Interaction Notes

Note 312

8 June 1977

Transients on Lossless Terminated Transmission Lines

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Abstract

This work contains a general exposition of the methods which are available in analyzing the transients on a lossless terminated line. After reviewing the well-known method based on the Γ-series expansion we present two alternative methods, one in the form of a Volterra integral equation and another corresponding to the singularity expansion method. For a resistively terminated line we have proved the identity between the Γ-series solution and the one obtained by the singularity expansion method. The application of these methods to more complicated terminations is illustrated by the case of a series RL termination. Weber's solution for a short-circuited line is compared with our solution. The importance of injecting the causality condition in our formulation for this class of problems is emphasized. The application of these methods to the treatment of the input current response of a thin biconical antenna is briefly outlined.

Contents

Acknowledgment

The author is grateful to Professor Albert E. Heins and Professor Charles B. Sharpe for many valuable discussions during the course of this study. The assistance of Mr. Soon K. Cho is appreciated. The encouragement which the author received from Dr. Carl E. Baum and Dr. David V. Giri is duly acknowledged.

I. Introduction

Transients on transmission lines is a classical problem in linear system analysis. Many authors have contributed significantly to the study of this problem. We like to mention particularly the work of Levinson [1], Bewley [2], Weber [3], Kuznetsov and Stratonovich [4]. Although the formulation for lossy lines terminated by an arbitrary load is known, a general solution seems to be not available because of the difficulty in evaluating some of the inverse Laplace transforms. For a lossy line terminated by a series RL load, the exact solution was found by Kuznetsov [4] with the aid of Lommel functions. When the line is lossless the analysis is considerably simpler. Even then no detailed treatment seems to be available for arbitrary terminations except for the case of a resistive load which is discussed in many standard books. It is therefore desirable to present a general treatment by which one can solve the problem for arbitrary termination in a systematic way. The work reported here is partly motivated by our desire to investigate the transient phenomena on biconical antennas which can be interpreted as a pair of biconical transmission lines terminated by a distributed load admittance [5,6]. Whatever method which we use for the transmission line problem is then equally applicable to analyze the transient response of a biconical antenna. Before the general methods are presented we like to review first the conventional treatment for a pair of lossless lines terminated by a resistive load.

II. Conventional Method of Treating a Lossless Line Terminated by a Resistive Load

We consider a pair of lossless lines terminated by a load impedance Z. The lines are assumed to be excited by a unit step voltage at the input end.

For convenience we introduce several normalized variables defined as follows:

$\xi = x/\ell$ ≡ normalized distance; $1 \geq \xi \geq 0$

$\tau = tc/\ell$ ≡ normalized time

ℓ ≡ length of the line (m)

c ≡ velocity of propagation on the lossless line, being equivalent to $1/(L'C')^{1/2}$ (m/s)

L', C' ≡ inductive (H/m) and capacitive (F/m) line constants, respectively

$s = (\Omega + j\omega)\ell/c$ ≡ normalized Laplace transform variable

$\omega = 2\pi f$ ≡ angular or radian frequency (rad/s)

f ≡ frequency (Hz)

(Note that in special cases $\Omega=0$)

In terms of these normalized variables we denote

$V(\xi,\tau)$ ≡ instantaneous line voltage (V)

$I(\xi,\tau)$ ≡ instantaneous line current (A)

$\tilde{V}(\xi,s)$ ≡ Laplace transform of $V(\xi,\tau)$ (Vs) or (V/Hz)

$= \mathcal{L}[V(\xi,\tau)] = \int_0^\infty V(\xi,\tau)e^{-s\tau}d\tau$

$\tilde{I}(\xi,s)$ Laplace transform of $I(\xi,\tau)$ (As) or (A/Hz)

$= \mathcal{L}[I(\xi,\tau)] = \int_0^\infty I(\xi,\tau)e^{-s\tau}d\tau$

For a unit step voltage applied at the input end we have

$$V(0,\tau) = u(\tau,0)$$

hence

$$\tilde{V}(0,s) = \int_0^\infty u(\tau - 0)e^{-st} = \frac{1}{s}$$

In terms of normalized variables ξ and s the line voltage and the line current in the Laplace transform domain can be written in the form

$$\tilde{V}(\xi,s) = \frac{e^{-\xi s} + \tilde{\Gamma}(s)e^{-(2-\xi)s}}{s[1 + \tilde{\Gamma}(s)e^{-2s}]} \tag{2.1}$$

$$Z_c\tilde{I}(\xi,s) = \frac{e^{-\xi s} - \tilde{\Gamma}(s)e^{-(2-\xi)s}}{s[1 + \tilde{\Gamma}(s)e^{-2s}]} \tag{2.2}$$

where $\tilde{\Gamma}(s)$ denotes the voltage reflection coefficient defined in the s-domain at the output end of the line, $\xi=1$, and Z_c denotes the characteristic impedance of the line, being equal to $(L'C')^{1/2}$. For convenience, we assume Z_c to be equal to unity in the subsequent analysis.

