EMP 3-37

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Interaction Notes
Note 337
1 April 1978

A Note on the Transverse Distributions of Surface Charge Densities on Multi-conductor Transmission Lines

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ABSTRACT

This note is concerned with a problem which occasionally arises in the general area of multiconductor transmission line theory. In the past, the method of moments has been applied [1, 2] for the computation of transverse charge distributions and the capacitive coefficient matrix for electrostatic systems formed by multiconductor transmission lines with prescribed voltages on each line. But classically, there has been an interest in the related problem of finding the transverse charge distributions given the net charge on each line [3,4]. When the net charges are prescribed, conformal mapping techniques have been successfully employed in determining the charge distributions for certain special cases e.g., the two wire problem [5] and a planar grating [6]. The integral equation formulation presented in this note for the charge distributions, is applicable to a general system of parallel conductors, not necessarily in the same plane, as long as their net charges are prescribed.

Contents

ACKNOWLEDGEMENT

We are grateful to our colleague Dr. T. K. Liu for useful discussions during the course of this work. Thanks are also due to Dr. C. E. Baum for his continued interest and guidance.

I. Introduction

A useful problem in analyzing a multiconductor transmission line or, equivalently, an electrostatic system of parallel cylindrical conductors is the determination of the transverse charge distributions as well as the capacitance coefficient matrix, when the potential on each conductor is assumed to be given. When the conductors are sufficiently far apart, the charge densities become uniform around each conductor and the elastance coefficient matrix [S] elements are given simply by [7],

$$S_{ii} = \frac{1}{2\pi\varepsilon_o} \ell n\left(\frac{c_{i0}^2}{a_i a_0}\right)$$

$$S_{ij} = \frac{1}{2\pi\varepsilon_o} \ell n\left(\frac{c_{i0} c_{j0}}{c_{ij} a_0}\right) \quad ; \ (i \neq j) \quad ; \text{ for } i,j = 1,2..N \qquad (1.1)$$

The above is valid for an electrostatic system of (N + 1) parallel conductors shown in Figures 1.1 and 1.2, when the conductor spacings are large compared to the radii. As indicated in Figure 1.2, a_i's are the radii and c_{ij}'s are the distances between the centers of the i th and j th conductors in a transverse plane z = constant. ε_o is the permitivity of the surrounding medium, assumed here to be free space without any loss of generality. However, when the conductors are not far apart and the proximity effects are

