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Diffraction Through a Circular Aperture in a Screen Separating Two Different Media

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DIFFRACTION THROUGH A CIRCULAR APERTURE IN A SCREEN SEPARATING TWO DIFFERENT MEDIA

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1.0 INTRODUCTION

The classic problem of the diffraction of a plane electromagnetic wave through a circular aperture has been treated by many authors. Exact solutions for arbitrary incident direction and arbitrary frequency have been given by Meixner and Andrejewski[(1)] (1950), Flammer[(2)] (1950), Lure[(3)] (1960), Nomura and Katsura[(4)] (1955). Some of these solutions were published by Bowman, Senior, Uslenghi[(5)] (1969). Thomas[(6)] (1969) derived the integral equations which would give the solution of the diffraction of a plane electromagnetic wave by a circular aperture in an infinite plane screen which separates two different media. He assumed a low frequency and gave an approximate numerical result for the special case of zero conductivitity. Butler and Umashankar (1976) generalized the problem to an aperture of arbitrary shape and described a numerical procedure by the method of moments to find a solution. In this paper, we shall derive an exact solution to the problem with arbitrary frequency and arbitrary media. We will specialize the result to the case of a circular aperture illuminated by a plane wave normally incident. The special case of the two media being equal reduces to the solution given by Flammer.

2.0 MATHEMATICAL FORMULATION OF THE PROBLEM

The problem at hand is illustrated in Figure 1.

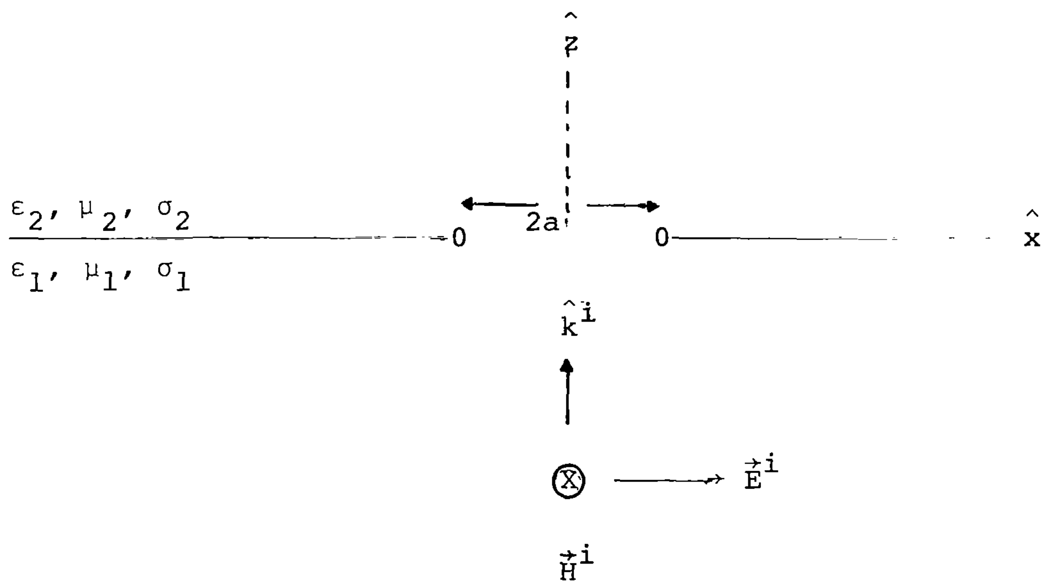

Figure 1 Incident Plane Wave On Circular Aperture

A plane wave is travelling in medium I with permitivity, permeability and conductivity ε_1, μ_1, and σ_1. The wave illuminates an infinite perfectly conducting screen at z=0 with a circular aperture whose center is at the origin. On the shadow side of the screen is a medium with permitivity, permeability, and conductivity given by ε_2, μ_2, σ_2.