The conventional method of determining $V(\xi,\tau)$ or $I(\xi,\tau)$ is to express (2.1) or (2.2) in a series using the expression

$$\frac{1}{1 + \tilde{\Gamma}(s)e^{-2s}} = \sum_{n=0}^{\infty} [-\tilde{\Gamma}(s)e^{-2s}]^n \tag{2.3}$$

Substituting (2.3) into (2.2), with $Z_c = 1$, we have

$$\tilde{I}(\xi,s) = \frac{1}{s}[e^{-\xi s} - \tilde{\Gamma}(s)e^{-(2-\xi)s}] \sum_{n=0}^{\infty} [-\tilde{\Gamma}(s)e^{-2s}]^n \tag{2.4}$$

For a resistive load $\tilde{\Gamma}(s)$ is a real constant which will be denoted by Γ and its value is given by

$$\Gamma = \frac{r - 1}{r + 1}$$

where r denotes the normalized terminal resistance. The inverse Laplace transform of (2.4) with $\tilde{\Gamma}(s) \equiv \Gamma$ yields

$$I(\xi,\tau) = \sum_{n=0}^{\infty} [(-\Gamma)^n \, u(\tau-2n-\xi) + (-\Gamma)^{n+1} \, u(\tau-2n-2+\xi)] \tag{2.5}$$

where $u(\tau-\tau_n)$ denotes a unit step function commencing at $\tau=\tau_n$.

Although (2.5) is known to be a valid solution by physical reasoning its derivation is considered to be unsatisfactory from the mathematical point of view because expansion (2.3) holds true only if $|\tilde{\Gamma}(s)e^{-2s}| < 1$, and in executing the inverse Laplace transform the contour of integration lies in the left-half plane where $|\tilde{\Gamma}(s)e^{-2s}|$ could exceed unity. This presentation is found in many books without justification. One way of removing this weak step is to expand the same function in terms of a finite sum with a remainder instead of as an infinite series. Thus, we write

$$\frac{1}{1 + \tilde{\Gamma}(s)e^{-2s}} = \sum_{n=0}^{N} [-\tilde{\Gamma}(s)e^{-2s}]^n + \frac{[-\tilde{\Gamma}(s)e^{-2s}]^{N+1}}{1 + \tilde{\Gamma}(s)e^{-2s}} \tag{2.6}$$

when substituting (2.6) into (2.2) the remainder would yield a term of the form

$$\frac{[-\tilde{\Gamma}(s)]^{N+2}}{s[1 + \tilde{\Gamma}(s)e^{-2s}]} e^{-s[2(N+2)-\xi]} \tag{2.7}$$

Because of the shifting theorem and the causality condition the inverse Laplace transform of (2.7) vanishes when $\tau < [2(N+2)-\xi]$. In other words, if one evaluates the series (2.5) up to $\tau < [2(N+2)-\xi]$ the remaining terms vanish identically. The importance of this remark is that (2.6) applies not only to resistive termination but to any termination. From now on we will designate the solution based on (2.6) as <u>the Γ-series solution</u>. In addition to the Γ-series method there are two alternative methods for treating the transients in an arbitrary terminated line. The discussion of these two methods is the main objective of this note.

