Interaction Notes

Note 324

On the Application of Symmetrization to the Transmission of Electromagnetic Waves Through Small Convex Apertures of Arbitrary Shape

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Abstract

The transmission of an electromagnetic wave through a small aperture in a perfectly conducting screen is examined from the viewpoint of symmetrization.

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I. INTRODUCTION

The question of how much of the electromagnetic energy that exists on one side of a wall can leak to the other side through a small opening in the wall has become, by virtue of its practical importance, a canonical problem in the theory of EMP (electromagnetic pulse) interactions [1].

As is well known, the earliest calculation of the transmission of an electromagnetic wave through a small circular aperture in a plane screen of perfect conductivity and zero thickness was performed by Lord Rayleigh. Using potential theory, he calculated the transmitted field of a plane harmonic wave normally incident on an electrically small circular aperture [2]. Years later Bethe derived expressions for the polarizabilities and effective dipole moments of small circular apertures. His results give the transmitted far field for any angle of incidence but not the transmitted near field [3]. Most recently Bouwkamp [4] and Meixner and Andrejewski [5,6] found an exact solution for both the near and far transmitted fields of a plane wave normally incident on a circular aperture.

Aperture problems can, at least in principle, be solved numerically, but they cannot be solved analytically unless the shape of the aperture happens to be simple enough to permit a separation of the variables and a scalarization of the electromagnetic field. However, from this it should not be inferred that if the aperture problem cannot be solved analytically, a numerical method is the only way to obtain a solution. Actually, as a preferable alternative, one can reformulate the problem so that upper and lower bounds on the true solution and not the true solution itself would have to be sought. Such a reformulation can be based on Levine and Schwinger's result that when the aperture is electrically small there is a variational principle for the upper bound and another variational principle for the lower bound [7,8]. However, this variational approach, which was used by Fikhmanas and Fridberg to find bounds on the electric and magnetic polarizabilities of electrically small apertures[9], does not lend itself to very easy calculation. Accordingly, it is of some interest to try a simpler method of sandwiching the true solution between upper and lower bounds.

In this note we shall examine how symmetrization, which has yielded interesting results in geometry and mathematical physics [10], may be used to establish two-sided bounds on the electric and magnetic polarizabilities of differently shaped convex apertures and thereby estimate their transmission properties in a simple economical manner.