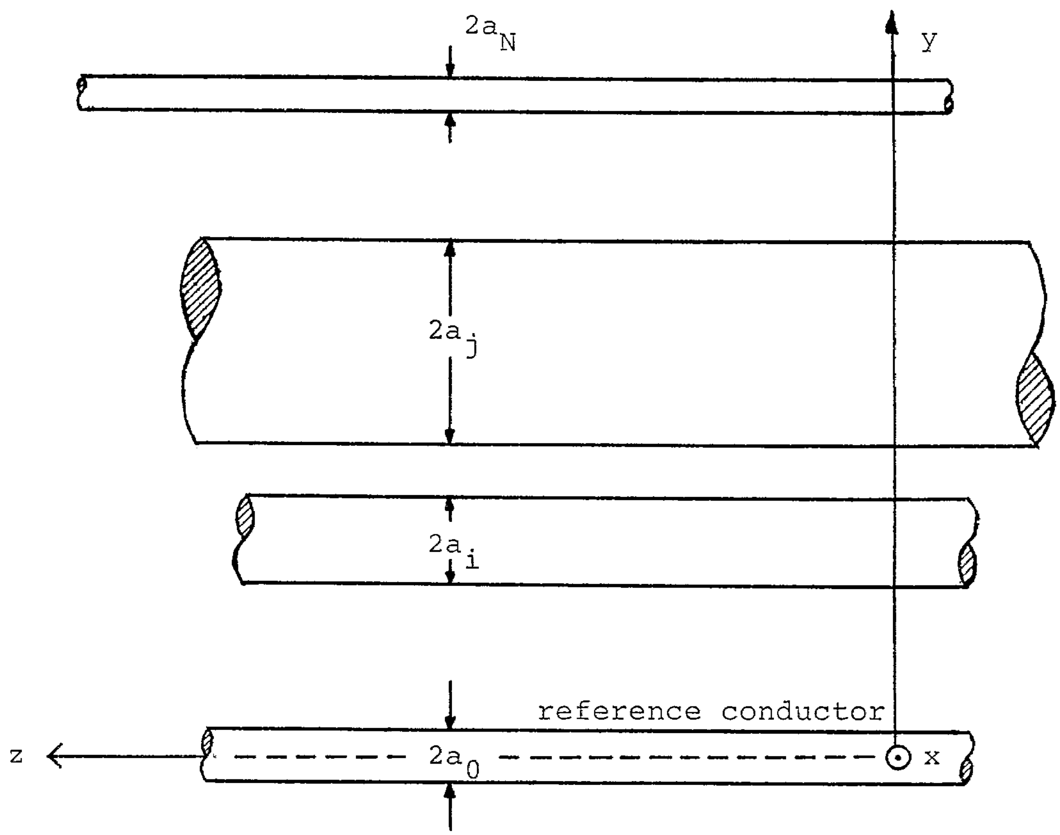


Figure 1.1 A system of (N + 1) bare conductors

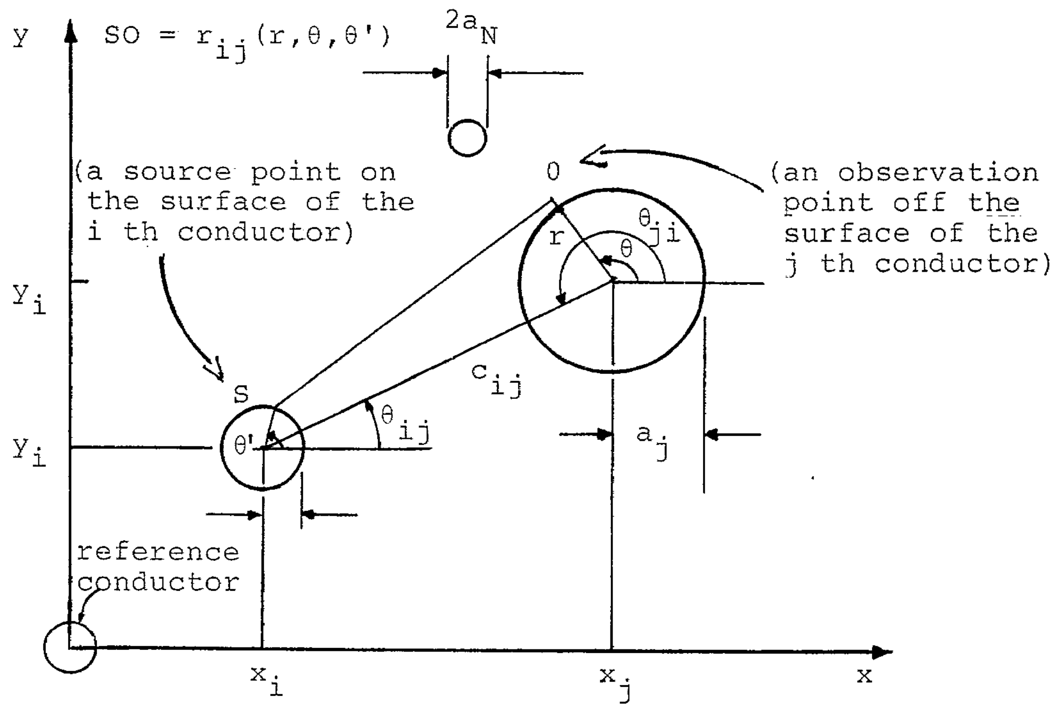


Figure 1.2 Geometry for computing a typical distance $r_{ij}(r,\theta,\theta')$

not negligible, no such closed form expressions are available (except in some special cases) for the charge distributions and the capacitance or elastance coefficient matrix elements. When proximity effects are present, one can identify two different problems, 1) the potential problem, i.e., when the given quantities are potentials on each conductor and 2) the charge problem, i.e., when the total per-unit-length charge on each wire is prescribed. In both problems, it is desired to compute the exact charge distribution around the circumference of each conductor in the bundle, as well as compute the potential and electric field in the region surrounding the wires. We will now elaborate on each of these problems separately in the following two sections.

II. Notational Details and Review of the Potential Problem

For a general system of (N + 1) parallel conductors with assigned potentials, an application of the method of moments has been found to be efficient in determining the coefficients of capacitance matrix elements [1]. With reference to Figure 1.1, let us designate the density of surface charge on the i th conductor by Fourier series expansions as,

$$\sigma_i(\theta) = \left[\sum_{\ell=0}^{\infty} \left\{ \sigma_{i\ell} \cos(\ell\theta) + \sigma'_{i\ell} \sin(\ell\theta) \right\} \right] \quad (C/m^2) \tag{2.1a}$$

$$= \sigma_{i0} \left[1 + \sum_{\ell=1}^{\infty} \left\{ \alpha_{i\ell} \cos(\ell\theta) + \beta_{i\ell} \sin(\ell\theta) \right\} \right] \tag{2.1b}$$

$$= \sigma_{i0} \left[1 + \tau_i(\theta) \right] = \sigma_{i0}\, t_i(\theta) \tag{2.1c}$$

where

$t_i(\theta)$ ≡ dimensionless transverse charge distribution

$\alpha_{i\ell}$ ≡ dimensionless Fourier cosine coefficients = $(\sigma_{i\ell}/\sigma_{i0})$

$\beta_{i\ell}$ ≡ dimensionless Fourier sine coefficients = $(\sigma'_{i\ell}/\sigma_{i0})$

σ_{i0} ≡ "dc term" in the Fourier series expansion, which is related to the total charge on the i th conductor.

Note that, in all of the above $i = 0, 1, 2, \ldots, N$ and the zeroth conductor is chosen to be the reference conductor for assigning the potentials according as,

$$V_i = (\phi_i - \phi_0) \quad \text{for} \quad i = 1, 2, \ldots, N. \tag{2.2}$$

The total charge per unit length, Q_i', of the i th conductor is then given by,

$$Q_i' = \int_0^{2\pi} \sigma_i(\theta)\ a_i\ d\theta = 2\pi a_i\ \sigma_{i0} \quad (\text{C/m}) \tag{2.3}$$

Thus, only the constant, or the dc term, in the Fourier Series expansion contributes to the net charge, and therefore, the charge density may be written as,

$$\sigma_i(\theta) = (Q_i'/2\pi a_i)\ t_i(\theta) \quad \text{for } i = 0, 1, 2, \ldots, N \tag{2.4}$$

We also assume the entire electrostatic system to be electrically neutral so that

$$Q_0' = -\sum_{i=1}^{N} Q_i' \tag{2.5}$$

Under the above described situation, the potential at any point $(\vec{r})$, just off the surface of the j th conductor, due to <u>all</u> of the charge distributions is given (for j = 0,1,...N) by [8],

$$\phi_j(\vec{r}) = \phi_j(r,\theta) = \frac{-1}{4\pi\varepsilon_o} \sum_{i=0}^{N} \int_0^{2\pi} \sigma_i(\theta')\ \ell n\left[r_{ij}^2(r,\theta,\theta')\right]\ a_i\ d\theta' \tag{2.6}$$

where θ and θ' are the polar angles measured anti-clockwise from the positive x axis along the j th and i th conductors respectively (see Figure 1.2). From purely geometrical considerations in Figure 1.2, we have

$$r_{ij}^2(r,\theta,\theta') = \Big[c_{ij}^2 + r^2 + a_i^2 - 2ra_i \cos(\theta-\theta')$$

$$-2c_{ij}\left\{a_i \cos(\theta'-\theta_{ij}) - r \cos(\theta-\theta_{ij})\right\}\Big] \quad (2.7)$$