III. Volterra Integral Equation Method

We consider the general case where $\tilde{\Gamma}(s)$ is a function of s. If (2.2), with $Z_c = 1$, is multiplied by $1 + \tilde{\Gamma}(s)e^{-2s}$ the following equation results

$$\tilde{I}(\xi,s) + \tilde{\Gamma}(s)e^{-2s}\tilde{I}(\xi,s) = \frac{1}{s}[e^{-\xi s} - \tilde{\Gamma}(s)e^{-(2-\xi)s}] \qquad (3.1)$$

By taking the inverse Laplace transform of (3.1) we obtain

$$I(\xi,\tau) = I_o(\xi,\tau) + \mathcal{L}^{-1}[-\tilde{\Gamma}(s)e^{-2s}\tilde{I}(s)] \qquad (3.2)$$

where

$$I_o(\xi,\tau) = I_{of}(\xi,\tau) + I_{ob}(\xi,\tau) \qquad (3.3)$$

with

$$I_{of}(\xi,\tau) = \mathcal{L}^{-1}[\frac{e^{-\xi s}}{s}] = u(\tau - \xi) \qquad (3.4)$$

$$I_{ob}(\xi,\tau) = \mathcal{L}^{-1}[-\frac{\tilde{\Gamma}(s)}{s} e^{-(2-\xi)s}] \qquad (3.5)$$

$I_{of}(\xi,\tau)$ represents the initial forward current wave propagating on the line and $I_{ob}(\xi,\tau)$ represents the first reflected wave or backward wave from the termination. For a given $\tilde{\Gamma}(s)$ we assume (3.5) can be evaluated, thus $I_{ob}(\xi,\tau)$ is considered to be a known function. On account of the convolution theorem in the Laplace transform (3.2) can be written in the form

$$I(\xi,\tau) = I_{ob}(\xi,\tau) + \int_o^\tau k(\tau - \tau')\, I(\xi,\tau')dt' \qquad (3.6)$$

where

$$k(\tau) = \mathcal{L}^{-1}[-\tilde{\Gamma}(s)e^{-2s}] \qquad (3.7)$$

Equation (3.6) with $I(\xi,\tau)$ as the unknown function corresponds to the Volterra integral equation of the second kind. Its solution is given by Picards' series [7], namely

$$I(\xi,\tau) = \sum_{n=0}^{\infty} I_n(\xi,\tau) \qquad (3.8)$$

where $I_o(\xi,\tau)$ is given by (3.3) and

$$I_n(\xi,\tau) = \int_0^\tau k(\tau - \tau')I_{n-1}(\xi,\tau')\, d\tau' \quad \text{for } n = 1,2,... \tag{3.9}$$

In the case $\tilde{\Gamma}(s)$ is a real constant, previously denoted Γ, we obtain from (3.5)

$$I_{ob}(\xi,\tau) = -\Gamma u(\tau - 2 + \xi)$$

hence

$$I_{ob}(\xi,\tau) = u(\tau - \xi) - \Gamma u(\tau - 2 + \xi) \tag{3.10}$$

and from (3.7) we have

$$k(\tau) = -\Gamma\delta(\tau - 2) \tag{3.11}$$

where $\delta(\tau-2)$ denotes the delta function defined at $\tau=2$. Substituting (3.10) and (3.11) into (3.9) we obtain the same expression given by (2.5). Of course, for a resistively terminated line it is entirely unnecessary to formulate the problem by this integral equation method as the method of Γ-series is much simpler. The integral equation method, however, is much more efficient and convenient for more complicated terminations. As an illustration we consider a series RL termination. In this case, we have

$$\tilde{\Gamma}(s) = \frac{\tilde{z}(s) - 1}{\tilde{z}(s) + 1}$$

where

$$\tilde{z}(s) = \frac{1}{Z_c}\left[R + s\left(\frac{c}{\ell}\right)L\right]$$

$$= r + \alpha s$$

$$r = R/Z_c\ , \quad \alpha = \frac{cL}{Z_c\ell} = \frac{L}{L'\ell}$$

$L' \equiv$ inductive line constant

The coefficient α is a measure of the load inductance in terms of the total line inductance. The reflection coefficient $\tilde{\Gamma}(s)$ can now be written in the form

$$\tilde{\Gamma}(s) = \frac{s - s_0}{s - s_1} \tag{3.12}$$

where

$$s_0 = -\left(\frac{r - 1}{\alpha}\right), \quad s_1 = -\left(\frac{r + 1}{\alpha}\right)$$

thus

$$\frac{\tilde{\Gamma}(s)}{s} = \frac{1}{s}\left(\frac{s - s_0}{s - s_1}\right) = \frac{\rho}{s} + \frac{1 - \rho}{s - s_1} \tag{3.13}$$

where

$$\rho = \frac{s_0}{s_1} = \frac{r - 1}{r + 1} \quad ;$$

using (3.5) and (3.7) one finds

$$I_{ob}(\xi,\tau) = -u(\tau - 2 + \xi)[\rho + (1 - \rho)e^{s_1(\tau-2+\xi)}] \tag{3.14}$$

$$k(\tau) = -\delta(\tau - 2) - u(\tau - 2)(1 - \rho)s_1 e^{s_1(\tau-2)} \tag{3.15}$$