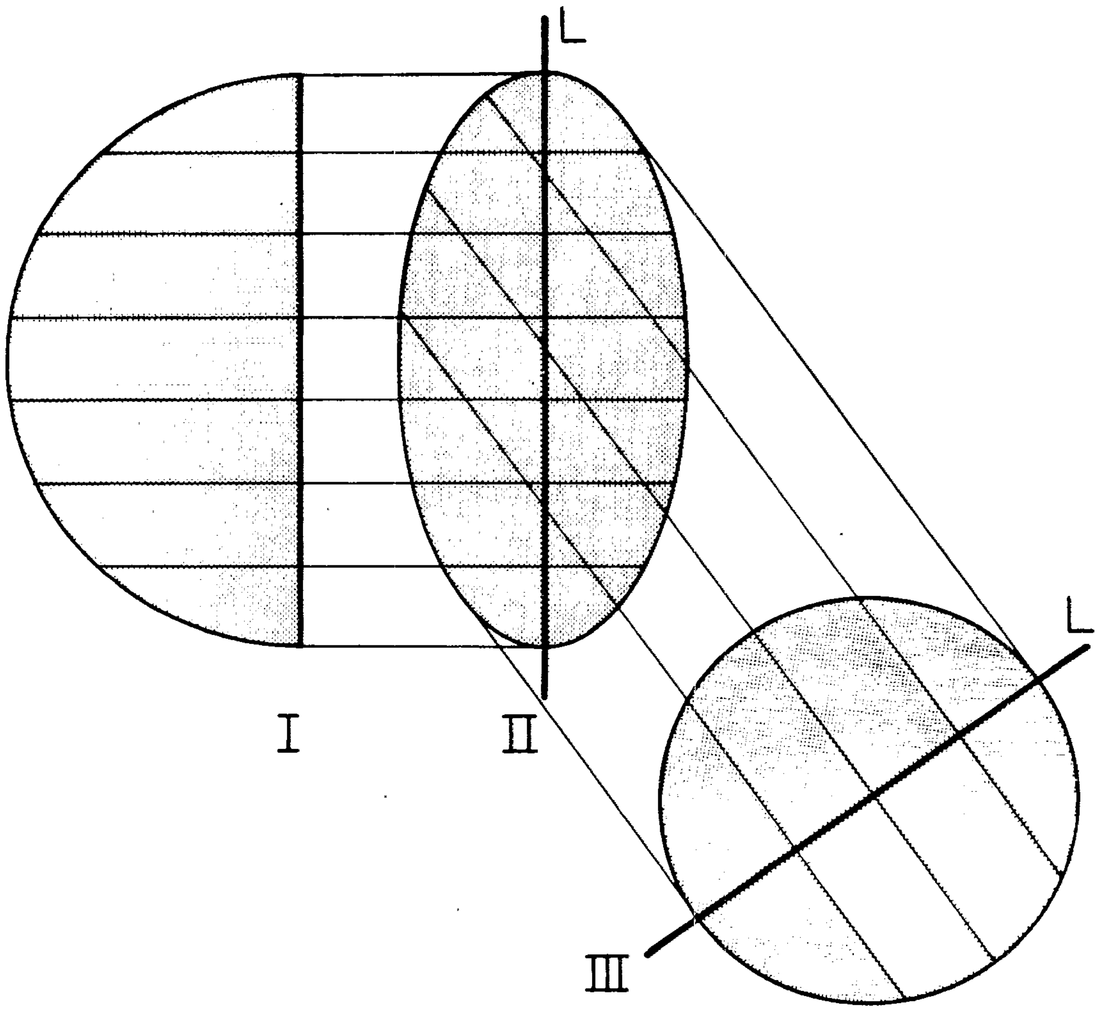
II. SYMMETRIZATION

Of the several kinds of symmetrization that have been invented we shall restrict our attention to the symmetrization of a plane figure with respect to a straight line. To symmetrize a plane figure with respect to a straight line L, we suppose the figure to consist of line segments that are parallel to each other and perpendicular to L (see Figure 1). We then shift each line segment along its own line until the line segment is bisected by L. The shifted line segments compose the symmetrized figure. For example, a semicircle of radius R, when symmetrized with respect to its bounding diameter, changes into an ellipse with semiaxes R and R/2. A further symmetrization can transform the ellipse into a circle of radius R/2. Symmetrization leaves the figure's area A unchanged and decreases, or, more accurately, never increases its perimeter P. For the case shown in Figure 1, the area is always $\pi R^2/2$ and the perimeter varies from $(2+\pi)R$ for the semicircle to πR for the circle.

As an instructive example, we apply the principle of symmetrization to the calculation of capacitance C. It is known that the symmetrization of a plane conducting plate decreases (i.e., never increases) the electrostatic capacity of the plate [10]. A plane figure symmetrized infinitely many times becomes a circle and, consequently, of all conducting plates of a given area the circular plate has the minimum capacity. Accordingly,

$$C \geq C_{in} \tag{1}$$

SYMMETRIZATION



- AREA UNCHANGED
- PERIMETER NEVER INCREASES

Fig. 1 Example of symmetrization of a plane figure with respect to a line L. The semi-circle of radius R is symmetrized with respect to its bounding diameter to produce an ellipse with semi-axes R and R/2. The ellipse, when symmetrized, becomes a circle or radius R/2. The area of each figure remains constant but the perimeter decreases with each symmetrization.

