

Interaction Notes

Note 358

On Statistical Error in EMP Interaction Paths

March 1979

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ABSTRACT

Within the context of a linear theory, we investigate the statistical uncertainties in the predicted internal EMP response due to coupling uncertainties in identified and unidentified points of entry (POEs). The uncertainties for the identified POEs can be quantified, leading to the consequent uncertainties in the predicted response. We also derive generally valid bounds for the relative error in the predicted response and the consequences of the central limit theorem. Motivated by a number of examples, we further estimate that the typical relative error should either decrease or, at worst, remain constant as the number of POEs increases. On the other hand, the influence of the unidentified POEs is very difficult to quantify and represents a major obstacle to overcome. Within the context of clearly stated assumptions and approximations, we discuss three aspects of this quantification. First, we show that as the number of POEs gets large, the effective error due to the unidentified POEs can become significant, if not dominate. Second, we indicate a rough way to incorporate an estimate of the unidentified POE contribution into an improved predicted response. Third, we suggest a method to analyze experimental data in order to extract parameters that characterize the unidentified POEs. In view of the crucial role of the unidentified POEs in the system interaction with a threat EMP environment, it is imperative that a statistical formalism be further developed for their inclusion in a predictive EMP theory.

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SECTION I  
INTRODUCTION AND SUMMARY

1. INTRODUCTION

One of the major divisions of EMP technology is the coupling, penetration, and propagation of EMP to critical internal points of various systems of interest, such as aircraft and satellites. This penetration can occur through deliberate points of entry (POEs), inadvertent POEs and through shielding. To develop a predictive capability for the internal response produced by a threat EMP, it is necessary to include a realistic estimate of the accuracy of the predictions. The development of this capability must involve a close interplay between theoretical analysis and experimental measurement.

The theoretical formulation for the internal response at a point is in terms of excitations at the various POEs and transfer functions from the individual POEs to the point of interest. Ideally, both sets of these quantities should be calculable from the presumed known EMP environment, and the known geometry and physical parameters. In actual practice, it is often necessary to resort to, for example, rough estimates, simplified model calculations, or measured values.

When a large complex system is under consideration, it might not be feasible to perform all the detailed theoretical calculations and numerous experimental measurements necessary for a complete deterministic analysis. In such a situation, a statistical approach may yield very useful, although less information without the expenditure of large amounts of time

and effort (ref. 1). There are a number of issues for which a statistical analysis has relevance. The most obvious is the fact that the necessary ingredients in making a prediction have errors and uncertainties associated with them so that the resulting prediction has a corresponding uncertainty. We must be able to calculate the latter, given the former (ref. 2). However, for a complicated system, there are some less obvious questions, which nevertheless are of interest. (Here, we can parameterize the degree of complexity by the number of POEs.) For example, given the fact that the predicted response is the result of a large number of contributions, do we expect that the individual errors in these contributions will accumulate to such an extent as to make prediction meaningless, i.e., as the number of POEs increases, what is the expected relationship between the predicted response and its corresponding uncertainty? Moreover, for a physical system with many POEs, it is likely that some will not be identified. We require techniques to estimate the number of POEs unidentified and their relative importance compared to the identified POEs. Although not fully answered, these issues will be addressed in the following sections.

This is an initial effort concerning various aspects of POE uncertainty, and, as such, it is largely exploratory. The main emphasis has been conceptual with a goal of developing

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1. Graham, W.R., and Mo, C.T.C., Probability Distribution of CW Induced Currents on Randomly Oriented Sub-Resonant Loops and Wires, R&D Associates, RDA-TR-105900-011, May 1976; also in IEEE Transaction on Antenna and Propagation, EMP Special Issue, February 1978; also as DNA-4162T (May 1976); also as AFWL EMP-IN-321 (June 1976).
  2. Mo, C.T.C., Statistical Relationship Between Testing and Prediction of EMP Interaction, R&D Associates, RDA-TR-105500-001, January 1977; also as AFWL EMP-IN-331 (January 1977).

ideas relevant to the area of investigation. The findings may not have immediate applicability, as would be the case for a manual with a list of formulas. Rather, we have started to delineate the problem area and made the first steps toward solutions.

The organization of this report is as follows. In subsection I.2, we detail the model used in this investigation; in subsection I.3, we summarize our findings and indicate a few of the important open questions. Section II is devoted to an investigation of the contribution to the response due to the identified POEs, while the contribution of the unidentified POEs is considered in Section III. Appendix A discusses a few explicit examples, the results of which are used to motivate some conclusions in the main text. Appendix B contains formulas for some moments of random variables.

## 2. THE MODEL

The physical situation under consideration is actually one of great complexity. For the purpose of analysis, we will introduce a number of simplifying assumptions into the basic relationship between response, excitation and transfer function. (The effect of penetration through shielding (ref. 3) will be ignored.) Typically, both the excitations and responses (e.g., electric fields) are vector quantities and, consequently, the transfer functions are dyadics. We will assume all three are scalars. Likewise, the quantities are complex numbers as the natural approach involves a frequency analysis of the driving EMP. We will assume all

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3. Karzas, W.J., and Mo, C.T.C., Linear and Nonlinear EMP Diffusion Through a Ferromagnetic Conducting Slab, R&D Associates, RDA-TR-109900-001, July 1975; also published as AFWL EMP Interaction Note No. 293; and in IEEE Transactions on Antennas and Propagation, EMP Special Issue, February 1978.

three are real numbers. Aside from simplifying the model, these assumptions are not fundamental in nature. The results of a statistical analysis are general, rather than specific and therefore, likely quite independent of the detailed algebraic structure of the model. What we learn here should be useful for more realistic models. In reality, the contribution from a given POE involves an integral over that POE. We will assume the POEs are small enough that the excitation and transfer function can be represented by single algebraic numbers. (This assumption is not essential, since an integral is just a sum, but we introduce it here for the sake of clarity and simplicity of the model.) If the excitations become strong enough, we would expect the response to become nonlinear. Here, we will restrict our attention to linear responses.

We will now incorporate the above assumptions into a model; namely, we assume the following real, linear, scalar, relationship:

$$R_i(\omega) = \sum_{j=1}^N T_{ij}(\omega) E_j(\omega). \quad (1)$$

The notation is as follows:

$N$  = number of POEs,

$j$  = label of  $j^{\text{th}}$  POE,  $1 \leq j \leq N$ ,

$i$  = label of  $i^{\text{th}}$  internal location of interest,

$\omega$  = angular frequency,

$R_i(\omega)$  = response of frequency  $\omega$ , at the  $i^{\text{th}}$  location,

$E_j(\omega)$  = excitation with frequency  $\omega$ , at the  $j^{\text{th}}$  POE,

$T_{ij}(\omega)$  = transfer function (matrix) connecting the  $j^{\text{th}}$  POE, driven at frequency  $\omega$ , to the  $i^{\text{th}}$  location.