Knowing $I_{ob}(\xi,\tau)$ and $k(\tau)$ we can find $I_1(\xi,\tau)$ using (3.9). The result gives

$$I_1(\xi,\tau) = -u(\tau - 2 + \xi)[\rho + (1 - \rho)e^{s_1(\tau-2+\xi)}]$$

$$-u(\tau - 4 + \xi)\{\rho^2 + [1 - \rho^2 + (1 - \rho)^2 s_1(\tau - 4 + \xi)]e^{s_1(\tau-4+\xi)}\} \tag{3.16}$$

The successive terms of $I_n(\xi,\tau)$ for $n \geq 2$ can be found accordingly.

If the Γ-series method were used the process is more tedious because one has to expand $[\tilde{\Gamma}(s)]^n/s$ in partial fraction that is quite involved as a result of the multiplicity of the poles contained in $[\tilde{\Gamma}(s)]^n$. Another advantage of the Volterra integral equation method is that once the first reflected wave is known the successive waves can be found based on this information alone.

This is due to the fact that the kernel $k(\tau)$ involved in the integral equation is related to the derivative of the first reflected wave. Since

$$\tilde{I}_{ob}(\xi,s) = -\frac{1}{s}\tilde{\Gamma}(s)e^{-(2-\xi)s}$$

and

$$\tilde{K}(s) = \mathcal{L}[k(\tau)] = -\tilde{\Gamma}(s)e^{-2s}$$

hence

$$\tilde{K}(s) = s\tilde{I}_{ob}(0,s)$$

it follows that

$$k(\tau) = \frac{\partial I_{ob}(0,\tau)}{\partial\tau}$$

where we interpret the derivative in a generalized sense that for a discontinuous unit step function

$$\frac{\partial u(\tau - \tau_n)}{\partial\tau} = \delta(\tau - \tau_n)$$

For example, from (3.14) one finds

$$\frac{\partial I_{ob}(0,\tau)}{\partial\tau} = -\delta(\tau - 2) - u(\tau - 2)[(1 - \rho)s_1e^{s_1(\tau-2)}]$$

which is the same as $k(\tau)$ given by (3.15). This completes our discussion of the integral equation method. Our next section deals with the singularity expansion method.

IV. The Singularity Expansion Method as Applied to a Resistive Termination

The terminology of this method was first suggested by Baum [8] in connection with his work dealing with the scattering of electromagnetic waves by objects. This method when applied to transmission lines was used by Weber [9] for the special case of a short-circuit terminated line. We shall comment on Weber's treatment at the end of this section. For our purpose we will apply this method to a resistive termination and show the analytic connection between the solution obtained by this method and the one based on the Γ-series method. For simplicity we consider just the input current to the line, corresponding to $\xi=0$. Equation (2.2), with $Z_c=1$ and $\tilde{\Gamma}(s)=\Gamma$ (real constant), becomes

$$\tilde{I}(0,s) = \frac{1}{s}\left[\frac{1 - \Gamma e^{-2s}}{1 + \Gamma e^{-2s}}\right] \tag{4.1}$$

To assure the fulfillment of the causality condition in the final solution we write $\tilde{I}(0,s)$ in the form

$$\tilde{I}(0,s) = \frac{1}{s}\left[1 - \frac{2\Gamma e^{-2s}}{1 + \Gamma e^{-2s}}\right] \tag{4.2}$$

In addition to $s=0$, the poles of $\tilde{I}(0,s)$ are given by the roots of the equation

$$1 + \Gamma e^{-2s} = 0 \tag{4.3}$$

Denoting these roots by s_n we find for $\Gamma < 0$,

$$s_n = \frac{1}{2}\,\ell n|\Gamma| + jn\pi,\ n = 0,\pm1,\pm2,\ldots \tag{4.4}$$

and for $\Gamma > 0$, we have

$$s_n = \frac{1}{2}[\ell n(\Gamma) + jn\pi],\ n = \pm1,\pm3,\ldots \tag{4.5}$$

Without loss of generality we assume Γ to be negative and not equal to -1 in the following discussion. The case of $\Gamma=-1$ requires a special treatment and will be discussed later.