We shall assume the incident wave is sinusoidal and suppress the factor $e^{-i\omega t}$. The general impulse can be found by Fourier Transforms. Maxwell's Equations are given by:

$$\nabla \times \vec{E} = i\omega\mu\vec{H}$$

$$\nabla \times \vec{H} = -i\omega\varepsilon^*\vec{E}$$

$$\nabla \cdot \vec{H} = \nabla \cdot \vec{E} = 0$$

$$\varepsilon^* = \varepsilon - \sigma/i\omega$$

Subscript 1 will indicate the incident medium and subscript 2 will indicate the medium on the shadow side of the screen. The incident wave will be travelling in the z direction with the electric field in the x direction. The incident electric and magnetic fields are given by

$$\vec{E}^i = E_o(\omega)e^{ik_1z}\hat{x}$$

$$\vec{H}^i = H_o(\omega)e^{ik_1z}\hat{y}$$

where

$$k_\ell = \frac{\omega}{c_\ell(\omega)}$$

$$c_\ell(\omega) = \frac{1}{(\varepsilon_\ell^*\mu_\ell)^{1/2}}$$

$$H_o(\omega) = E_o(\omega)/Z_1$$

$$Z_\ell(\omega) = \left(\mu_\ell/\varepsilon_\ell^*\right)^{1/2}$$

$$\varepsilon_\ell^*(\omega) \quad \varepsilon_\ell - \sigma_\ell/i\omega$$

$\ell = 1,2$

Real parts of Square Roots are Positive.

The boundary conditions are the following:

1. The scattered wave satisfies the ratiation boundary condition at large distances from the aperture.

2. The tangential electric field is zero on the screen.

3. The tangential electric and magnetic fields are zero at the aperture.

4. The tangential electric field goes to zero as the rim is approached from the aperture.

There is a unique solution of Maxwell's Equations which satisfy the above boundary conditions.

Let us introduce oblate spheroidal coordinates defined by

$$x = a\sqrt{(1+\xi^2)(1-\eta^2)}\cos\phi$$

$$y = a\sqrt{(1+\xi^2)(1-\eta^2)}\sin\phi$$

$$z = a\xi\eta \text{ sign } z$$

so that $\xi = 0$ is the equation of the surface of the aperture and $\eta = 0$ is the surface of the screen.

The range of the variables is

$$0 \le \xi < \infty$$

$$0 \le \eta \le 1$$

If there were no aperture, the wave would be relfected so as to satisfy the boundary condition at the screen..

$$\vec{E}^r = -E_o(\omega)e^{-ik_1 z}\,\hat{x}$$

$$\vec{H}^r = H_o(\omega)e^{-ik_1 z}\,\hat{y}$$

The fields on the incident side with an aperture present are given by

$$\vec{E}_1 = \vec{E}^i + \vec{E}^r + \vec{E}_1^s$$

$$\vec{H}_1 = \vec{H}^i + \vec{H}^r + \vec{H}_1^s$$

where $\vec{E}^s$ and $\vec{H}^s$ are the scattered electric and magnetic fields, on the incident side of the screen. The mathematic description of the boundary conditions is:

$$E_{1x}(\xi,o,\phi) = E_{1y}(\xi,o,\phi) \overset{0}{=} E_{2x}(\xi,o,\phi) = E_{2y}(\xi,o,\phi) = 0$$

$$E_{1x}(o,\eta,\phi) = E_{2x}(o,\eta,\phi) \qquad E_{1y}(o,\eta,\phi) = E_{2y}(o,\eta,\phi)$$

$$H_{1x}(o,\eta,\phi) = H_{2x}(o,\eta,\phi) \qquad H_{1y}(o,\eta,\phi) = H_{2y}(o,\eta,\phi)$$

$$\lim_{\eta\to 0} \phi\cdot E\,(o,\eta,\phi) = 0$$

$$\lim_{\xi\to\infty} E_2(\xi,\eta,\phi)\; e^{-ik_2\xi}\,\xi = \text{constant}$$

$$\lim_{\xi\to\infty} \vec{E}_1^s(\xi,\eta,\phi)\; e^{-ik_1\xi}\,\xi = \text{constant}$$