For ease in writing, we will usually drop the reference to frequency ( $\omega$ ) and location ( $i$ ) so that Equation 1 becomes

$$R = \sum_{j=1}^N T_j E_j. \quad (2)$$

It is to be noticed that the transfer function,  $T_j$ , itself might involve structure due to multiple conduction paths from the  $j^{\text{th}}$  POE to the point of interest. However, within the context of a linear theory, i.e., no interaction among POEs is assumed, the net transfer function can be considered as a single algebraic number. The extent to which Equation 2 deviates from an actual physical situation is indicated by the various input assumptions discussed above.

### 3. SUMMARY


The findings of this report can be separated into two relatively loosely distinctive pieces. The first piece concerns explicit results. Included here is the calculation of the uncertainty in the predicted response (due to the identified POEs) arising from the individual uncertainties in the excitations and transfer functions. We also derive bounds on moments of the distribution of the predicted response. Finally, the implications of the central limit theorem, for a simple case, are presented.

The second piece of the findings can be characterized as expected behavior of the response based on reasonable assumptions. Here we exhibit that the relative error of the response, typically, should not grow with increasing numbers of POEs. At worst, the relative error should be a constant characterized by the input relative errors. As for unidentified POEs, we first estimate the relative importance of these based on extrapolation of the identified POE behavior



and suggest that when a large number of POEs are involved, the error resulting from a relatively small fraction of them being unidentified can become very significant, if not dominate. Then a crude method is suggested to quantify and improve errors in the predicted response by incorporating an estimate of the contribution due to the unidentified POEs. Finally, by pushing the data harder than it probably should be, we calculate a set of parameters that should represent, approximately, the number and distribution of the unidentified POEs. The goal here has been to try to reasonably parameterize our ignorance of the unidentified POEs and then to organize the data as to extract meaningful information about these parameters.

There are many questions remaining. The restrictions of the model to scalar and real quantities could be removed with a corresponding increase in algebraic complexity. In particular, the application to a given situation could be effected quite straightforwardly, i.e., the uncertainty in the predicted response due to the input uncertainties, but the general estimates will be more difficult to obtain. Similar remarks apply to larger POEs and composite transfer functions. The incorporation of nonlinear effects is very important for the evaluation of the response in the event of a threat EMP. However, the theoretical foundations of such nonlinear effects must first be better understood before we could hope to statistically model them. Finally, the whole question of unidentified POEs needs further investigation. Since the area is very nebulous, it is difficult to quantify its effects and yet, they could be of major importance for predicting the response of a system. The material in Section III is exploratory and merely a start. It is characterized by a lack of mathematical rigor and consequently by a large amount of speculation. There are many formal mathematical



questions here that need to be investigated. Where completely rigorous answers are either impossible or not reasonably attainable, some reasonably plausible input assumptions would be very useful. However, apart from these purely mathematical questions, comparison with some experimental data would be needed in suggesting fruitful theoretical developments.

SECTION II  
IDENTIFIED POEs

1. UNCERTAINTY IN PREDICTED RESPONSE

For this section, we assume we have identified  $n$  ( $\leq N$ ) of the POEs. For each POE, we assume that the transfer function is known to within some accuracy,

$$T_j = t_j \pm \delta_j, \quad 1 \leq j \leq n, \quad (3)$$

where  $t_j$  = the predicted or measured value,

$\delta_j$  = its uncertainty range.

As such, there are two possible sources for the value of the transfer function. One source is due to a measurement of it in a preliminary experiment in which the  $j^{\text{th}}$  POE is locally excited and the corresponding response measured. The other source is due to a theoretical calculation of the transfer function. This typically requires simplifying the geometry and the calculations involved so that the result has an estimated error. For the purposes of this report, the value of the transfer function from either source is acceptable. We will not consider the comparison of theory and experiment for this input data. Statistically, Equation 3 is viewed as follows. The transfer function  $T_j$ , is a (confidence) random variable, with the measured, or calculated value  $t_j$ , as an estimator of the mean of its distribution and  $\delta_j$  as a given input estimate of the standard deviation of that distribution. Likewise, the excitation at the  $j^{\text{th}}$  POE, for the actual experiment of interest, is assumed known within certain errors,

$$E_j = e_j \pm \sigma_j, \quad 1 \leq j \leq n. \quad (4)$$

Again, the source of these numbers is either measurement or calculation while the statistical interpretation is the same as for Equation 3.

For a given experiment, Equations 3 and 4 are known quantities. Then, based on our model, but only arising from the identified POEs, we can predict the expected response in this experiment. This response (note  $n$ , not  $N$ ),

$$R = \sum_{j=1}^n T_j E_j, \quad (5)$$

is a random variable with a mean or expected value  $\gamma$  (i.e., our prediction) and a standard deviation  $\sigma$  (i.e., the associated error). Assuming the  $T_j$  and  $E_j$  are independent, we find the mean to be

$$\gamma = E\{R\} = \sum_{j=1}^n t_j e_j, \quad (6)$$

while the standard deviation is given by

$$\sigma^2 = E\{R^2\} - [E\{R\}]^2 = \sum_{j=1}^n (t_j^2 \sigma_j^2 + e_j^2 \delta_j^2 + \delta_j^2 \sigma_j^2). \quad (7)$$

For the special case in which all the products,  $T_j E_j$  have identical distributions ( $\mu_0 = \text{mean}$ ,  $\sigma_0 = \text{standard deviation}$ ), we have the simple results,

$$\gamma = n \mu_0, \quad (8)$$

$$\sigma^2 = n \sigma_0^2. \quad (9)$$

In many experimental situations, it is reasonable to assume that the relative errors in  $T_j$  and  $E_j$  are independent of  $j$ ; i.e.,

$$\delta_j = \beta |t_j|, \quad (10)$$

$$1 \leq j \leq n.$$

$$\sigma_j = \beta' |e_j|, \quad (11)$$

Consequently, Equation 7 becomes

$$\sigma^2 = (\beta^2 + \beta'^2 + \beta^2 \beta'^2) \sum_{j=1}^n t_j^2 e_j^2 \equiv \Delta^2 \sum_{j=1}^n t_j^2 e_j^2, \quad (12)$$

which also defines the "relative error"  $\Delta$  in the product  $t_j e_j$ .

When the values  $t_j$  and  $e_j$  become too small, Equations 10 and 11 probably become invalid as there will always be some additive residual error. In this case, Equation 12 becomes an estimate of the standard deviation which still should be reasonable as long as not all values are small or comparable to those residual errors.