The expression for $\tilde{I}(0,s)$ as given by (4.2) can now be expanded into a residue series in terms of the poles of that function. We consider the function $(1 + \Gamma e^{-2s})^{-1}$ which satisfies the criteria as required by Mittag-Leffler theorem [10]. Then its expansion is given by

$$\frac{1}{1 + \Gamma e^{-2s}} = \frac{1}{1 + \Gamma} + \sum_{n=0}^{\pm\infty} \frac{1}{2} \left(\frac{1}{s - s_n} + \frac{1}{s_n}\right) \tag{4.6}$$

hence

$$\frac{1}{s(1 + \Gamma e^{-2s})} = \frac{1}{(1 + \Gamma)s} + \sum_{n=0}^{\pm\infty} \frac{1}{2s_n(s - s_n)} \tag{4.7}$$

Substituting (4.7) into (4.2), we have

$$\tilde{I}(0,s) = \frac{1}{s} - 2\Gamma e^{-2s}\left[\frac{1}{(1 + \Gamma)s} + \sum_{n=0}^{\pm\infty} \frac{1}{2s_n(s - s_n)}\right] \tag{4.8}$$

The inverse Laplace transform of (4.8) yields

$$I(0,\tau) = u(\tau - 0) - 2\Gamma u(\tau - 2)\left[\frac{1}{1 + \Gamma} + \sum_{n=0}^{\pm\infty} \frac{1}{2s_n} e^{s_n(\tau-2)}\right] \tag{4.9}$$

If we let

$$s_n = \alpha + j\beta_n$$

where

$$\alpha = \frac{1}{2} \ln|\Gamma|$$

$$\beta_n = n\pi, \quad n = 0,\pm1,\pm2,\ldots$$

then (4.9) can be written in the form

$$I(0,\tau) = u(\tau - 0) - 2\Gamma u(\tau - 2)\left[\frac{1}{1 + \Gamma}\right.$$

$$\left. + e^{\alpha(\tau-2)} \sum_{n=0}^{\infty} \left(\frac{2 - \delta_{n,0}}{2}\right) \frac{\alpha\cos[\beta_n(\tau-2)] + \beta_n \sin[\beta_n(\tau-2)]}{\alpha^2 + \beta_n^2}\right] \tag{4.10}$$

here $\delta_{n,0}$ denotes the Kronecker delta.

According to the Γ-series method, for $\xi = 0$, (2.5) reduces to

$$I(0,\tau) = u(\tau - 0) - 2\Gamma u(\tau - 2) + 2\Gamma^2 u(\tau - 4) + \cdots \tag{4.11}$$

Equations (4.10) and (4.11) would be equivalent only if the series contained in the summation sign of (4.10) is proportional to $e^{-\alpha(\tau-2)}$ with the constant of proportionality determined by the time interval in which the series represents. The proof of the identity between (4.10) and (4.11) is shown as follows: we recognize that $\cos\beta_n(\tau-2)$ and $\sin\beta_n(\tau-2)$ are two orthogonal sets of functions with a periodicity equal to 2, thus we let

$$e^{-\alpha(\tau-2)} = \sum_{n=0}^{\infty} \left[a_n \cos[\beta_n(\tau - 2)] + b_n \sin[\beta_n(\tau - 2)]\right]$$

for $[2(m+1)] > \tau > 2m$. One finds

$$a_n = -\left(\frac{2 - \delta_{n,0}}{2}\right)\frac{(1 + \Gamma)}{(-\Gamma)^m} \frac{\alpha}{\alpha^2 + \beta_n^2}$$

$$b_n = -\frac{(1 + \Gamma)}{(-\Gamma)^m} \frac{\beta_n}{\alpha^2 + \beta_n^2}$$

hence,

$$\sum_{n=0}^{\infty} \left(\frac{2 - \delta_{n,0}}{2}\right) \frac{\alpha\cos[\beta_n(\tau-2)] + \beta_n \sin[\beta_n(\tau-2)]}{\alpha^2 + \beta_n^2} = -\frac{(-\Gamma)^m}{1 + \Gamma} e^{-\alpha(\tau-2)}$$

$$\text{for } [2(m+1)] > \tau > 2m \tag{4.12}$$

In view of (4.12), we can write (4.10) in the form

$$I(0,\tau) = u(\tau - 0) - 2\Gamma u(\tau - 2)\left[\frac{1 - (-\Gamma)^m}{1 + \Gamma}\right]$$

$$[2(m+1)] > \Gamma > 2m... \qquad (4.13)$$

If we let m take the successive values 1,2,3,... (4.13) indeed is identical to (4.11). Of course, it is not easy to recognize that the series obtained by the singularity expansion method as given by (4.10) is an alternative representation of the Γ-series solution without such a detailed analysis. For a non-resistive termination the poles are more complicatedly distributed. In fact for most of the cases there is no closed form solution for these poles; the proof of the identity between the Γ-series solution and the one obtained by the singularity expansion method would be extremely difficult. Based on what we have discussed for the resistively terminated case, we have the confidence that these alternative representations must be equivalent.