By letting the point $\vec{r}$ fall on the surface of the j th conductor, i.e., $r = |\vec{r}| = a_j$, one has the voltage integral equation (VIE)

$$\phi_j(a_j,\theta) = \frac{-1}{4\pi\varepsilon_o} \sum_{i=0}^{N} \int_0^{2\pi} \sigma_i(\theta') \,\ell n\left[r_{ij}^2(a_j,\theta,\theta')\right] a_i \, d\theta' \quad (2.8)$$

The conventional procedure in solving the potential problem is to cast the above equation into matrix form by choosing a sufficient number of basis functions (typically sines and cosines like in equation (2.1b)) to represent the unknown function $\sigma_i(\theta')$ and, usually, the same number of match points on each of the conductors. This procedure results in a matrix of matrices (or a super matrix) of the following form

$$\begin{bmatrix} \Psi_0 \\ \Psi_1 \\ \Psi_2 \\ \vdots \\ \Psi_N \end{bmatrix} = \begin{bmatrix} \lambda_{00} & \lambda_{01} & \cdots & \lambda_{0N} \\ \lambda_{10} & \lambda_{11} & \cdots & \lambda_{1N} \\ \lambda_{20} & \lambda_{21} & \cdots & \lambda_{2N} \\ \vdots & & & \\ \lambda_{N1} & \lambda_{N2} & & \lambda_{NN} \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_N \end{bmatrix} \quad (2.9)$$

where each element in the above matrix is a matrix given by

$$[\Psi]_i = \begin{bmatrix} \phi_i \\ \phi_i \\ \cdot \\ \cdot \\ \cdot \\ \phi_i \\ \phi_i \\ \phi_i \\ \cdot \\ \cdot \\ \cdot \\ \phi_i \end{bmatrix}_{1\times(2P+1)} ; \qquad [\gamma]_i = \begin{bmatrix} \sigma_{i0} \\ \sigma_{i1} \\ \cdot \\ \cdot \\ \cdot \\ \sigma_{iP} \\ \sigma'_{i1} \\ \sigma'_{i2} \\ \cdot \\ \cdot \\ \cdot \\ \sigma'_{iP} \end{bmatrix}_{1\times(2P+1)} \quad \text{for } i = 0,1,\ldots N \tag{2.10}$$

$$\text{and } [\lambda]_{ij} = \begin{bmatrix} d^{ij}_{11} & \cdots\cdots & d^{ij}_{1,2P+1} \\ \vdots & & \vdots \\ d^{ij}_{2P+1,1} & & d^{ij}_{2P+1,2P+1} \end{bmatrix}_{(2P+1)\times(2P+1)} \quad \left\{ \begin{matrix} \text{for } i=0,1,..N \\ \text{and } j=0,1,..N \end{matrix} \right\} \tag{2.11}$$

We have assumed that each of the charge distributions is expanded into P number of cosine and P number of sine terms, so that there is a total of (2P + 1) unknown coefficients, including the dc term, associated with each of the (N + 1) conductors. For instance, the (2P + 1) unknown coefficients associated with the i th conductor are $\left[\sigma_{i0}, \sigma_{i1}, \sigma_{i2} \cdots \sigma_{iP}, \sigma'_{i1}, \sigma'_{i2} \cdots \sigma'_{iP}\right]$ of equation (2.1a). Next, the linear system of equations (2.9) have their solution given by

$$[\gamma] = [\lambda]^{-1} [\Psi] = [\Gamma] [\Psi] \tag{2.12}$$

Since the potentials on various conductors are prescribed, and the λ matrix is filled by integrals of the form

$$\int_0^{2\pi} \ln \left[r_{ij}^2 (a_j, \theta_k, \theta') \right] d\theta' \quad \text{or} \quad \int_0^{2\pi} \cos(\ell\theta_k) \ln \left[r_{ij}^2 (a_j, \theta_k, \theta') \right] d\theta'$$

$$\text{or} \int_0^{2\pi} \sin(\ell\theta_k) \ln \left[r_{ij}^2 (a_j, \theta_k, \theta') \right] d\theta' ,$$

the problem is formally completed, and we have the solution which consists of the Fourier cosine and sine coefficients of all the charge distributions. As a by-product in the above procedure, we also obtain the generalized capacitive coefficient matrix $[\mathcal{K}]_{(N+1)\times(N+1)}$. From a knowledge of the $[\mathcal{K}]$ matrix and by imposing the charge neutrality of equation (2.5), one gets the capacitive coefficient matrix $[K]_{N\times N}$ after referencing all the potentials with respect to the zeroth conductor. The [K] matrix essentially relates the per-unit-length charge Q_i' of equation (2.3) on each wire with the potentials referenced to the zeroth conductor ($V_i = \phi_i - \phi_0$), according as

$$\begin{bmatrix} Q_1' \\ Q_2' \\ Q_3' \\ \vdots \\ Q_N' \end{bmatrix} = \begin{bmatrix} K_{11} & \cdots & K_{1N} \\ K_{21} & \cdots & K_{2N} \\ K_{31} & \cdots & K_{3N} \\ \vdots & & \vdots \\ K_{N1} & & K_{NN} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_N \end{bmatrix} \tag{2.13}$$

As a practical matter, it is noted that the $[\mathcal{K}]$ matrix elements are easily related to the $[\lambda]$ matrix elements, and that $[\mathcal{K}]$ and [K] matrices are both symmetric. Furthermore an efficient procedure for computing [K] from $[\mathcal{K}]$ may be found in Ref. [7]. This completes our brief review of the method of moments as applied to the potential problem.