3.0 DERIVATION OF THE SOLUTION

Let us introduce a vector potential $\vec{\Pi} = \alpha\Pi_x\hat{x} + \beta\Pi_z\vec{z}$. The electric and magnetic fields are then given by [5]

$$\vec{H}_2 = -i\omega\varepsilon_2^* \nabla \times \vec{\Pi}_2 \qquad \vec{H}_1^{(s)} = -i\omega\varepsilon_1^* \nabla \times \vec{\Pi}_1$$

$$\vec{E}_2 = \nabla \times \nabla \times {}_2 \qquad \vec{E}_1^{(s)} = \nabla \times \nabla \times {}_1$$

Π_x and Π_z both satisfy the wave equation but with different velocities in different regions. Thus,

$$\nabla^2\Pi_{1x} = -k_1^2\Pi_{1x} \qquad \nabla^2\Pi_{1z} = -k_1^2\,\Pi_{1z}$$

$$\nabla^2\Pi_{2x} =--k_2^2\Pi_{2x} \qquad \nabla 2\Pi_{2z} = -k_2^2\,\Pi_{2z}$$

All of Maxwell's Equations are automatically satisfied. If Π_{1x} and Π_{2x} and $\frac{\partial\Pi_{2z}}{\partial z}$ and $\frac{\partial\Pi_{2z}}{\partial z}$ are chosen to be zero at $\eta=0$, then all of the screen boundary are satisfied. A separable solution of the wave equation in Oblate Spheroidal coordinates is

$$R_{mn}^{(p)}(-ic,\, i\xi)\; S_{mn}^{(q)}(-ic,\eta)\;\; \Phi_m^{(k)}(\phi)$$

$p=1,2,3,4, \qquad q=1,2 \qquad k=1,2$

where

$$\Phi_m^{(1)}(\phi) = \cos m\phi$$

$$\Phi_m^{(2)}(\phi) = \sin m\phi$$

where $R_{mn}^{(p)}(-ic, i\xi)$ are the radial spheroidal wave functions, (see Abramowich[8]) and where $S_{mn}^{(q)}(-ic,\eta)$ are the angular wave functions (see appendix). If the fields are to be finite on the axis, $(\eta-1)$, then $q=1$, and also η is an integer. If the fields are to satisfy the radiation boundary conditions at $\xi=\infty$, then $p=3$.

The functions $S_{mn}^{(1)}(-ic,\eta) = \sum_{r=0,1}^{\infty}{}' f_r^m(c) P_{m+n}^m(\eta)$

where the sume is over the odd integers if n-m is odd and over the even integers if n-m is even. $P_{m+r}^m(\eta)$ is the Associated Legendre's Function. If this is to vanish at $\eta=0$, then n-m must be odd. If the derivative of this is to vanish at $\eta=0$, then n-m must be even.

Let us find Π_x satisfying all the boundary conditions except the rim condition

$$\Pi_{1x} = \Sigma(A_{mn} \cos m\phi + B_{mn} \sin m\phi)\, R_{mn}^{(3)}(-ic, i\xi)\, S_{mn}^{(1)}(-ic, \eta)$$

$$\Pi_{2x} = \Sigma(C_{mn} \cos m\phi + D_{mn} \sin m\phi)\, R_{mn}^{(3)}(-ic_2, i\xi)\, S_{mn}^{(1)}(-ic, \eta)$$

where the sum is such that n-m is an odd integer. The scattered fields are given by

$$E_{\ell x} = \frac{\partial^2 \Pi_{\ell x}}{\partial x^2} + k_\ell^2 \Pi_{\ell x} \qquad E_{\ell y} = \frac{\partial^2 \Pi_{\ell x}}{\partial x \partial y} \qquad E_{\ell z} = \frac{\partial^2 \Pi_{\ell x}}{\partial x \partial z}$$

$$H_{\ell x} = 0 \qquad H_{\ell y} = -i\omega\varepsilon_\ell \frac{\partial \Pi_{\ell x}}{\partial z} \qquad H_{\ell z} = i\omega\varepsilon_\ell \frac{\partial \Pi_{\ell x}}{\partial y}$$

The continuity of E is assured at $\xi=0$, if $\Pi_{1x} = \Pi_{2x}$ at $\xi=0$.

