{4}Y_dZ_d)\cosh\gamma a + Y_dZ_o\sinh\gamma a$$

$$T_{12} = -Z_d(1 + \tfrac{1}{4}Y_dZ_d)\cosh\gamma a - Z_o(1 + \tfrac{1}{4}Y_dZ_d)\sinh\gamma a$$

$$T_{21} = -Y_d\cosh\gamma a - \frac{1}{Z_o}(1 + \tfrac{1}{4}Y_dZ_d)\sinh\gamma a \qquad (19)$$

It is easy to show that T is also unimodular:

$$\det T = T_{11}T_{22} - T_{12}T_{21} = 1 \qquad (20)$$

Using the transmission matrix method, the calculation of voltage and current propagation along a finite transmission line with discrete point loads becomes a simple matter of 2×2 matrix multiplications.

SECTION IV

DISPERSION RELATION

Suppose in Fig. 2 the transmission line section between reference planes A and C is repeated an infinite number of times to the right and to the left. One obtains in this way an infinite periodically-loaded transmission line of period a. If, at a given frequency ω, a wave is to propagate down this structure, the voltage and current in reference plane C can differ from those at plane A by at most a phase factor. This is because planes A and C are separated by exactly one period, and are hence physically equivalent. Let this phase factor be denoted by $\exp(-jka)$, that is,

$$\begin{pmatrix} V(C) \\ I(C) \end{pmatrix} = e^{-jka} \begin{pmatrix} V(A) \\ I(A) \end{pmatrix} \tag{21}$$

This result actually follows from Floquet's theorem (7), and k can be identified with the propagation constant of an eigenmode. Relation (21) is compatible with relation (17) only if the phase factor is an eigenvalue of the transmission matrix T. One therefore requires that

$$\begin{vmatrix} T_{11} - e^{-jka} & T_{12} \\ T_{21} & T_{22} - e^{-jka} \end{vmatrix} = 0 \tag{22}$$

This condition, by (20), becomes

$$e^{-j2ka} - (T_{11} + T_{22})e^{-jka} + 1 = 0 \tag{23}$$

The solution is

$$\cos ka = \tfrac{1}{2}(T_{11} + T_{22}) \tag{24}$$

Upon substitution of the matrix elements from (19), equation (24) becomes

$$\cos ka = (1 + \tfrac{1}{2}Y_d Z_d)\cosh \gamma a + \frac{1}{2Z_o}(Z_d + \tfrac{1}{4}Y_d Z_d^2 + Y_d Z_o^2)\sinh \gamma a \qquad (25)$$

The quantities Y_d , Z_d , Z_o and γ are all functions of the frequency ω . Therefore (25) is a dispersion relation connecting ω with the propagation constant k . It agrees with the general form (10). If k is a solution of (25) at a given ω , then so are $-k$ and

$$k_n = k + \frac{2n\pi}{a} \ , \qquad n = \pm 1, \pm 2, \ldots \qquad (26)$$

as well as their negatives $-k_n$, because of the evenness and periodicity of the cosine function.

SECTION V

PASSBANDS AND STOPBANDS

In many practical situations the resistive elements in the line are negligible, so that all the impedances and admittances are expressible in terms of inductances and capacitances L , C , L_d and C_d :

$$Y = j\omega C \quad , \qquad Z = j\omega L$$

$$Y_d = j\omega C_d , \qquad Z_d = j\omega L_d \tag{27}$$

From these, one has

$$Z_o = \sqrt{\frac{L}{C}} \quad , \qquad \gamma = j\omega \sqrt{LC} \tag{28}$$

In this case, relation (25) becomes

$$\cos ka = (1 - \tfrac{1}{2}\omega^2 L_d C_d)\cos(\omega a\sqrt{LC})$$

$$-\tfrac{1}{2}\omega L_d\sqrt{\frac{C}{L}}\,(1 - \tfrac{1}{4}\omega^2 L_d C_d + \frac{C_d L}{L_d C})\sin(\omega a\sqrt{LC}) \tag{29}$$