III. Formulation of the Charge Problem

In this section, we consider the problem of determining the transverse charge distributions on all of the (N + 1) conductors given the total per-unit-length charges on them. The starting point is once again the potential $\phi_j(\vec{r})$ at an observation point located just off the surface of the j th conductor (see Figure 1.2), due to all of the charge distributions. This was given by equation (2.6), for $j = 0, 1, 2, \ldots, N$, as

$$\phi_j(\vec{r}) = \phi_j(r,\theta) = \frac{-1}{4\pi\varepsilon_o} \sum_{i=0}^{N} \int_0^{2\pi} \sigma_i(\theta') \, \ell n\left[r_{ij}^2(r,\theta,\theta')\right] a_i \, d\theta' \tag{3.1}$$

It is recalled that in the potential problem of the preceding section, this equation was specialized to the j th conductor surface ($r = a_j$), matrisized and solved for the Fourier coefficients of $\sigma_i(\theta)$. However, for the problem at hand, we eliminate the potentials by considering the normal E-field on the j th conductor and equating it to the surface charge, i.e.,

$$E_r(a_j,\theta) = \left.\frac{-\partial\phi_j}{\partial r}\right|_{r=a_j} = \frac{\sigma_j(\theta)}{\varepsilon_o} \tag{3.2}$$

Using equation (3.1) in (3.2),

$$\sigma_j(\theta) = \frac{1}{4\pi} \sum_{i=0}^{N} \frac{\partial}{\partial r} \left.\left[\int_0^{2\pi} \sigma_i(\theta') \, \ell n\left\{r_{ij}^2(r,\theta,\theta')\right\} a_i \, d\theta'\right]\right|_{r=a_j} \tag{3.3}$$

Separating the singular term on the right side

$$\sigma_j(\theta) = \left[\frac{1}{4\pi} \frac{\partial}{\partial r} \int_0^{2\pi} \sigma_j(\theta') \; \ell n \left\{ r_{jj}^2(r,\theta,\theta') \right\} a_j \; d\theta' \right.$$

$$\left. + \frac{1}{4\pi} \sum_{\substack{i=0 \\ i \neq j}}^{N} \frac{\partial}{\partial r} \int_0^{2\pi} \sigma_i(\theta') \left\{ \ell n \; r_{ij}^2(r,\theta,\theta') \right\} a_j \; d\theta' \right]_{r=a_j}$$

(3.4)

The singular term can be simplified as follows. The singular kernel is

$$\left. \frac{\partial}{\partial r} \ell n(r_{jj}^2) \right|_{r=a_j} = \left. \frac{\partial}{\partial r} \ell n\left[r^2 + a_j^2 - 2ra_j \cos\,(\theta-\theta') \right] \right|_{r=a_j}$$

$$= 2 \lim_{r \to a_j} \left\{ \frac{r - a_j \cos(\theta-\theta')}{r^2 + a_j^2 - 2ra_j \cos(\theta-\theta')} \right\}$$

The singularity appears when $\theta' \to \theta$. For values of θ' not equal to θ, the kernel is given exactly by $(1/a_j)$. Near the singular point $(\theta' \to \theta)$, using $\cos(\theta-\theta') \simeq 1 - [(\theta-\theta')^2/2]$, this singular kernel becomes

$$= 2 \lim_{r \to a_j} \left[\frac{1}{2a_j} + \frac{r - a_j}{(r-a_j)^2 + a_j^2 \; (\theta-\theta')^2} \right]$$

$$= \frac{1}{a_j} \left[1 + 2\pi \lim_{\varepsilon \to 0} \frac{1}{\pi} \frac{\varepsilon}{[(\theta-\theta')^2 + \varepsilon^2]} \right]$$

$$= \left[\frac{1 + 2\pi \; \delta(\theta-\theta')}{a_j} \right]; \; \left\{ \begin{array}{l} \text{using equation (2.4)} \\ \text{of Ref. [9].} \end{array} \right\} \qquad (3.5)$$

This form of the singular kernel is now valid for the entire range of θ' $(0 \leq \theta' \leq 2\pi)$. It is noted that Smith [10] found a similar behaviour of the singular kernel in solving for the high frequency current distributions in the context of skin depth resistance for a system of parallel coplanar conductors carrying equal currents in the same direction. Some fairly standard methods of kernel investigations often used at high frequencies are seen useful in the present electrostatic problem.

The nonsingular part contained in the summation term of equation (3.4), denoted here by $N_{ij}(\theta,\theta')$, is given by

$$N_{ij}(\theta,\theta') = \frac{\partial}{\partial r} \ln \left\{ r_{ij}^2(r,\theta,\theta') \right\} \Bigg|_{r=a_j}$$

$$= 2\left[\frac{a_j - a_i \cos(\theta-\theta') + c_{ij} \cos(\theta-\theta_{ij})}{r_{ij}^2(a_j,\theta,\theta')} \right] \qquad (3.6)$$

Upon using the results of the singular and nonsingular kernels in equation (3.4), we have

$$\sigma_j(\theta) = \frac{1}{4\pi} \int_0^{2\pi} \sigma_j(\theta') \left[1 + 2\pi\delta(\theta-\theta')\right] d\theta'$$

$$+ \frac{1}{4\pi} \sum_{\substack{i=0 \\ i \neq j}}^{N} \int_0^{2\pi} \sigma_i(\theta') \, N_{ij}(\theta,\theta') \, d\theta' \qquad (3.7)$$