The continuity of H is assured at $\xi=0$, if

$$2H_o - i\omega\varepsilon_1^* \frac{\partial \Pi_{1x}}{\partial z} = i\omega\varepsilon_2^* \frac{\partial \Pi_{2x}}{\partial z} \quad \text{at } \xi=0$$

The partial with respect to z is given by

$$\frac{\partial}{\partial z} = \pm\left[\frac{\eta(1+\xi^2)}{a(\xi^2 + \eta^2)} \frac{\partial}{\partial \xi} + \frac{\xi(1-\eta^2)}{a^2(\xi^2 + \eta^2)} \frac{\partial}{\partial \eta}\right]$$

where the plus sign is used if $z>0$ and the minus sign if $z<0$.

It follows that

$$\frac{-2H_o a\eta}{i\omega} = \varepsilon_2 \frac{\partial \Pi_{2x}}{\partial \xi} + \varepsilon_1 \frac{\partial \Pi_{1x}}{\partial \xi} \quad \text{at } \xi=0$$

Substituting the series in the above two boundary conditions leads to

$B_{mn} = D_{mn} = 0$ and $m = 0$ so that

$$\Sigma A_{on} R_{on}^{(3)}(-ic_1, io) S_{on}^{(1)}(-ic_1, \eta) = \Sigma C_{on} R_{on}^{(3)}(-ic_2, io) S_{on}^{(1)}(-ic_2, \eta)$$

$$\varepsilon_1 \Sigma A_{on} R_{on}^{(3)'}(-ic_1, io) S_{on}^{(1)}(-ic_1, \eta) + \varepsilon_2 \Sigma C_{on} R_{on}^{(3)'}(-ic_2, io) S_{on}^{(1)}(-ic_2, \eta)$$

$$= \frac{-2H_o a\eta}{i\omega} = \frac{-2H_o a}{i\omega} P_1(\eta)$$

where the prime indicates a derivative with respect to ξ. Since the S_{on} functions an orthogonal, we can multiply both sides by S_{on} and integrate

$$A_{on} = \frac{+1}{R_{on}^{(3)}(-ic_1, io) I_n} \sum_{\substack{\bar{n}=1 \\ odd}}^{\infty} C_{o\bar{n}} R_{o\bar{n}}^{(3)}(-ic_2, io) I_{n\bar{n}}$$

and

$$A_{on} = \frac{1}{R_{on}^{(3)'}(-ic_1, io)\varepsilon_1 I_n} \left\{ - \frac{2H_o a f_1^{01}}{j\omega 3} - \varepsilon_2 \sum_{\substack{\bar{n}=1 \\ odd}}^{\infty} C_{o\bar{n}} R_{o\bar{n}}^{(3)'}(-ic_2, io) I_{n\bar{n}} \right\}$$

Eliminate A_{on} and obtain

$$\Sigma C_{o\bar{n}} E_{n\bar{n}} = F_n$$

where

$$E_{n\bar{n}} = \left[\varepsilon_1 R_{o\bar{n}}^{(3)}(-ic_1, io) R_{on}^{(3)'}(-ic_2, io) \right.$$

$$\left. + \varepsilon_2 R_{on}^{(3)}(-ic_2, io) R_{o\bar{n}}^{(3)'}(-ic_1, io) \right] J_{n\bar{n}}$$

$$I_{n\bar{n}}(c_1, c_2) = I_{n\bar{n}} = \int_o^1 S_{on}^{(1)}(-ic_1, \eta) S_{o\bar{n}}^{(1)}(-ic_2, \eta) d\eta \qquad (n, \bar{n} = \text{odd})$$

$$F_n = -\frac{2}{3} aH_o f_1^{01}(c_1) R_{on}^{(3)}(-ic_1, io)/i\omega$$

$$I_n = I_{nn}(c_1, c_1)$$

The solution of this infinite set of equations can be substituted in the equation for A_{on}, Π_{2x}, $\vec{E}_2$, and $\vec{H}_2$ to give the diffracted fields. However, the rim condition is not satisfied ($E\phi \to \infty$).