The right-hand side is an even function of the frequency. Equation (29) can be put into a simpler form by introducing an amplitude $A(\omega)$ and a phase $\phi(\omega)$ such that

$$A(\omega)\cos\phi(\omega) = 1 - \tfrac{1}{2}\omega^2 L_d C_d$$

$$A(\omega)\sin\phi(\omega) = \tfrac{1}{2}\omega L_d\sqrt{\frac{C}{L}}\left(1 - \tfrac{1}{4}\omega^2 L_d C_d + \frac{C_d L}{L_d C}\right) \tag{30}$$

Then (29) becomes

$$\cos ka = A(\omega)\cos(\omega a\sqrt{LC} + \phi(\omega)) \tag{31}$$

The right-hand side is a quasi-sinusoidal function with a modulated amplitude.

After some algebra it is found that

$$A^2(\omega) = 1 + \frac{\omega^2 L_d^2 C}{4L}\left(1 - \tfrac{1}{2}\omega^2 L_d C_d - \frac{C_d L}{L_d C}\right)^2 \qquad (32)$$

which is greater than 1 . From this one concludes that there exist an infinite number of frequency ranges over which the right-hand side of (31) is between the limits 1 and -1 , and other ranges over which the expression is outside those limits. Equation (31) shows that k is real in the first case, implying propagation; it is imaginary in the second case, indicating attenuation. Consequently one has an infinite sequence of alternating passbands and stop-bands. The band boundaries are determined by the condition

$$\cos ka = \pm 1 \qquad (33)$$

or

$$k = \frac{n\pi}{a} \quad , \quad n = \pm 1, \pm 2, \ldots \qquad (34)$$

This condition corresponds to a total reflection of the waves at the loads.

SECTION VI

PHASE VELOCITY AND GROUP VELOCITY

Let the right-hand side of the dispersion relation (29) be denoted by $F(\omega)$:

$$\cos ka = F(\omega) \tag{35}$$

If, at a given frequency ω in a passband, k is a real solution of (35), then so are $-k$ and the members of the infinite sequence

$$k_n = k + \frac{2n\pi}{a}, \quad n = \pm 1, \pm 2, \ldots \tag{36}$$

as well as their negatives $-k_n$. The waves on the transmission line therefore consist of an infinite number of spatial harmonics traveling to the right or to the left. Each harmonic has a phase velocity

$$v_{ph} = \pm \frac{\omega}{k_n} \tag{37}$$

which is different for different n's . The $\pm$ sign refers to the two possible directions of propagation. Thus, at a given frequency, there is no unique phase velocity.

There is, however, a unique group velocity. The group velocity is the velocity at which energy is transmitted by the waves, and is a very useful quantity in analyzing propagation characteristics. It is given by

$$v_g = \frac{\partial \omega}{\partial k} \tag{38}$$

From (35) one finds that

$$v_g = \pm \frac{a\sqrt{1 - F^2(\omega)}}{\frac{\partial F(\omega)}{\partial \omega}} \tag{39}$$

which has a uniquely defined magnitude at a given frequency. Again the $\pm$ sign refers to the two possible directions of propagation. By (33) and (35), $F(\omega) = \pm 1$ at the band boundaries. Therefore one has the general result that the group velocity vanishes at band boundaries.

SECTION VII

ILLUSTRATIVE EXAMPLE

In this section a numerical example will be worked out to illustrate the general results derived above. Consider the dispersion relation (29) applicable to the case of negligible resistive circuit elements. It is convenient to introduce three dimensionless parameters

$$\Omega = \omega a\sqrt{LC}$$

$$\alpha = \frac{L_d C_d}{a^2 LC}$$

$$\beta = \frac{C_d L}{L_d C} \tag{41}$$

Hence Ω serves as a dimensionless frequency variable, and α and β are parameters characterizing the periodic loads. Then (29) becomes

$$\cos ka = (1 - \tfrac{1}{2}\alpha\Omega^2)\cos \Omega - \tfrac{1}{2}\Omega\sqrt{\frac{\alpha}{\beta}}\,(1 + \beta - \tfrac{1}{4}\alpha\Omega^2)\sin \Omega \tag{42}$$