Using equations (2.3), (2.4) and the sifting property of the Dirac delta function [9], we have

$$\sigma_j(\theta) = \frac{\sigma_{j0}}{2} + \frac{\sigma_j(\theta)}{2} + \frac{1}{4\pi}\left[\sum_{\substack{i=0\\ i\neq j}}^{N} a_i \int_0^{2\pi} \sigma_i(\theta')\, N_{ij}(\theta,\theta')\, d\theta'\right]$$

or

$$\sigma_j(\theta) = \sigma_{j0} + \frac{1}{2\pi} \sum_{\substack{i=0\\ i\neq j}}^{N} \int_0^{2\pi} \sigma_i(\theta')\, G_{ij}(\theta,\theta')\, d\theta' \qquad (3.8)$$

where the dimensionless non-singular kernel G_{ij} is given by

$$G_{ij}(\theta,\theta') = a_i N_{ij}(\theta,\theta') \qquad (3.9)$$

In terms of the total charges per unit length and the dimensionless distributions of charge $t_i(\theta)$ defined by equation (2.4), the charge integral equation (CIE) of equation (3.8) becomes

$$\left[Q_j' \, t_j(\theta) = Q_j' + \frac{1}{2\pi} \sum_{\substack{i=0\\ i\neq j}}^{N} Q_i' \int_0^{2\pi} t_i(\theta')\, M_{ij}(\theta,\theta')\, d\theta'\right] \qquad (3.10)$$

$$\text{for} \quad j = 0, 1, 2, \ldots, N$$

This is a system of CIE which may be solved for the N unknown transverse distributions of charges $t_i(\theta)$, given the N total per-unit length charges Q_i on each of the N wires which exclude the reference conductor. It is recalled that in the above CIE,

$$t_i(\theta) = \left[1 + \sum_{\ell=1}^{\infty} \alpha_{i\ell} \cos(\ell\theta) + \sum_{\ell=1}^{\infty} \beta_{i\ell} \sin(\ell\theta)\right] \tag{3.11a}$$

$$M_{ij}(\theta,\theta') = \frac{a_j}{a_i} G_{ij}(\theta,\theta') = a_j N_{ij} (\theta,\theta')$$

$$= 2\left[\frac{a_j^2 - a_i a_j \cos(\theta-\theta') + c_{ij} \cos(\theta-\theta_{ij})}{r_{ij}^2(a_j,\theta,\theta')}\right] \tag{3.11b}$$

$$r_{ij}^2(a_j,\theta,\theta') = \left[c_{ij}^2 + a_j^2 + a_i^2 - 2a_i a_j \cos(\theta-\theta') - 2c_{ij}\left\{a_i \cos(\theta'-\theta_{ij}) - a_j \cos(\theta-\theta_{ij})\right\}\right] \tag{3.11c}$$

It is observed that if one were to solve the CIE of (3.10) numerically, the upper limits of infinity in the summations of equation (3.11a) would be replaced by a suitably chosen and large enough integer. Also, although there are $(N + 1)$ conductors, only N number of charge per unit length Q_i' for $i = 1, 2, \ldots N$ need be prescribed in view of the fact that the entire system is neutral. That is, the zeroth conductor carries a total charge per unit length equal to negative of the sum of the net charges per unit length on the remainder of the conductors.

At this stage, we may conclude the formulation of the CIE and consider a simple illustrative example of a balanced two wire transmission line $(N = 1)$ in the following subsection.

A. An Illustrative Example

The transverse charge distributions on a balanced two wire transmission line, or a 1-line (see Figure 3.1) is solved here using the CIE. It is our belief that this approach has not been used previously for a balanced transmission line, and a closed form analytical solution exists for this case, by conformal mapping techniques or otherwise, to verify the numerical solution. The CIE leads to a coupled pair of integral equations for $t_o(\theta)$ and $t_1(\theta)$ given by

$$t_0(\theta) = 1 - \frac{1}{2\pi} \int_0^{2\pi} t_1(\theta') \, U_{10}(\theta,\theta') \, d\theta' \tag{3.12a}$$

$$t_1(\theta) = 1 - \frac{1}{2\pi} \int_0^{2\pi} t_0(\theta') \, U_{01}(\theta,\theta') \, d\theta' \tag{3.12b}$$

where the kernels for this case of unequal radii are given by

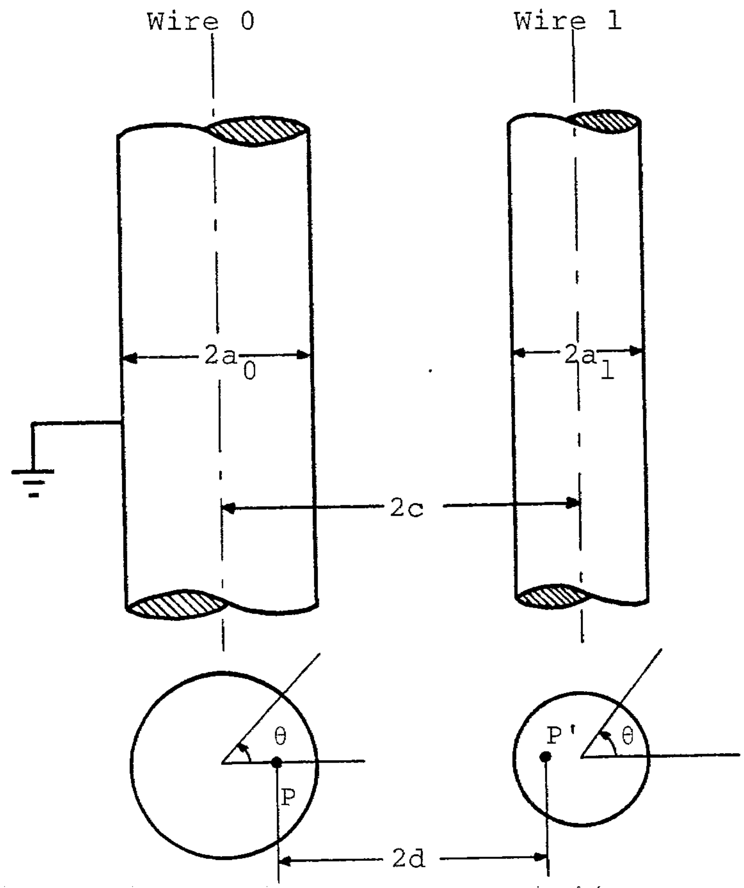
$$U_{10}(\theta,\theta') = \left[\left\{ a_1^2 - a_1 a_0 \cos(\theta-\theta') + 2a_1 c \cos(\theta) \right\} \Big/ r_{10}^2 \right] \tag{3.13a}$$

$$U_{01}(\theta,\theta') = \left[\left\{ a_0^2 - a_1 a_0 \cos(\theta-\theta') + 2a_0 c \cos(\theta) \right\} \Big/ r_{01}^2 \right] \tag{3.13b}$$

$$r_{01}^2 = r_{10}^2 = \Big[4c^2 + a_0^2 + a_1^2 - 2a_0 a_1 \cos(\theta-\theta') - 2c \left\{ a_1 \cos(\theta') - a_0 \cos(\theta) \right\} \Big] \tag{3.13c}$$

