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A GEOMETRIC THEORY OF NATURAL OSCILLATION FREQUENCIES IN EXTERIOR SCATTERING PROBLEMS

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ABSTRACT

The representation of the transient electromagnetic response of finite size, smooth, objects composed of simple media in terms of a complex exponential series is the central ingredient of the singularity expansion method (SEM). The exponential terms correspond to the complex natural frequencies associated with the object geometry. That such a simple series can predict the force free response of complicated objects begs the question "is there not a corresponding more direct method to compute the natural frequencies in exterior scattering problems."

To partially answer this question a geometric ray optics method, which because of its asymptotic nature is particularly suited to compute the higher order resonances, is described.

The idea is, in general terms, to consider a smooth, convex object with a surface impedance boundary condition. Complex resonant frequencies are computed from closed path integrals over the surface. It is hypothesized that the paths Γ_i represent surface geodesics whose definition includes electromagnetic inertial effects.

The method when applied to a sphere is shown to reduce to the well known uniform asymptotic expansion of the spherical Hankel functions. In this example a comparison of the asymptotic and exact results for the natural oscillation frequencies of a sphere is given.

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Introduction

The complex poles in the s domain which determine the complex resonant frequencies of finite convex scattering geometrics have been determined numerically [1]. From theoretical studies it has been observed that the complex frequencies or pole locations depend only upon the scattering obstacle and not upon the incident waveform [2,3]. In reference [1], Tesche observed that the approximate distance between adjacent poles in the s plane have the property that

$$\mathrm{Im}\,(\Delta\, s) \cong \pi\, c/_{L} \tag{1}$$

where L is a characteristic length of the body and c is the speed of light. Physically, one expects that the exterior frequencies must be complex to account for ray divergence and bending losses.

These observations suggest that the exterior resonance phenomenon is a form of damped periodic motion. Similar systems are cavity resonance and Bohr orbits. Thus (1) can be considered to be related to a phase reinforcement condition which requires that closed ray paths around the object corresponding to resonant frequencies effectively contain an integer number of wavelengths. This, of course, is the principle of Bohr orbits of the Old Quantum Theory [4].

An asymptotic geometric theory should apply to electrically large objects and improve as the object dimensions, scaled in wavelengths, or mode index n becomes large. The quasi-periodic nature of observed transient responses from such objects suggest the candidate theories of transverse resonances [5], the Wentzel-Kramers-Brillouin or

WKB approximation [6], and a suitable modification of the Bohr-Sommerfeld-Wilson condition of the Old Quantum Theory [4].

Theory

To motivate the development to follow, it is useful to outline the elegant geometric treatment of interior resonances of Keller and Rubinow [7]. Many of their basic ideas and topological concepts, as well as their comparative accuracy, can be expected to carry over to the more difficult external problem. Their idea is to solve the scalar wave equation

$$(\nabla^2 + k^2 n^2)\ \psi(\bar{x}) = 0 \tag{2}$$

on the interior of a smooth closed surface S where ψ is assumed to satisfy Neumann or Dirichlet conditions on S. If we assume a solution of the form

$$\psi(\bar{x}) = A\ (\bar{x})\, e^{ik_o S(\bar{x})} \tag{3}$$

it follows from substituting (3) into (2) that the phase function $S(\bar{x})$ and the amplitude function $A(\bar{x})$ satisfy the eikonal and transport equations

$$(\nabla\ S)^2 = n^2 \tag{4}$$

$$\nabla^2\ S + 2\ \bar{\nabla}\ S \cdot \ln A = 0\ . \tag{5}$$

If the local ray trajectory has a unit tangent vector $\hat{t}$ then the local wave vector $\bar{k}$ is given by

$$\bar{k} = k_o\ \bar{\nabla}\ S \quad , \qquad \nabla\ S = \hat{t}\ n(\bar{x}) \tag{6}$$

The resonance condition is

$$\oint \bar{k} \cdot \overline{dl} = 2\pi\,(n + m/4 + b/2) \tag{7}$$

where n, m, and b are integers. In particular, m and b are the number of times the ray hits the caustic or boundary respectively. (This is for Dirichlet conditions on S; for the Neumann condition, b is identically zero.) Condition (7) holds for every independent closed curve on the covering space. Keller and Rubinow consider the example of a resonant ray inside a perfectly conducting circular surface of radius a. Using geometric optics, one finds that there exists a caustic surface of radius $a_o < a$ as shown in Figure 1.