Figure 1

where C denotes the electrostatic capacitance of a plane conducting plate and C_{in} denotes the electrostatic capacitance of the circular plate of radius r_{in}, that has been obtained by completely symmetrizing the original plate. This places a lower bound on C. To obtain an upper bound, we invoke the conjecture that of all plates with a given perimeter, the circular plate has the maximum capacitance [10]. Thus we find

$$C_{out} \geq C \qquad (2)$$

where C_{out} is the electrostatic capacitance of a circular plate of radius r_{out}, whose perimeter is equal to that of the perimeter of the original plate. From (1) and (2) it follows that

$$C_{in} \leq C \leq C_{out} \qquad (3)$$

Since we have

$$r_{in} = (A/\pi)^{1/2} \qquad (4)$$

$$r_{out} = P/2\pi \qquad (5)$$

and the electrostatic capacitance of a circular plate (disk) in MKS units is given by

$$C = 8\varepsilon_0 a \qquad (6)$$

where a is the radius of the disk and ε_0 is the dielectric constant of free space, upon replacing a by r_{in} and r_{out} we obtain from (3)-(6) that the capacitance C of a plate of area A and perimeter P is delimited by

$$(A/\pi)^{1/2} \leq C/8\varepsilon_0 \leq P/2\pi \qquad (7)$$

Here $\varepsilon_0 = (36\pi)^{-1} \times 10^{-9}$ farads per meter.

Both Maxwell [11] and Rayleigh [12] made unproven statements concerning bounds on the capacitance of plates, which agree with (7). Moreover, the capacitance of an elliptic plate of eccentricity e, as given by

$$C_{ellipse}/8\varepsilon_0 = (A\pi)^{1/2}(1-e^2)^{-1/4}/2K(e^2) \underset{e\to o}{\longrightarrow} (A/\pi)^{1/2}(1+e^2/64) \qquad (8)$$

where $K(e^2)$ is the complete elliptic integral of the first kind [12], clearly satisfies the left side of (7). To show that it also satisfies the right side we only need to recall that for an ellipse

$$P_{ellipse}/2\pi = 2(A/\pi)^{1/2}E(e^2)(1-e^2)^{-1/4}/\pi \underset{e\to o}{\longrightarrow} (A/\pi)^{1/2}(1+3e^2/64) \qquad (9)$$

where $E(e^2)$ is the complete elliptic integral of the second kind.

By virtue of the apparent validity of (7) for the capacitance of plates of arbitrary size and shape we are led to believe that other quantities of physical interest may be similarly sandwiched between bounds involving only the purely geometric parameters A and P.

III. POLARIZABILITIES AND TRANSMISSION COEFFICIENTS OF SMALL APERTURES

Let us now consider the transmission of electromagnetic energy through an electrically small aperture which is located in a plane screen of perfect conductivity and zero thickness. Since the aperture is small, the fields on the shadow side of the screen appear to emanate from dipoles located in the aperture. These electric and magnetic dipoles, having moments $\underline{p}$ and $\underline{m}$, radiate in free space and are linearly related to the incident traveling wave through the vector electric polarizability with components α_i and the dyadic magnetic polarizability with components β_{ij}. That is,

$$p_i = \varepsilon_o \alpha_i E_i^{inc} \qquad (i = 1,2,3) \qquad (10)$$

$$m_i = \mu_o \sum_{j=1}^{2} \beta_{ij} H_j^{inc} \qquad (i = 1,2) \qquad (11)$$

where $\mu_o = 4\pi \times 10^{-7}$ henries per meter. The incident electric and magnetic fields are plane waves of the form $\underline{E}^{inc} \exp i\,(\underline{k}\cdot\underline{r}-\omega t)$ and $\underline{H}^{inc} \exp i(\underline{k}\cdot\underline{r}-\omega t)$ where $\underline{r}$ is the position vector, $\underline{k}$ is the wave vector and ω is the frequency.

For a circular aperture of radius a the polarizabilities are given by the simple expressions

$$\alpha_i^{circle} = \frac{8}{3} a^3 \delta_{i3} \qquad (i = 1,2,3) \qquad (12)$$

$$\beta_{ij}^{circle} = \frac{16}{3} a^3 \delta_{ij} \qquad (i,j = 1,2) \qquad (13)$$

where $$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

The values 1, 2 and 3 correspond respectively to the directions $\hat{e}_{||}$, $\hat{e}_{\perp}$ and $\hat{e}_n$. The aperture plane is defined by the unit vectors $\hat{e}_{||}$ and $\hat{e}_{\perp}$ and the normal (pointing toward the shadow side) is defined by $\hat{e}_n = \hat{e}_{||} \times \hat{e}_{\perp}$ (see Figure 2). The polarizabilities are defined here for incident traveling waves and for dipoles radiating in free space. For short circuit incident fields and for dipoles radiating in the presence of a conducting wall, all values of the polarizabilities should be divided by the numeric 4.