Take a specific set of values for α and β :

$$\alpha = 0.2 \ , \quad \beta = 0.5 \tag{43}$$

This choice for β is quite typical of practical situations. The choice for α corresponds to a fairly heavy loading of the line. It will be seen below that the loading (43) produces at low frequencies a reduction of the group velocity to about 2/3 the uniform line value.

The passband and stopband structures can be found graphically by plotting out the right-hand side of the dispersion relation (42) as a function of the dimensionless frequency variable Ω , as is done in Fig. 3. The curve oscillates with ever increasing amplitude as $\Omega \to \pm \infty$. Those portions of the curve lying

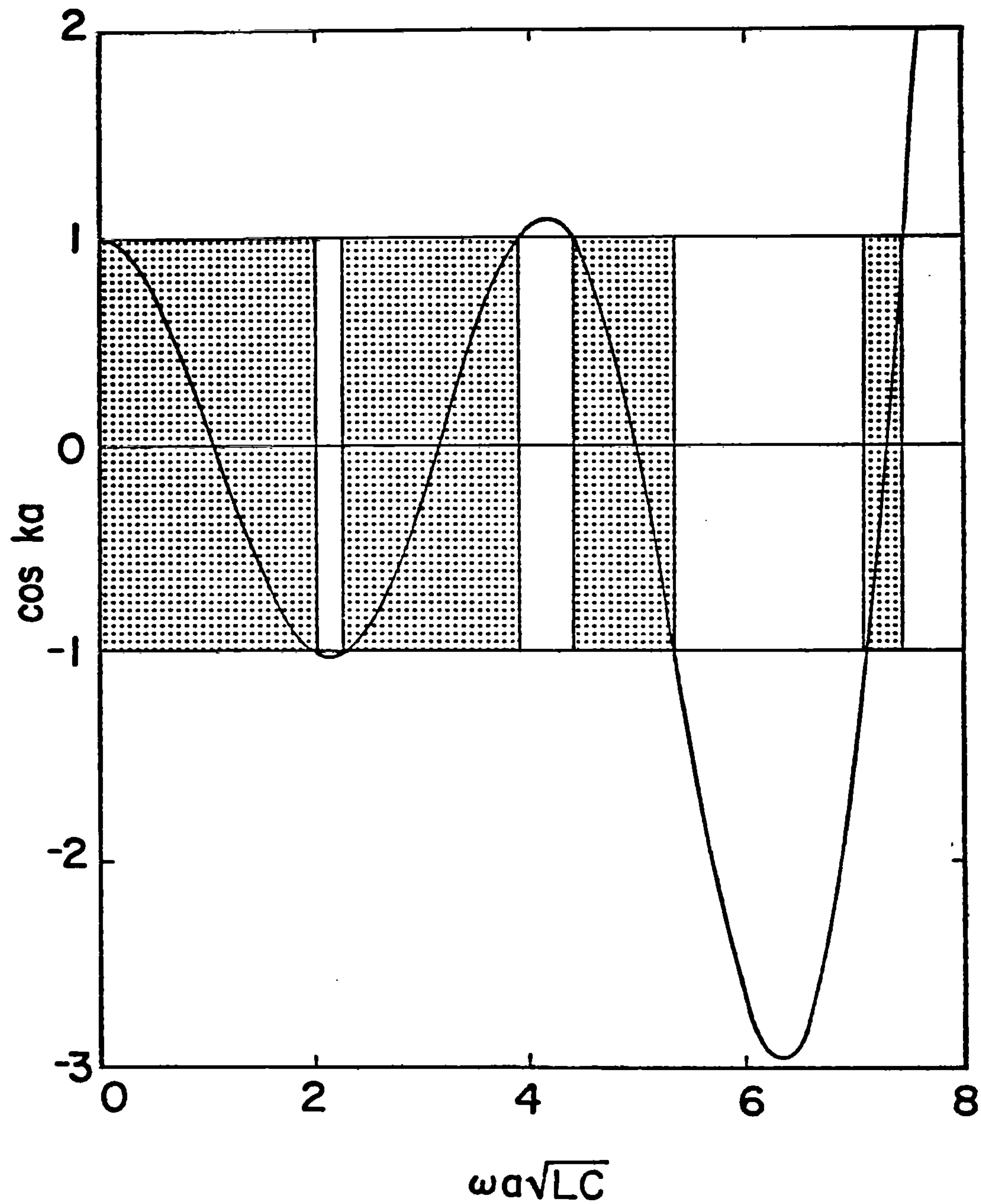

Figure 3. Graphical solution of the dispersion relation (42) of a periodically loaded transmission line in the (cos ka, ω) plane. The frequency ranges for which $-1 \leq \cos ka \leq 1$ correspond to the passbands (dotted areas).

between the horizontal lines $\cos ka = 1$ and -1 correspond to the passbands. They are indicated by the dotted areas in the figure.

Each intersection of the curve in Fig. 3 with a horizontal line yields a pair of coordinates $(\cos ka , \omega)$ which form a solution of the dispersion relation. Each value of $\cos ka$ corresponds to an infinite sequence of propagation constants $\pm k_n$ according to (9). In a passband, these constants are real. To study the