## 2. BOUNDS

It is interesting to derive bounds on the relative error for the predicted response. Also, under many circumstances, the expected value of the absolute value of the predicted response is a useful quantity to consider, e.g., the total peak induced current in a wire bundle. Based on the inequality

$$E\{[|R| - E\{|R|\}]^2\} \geq 0, \quad (13)$$

we easily see that

$$\sqrt{E\{R^2\}} \geq E\{|R|\} \geq |E\{R\}|. \quad (14)$$

Consequently, we have the bounds

$$\frac{\sigma}{|E\{R\}|} \geq \frac{\sigma}{E\{|R|\}} \geq \frac{\sigma}{\sqrt{E\{R^2\}}}, \quad (15)$$

which, in the simple case embodied in Equations 8 and 9, becomes

$$\frac{\sigma_o}{|\mu_o|\sqrt{n}} \geq \frac{\sigma}{E\{|R|\}} \geq \frac{\sigma_o}{|\mu_o|\sqrt{n}} \frac{1}{\sqrt{1 + \left(\frac{\sigma_o}{\mu_o\sqrt{n}}\right)^2}}. \quad (16)$$

The standard deviation of the absolute value of the response,  $\tilde{\sigma}$ , is a quantity of interest which, however, is not readily calculable. In fact, all we can say in general is that the distribution for the absolute value of the response is tighter than for the response itself,

$$\tilde{\sigma} \leq \sigma, \quad (17)$$

and thus

$$\frac{\tilde{\sigma}}{E\{|R|\}} \leq \frac{\sigma}{E\{|R|\}}. \quad (18)$$

That is, its upper bound is bounded by Equation 15. For the simple case considered above, Equation 18 implies

$$\frac{\gamma}{E\{|R|\}} \leq \frac{\sigma_o}{|\mu_o|\sqrt{n}}. \quad (19)$$

### 3. EXPECTED BEHAVIOR

#### a. General Comments

For any given experiment,  $\gamma$ , Equation 6, and  $\sigma$ , Equation 7 or 12, can be calculated. However, it is useful to attempt to obtain some rough indications of the behavior of  $\gamma$  and  $\sigma$ , not in a single given experiment, but within a class of experiments. The experiments themselves are related to each other in some way; e.g., a given system is illuminated from various orientations but with the same source strength. The experiments are labeled by a set of integers,  $\alpha \leq N_2$ . Each experiment, then, has a predicted value for the response,  $\gamma_\alpha$ , and an associated error squared  $\sigma_\alpha^2$ . In the following, it will be convenient to view these quantities as sampled values of two respective random variables with certain distributions. Let us call these random variables X and Y.

We will now discuss two questions concerning this set of experiments. First, if we were to perform a further experiment of this class, labeled  $N_2 + 1$ , what is the likely value of the true response and its uncertainty, without and with another prediction (or measurement)? Second, is it possible to say anything about the relative error in this class of experiments? The first question can be discussed in general while the second cannot. Some degree of insight is imparted for both questions by consideration of a few examples. The individual responses are composed of  $t_j e_j$ , and the examples are characterized by the different ways in which these products can occur. We will briefly discuss the examples and then answer the two questions we have raised.

b. Examples

The first type of example is motivated by considering a situation in which there is some degree of correlation between  $t_j$  and  $e_j$ . Here, think of ordering the  $t_j$  from minimum to maximum; i.e., some POEs, due to the fact that they are closer to the point of interest or are physically larger, are more important than others. A degree of correlation then means that the  $e_j$  vary in some more or less systematic way, closely related to the  $t_j$ . This can result from the fact that the system has been excited by an EMP from a definite direction. Examples 1 and 2 in Appendix A are of this type. The dominant feature here is that most sums scale with  $n$ .

The second type of example has a lack of correlation between  $t_j$  and  $e_j$ . Here we assume that the signs and magnitudes of these quantities fluctuate so much that the product can be considered as random between minimum and maximum values. Example 3 has fluctuations which are symmetric with respect to zero,

$$E\{X\} = 0, \quad (20)$$

while Example 4 has a bias so Equation 20 is not satisfied. These examples can be characterized by the magnitude of the quantity  $\sqrt{n}|\mu_o|/\sigma_o \equiv \xi$

where  $\mu_o$  = the mean, and

$\sigma_o$  = the standard deviation of the distribution for the individual terms in the sum for X.

Example 3 has  $\xi \longrightarrow 0$ , while Example 4 has  $\xi \gg 1$ .



In the following, we will make use of the quantities  $E\{|X|\}$  and  $\sigma_{|X|}$  (the mean and standard deviation of the distribution for  $|X|$ ) and  $E\{Y\}$ . For simplicity, we will assume  $\sigma_\alpha$  is given by Equation 12. We list here the results for the various examples, for large  $n$ ;

$$E\{|X|\} = \lambda_1 n T E, (1, 2, 4), \quad (21)$$

$$E\{|X|\} = \lambda_2 \sqrt{n} T E, (3), \quad (22)$$

$$\sigma_{|X|} = \lambda_1' n T E, (1, 2), \quad (23)$$

$$\sigma_{|X|} = \lambda_2' \sqrt{n} T E, (3, 4), \quad (24)$$

$$E\{Y\} = \lambda_3 \Delta^2 n T^2 E^2, (1, 2, 3, 4). \quad (25)$$

Occurring here are  $T$  and  $E$  which are the respective maximums of  $|t_j|$  and  $|e_j|$  while all the  $\lambda$ 's are constants of order one, depending on the example.

### c. Implications for Further Experiments

We now consider experiment  $N_2 + 1$  which belongs to the same class of experiments under consideration. If no effort is made to specifically predict the response, we could expect that the magnitude of the response should be  $E\{|X|\}$ , within an uncertainty range. This latter range reflects both the fact that  $|X|$  has a spread in its true value as well as the fact that the individual products that compose  $X$  have uncertainties. Based on Equation B-5 in Appendix B, the realization of the response for the  $N_2 + 1$  experiment is expected to be

$$|x_{N_2 + 1}| \approx E\{|X|\} \pm \sqrt{\sigma_{|X|}^2 + E\{Y\}}. \quad (26)$$

This is approximately true in general, subject only to the approximation that sample quantities (based on experiments  $\alpha$ ) are used for population quantities (of the random variable  $X$ ). This approximation can be shown to be valid when  $n$  is large.

For this case of no further prediction for the  $N_2 + 1$  experiment, the uncertainty range for  $|x_{N_2 + 1}|$  for our examples behaves as, concentrating on the  $n$  dependence,

$$\sqrt{\sigma^2_{|X|} + E\{Y\}} \sim \begin{cases} n, & (1, 2), \\ \sqrt{n}, & (3, 4). \end{cases} \quad (27)$$

The examples with randomness (3, 4) show a  $\sqrt{n}$  behavior since the distribution of  $|X|$  is relatively narrow (see Equation 24). On the other hand, the examples that have a degree of correlation (1, 2) have a  $n$  behavior due to the fact that the distribution of  $|X|$  scales with  $n$ .

If the  $N_2 + 1$  response is predicted but not the uncertainty range, we can improve on Equation 26. This is due to the fact that the spread in  $|X|$  in the previous  $N_2$  experiments is no longer useful as an indicator of where the  $x_{N_2 + 1}$  is, and all we have to consider is the individual uncertainties in the new products. We then have

$$|x_{N_2 + 1}| \approx \gamma_{N_2 + 1} \pm \sum \sigma_\alpha / n. \quad (28)$$

The uncertainty range here, for all examples is less than in Equation 26. Notice that nothing based on the information  $\{X_i\}_{i=1}^{N_2}$  can be said of the new relative error, precisely because the new prediction adds a new piece of information to the set.