For elliptic apertures with semi-axes a and b along $\hat{e}_{||}$ and $\hat{e}_{\perp}$ respectively we have

$$\alpha_i^{\text{ellipse}} = \frac{4\pi}{3} \frac{ab^2}{E(e^2)} \delta_{i3} \tag{14}$$

$$\beta_{ij}^{\text{ellipse}} = \begin{cases} \dfrac{4\pi}{3} \dfrac{ab^2e^2}{(1-e^2)[K(e^2)-E(e^2)]} \delta_{i1} & (15) \\[2ex] \dfrac{4\pi}{3} \dfrac{ab^2e^2}{E(e^2)-(1-e^2)K(e^2)} \delta_{i2} & (16) \end{cases}$$

where $e = (1-b^2/a^2)^{1/2}$ is the eccentricity of the ellipse and $K(e^2)$ and $E(e^2)$ are elliptic integrals of the first and second kind [13].

The transmission coefficient τ is defined as the ratio of the total far-field power transmitted through the aperture divided by the total power incident on the aperture. For the case where the principal axes of magnetic polarizability dyadic correspond to $\hat{e}_{||}$ and $\hat{e}_{\perp}$ we find

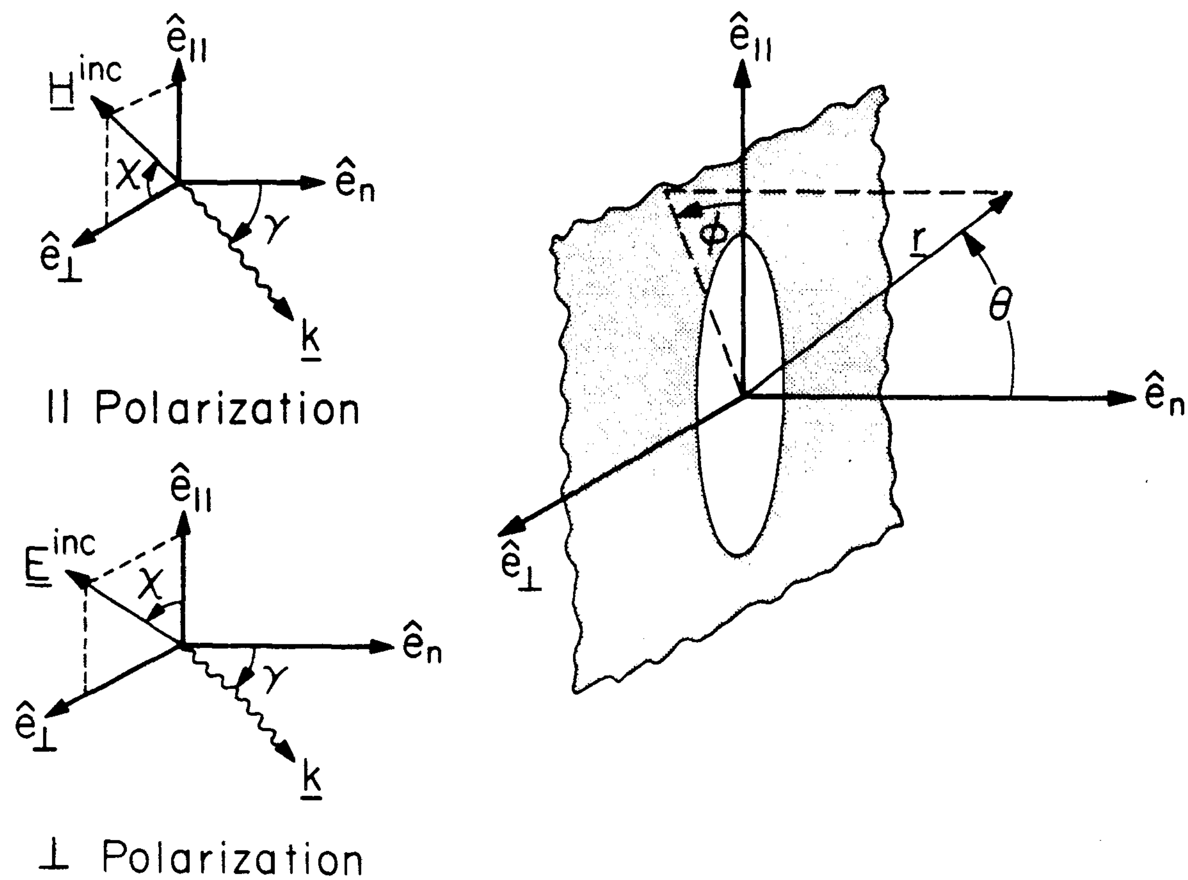


Fig. 2 Unit vectors $\hat{e}_{||}$ and $\hat{e}_\perp$ lie in the aperture plane, and $\hat{e}_n$. For || polarization $\underline{H}^{inc}$ is always parallel to the aperture plane and makes angle χ with respect to $\hat{e}_\perp$. For $\perp$ polarization $\underline{E}^{inc}$ is always parallel to the aperture plane and makes angle χ with respect to $\hat{e}_{||}$

Figure 2

$$\tau = \frac{k^4}{12\pi A}\left[\alpha_3^2 \sin^2\gamma \binom{0}{1} + (\beta_{11}^2 \sin^2\chi + \beta_{22}^2 \cos^2\chi)\binom{\cos^2\gamma}{1}\right] \tag{17}$$

for $\binom{\perp}{||}$ polarization [14]. Here γ is the angle of incidence, i.e. the angle between $\underline{k}$ and $\hat{e}_n$, and χ is the angle between $\underline{H}^{inc}$ and $\hat{e}_\perp$ for parallel polarization and is the angle between $\underline{E}^{inc}$ and $\hat{e}_{||}$ for perpendicular polarization (see Figure 2).