Finally, we like to comment on the treatment given by Weber [9] for a short-circuited line ($\Gamma = -1$). The function which Weber analyzed corresponds to the voltage distribution along the line for a step input voltage excitation. In Laplace-transform domain, the function which he considered is

$$\tilde{V}(\xi,s) = \frac{e^{\xi s} - e^{-(2-\xi)s}}{s(1 - e^{-2s})} \qquad (4.14)$$

The residue series representation of (4.14) was obtained by Weber without following the discipline as demanded by Mittag-Leffler theorem. Although his result is correct the procedure which leads to his solution is, strictly speaking, not justified for many irrational functions. The final solution which Weber obtained is of the form

$$V(\xi,\tau) = u(\tau - 0)\left[\sum_{n=1}^{\infty} \frac{2\sin[n\pi\xi]}{n\pi} + \sum_{n=1}^{\infty} \frac{\sin[n\pi(\tau - \xi)]}{n\pi}\right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{\sin[n\pi(\tau + \xi)]}{n\pi}\right] \tag{4.15}$$

The solution clearly represents D'Alembert's solution for the one-dimensional wave equation. From the point of view of transient analysis it does not explicitly exhibit the causality condition: $V(\xi,\tau) = 0$ for $\tau < \xi$. Actually (4.15) is a Fourier series expansion of the periodic wave shown in figure 4.1. The function indeed is vanishing for $\xi > \tau > 0$. In contrast to Weber's presentation we treat (4.14) as consisting of two terms, i.e., we let

$$\tilde{V}(\xi,s) = \tilde{V}_1(\xi,s) + \tilde{V}_2(\xi,s) \tag{4.16}$$

where

$$\tilde{V}_1(\xi,s) = \frac{e^{-\xi s}}{s(1 - e^{-2s})}$$

$$\tilde{V}_2(\xi,s) = -\frac{e^{-(2-\xi)s}}{s(1 - e^{-2s})} \tag{4.17}$$

By applying Mittag-Leffler theorem to the function

$$\frac{1}{1 - e^{-2s}} - \frac{1}{2s}$$

which has no pole at the origin, a condition required by that theorem, we obtain

$$\frac{1}{1 - e^{-2s}} - \frac{1}{2s} = \frac{1}{2}\left[1 + \sum_{n=\pm 1}^{\pm\infty} \frac{1}{s - s_n}\right] \tag{4.18}$$

where

$$s_n = jn\pi$$

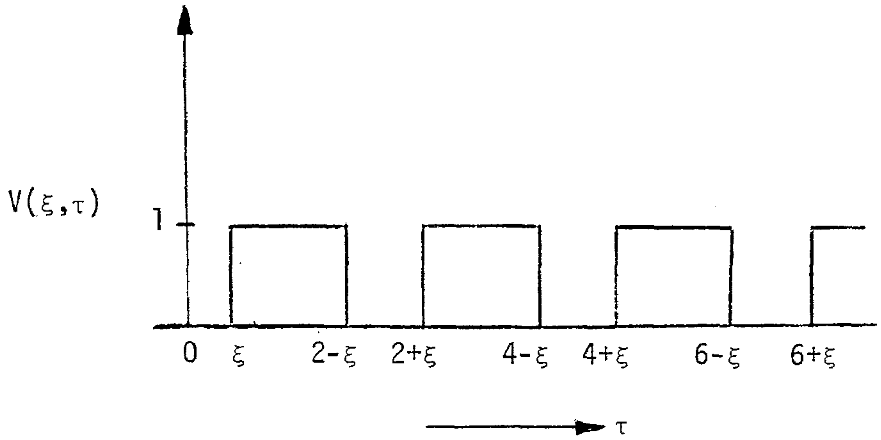

Figure 4.1: $V(\xi,\tau)$ for a short-circuit termination.