#### d. Relative Error

For each experiment  $\alpha$ , the relative error is  $\sigma_\alpha/|\gamma_\alpha|$  which is a realization of the random variable  $Z = \sqrt{Y}/|X|$ . Possible values of this random variable can be arbitrarily large so its expected value is not particularly useful. However, a bound on the probability of large values would be useful. Unfortunately, a tight bound, independent of distributions for the random variables  $X$  and  $Y$ , is very difficult to obtain. (A loose bound is given by the Chebyshev inequality (ref. 4).) Consequently, we will use our examples in Appendix A in order to get an indication of some reasonable behavior. Explicit calculation for large  $n$  and values of  $c$  somewhat larger than 1,  $c > 1$ , show that

$$P\{Z > c\Delta/\sqrt{n}\} \approx 1/c, \quad (1, 2, 4), \quad (29)$$

$$P\{Z > c\Delta\} \approx .6/c, \quad (3). \quad (30)$$

Therefore, for these examples, the probability of large relative errors is comparatively small. To expect similar behavior to hold for an actual experiment is, of course, hardly guaranteed. However, for a situation of great complexity without much symmetry, it would be anticipated that the data would have characteristics similar to one of the examples so that either Equation 29 or 30 (probably with different constants) should be a reasonable approximation to what is observed. The inference to be drawn from these equations is that the probability is small that the relative error grows with increasing  $n$ .

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4. Feller, W., An Introduction to Probability Theory and Its Applications, Volume I, Wiley and Sons, New York, p. 183, 1971.

#### 4. CENTRAL LIMIT CASE

There is at least one circumstance in which the standard deviation of the absolute value of the predicted response,  $\tilde{\sigma}$ , can be easily calculated. This is the case of  $n$  becoming large. Then, according to the central limit theorem,\* the distribution for the random variable  $R$ , Equation 5, will become normal with mean  $\mu$  and standard deviation  $\sigma$ . We can now calculate the expected value of the absolute value of the response,

$$E\{|R|\} = |\mu| + \sqrt{2/\pi} \left[ \sigma e^{-\mu^2/2\sigma^2} - |\mu| \sqrt{2\pi} \phi(-|\mu|/\sigma) \right] > |\mu| \quad (31)$$

where

$$\phi(x) = 1/\sqrt{2\pi} \int_{-\infty}^x dt e^{-t^2/2}. \quad (32)$$

Consequently, we have

$$\tilde{\sigma}^2 = \sigma^2 + \mu^2 - [E\{|R|\}]^2, \quad (33)$$

and the relative error is

$$\frac{\tilde{\sigma}}{E\{|R|\}} = \sqrt{\frac{\sigma^2 + \mu^2}{[E\{|R|\}]^2} - 1}. \quad (34)$$

This limit becomes particularly simple when all the products have identical distributions so that Equations 8 and 9 hold. Then the relative error is

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\* See ref. 4, Chapter 10.

$$\frac{\tilde{\sigma}}{E\{|R|\}} = \left[ \frac{1 - 2\sqrt{2/\pi} \xi f(\xi) - 2/\pi f^2(\xi)}{(\sqrt{2/\pi} f(\xi) + \xi)^2} \right]^{1/2} \quad (35)$$

where, as above,

$$\xi = \sqrt{n} |\mu_0|/\sigma_0 \quad (36)$$

and

$$f(\xi) = e^{-\xi^2/2} - \xi \sqrt{2\pi} \phi(-\xi). \quad (37)$$

The relative error, Equation 35, is a monotonically decreasing function of  $\xi$  with the limits,

$$\frac{\tilde{\sigma}}{E\{|R|\}} = \begin{cases} \sqrt{\pi/2} - 1, & \xi \ll 1, \\ 1/\xi, & \xi \gg 1. \end{cases} \quad (38)$$

The interesting result that the relative error is confined to a small interval,

$$0 \leq \frac{\tilde{\sigma}}{E\{|R|\}} \leq 0.7555, \quad (39)$$

is consistent with Equations 29 and 30. This simple behavior could well be a very good first approximation to many complicated systems.

##### 5. SQUARE OF THE RESPONSE

We end our discussion of the identified POEs by a brief consideration of the square of the response as the primary variable. The virtue of such an approach is that we would

then be dealing with an explicitly non-negative quantity and do away with the hardship of taking absolute values. In terms of the products,  $R_j = T_j E_j$ , the random variable under consideration is

$$R^2 = \left( \sum_{j=1}^n R_j \right)^2. \quad (40)$$

Let the various moments of  $R_j$  be

$$\mu_{kj} \equiv E(R_j^k). \quad (41)$$

The expected value of  $R^2$  is then

$$E(R^2) = \sum_j (\mu_{2j} - \mu_{1j}^2) + \left( \sum_j \mu_{1j} \right)^2 \quad (42)$$

while the standard deviation is given by a somewhat more complicated equation which can be found in Appendix B, Equation B-18. Apart from the greater algebraic complexity, the main disadvantage to the use of  $R^2$  is that up to the fourth moment of the input  $R_j$ 's is required and therefore, also for  $T_j$  and  $E_j$ . Moments beyond the second are not usually measured or estimated and would require more effort. If the  $R_j$  are assumed to be normally distributed, then, of course, the resulting equations are somewhat simpler. However, to the extent that the moments can be considered input data, the use of  $R^2$  could be quite fruitful. A full statistical analysis, based on  $R^2$ , has not, as yet, been done.

SECTION III  
UNIDENTIFIED POES

1. GENERAL REMARKS

The question of unidentified POEs is one that is very difficult to quantify. We do not know the number unidentified nor their distributions. Any measurement that bears on this question does not concern only one but an aggregate of them. The techniques required to extract information about the unidentified POEs are not readily at hand. In fact, there does not seem to be any rigorous mathematical methods available at all for this problem. Consequently, we must resort to various assumptions, approximations, estimates, and expectations. Which particular mixture is used is determined to a large degree by the particular question posed concerning the unidentified POEs. In the following subsections, we will state the question of interest and the assumptions used as inputs.

The contribution of the unidentified POEs can be determined by rewriting Equation 2 as

$$R - \sum_{j=1}^n T_j E_j = \sum_{j=n+1}^N T_j E_j, \quad (43)$$

or, in an obvious notation,

$$R - R^I = R^U. \quad (44)$$

A combination of experiment and prediction, then, can be viewed as a determination of  $R^U$ . The representative value of the random variable  $R$  is the measured value,  $\gamma^M$ , which

has a certain error,  $\sigma^M$ . The representative value of  $R^I$  is just the predicted response we discussed in Section II,  $\gamma \pm \sigma$ . Consequently, the representative value of  $R^U$ ,  $\gamma^U$ , is

$$\gamma^U = \gamma^M \pm \sigma^M - (\gamma \pm \sigma). \quad (45)$$

In principle, if (1) all the POEs have been identified and (2) the  $T_j$  and  $E_j$  and  $R$  have been accurately measured, then  $\gamma^U$  in Equation 45 is zero and the calculational errors,  $\sigma$ , in the prediction procedure of  $\gamma$  can be statistically calibrated by the measurements (ref. 2). In practice, if the errors in (2) are large enough, unidentified POEs cannot be ruled out, except at some level of importance. If Equation 45 is not zero, within errors, the fault can be with (1) or (2), or both. Here we will concentrate on the fault with (1)--the unidentified POEs. The idea is to extract or "squeeze-out" information about  $\gamma^U$  using known information in the measured and predicted responses.