IV. BOUNDS ON POARIZABILITIES AND TRANSMISSION COEFFICIENT

Imitating the procedure we followed to establish bounds on the capacitance of plates, we now construct bounds on the mean magnetic polarizability β_m of a convex aperture by replacing the radius a, which appears in expression (13) for the polarizability of a circular aperture, by r_{in}(4) and r_{out} (5) of the aperture. Thus we get

$$\frac{16}{3}\left(\frac{A}{\pi}\right)^{3/2} \leq \beta_m \leq \frac{16}{3}\left(\frac{P}{2\pi}\right)^{3} \qquad (18)$$

where by definition $\beta_m = (\beta_{11} + \beta_{22})/2$.

To test the plausibility of (18) we examine several special cases. For the elliptic aperture of small eccentricity ($e \ll 1$), (18) becomes

$$\frac{16}{3}\left(\frac{A}{\pi}\right)^{3/2} \leq \beta_m \leq \frac{16}{3}\left(\frac{A}{\pi}\right)^{3/2}\left(1 + \frac{9}{64}e^4\right), \qquad (19)$$

(15) and (16) yield

$$\beta_m = \frac{16}{3}\left(\frac{A}{\pi}\right)^{3/2}\left(1 + \frac{3}{32}e^4\right), \qquad (20)$$

and thus we clearly see that (18) is satisfied in the case of mildly eccentric ellipses. It can also be shown that (18) holds true for elliptic apertures of arbitrary eccentricity ($0 \leq e \leq 1$) and for other convex apertures such as the rectangular and the rhombical aperture [14,15]. The fact that these test cases are in complete agreement with (18) leads us to believe that the assertion (18) is valid for all convex apertures.

Accepting the general validity of (18) and recalling that symmetrization reduces P without changing A we conclude that of all convex apertures of fixed area A the circular aperture possesses the smallest mean magnetic polarizability.