Let us look for another solution involving Π_2.

$$\Pi_{1z} = \Sigma(G_{mn} \cos m\phi + H_{mn} \sin m\phi) R_{mn}^{(3)}(-ic_1, i\xi) S_{mn}^{(1)}(-ic_1, \eta)$$

$$\Pi_{2z} = \Sigma(J_{mn} \cos m\phi + K_{mn} \sin m\phi) R_{mn}^{(3)}(-ic_2, i\xi) S_{mn}^{(1)}(-ic_2, \eta)$$

where the sum is such that n-m is even.

The fields are given by

$$E_{\ell x} = \frac{\partial^2 \Pi_{\ell z}}{\partial x \partial z} \qquad E_{\ell y} = \frac{\partial^2 \Pi_{\ell z}}{\partial y \partial z} \qquad E_{\ell z} = -\frac{\partial^2 \Pi_{\ell z}}{\partial x^2} - \frac{\partial^2 \Pi_{\ell z}}{\partial y^2} + k^2 \Pi_{\ell z}$$

$$H_x = 0 \qquad H_y = i\omega\varepsilon_\ell \frac{\partial \Pi_{\ell z}}{\partial x} \qquad H_{\ell z} = -i\omega\varepsilon_\ell \frac{\partial \Pi_{\ell z}}{\partial y}$$

The continuity boundary conditions are

$$\frac{\partial \Pi_{1z}}{\partial \xi} = \frac{\partial \Pi_{2z}}{\partial \xi} \qquad \text{at} \qquad \xi = 0$$

$$\varepsilon_2 \Pi_{2z} - \varepsilon_1 \Pi_{1z} = \frac{2H_o x}{i\omega} = \frac{2H_o a}{i\omega} \sqrt{1-\eta^2} \cos\phi = -\frac{2H_o a P_1^1(\eta) \cos\phi}{i\omega} \quad \text{at } \xi=0$$

In order that both sides have the same θ dependence, choose

$$H_{mn} = K_{mn} = 0 \qquad \text{and } m=1$$

It follows that

$$\sum_{\substack{n=1\\ \text{odd}}}^{\infty} G_{1n} R_{1n}^{(3)\prime}(-ic_1, io) S_{1n}^{(1)}(-ic_1, o) = \sum_{\substack{n=1\\ \text{odd}}}^{\infty} J_{1n} R_{1n}^{(3)\prime}(-ic_2, io) S_{1n}^{(1)}(-ic_2, o)$$

$$-\varepsilon_1 \sum_{\substack{n=1\\ \text{odd}}}^{\infty} G_{1n} R_{1n}^{(3)}(-ic_1, io) S_{1n}^{(1)}(-ic_1, o) + \varepsilon_1 \sum_{\substack{n=1\\ \text{odd}}}^{\infty} J_{1n} R_{1n}^{(3)}(-ic_2, io)$$

$$\cdot S_{1n}^{(1)}(-ic_2, o) = -\frac{2H_o a P_1^1(\eta)}{i\omega}$$

solve both equations for G_{1n}

$$G_{1n} = \frac{-1}{R_{1n}^{(3)\prime}(-ic_1, io)L_n} \sum_{\bar{n}=1,\ \text{odd}}^{\infty} J_{1\bar{n}} R_{1\bar{n}}^{(3)\prime}(-ic_2, io)L_{n\bar{n}}$$

$$G_{1n} = \frac{1}{-\varepsilon R_{1n}^{(3)}(-ic_1, io)L_n} \left[-\frac{4H_o a}{3i\omega} - \varepsilon_2 \sum_{\bar{n}=1,\ \text{odd}}^{\infty} J_{1\bar{n}} R_{1\bar{n}}^{(3)}(-ic_2, io)\ L_{n\bar{n}} \right]$$