For the present purpose of illustration, it suffices to consider the case of equal radii $a_0 = a_1 = a$ in which case

$$t_0(\theta) = t_1(\pi-\theta) \tag{3.14}$$



Potential	0	V_1
Charge/unit length	$Q'_0 = -Q'_1$	Q'_1
Charge density	$\left[\sigma_0(\theta) = \left(\frac{Q'_0}{2\pi a_0}\right) t_0(\theta)\right]$	$\left[\sigma_1(\theta) = \left(\frac{Q'_1}{2\pi a_1}\right) t_1(\theta)\right]$
$[\text{If } (a_0 = a_1 = a]^*$		
Charge/unit length	$Q'_0 = -Q'_1$	Q'_1
Charge density	$\left[\sigma_0(\theta) = \left(\frac{-Q'_1}{2\pi a}\right) t_1(\pi-\theta)\right]$	$\left[\sigma_1(\theta) = \left(\frac{Q'_1}{2\pi a}\right) t_1(\theta)\right]$

* Note that for this case $t_0(\theta) = t_1(\pi-\theta)$

Figure 3.1 A balanced two wire transmission line or a 1-line and we need only solve one integral equation instead of a coupled pair and it is given by

$$t_1(\theta) = 1 - \frac{1}{2\pi} \int_0^{2\pi} t_1(\pi-\theta) \; E(\theta,\theta') \; d\theta' \tag{3.15}$$

where the kernel specialized for the equal radii case $(a_1 = a_0 = a)$ is given by

$$E(\theta,\theta') = \left[\frac{\sin^2\left(\frac{\theta-\theta'}{2}\right) + \Delta \cos(\theta)}{\sin^2\left(\frac{\theta-\theta'}{2}\right) + \Delta \cos(\theta) - \Delta \cos(\theta') + \Delta^2} \right] \tag{3.16}$$

with $\Delta \equiv$ (half separation/radius) = (c/a) in Figure 3.1.

Furthermore, because of the symmetry in the problem, we need only solve for $t_1(\theta)$ in the range $0 \leq \theta \leq \pi$, so that the CIE of (3.15) becomes

$$\left[t_1(\theta) = 1 - \frac{1}{2\pi} \int_0^{\pi} t_1(\theta) \; S(\theta,\theta') \; d\theta' \right] \tag{3.17}$$

where the sum kernel $S(\theta,\theta')$ is given by

$$S(\theta,\theta') = E(\theta,\theta'-\pi) + E(\theta,\pi-\theta')$$

$$= \left\{ \left[\frac{\cos^2\left(\frac{\theta-\theta'}{2}\right) + \Delta \cos(\theta)}{\cos^2\left(\frac{\theta-\theta'}{2}\right) + \Delta \cos(\theta) + \Delta \cos(\theta') + \Delta^2} \right] + \left[\frac{\cos^2\left(\frac{\theta+\theta'}{2}\right) + \Delta \cos(\theta)}{\cos^2\left(\frac{\theta+\theta'}{2}\right) + \Delta \cos(\theta) + \Delta \cos(\theta') + \Delta^2} \right] \right\} \tag{3.18}$$

This is recognized as a Fredholm integral equation of the second kind and has been solved numerically for several values of Δ.

An alternate analytical procedure in solving for the transverse charge distribution on a balanced two wire transmission line consists of determining the effective separation or the charge centroid separation 2d of Figure 3.1. This is available, e.g., [11, 12].

$$d = c\sqrt{1 - \left(\frac{a_0 + a_1}{2c}\right)^2}\sqrt{1 - \left(\frac{a_0 - a_1}{2c}\right)^2} \qquad (3.19)$$

For the special case of $a_0 = a_1 = a$, this becomes

$$d = \sqrt{c^2 - a^2} \qquad (3.20)$$

By placing two line charges at P and P' of Figure (3.1), one can derive the charge densities on both the wires and they are given by

$$\left[\sigma_1^{(a)}(\theta) = \left(\frac{Q_1'}{2\pi a}\right) t_1^{(a)}(\theta)\right] \qquad (3.21a)$$

$$\left[\sigma_0^{(a)}(\theta) = \left(\frac{Q_0'}{2\pi a}\right) t_0^{(a)}(\theta) = \left\{\frac{-Q_1'}{2\pi a} t_1^{(a)}(\pi-\theta)\right\}\right] \qquad (3.21b)$$

where the dimensionless transverse charge distribution is seen to be

$$t_1^{(a)}(\theta) = \left[\frac{1 + A\cos(\theta)}{1 + 2A\cos(\theta) + A^2} - \frac{1 + B\cos(\theta)}{1 + 2B\cos(\theta) + B^2}\right] \qquad (3.21c)$$

where A and B are geometrical constants given by

$$\left[A = \Delta - \sqrt{\Delta^2 - 1}\right] \; ; \; \left[B = \Delta + \sqrt{\Delta^2 - 1}\right] \qquad (3.21d)$$

and Δ has been defined in conjunction with equation (3.16). The superscript (a) in the above denotes the analytical solution. The numerical solution for $t_1(\theta)$ by solving the CIE of equation (3.15) on a desk-top computer using pulse basis functions in a method of moments is compared with this analytical solution $t_1^{(a)}(\theta)$ in Figure 3.2. The agreement is seen to be very good except for small values of Δ near $\theta = 180°$. This is attributed to the particular numerical procedure used in solving equation (3.15). The accuracy of the numerical solution can certainly be improved by considering larger matrix sizes and efficient integration routines, but this is not clearly warranted.