There are now a number of questions that can be addressed. First, is it possible to estimate the relative sizes of  $\gamma^U$  and, say  $\gamma$ ? Second, if something is learned about  $\gamma^U$  from a given experimental situation, is it possible to use that information in a similar experiment in order to make a better prediction by incorporating, somewhat, the effect of the unidentified POEs? Third, is it possible to extract information about the number and distribution of unidentified POEs from knowing something about  $\gamma^U$ ? These are the questions that will concern us in the following subsections.

## 2. ESTIMATE OF RELATIVE IMPORTANCE OF UNIDENTIFIED POES

If we define  $k = N - n$ , the contribution due to the unidentified POEs, Equation 43, is



$$R_k^U = \sum_{i=1}^k T_i^U E_i^U. \quad (46)$$

None of the quantities on the right side of this equation are measured (individually) or calculated. However, we would, as a first approximation, expect their distribution to be similar to the identified POE contribution. In Section II, the two possibilities discussed would imply (cf. Appendix A)

$$E\{|R_k^U|\} = c' \begin{cases} k \hat{T} \hat{E}, \\ \sqrt{k} \hat{T} \hat{E}, \end{cases} \quad (47)$$

where  $\hat{T}$  ( $\hat{E}$ ) = the maximum of  $|t_i^U|$  ( $|e_i^U|$ ) for the unidentified POEs,

$c'$  = a constant of order unity.

Although  $k$ , in any given situation, is some definite number, we do not know it. We can parameterize this lack of knowledge by introducing a probability distribution for the number of unidentified POEs, assuming for simplicity that the sampling realization of the  $k$  unidentified POEs is independent and is from the same one population. Therefore, let us define  $p_k$  as the probability that exactly  $k$  POEs are unidentified. Thus, the expected value of  $|R^U|$  is

$$\begin{aligned} E\{|R^U|\} &= \sum_{k=1}^{\infty} p_k E\{|R_k^U|\} \\ &\approx c' \hat{T} \hat{E} \sum_{k=1}^{\infty} p_k \left( \frac{k}{\sqrt{k}} \right) \\ &\equiv c' \hat{T} \hat{E} \left( \frac{\bar{k}}{\sqrt{\bar{k}}} \right) \end{aligned} \quad (48)$$

where we have incorporated the assumption of Equation 47 and we have identified the sums as average values of  $k$  and  $\sqrt{k}$ .

We next inquire about the magnitudes of  $\hat{T}$  and  $\hat{E}$ . It would be expected that all POEs with large transfer functions would be identified. Therefore, we expect that the maximum of the unidentified ( $\hat{T}$ ) should be smaller, or even considerably so, than that of the identified ( $T$ ),

$$\hat{T} = \epsilon T, \quad \epsilon < 1. \quad (49)$$

On the other hand, the maximum excitations ( $E$  and  $\hat{E}$ ) should be comparable, say

$$E = \hat{E} \quad (50)$$

Incorporating the estimates of  $\hat{T}$  and  $\hat{E}$  into Equation 48, we have (compare Equations 21 and 22)

$$E\{|R^U|\} = c'' \epsilon \left( \frac{\bar{k}/n}{\sqrt{\bar{k}/\sqrt{n}}} \right) E\{|X|\}, \quad (51)$$

where  $c''$  = another constant.

The presence of unidentified POEs acts as a source of error in the predicted response. According to Equation 51, we find that the effective relative error is

$$\frac{E\{|R^U|\}}{E\{|X|\}} = c'' \epsilon \left( \frac{\bar{k}/n}{\sqrt{\bar{k}/\sqrt{n}}} \right). \quad (52)$$

If we knew the distribution for  $k$ , we could calculate the  $k$  averages that occur in Equation 52. For example, consider the case that the probability of the unidentified POEs can be characterized by a Poisson distribution with an average rate  $\lambda$ ; i.e., the predictor has an average ignorance of ignoring  $\lambda$  POEs. There are then two simple limits we can consider explicitly. The first consists of assuming that  $\lambda$  is small compared to one; i.e., almost all POEs have been identified. Then we have

$$\begin{aligned}\bar{k} &= \lambda, \\ \sqrt{\bar{k}} &\approx \lambda.\end{aligned}\tag{53}$$

Therefore, to the extent that  $\lambda$  does remain small, the effective error, Equation 52, decreases with  $n$  faster than the relative error in the predicted response, as implied by Equations 29 and 30.

As indicated above, the parameter,  $\lambda$ , that occurs in the assumed Poisson distribution is equal to the expected number of unidentified POEs. If  $n$  is quite large, it is unreasonable to assume  $\lambda$  being small; it is very likely that some fraction of POEs will remain unidentified. Consequently, the second limit consists of the more reasonable assumption that  $\lambda$  grows with  $n$ . If we define  $p$  as the expected proportion identified,  $n = pN$ , then an estimate of  $\lambda$  is

$$\lambda = qN \approx q/p n, \quad q = 1 - p.\tag{54}$$

When  $\lambda$  is large, the sums in Equation 52 become

$$\begin{aligned}\bar{k} &= \lambda, \\ \sqrt{\bar{k}} &\approx \sqrt{\lambda}.\end{aligned}\tag{55}$$

The effective relative error, Equation 52, is now modified to read

$$\frac{E\{|R^U|\}}{E\{|X|\}} = c^n \varepsilon \left( \frac{q/p}{\sqrt{q/p}} \right). \quad (56)$$

Consequently, for this more reasonable case of a fraction of the POEs remaining unidentified, the unidentified POEs should either become the dominate source of error for large  $n$ , or of comparable size to the error in the predictive response, as can be readily seen by comparing with Equations 29 and 30.

In making the error estimate of Equation 56, we have assumed:

1. Similar distributions of transfer functions and excitations for identified and unidentified POEs.
2. Distributions similar to those of Appendix A.
3. The Poisson distribution for the number of unidentified POEs.
4. An expected proportion identified,  $p$ .
5. A measure of relative importance of an unidentified to an identified POE,  $\varepsilon$ .

### 3. IMPROVEMENT IN PREDICTED RESPONSE

We will now start to make use of the assumed result that a determination of the contribution of the unidentified POEs, Equation 45, is available. How can we make use of this information to improve the value of the predicted response in a similar circumstance? To do this, again consider the series of experiments, labeled by the set of integers  $\{\alpha\}$  considered in Section II, in which a given system is illuminated from

various orientations but with the same excitation source strength. For each of the experiments, we measure the response at a given point,  $\gamma_{\alpha}^M \pm \sigma_{\alpha}^M$ , and we calculate the predicted response there,  $\gamma_{\alpha} \pm \sigma_{\alpha}$ . Using these results, we can calculate, by Equation 45, the value of the contribution of the unidentified POE,  $\gamma_{\alpha}^U \pm \sigma_{\alpha}^U$ , for each experiment. These numbers, including errors ( $\sigma_{\alpha}^U$ ), are sample values assumed by the random variable  $R^U$ . As such, the collection of these numbers constitutes a sample distribution of the random variable  $R^U$ . The sample distribution has a sample mean,  $\hat{\mu}$ , and sample standard deviation,  $\hat{\sigma}$ . (See Appendix B, Equations B-1 and B-5, for formulas to calculate these quantities.) The sample mean and sample standard deviation are random variables which, themselves, have spread. If the number of sample points is large, the sample parameters and the corresponding population parameters are expected to be in close accord. (For example, if  $R^U$  is normally distributed, the spreads can be explicitly calculated.) Given the fact that the sample distribution is the only information we have about  $R^U$ , we can use the sample quantities,  $\hat{\mu}$  and  $\hat{\sigma}$ , to parameterize the distribution for the unidentified POEs, keeping clearly in mind the caveat that sample quantities are not population quantities.