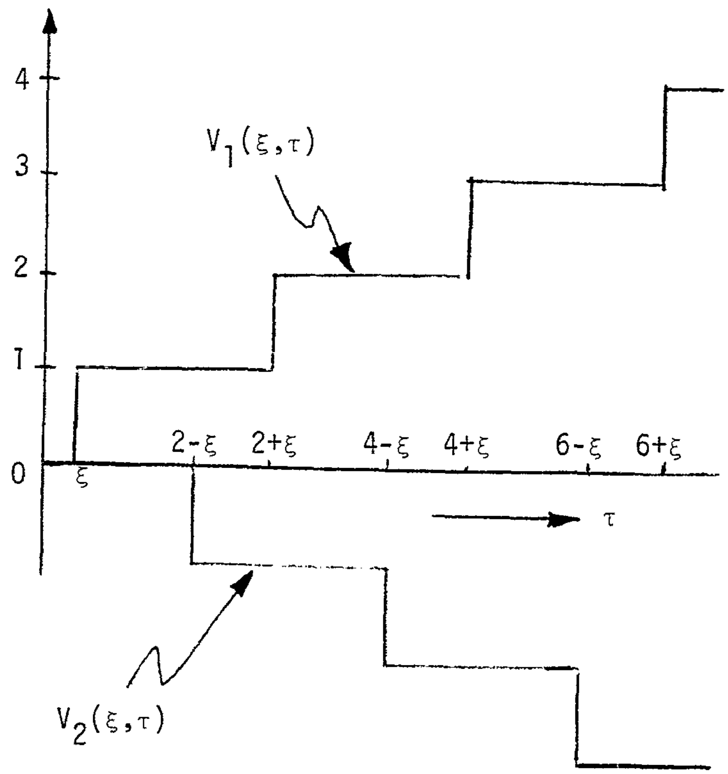

Figure 4.2: $V_1(\xi,\tau)$ and $V_2(\xi,\tau)$ for a short-circuit termination.

hence

$$\frac{1}{s(1 - e^{-2s})} = \frac{1}{2}\left[\frac{1}{s^2} + \frac{1}{s} + \sum_{n=\pm 1}^{\pm\infty} \frac{1}{s_n(s - s_n)}\right] \tag{4.19}$$

As a result of the shifting theorem we obtain,

$$V_1(\xi,\tau) = u(\tau - \xi)\left[\frac{1}{2} + \frac{1}{2}(\tau - \xi) + \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin[n\pi(\tau - \xi)]\right]\ldots \tag{4.20}$$

$$V_2(\xi - \tau) = u(\tau - 2 + \xi)\left[\frac{1}{2} + \frac{1}{2}(\tau - 2 + \xi) + \sum_{n=1}^{\infty} \sin[n\pi(\tau - 2 + \xi)]\right] \tag{4.21}$$

Except for the negative sign, $V_2(\xi,\tau)$ is merely a delayed reproduction of $V_1(\xi,\tau)$. It is observed that because of the step function $u(\tau - \xi)$ contained in $V_1(\xi,\tau)$, the causality condition is automatically met. It can be shown that our solution is actually equivalent to Weber's because the function $f(\tau) = 1 - \tau$, $2 \geq \tau \geq 0$ has a Fourier series representation given by

$$1 - \tau = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(n\pi\ \tau) \tag{4.22}$$

Eqs. (4.20) and (4.21) are shown in figure 4.2. The sum of the two functions yields again the periodic square wave shown in figure 4.1.

V. Summary

In this note we have examined several distinct methods of analyzing transients on lossless transmission lines with arbitrary terminations. It appears that the integral equation is potentially more appealing because from the information of the first reflected wave it is possible to construct the kernel of the integral equation and subsequently to find the complete solution based on quadrature. The singularity expansion method, on the other hand, does furnish the complete solution without iteration, provided that the singularities of the response function are available. Unfortunately, even for a simple series RL termination it is necessary to solve a transcendental equation to determine the numerical values of these singularities. For a resistively terminated load we have shown that the solutions obtained by these different methods are analytically equivalent. This establishes the foundation that for an arbitrarily terminated line all these methods are equivalent. The methods discussed here are equally applicable to the transient analysis of small-angle biconical antennas. The only difference is that the terminal impedance or admittance function involves exponential integral functions, hence the determination of the singularities then becomes more laborious. This work will be reported elsewhere in a separate note.

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