Eliminate G_{1n} and obtain

$$\sum_{\bar{n}=1,\ \text{odd}}^{\infty} J_{1\bar{n}} M_{n\bar{n}} = N_n$$

where

$$M_{n\bar{n}} = L_{n\bar{n}} \left[\varepsilon_1 R_{1\bar{n}}^{(3)\prime}(-ic_2, io) R_{1n}^{(3)}(-ic_1, io) + \varepsilon_2 R_{1\bar{n}}^{(3)}(-ic_2, io) R_{1n}^{(3)\prime}(-ic_1, io) \right]$$

$$L_{n\bar{n}} = \int_o^1 S_{1n}(-ic_1, o) S_{1\bar{n}}(-ic_2, o)\, d\eta$$

$$N_n = -\frac{4aH_o f_o^{1n}}{3i\omega} R_{1n}^{(3)\,\prime}(-ic_1, io)$$

$$L_n = L_{nn}(c_1, c_1)$$

This again gives an infinite set of linear equations to solve for J_{1n}. Substitution of these constants in the above equations will give the fields. To get a unique solution, we take

$$\vec{\Pi} = \alpha\Pi_x\hat{x} + \beta\Pi_z\hat{z} \qquad \text{where } \alpha + \beta = 1$$

This will satisfy all the boundary conditions except the condition that $E_\phi = 0$ on the rim.

E_ϕ is given by [(2)]

$$E_\phi = c\phi \left\{ \frac{-\alpha(1-\eta^2)^{\frac{1}{2}}(\xi^2+1)^{\frac{1}{2}}}{ka^2(\xi^2+\eta^2)} \right\} \Sigma' c_{on} \left\{ \frac{R_{on}^{(3)}(-ic_2, io)}{(\xi^2+1)^{\frac{1}{2}}} \frac{d}{d\eta}\left[(1-\eta^2)^{\frac{1}{2}} \frac{dS_{on}}{d\eta}\right] \right.$$

$$+ \frac{S_{on}^{(1)}(-ic_2,\eta)}{(1-\eta^2)^{\frac{1}{2}}} \frac{d}{d\xi}\left[(\xi^2+1)^{\frac{1}{2}} \frac{dR_{on}^{(3)}}{d\xi}(-ic_2, i\xi)\right]$$

$$+ \beta \frac{(1-\eta^2)^{\frac{1}{2}}(\xi^2+1)^{\frac{1}{2}}}{ka^2(\xi^2+\eta^2)} \Sigma' J_{1n} \frac{\xi R_{1n}^{(3)}(-ic_2, i\xi) S_{1n}'(-ic_2,\eta)}{\xi^2+1}$$

$$\left. + \frac{\eta}{1-\eta^2} S_{1n}^{(1)}(-ic_2, \eta) \frac{dR_{1n}^{(3)}}{d\xi}(-ic_2, i\xi) \right\}\Bigg\}$$

The differential equations for R and S are

$$\frac{d}{d\xi} \quad (\xi^2+1) \; \frac{dR_{mn}}{d\xi} + \left(-\lambda_{mn} + c^2\xi^2 + \frac{m^2}{\xi^2+1} \right) R_{mn} = 0$$

$$\frac{d}{d\eta} \quad (1-\eta^2) \; \frac{dS_{mn}}{d\eta} \quad \left(\lambda_{mn} + c^2\eta^2 \quad - \frac{m^2}{1-\eta^2} \right) S_{mn} = 0$$

Since $S_{on}^{(1)}(-ic_2, \eta) = \sum_{\substack{r=1 \\ odd}}^{\infty} f_r^{on}(c_2) \; P_r(\eta)$

and $P_r(o) = 0$

it follows that $S_{on}^{(1)}(-ic_2, \eta) = 0$.