As may be expected, the charge densities concentrate on the inside (i.e., $\theta = 180°$ for wire 1 and $\theta = 0$ for wire 0) as Δ gets smaller or the wires brought closer together. For $\Delta > 5$, the "thin wire" approximation or the rotational symmetry holds resulting in $t_1(\theta) \simeq 1$. But for $\Delta \leq 5$, the transverse or angular distribution of charge departs significantly from the value of unity.

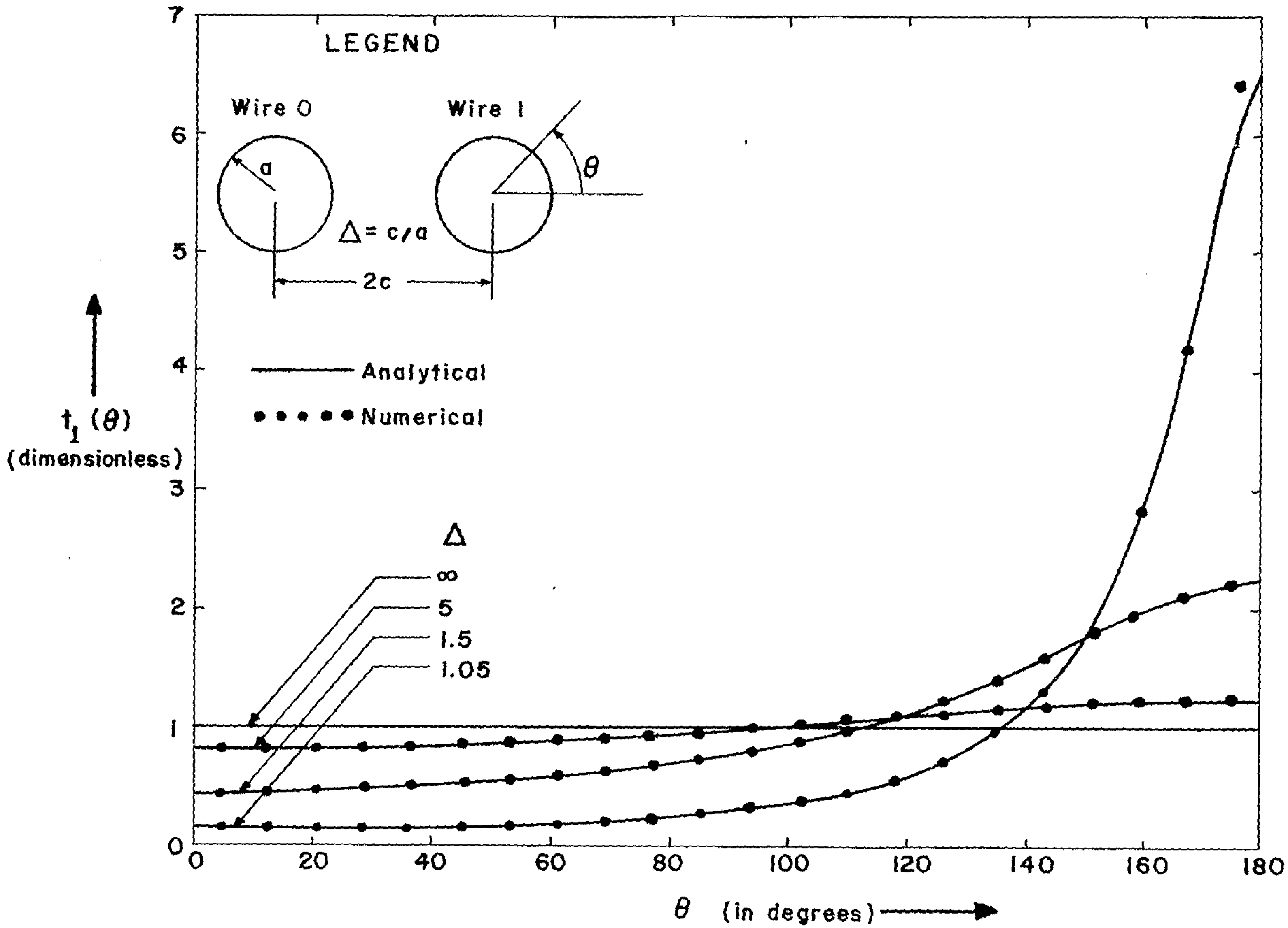


Figure 3.2 Electrostatic surface charge density as a function of the azimuthal angle on a balanced two-wire transmission line

Furthermore, one may also determine analytically the Fourier cosine coefficients (sine coefficients vanish because of symmetry) by performing a Fourier series analysis of the analytical solution, by setting

$$t_1^{(a)}(\theta) = 1 + \sum_{\ell=1}^{\infty} \beta_\ell^{(a)} \cos(\ell\theta) \qquad (3.22)$$

and

$$\beta_\ell^{(a)} = \frac{2}{\pi} \int_0^\pi t_1^{(a)}(\theta) \cos(\ell\theta)\, d\theta \qquad (3.23)$$

Substituting for $t_1^{(a)}(\theta)$ from equation (3.21c), we have

$$\beta_\ell^{(a)} = \frac{2}{\pi} \int_0^\pi \cos(\ell\theta) \left[\frac{1 + A\cos(\theta)}{1 + 2A\cos(\theta) + A^2} - \frac{1 + B\cos(\theta)}{1 + 2B\cos(\theta) + B^2}\right] d\theta \qquad (3.24)$$

One can perform the integrals, using tabulated integrals [13] (note that $A < 1$ and $B > 1$), to obtain

$$\begin{aligned} \beta_\ell^{(a)} &= (-1)^\ell \left[A^\ell + B^{-\ell}\right] \\ &= (-1)^\ell \left[\left\{\Delta - \sqrt{\Delta^2 - 1}\right\}^\ell + \left\{\Delta + \sqrt{\Delta^2 - 1}\right\}^{-\ell}\right] \end{aligned} \qquad (3.25)$$

It may be seen here that for large Δ (say $\Delta > 5$) all the coefficients become small compared with unity resulting in $t_1^{(a)}(\theta) \simeq 1$. Good agreement was also obtained between analytical β's and numerical β's obtained by solving equation (3.15) using cosinusoidal basis functions in a method of moments. These results are presented in Table 1.