Having arrived at  $\hat{\mu}$  and  $\hat{\sigma}$ , we still cannot use these quantities in a different, but similar, experiment. The main obstacle to their use is the fact that their values depend explicitly on the overall strength of the excitation source of the experiments. A different experiment will employ a different strength, but the method of normalization is not immediately clear. One possibility is to note, within the context of a linear theory, that the ratio of the contribution due to the unidentified POEs and the identified POEs is independent of the overall scale of excitation. We next

have to decide on the standard predicted response to be used in the ratio. Since the unidentified POEs are characterized by  $\hat{\mu} \pm \hat{\sigma}$ , in this series of experiments, the corresponding quantity for the predicted response should be used, namely  $E\{|X|\} \pm \sqrt{E\{Y\}}$ , (cf. Equations 21, 22, and 25). We can then define a measure of the relative importance of the two contributions by

$$\rho = \frac{\hat{\mu} \pm \hat{\sigma}}{E\{|X|\} \pm \sqrt{E\{Y\}}}, \quad (57)$$

a quantity independent of the scale of excitation. Notice that this, by its very nature, may be a reasonable approximation for the quantity  $\epsilon q/p$  or  $\epsilon \sqrt{q/p}$  and thus may be used to roughly calibrate the ignorance parameter  $\lambda$  of Equation 54.

We can now use  $\rho$  in an effort to improve the predicted response in a similar experiment. What we mean by a similar experiment is that the same personnel and equipment are involved so that the ratio of unidentified to identified POEs should remain about constant and the measuring and computing capability should remain about constant. The idea is that the experiment statistically calibrates the contribution due to unidentified POEs. Removing the scale dependence, we expect its statistically calibrated relative contribution to lie near  $\rho$ . Consequently, we alter our predicted response by

$$\gamma \pm \sigma \rightarrow (\gamma \pm \sigma) (1 + \eta \rho), \quad (58)$$

where

$$\eta = \text{sign of } \gamma \quad (59)$$

since  $\rho$  is normalized with a necessarily positive quantity. This can be, at best, a rough first-order approximation to the inclusion of the effects of the unidentified POEs. Nevertheless it is clearly desirable to have some means of incorporating the unidentified POEs into a predictive capability, particularly when applied to a threat EMP environment where direct measurement is not feasible. A more comprehensive quantification of such, necessarily probabilistic in nature, is imperatively needed and is yet to be obtained.

In arriving at the scaled alteration in the predicted response, Equation 58, we have used the following approximations:

1. The sample mean and standard deviation are equated to population mean and standard deviation, respectively.
2. The ratio of contributions is independent of the strength of excitation.
3. The relevance of  $\rho$  to a similar experiment.
4. PARAMETERS OF THE UNIDENTIFIED POEs

It would be useful for the experimenter to estimate the "average" number of unidentified POEs and their relative importance. As indicated above, no rigorous method of obtaining this information is readily at hand. However, within the context of a number of assumptions, it is possible to calculate numbers that should give indications of this information. Above, we introduced a sample distribution of the random variable  $R^U$  and noted that we could calculate a sample mean and a sample standard deviation (equivalently, the first and second moments). Of course, it is possible to calculate any moment of the sample distribution. (Formulas for the first four are contained in Appendix B, Equations B-1 through B-4.) When the number of sample points is large,

these sample moments can be shown to be good approximations to the corresponding population moments. With the theoretical objections clearly in mind, we will use the former to represent the latter.

We now have to relate these moments to a simple model to exhibit how the unidentified POEs contribute in the series of experiments  $\{\alpha\}$ . Since all the experiments have the same strength of excitation but random orientations, the contribution of a given unidentified POE should be random. The idea is to use the experimental results to estimate and calibrate the unknown distribution of the effect of the unidentified POEs. As a simple example to illustrate the ideas, consider a case that this distribution can be approximated by a normal one with  $\mu_o = \text{mean}$ ,  $\sigma_o = \text{standard deviation}$ . We next assume that all unidentified POEs are identically distributed. Consequently, we use two numbers  $(\mu_o, \sigma_o)$  to parameterize the contribution of an unidentified POE. If, as above, we further assume that the number of unidentified POEs has a Poisson distribution characterized by a parameter  $\lambda$ , then a calibrated knowledge of the numbers  $\mu_o$ ,  $\sigma_o$ , and  $\lambda$  would tell us a great deal about the effects of unidentified POEs in a similar experiment.

We can calculate, in this model, the various moments of  $R^U$  in terms of  $\mu_o$ ,  $\sigma_o$ , and  $\lambda$ . These formulas are contained in Appendix B, Equations B-11 through B-14. Equating the sample moments to the calculated ones, we solve for the three parameters. For example, if  $\mu_o$  were to be zero, we could solve

$$E\{(R^U)^2\} = \sigma_o^2 \lambda, \quad (60)$$

$$E\{(R^U)^4\} = 3 \sigma_o^4 (\lambda^2 + \lambda), \quad (61)$$



for  $\sigma_0$  and  $\lambda$ . If  $\mu_0 \neq 0$ , the first three moments determine the three parameters. Mathematical pitfalls notwithstanding, this program should yield numbers of sufficient validity to be useful in the evaluation of EMP test data.

The approximations used in obtaining these estimates of  $\mu_0$ ,  $\sigma_0$  and  $\lambda$  are as follows:

1. Sample moments are used for population moments.
2. All unidentified POEs are independent and are sampled from an identical and normal distribution.
3. The number of unidentified POEs has a Poisson distribution.

## APPENDIX A

We here consider four examples of the calculation of  $\gamma$  and  $\sigma$ , two with correlation (1 and 2) and two with randomness (3 and 4). Notice that the index  $j$  is defined slightly different from that in the main text for Examples 1 and 2.