Put $\eta=0$ in the differential equation and get

$$S_{on}^{(1)''}(-ic_2, o) = 0$$

Expand in a Taylor Series

$$S_{on}^{(1)}(-ic_2, \eta) = S_{on}^{(1)'}(-ic_2, o)\eta + S_{on}^{(1)'''}(-ic_2, o) \frac{\eta^3}{6} + \ldots$$

Further expansion leads to

$$\frac{d}{d\eta}\ (1-\eta^2)^{\frac{1}{2}}\frac{dS_{on}^{(1)}}{d\eta}$$

$$= \left[S'''(-ic_2,\ o) - S'(-ic_2,\ o)\right]\eta + O(\eta^3)$$

and $\frac{d}{d\xi}\ (\xi^2+1)^{\frac{1}{2}}\ \frac{dR_{om}^{(3)}}{d\xi} = R_{on}^{(3)\,''}(-ic_2,\ io) + 0(\xi^2)$

Using the differential equations

$$R_{on}^{(3)\,''}(-ic_2,\ io) = \lambda_{on}R_{on}^{(3)}(-ic_2,\ io)$$

$$S_{on}^{(1)\,'''}(-ic_2,\ io) = (2-\lambda_{on})S_{on}^{(1)\,'}(-ic_2,\ io)$$

Collect the highest power of η in the expression for E_ϕ.

$$E_\phi = \frac{c\phi}{ka^2\eta} \left\{ -\alpha\ \Sigma C_{on} \left\{ R_{on}^{(3)}(0)(1-\lambda_{on})S_{on}^{(1)'}(0) + S_{on}^{(1)'}(0)\lambda_{on}R_{on}^{(3)}(0) \right. \right.$$

$$\left. + \beta\ \Sigma\ J_{1n}S_{1n}^{(1)}(0)R_{1n}^{(3)'}(0) \right\}$$

$$= \frac{c\phi}{k^2a\eta} \left\{ -\alpha\ \Sigma\ C_{on}R_{on}^{(3)}(0)S_{on}^{(1)'}(0) + \beta\ \Sigma\ J_{1n}R_{1n}^{(3)'}(0)S_{1n}^{(1)}(0) + 0(\eta^2) \right\}$$

This is infinite as $\eta \to 0$ unless α and β are chosen so that

$$\alpha\ \Sigma C_{on}R_{on}^{(3)}(0)S_{on}^{(1)'}(0)$$

$$= \beta\ \Sigma J_{1n}R_{1n}^{(3)'}(0)S_{1n}^{(1)}(0)$$

In this case

$$\alpha = \frac{\Sigma J_{1n} R_{1n}^{(3)\,'}(0) S_{1n}^{(1)}(0)}{\Sigma J_{1n} R_{1n}^{(3)\,'}(0) S_{1n}^{(1)}(0) + \Sigma C_{on} R_{on}^{(3)}(0) S_{on}^{(1)\,'}(0)}$$

and $\beta = 1-\alpha$ and $E_\phi = 0$ at $\xi = \eta = 0$.

In the special case when $\varepsilon_1^* = \varepsilon_2^*$ and $\mu_1 = \mu_2$, then $I_{n\bar{n}} = I_n \delta_n^{\bar{n}}$

and $L_{n\bar{n}} = \delta_n^{\bar{n}} L_n$

$$L_{n\bar{n}} = 2\varepsilon^* R_{on}^{(3)}(0) R_{on}^{(3)\,'}(0) \delta_n^{\bar{n}} I_n$$

$$A_{on} = C_{on} = \frac{iaH_o f_1^{on}}{3\omega\varepsilon^* R^{(3)\,'}(0) I_n}$$

$$M_{n\bar{n}} = 2\varepsilon R_{1n}^{(3)\,'}(0) R_{1n}^{(3)}(0) \delta_n^{\bar{n}} I_n$$

$$-G_{1n} = J_{1\bar{n}} = \frac{2E_o a i f_o^{1n}}{3kI_n R_{1n}^{(3)}(0)}$$

These check the values given by Flammer[2]. Thus, we have found a solution which generalizes that of Flammer to the case of arbitrary medium.

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