phase velocity and group velocity, it is convinient to plot the solutions in the (k , ω) plane. This is done in Fig. 4. Each continuous horizontal curve corresponds to a passband. The passbands are separated by gaps representing the stopbands. It is seen that, as the frequency increases, the passbands contract while the stopbands widen. Thus most of the low-frequency waves are passed, while most of the high-frequency ones are stopped. This is not surprising since four-terminal networks of the type shown in Fig. 1 function as a low-pass filter when connected in tandem.

The phase velocity at frequency ω, as defined in (37), is proportional to the slope of the straight line drawn from the origin to a point with ordinate ω on a curve of Fig. 4. On the other hand, the group velocity is proportional to the slope of the curve at that point. From Fig. 4, it is seen that the group velocity is greatest at low frequencies. One can find the low-frequency limit of the group velocity by expanding (42) in ascending powers of ω. The result is

$$\omega \approx \pm \frac{k}{\sqrt{LC}} \left[1 + \alpha + \sqrt{\frac{\alpha}{\beta}}\,(1 + \beta)\right]^{-\frac{1}{2}} , \qquad k,\omega \to 0 \tag{44}$$

Therefore, by (43), the group velocity at low frequencies is

$$v_g = \frac{\partial\omega}{\partial k} \approx \pm \frac{0.68}{\sqrt{LC}} \tag{45}$$

showing a reduction of some 32% from the uniform line value by the loading.