Example 1: Let  $j = -L, \dots, L - 1$ ,  $n = 2L$ , and we have for the transfer functions,

$$t_j = 2/L T \begin{cases} (L/2 - j), & 0 \leq j \leq L - 1, \\ (L/2 + j), & -L \leq j \leq 0, \end{cases} \quad (\text{A-1})$$

while, for the excitations, ( $\alpha \geq 0$ )

$$e_j = 2/L E \begin{cases} (L/2 + \alpha - j), & \alpha \leq j \leq L - 1, \\ (L/2 - \alpha + j), & -L + \alpha \leq j \leq \alpha, \\ (-3L/2 + \alpha - j), & -L \leq j \leq -L + \alpha. \end{cases} \quad (\text{A-2})$$

Figure A-1 shows a graphical presentation of these behaviors. Here the maximum (in absolute value) transfer function occurs for  $j = 0$  and  $-L$  while the maximum (in absolute value) excitation occurs for  $j = \alpha$ ,  $-L + \alpha$ . The experiments are labeled by  $\alpha$ . The calculation of the respective  $\gamma_\alpha$  and  $\sigma_\alpha$  involve appropriate sums, which can be done. However, if  $L$  is large, the sums can be accurately approximated by integrals with a new parameter  $y$ ,

$$y = 2/L \alpha. \quad (\text{A-3})$$

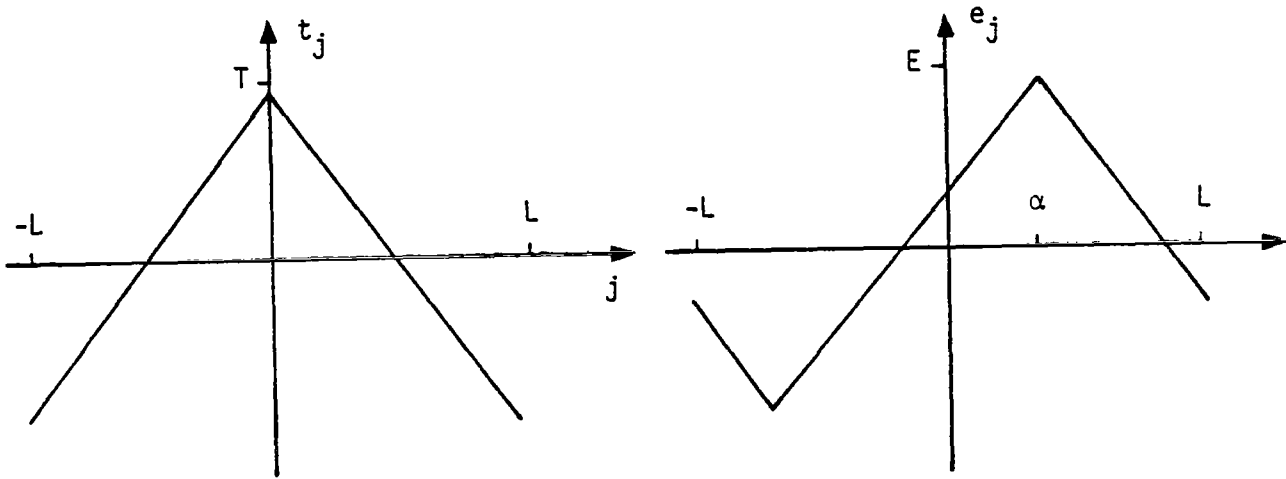


FIGURE A-1. VARIATION OF  $t_j$  AND  $e_j$  FOR EXAMPLE 1.

We then find ( $y \geq 0$ ),

$$\gamma_{\alpha} = T E \frac{1}{3} L (1 - y) (2 + 2y - y^2), \quad (\text{A-4})$$

$$\sigma_{\alpha}^2 = \Delta^2 T^2 E^2 \frac{1}{30} L [12 - 10y^2 (2 - y)^2], \quad (\text{A-5})$$

where  $\Delta$  is defined in Equation 12. The quantities of interest used in Section II are

$$E\{|X|\} = 5/24 n E T, \quad (\text{A-6})$$

$$\sigma_{|X|}^2 = n^2 E^2 T^2 [2/45 - (5/24)^2], \quad (\text{A-7})$$

$$E\{Y\} = \Delta^2 \frac{1}{9} n E^2 T^2. \quad (\text{A-8})$$

Finally, for a number  $c$  somewhat larger than 1,  $c > 1$ , we find the probability statement

$$P\{\sqrt{Y}/|X| > c\Delta/\sqrt{n}\} = 2/(\sqrt{5}c) + O(1/c^3). \quad (\text{A-9})$$

Example 2: Let  $j = -L, \dots, L - 1$ ,  $n = 2L$ , and now we have

$$t_j = T \cos 2\pi (j - \alpha)/L; \quad (\text{A-10})$$

$$e_j = E L/2\pi \frac{1}{j} \sin 2\pi j/L. \quad (\text{A-11})$$

For  $L$  large, we find the leading behavior to be

$$\gamma_{\alpha} = E T L/4 \cos 2\pi \alpha/L, \quad (\text{A-12})$$

$$\sigma_{\alpha}^2 = \Delta^2 E^2 T^2 L/4. \quad (\text{A-13})$$

We now have

$$E\{|X|\} = 1/4\pi n E T, \quad (\text{A-14})$$

$$\sigma_{|X|}^2 = n^2 E^2 T^2 [1/128 - (1/4\pi)^2], \quad (\text{A-15})$$

$$E\{Y\} = \Delta^2 1/8 n E^2 T^2, \quad (\text{A-16})$$

while the probability condition becomes

$$P\{\sqrt{Y}/|X| > c\Delta/\sqrt{n}\} = \sqrt{8}/(\pi c) + O(1/c^3). \quad (\text{A-17})$$

Example 3: Assume that the possible values of the individual products  $t_j e_j$  are

$$T E (1, 1/2, 0, -1/2, -1), \quad (\text{A-18})$$

all with equal probability. Also assume that different individual terms are independently distributed. The possible values for  $\gamma_{\alpha}$  are

$$\gamma_{\alpha} = \{n, n - 1/2, \dots -n\} T E \quad (\text{A-19})$$

with a distribution that depends on  $n$ . The calculation of  $E\{|X|\}$  for a large  $n$  is, in general, very involved. However, explicit calculations for  $n = 4, 5,$  and  $6$  show that

$$E\{|X|\} = 17/30 T E \sqrt{n} \quad (A-20)$$

where the factor 17/30 is just a reasonable approximation\* while

$$\sigma_{|X|}^2 = n T^2 E^2 [1/2 - (17/30)^2], \quad (A-21)$$

$$E\{Y\} = \Delta^2 T^2 E^2 1/2 n. \quad (A-22)$$

A rough estimate of the probability statement, again based on explicit calculations, is

$$P\{\sqrt{Y}/|X| > c\Delta\} = .6/c. \quad (A-23)$$

The calculations here are tedious. However, we get a similar conclusion from another example of this class for which explicit results can be obtained. If we were to replace Equation A-18 with

$$T E \{1, -1\}, \quad (A-24)$$

it is straightforward to show, for large n, that

$$P\{\sqrt{Y}/|X| > c\Delta\} = \sqrt{2/\pi} 1/c. \quad (A-25)$$

Example 4: If we assume that the possible values of the individual products  $t_j e_j$  are not symmetric with respect to zero, but, for example, are

$$T E \{3/2, 1, 1/2, 0, -1/2\}, \quad (A-26)$$

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\*The central limit theorem shows that the correct factor is  $1/\sqrt{\pi}$ .

then  $E\{|X|\}$  grows with  $n$ . For the case above, we find\*, for large  $n$ , that

$$E\{|X|\} = 1/2 T E n, \quad (A-27)$$

and

$$\sigma_{|X|}^2 = 1/2 n T^2 E^2, \quad (A-28)$$

$$E\{Y\} = \Delta^2 T^2 E^2 3/4 n. \quad (A-29)$$

The probability estimate for  $\sqrt{Y}/|X|$  is extremely difficult to get in this case. However, we expect very small probabilities for large values of this random variable. We can get an idea of the behavior by briefly considering another example of this class (random but with a bias). Let Equation A-26 be replaced by

$$T E \{-1, 3\}. \quad (A-30)$$


It is then possible to prove (ref. 5), for large  $n$  and moderate  $c$ ,  $c > 1$ , that

$$\begin{aligned} P\left\{\sqrt{Y}/|X| > \sqrt{5\Delta}/\sqrt{n} (1 + 8/5 c/\sqrt{n})\right\} &= 1 - \Phi(c) \\ &= 1/\sqrt{2\pi} 1/c e^{-c^2/2}. \end{aligned} \quad (A-31)$$

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\*The central limit theorem shows that the approach to Equation A-27 to be  $E\{|X|\} = TE \{1/2 n + \sqrt{4/n\pi} e^{-n/4}\}$ .