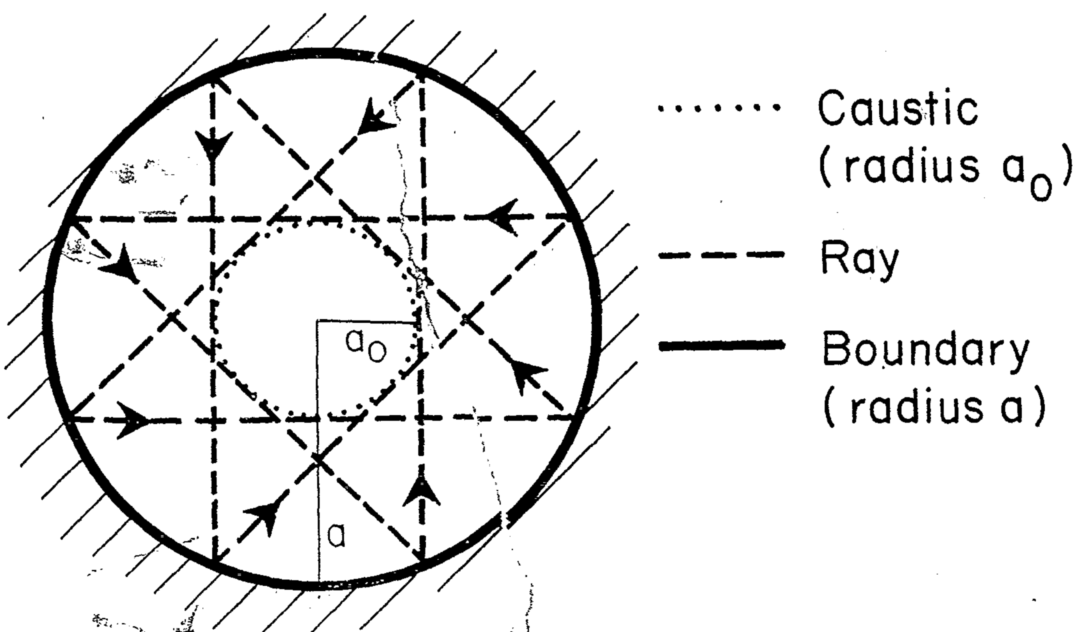
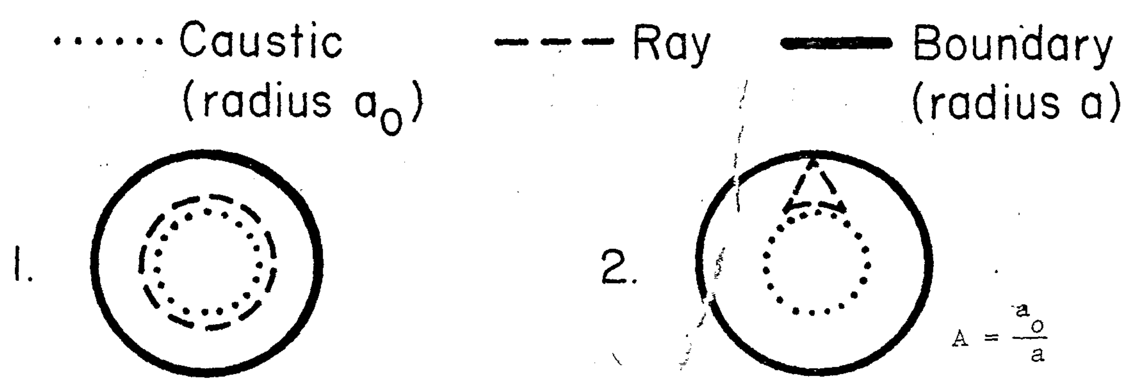


Figure 1. Interior Resonant Ray

If one accounts for the number of ray congruencies, and the independent closed curves, by successive elimination of the parameters such as the caustic radius a_o in Figure 1, the desired wave number k and hence the complex frequency s can be obtained.

The independent paths on the covering cannot be deformed into each other without crossing a singularity of the field. Thus, Equation (7) is actually a residue theorem in disguise. This characterization of the resonance condition can be used directly in target discrimination as we will discuss in conclusion of the paper.