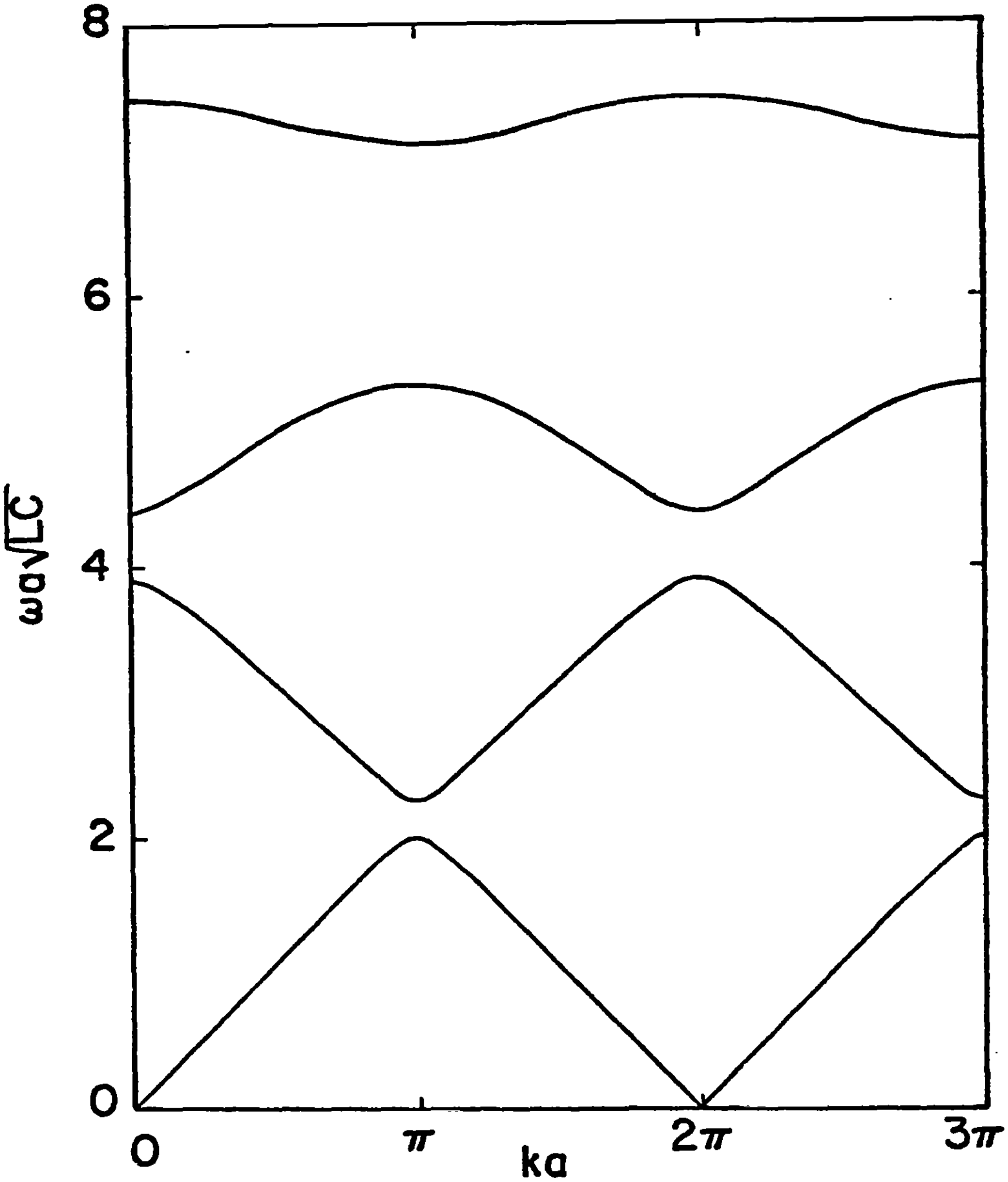

Figure 4. Plot of frequency ω as a function of propagation constant k . showing the passband-stopband structures. The curves for negative k and ω can be obtained by reflections with respect to the horizontal and vertical coordinate axes.

As the frequency is increased, the passbands become progressively narrower. In Fig. 4, they are seen to approach the limit of a horizontal line. The group velocity therefore tends to zero, implying zero energy transmission. In this limit, the excitation on the line consists of standing waves in the uniform sections of the line, bouncing back and forth between two loads.

BIBLIOGRAPHY

L. Brillouin, Wave Propagation in Periodic Structures, Dover Publications, New York, 1953.

R.E. Collin, Field Theory of Guided Waves, McGraw - Hill Book Company, New York, 1960, Chapter 9.

L.A. Pipes, Applied Mathematics for Engineers and Physicists, McGraw - Hill Book Company, New York, 1958, Chapters 7 and 10.

J.C. Slater, Microwave Electronics, D. van Nostrand Company, Princeton, 1950, Chapter VIII.