5. Handbook of Mathematical Functions, edited by Abramowitz, M. and Stegun, I.A., Dover, 1972, p. 945, Equation 26.5.21.



We expect a similar behavior for the above sample, i.e., the probability rapidly goes to zero for values larger than some  $c/\sqrt{n}$ .



## APPENDIX B

A sample distribution of the random variable  $R^U$  is given by the set of numbers  $(\gamma_\alpha^U \pm \sigma_\alpha^U)$ . We incorporate the errors by assuming that the individual points are spread out according to a normal distribution with mean  $\gamma_\alpha^U$  and standard deviation  $\sigma_\alpha^U$ . If we have  $N$  points and they have been independently sampled, the first few sample moments of the sample distribution are (we here delete the superscript  $U$ )

$$\hat{E}\{R\} = 1/N \sum_{\alpha} \gamma_{\alpha}, \quad (\text{B-1})$$

$$\hat{E}\{R^2\} = 1/N \sum_{\alpha} (\gamma_{\alpha}^2 + \sigma_{\alpha}^2), \quad (\text{B-2})$$

$$\hat{E}\{R^3\} = 1/N \sum_{\alpha} \gamma_{\alpha} (\gamma_{\alpha}^2 + 3 \sigma_{\alpha}^2), \quad (\text{B-3})$$

$$\hat{E}\{R^4\} = 1/N \sum_{\alpha} (\gamma_{\alpha}^4 + 6 \gamma_{\alpha}^2 \sigma_{\alpha}^2 + 3 \sigma_{\alpha}^4), \quad (\text{B-4})$$

where carot = sample quantities.

We notice that the inclusion of the errors leads to a wider distribution than if the errors were not present, i.e., the distribution of the true values  $\gamma^U$  themselves. The standard deviation of the sample,  $\hat{\sigma}$ , is given by

$$\hat{\sigma}^2 = \hat{E}\{R^2\} - [\hat{E}\{R\}]^2. \quad (\text{B-5})$$

The model for the calculation of  $R^U$  is to randomly select  $k$  numbers from a normally distributed ( $\mu_0$  = mean,  $\sigma_0$  = standard deviation) random variable and to assume that the number selected,  $k$ , is distributed according to a Poisson distribution ( $p_0 = e^{-\lambda}$ ).

That is, we take

$$R_L^U \equiv \sum_{k=1}^L R_k^U \quad (\text{B-6})$$

in which

$$R_k^U \in N(\mu_0, \sigma_0),$$

$$L \in \text{Poisson}(\lambda). \quad (\text{B-7})$$

The calculation of the moments of such a random variable is then

$$E\{R\} = \mu_0 \sum_{k=1}^{\infty} k p_k, \quad (\text{B-8})$$

$$E\{R^2\} = \sigma_0^2 \sum_{k=1}^{\infty} k p_k + \mu_0^2 \sum_{k=1}^{\infty} k^2 p_k, \quad (\text{B-9})$$

$$E\{R^3\} = \mu_0^3 \sum_{k=1}^{\infty} k^3 p_k + 3\mu_0 \sigma_0^2 \sum_{k=1}^{\infty} k^2 p_k, \quad (\text{B-10})$$

$$E\{R^4\} = \mu_0^4 \sum_{k=1}^{\infty} k^4 p_k + 6\mu_0^2 \sigma_0^2 \sum_{k=1}^{\infty} k^3 p_k + 3\sigma_0^4 \sum_{k=1}^{\infty} k^2 p_k. \quad (\text{B-11})$$

Finally, making use of the moments of the Poisson distribution,

$$\sum_{k=1}^{\infty} k^n p_k = e^{-\lambda} (\lambda d/d\lambda)^n (e^{\lambda} - 1). \quad (\text{B-12})$$

we have,

$$E\{R\} = \mu_0 \lambda, \quad (\text{B-13})$$

$$E\{R^2\} = \mu_0^2 \lambda^2 + \lambda (\mu_0^2 + \sigma_0^2), \quad (\text{B-14})$$

$$E\{R^3\} = \mu_0^3 \lambda (\lambda^2 + 3\lambda + 1) + 6\mu_0 \sigma_0^2 \lambda (\lambda + 1), \quad (\text{B-15})$$

$$E\{R^4\} = \mu_0^4 \lambda (\lambda^3 + 6\lambda^2 + 7\lambda + 1) + 6\mu_0^2 \sigma_0^2 \lambda (\lambda^2 + 3\lambda + 1) + 3\sigma_0^4 \lambda (\lambda + 1). \quad (\text{B-16})$$

For the random variable  $R^2$ , Equation 40, we have its mean and standard deviation as (all sums run from 1 to n)

$$E\{R^2\} = \sum_i (\mu_{2i} - \mu_{1i}^2) + \left( \sum_i \mu_{1i} \right)^2, \quad (\text{B-17})$$

$$\sigma^2 = E\{R^4\} - [E\{R^2\}]^2 \quad (\text{B-18})$$

$$\begin{aligned} &= \sum_i \left( \mu_{4i} - 6\mu_{1i}^4 - 4\mu_{1i}\mu_{3i} + 12\mu_{1i}^2\mu_{2i} - 3\mu_{2i}^2 \right) \\ &+ \left( \sum_i \mu_{1i} \right) \sum_j \left( 4\mu_{3j} - 12\mu_{1j}\mu_{2j} + 8\mu_{1j}^3 \right) \\ &+ \left( \sum_i \mu_{1i} \right)^2 \sum_j \left( 4\mu_{2j} - 4\mu_{1j}^2 \right) \\ &+ \left( \sum_i \mu_{1i}^2 \right) \sum_j \left( 2\mu_{1j}^2 - 4\mu_{2j} \right) \\ &+ 2 \left( \sum_i \mu_{2i} \right)^2 \end{aligned}$$

where the individual moments are defined in Equation 41. If the individual  $R_j$  are normally distributed  $(\mu_j, \sigma_j)$ , the above equations simplify to

$$E\{R^2\} = \sum_i \sigma_i^2 + \left(\sum_i \mu_i\right)^2, \quad (\text{B-19})$$

$$\sigma^2 = 4 \left(\sum_i \mu_i\right)^2 \sum_j \sigma_j^2 + 2 \left(\sum_i \sigma_i^2\right)^2. \quad (\text{B-20})$$

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