	Δ = 5		Δ = 2		Δ = 1.5		Δ = 1.05	
ℓ	$\beta_\ell^{(analy)}$	$\beta_\ell^{(numer)}$	$\beta_\ell^{(analy)}$	$\beta_\ell^{(numer)}$	$\beta_\ell^{(analy)}$	$\beta_\ell^{(numer)}$	$\beta_\ell^{(analy)}$	$\beta_\ell^{(numer)}$
0	1	1	1	1	1	1	1	1
1	$-.2020$	$-.2020$	$-.5358$	$-.5359$	$-.7639$	$-.7639$	$-.1459\times10^{-1}$	$-.1454\times10^{-1}$
2	$.2041\times10^{-1}$	$.2041\times10^{-1}$	$.1435$	$.1435$	$.2917$	$.2917$	$.1065\times10^{-1}$	$.1058\times10^{-1}$
3	$-.2061\times10^{-2}$	$-.2061\times10^{-2}$	$-.3847\times10^{-1}$	$-.3847\times10^{-1}$	$-.1114$	$-.1114$	$-.7775$	$-.7694$
4	$.2082\times10^{-3}$	$.2082\times10^{-3}$	$.1030\times10^{-1}$	$.1031\times10^{-1}$	$.4257\times10^{-1}$	$.4257\times10^{-1}$	$.5674$	$.5582$
5	$-.2104\times10^{-4}$	$-.2104\times10^{-4}$	$-.2762\times10^{-2}$	$-.2762\times10^{-2}$	$-.1626\times10^{-1}$	$-.1626\times10^{-1}$	$-.4141$	$-.4038$
6	$.2125\times10^{-5}$	$.2125\times10^{-5}$	$.7401\times10^{-3}$	$.7401\times10^{-3}$	$.6211\times10^{-2}$	$.6211\times10^{-2}$	$.3022$	$.2909$
7	$-.2147\times10^{-6}$	$-.2147\times10^{-6}$	$-.1983\times10^{-3}$	$-.1983\times10^{-3}$	$-.2372\times10^{-2}$	$-.2372\times10^{-2}$	$-.2206$	$-.2084$
8	$.2169\times10^{-7}$	$.2174\times10^{-7}$	$.5314\times10^{-4}$	$.5314\times10^{-4}$	$.9062\times10^{-3}$	$.9062\times10^{-3}$	$.1610$	$.1483$
9	$-.2191\times10^{-8}$	$-.2218\times10^{-8}$	$-.1423\times10^{-4}$	$-.1424\times10^{-4}$	$-.3461\times10^{-3}$	$-.3461\times10^{-3}$	$-.1175$	$-.1045$
10	$.2213\times10^{-9}$	$.2808\times10^{-9}$	$.3815\times10^{-5}$	$.3815\times10^{-5}$	$.1322\times10^{-3}$	$.1322\times10^{-3}$	$.8576\times10^{-1}$	$.7297\times10^{-1}$

Table 1. Comparison of analytical and numerical values of Fourier coefficients of transverse charge densities on a balanced two wire transmission line.

In concluding this section, our example of a balanced two wire transmission line (or a 1-line) has successfully provided a test case for the system of CIE's.

From this relatively simple example, we have shown that the CIE provides a method for calculating the charge distributions on a number of conductors, given the total charge on each. Although this method has been illustrated with only two, identical conductors which can be treated by other techniques, it can be easily employed for three or more conductors for which no analytical solution exists.

IV Summarizing Remarks

Kellogg [3], in his classical work on the theory of potentials proves an existence theorem quoted below.

"Given either the constant values of the potential on the conductors $R_1, R_2, \ldots R_k$, or, given the total charge on each of them, it is possible to determine the densities of charges in equilibrium on the conductors, producing, in the first case, a potential with the given constant values on the conductors, or having, in the second case, the given total charges, on the conductors."

In this note, we address the second case (charge problem) of the above quotation and formulate a system of integral equations for the densities of charge. By way of an interesting example, we consider a balanced two wire transmission line for which the charge densities are analytically known by the method of conformal mapping or otherwise. It is true however that in the context of a resurgence of interest in multiconductor transmission lines due to an application in the EMP area, the first case of Kellog's quotation above (potential problem) is most often used, since it also can yield the capacitive coefficients matrix as a by-product. It is observed however that in a realistic situation, where the potentials are prescribed and the elements of capacitive coefficients matrix have been experimentally measured, CIE provides an alternate way of determining the charge densities.

Furthermore Smith's [10] work on the proximity effect develops a procedure for determining the transverse high frequency current distributions for systems of parallel and coplanar conductors carrying equal currents. Essentially, this note extends this work into an electrostatic application of multiconductor transmission lines not restricted to a coplanar situation, where the individual wires are required to be parallel and carrying pre-assigned amounts of net charges. It is also our belief that analytical solutions for charge densities on general systems of conductors with more than two conductors do not exist except in some special cases, e.g., planar grating [6]. Other special cases of interest that have been treated in the past include computation of charge densities on a cylindrical test body with its axis parallel to the plates of a two-plate-transmission line type of EMP simulator [14, 15]. Also, Marin [16] has calculated the charge distribution on a grid of rods replacing one of the conducting plates in a parallel plate transmission line.

In conclusion, the usefulness of determining the angular distribution of charge densities lies in the computation of the field coupling parameters when the multi-conductor transmission line is illuminated by an external field.

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