The electric polarizability contributes to transmission through small apertures only when the incident wave is obliquely incident and polarized parallel to the plane of incidence. To construct bounds for the electric polarizability we note that, for a circular aperture of radius a and area A, (12) can be written as

$$\alpha_i^{circle} = \frac{8}{3\pi^2} \frac{A^2}{a} \delta_{i3}. \tag{21}$$

Then by replacing the radius a of this expression by r_{in}(4) and r_{out}(5) we arrive at

$$\frac{16}{3\pi} \frac{A^2}{P} \leq \alpha_3 \leq \frac{8}{3} \left(\frac{P}{\pi}\right)^{3/2} \tag{22}$$

To test the plausibility of (22) we again consider the case of a mildly eccentric ellipse ($e << 1$). In this case (22) becomes

$$\frac{8}{3} \left(\frac{A}{\pi}\right)^{3/2} \left(1 - \frac{3}{64} e^4\right) \leq \alpha_3^{ellipse} \leq \frac{8}{3} \left(\frac{A}{\pi}\right)^{3/2} \tag{23}$$

and from (14) we have

$$\alpha_3^{ellipse} = \frac{8}{3} \left(\frac{A}{\pi}\right)^{3/2} \left(1 - \frac{3}{64} e^4\right). \tag{24}$$

Obviously, expression (24) is equal to the lower bound in (23). Furthermore, with the aid of (14) it can be verified that the lower bound in (22) is precisely the value of the electric polarizability of ellipses of arbitrary eccentricity [9,15]. Also we note that the electrical polarizabilities of rectangular and rhombical apertures satisfy (22) [14,15].

Assuming the validity of (22) and invoking symmetrization, we find that of all convex apertures of fixed area the circular aperture possesses the largest electric polarizability.

The bounds that have been proposed for the electric (22) and mean magnetic (18) polarizabilities can be used to obtain bounds on the transmission coefficient (17). In some modern applications the quantity of interest is the upper bound for the case where the incident wave is directed and polarized to maximize the transmission through the given aperture. Clearly, maximum possible transmission through a given aperture occurs when the incident wave is parallel polarized and is made to fall on the aperture at grazing incidence. To find the upper bound for maximum possible transmission we use (22) and note that $r_{out} \geq r_{in}$. Thus

$$\alpha_3^2 \sin^2\gamma \leq \frac{64}{9} \left(\frac{A}{\pi}\right)^3 \leq \frac{64}{9} \left(\frac{P}{2\pi}\right)^6 . \qquad (25)$$

Moreover, in view of (18) we can write

$$\beta_{11}^2 \sin^2\chi + \beta_{22}^2 \cos^2\chi \leq \frac{1024}{9} \left(\frac{P}{2\pi}\right)^6 \qquad (26)$$

Substituting (25) and (26) into (17)we thus obtain the following expression for the maximum possible transmission through a small aperture of area A and perimeter P:

$$\tau \leq \frac{68(P/\lambda)^6}{27\pi^3(A/\lambda^2)} \qquad (27)$$

where $\lambda = 2\pi/k$ is the wavelength of the incident radiation.

Since symmetrization reduces P and keeps A unchanged we see from (27) that the maximum possible transmission decreases as the aperture is symmetrized. That is, the maximum possible transmission decreases as the shape of the aperture approaches that of a circle [16].

V. CONCLUSIONS

By delimiting the polarizabilities of a small convex aperture of arbitrary shape and given area we have found upper and lower bounds on its transmission coefficient. Symmetrizing the aperture we see that the maximum possible tranmission decreases as the shape of the aperture approaches that of a circle. For example, the maximum possible transmission decreases as the shape of the aperture is changed from that of an equilateral triangle to that of a square and finally to that of a circle.

The bounds are simple to evaluate from a knowledge of the aperture's area and perimeter and therein lies the desirability and economy of this method.

It appears that this method of estimation can be generalized to handle other boundary-value problems and thus provide information as to how their solutions are modified when there is a change of shape.

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