In the example of Figure 1 there are two independent paths in the covering labeled 1 and 2 in Figure 2.



$$A = \frac{a_o}{a}$$

$$k_o 2\pi a_o = 2\pi m \qquad (8)$$

$$k_o(2\sqrt{a^2-a_o^2} - 2a_o\cos^{-1}(A)) \qquad (9)$$
$$= 2\pi(n + 3/4)$$

Figure 2. The Independent Paths

Eliminate a_o from Equations (8) and 9) and solve for ka. The resulting expression for ka is an asymptotic formula for j_{nm}, the nth zero of the Bessel function J_m.

The following table from Reference 2 compares approximate and exact zeros.

Table 1. Interior Resonance Comparison

m	n	Approximate	Exact	Fractional Error
0	1	2.356	2.405	0.0204
0	2	5.498	5.520	0.0040
0	3	8.639	8.654	0.0017
0	4	11.781	11.792	0.0009
1	1	3.795	3.802	0.0097
1	2	6.997	7.016	0.0027
1	3	10.161	10.173	0.0012
1	4	13.311	13.324	0.0010
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

As can be seen, the asymptotic solution is surprisingly accurate even for the lowest order modes.

In the exterior problem the situation becomes more complicated. The role of the caustic and the boundary are interchanged and the ray paths become curved. In Figure 3, evanescent rays are shown trapped between the surface S_o and their respective caustics. When curvature is encountered, some of the energy radiates away from the object.

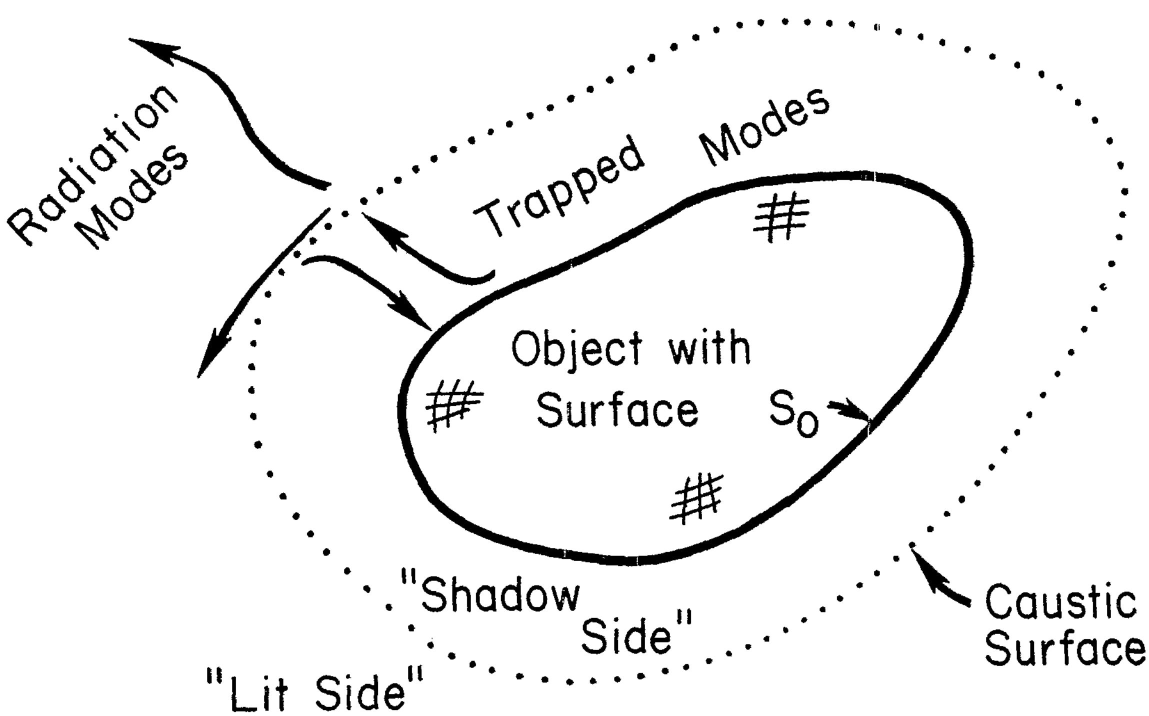


Figure 3. Exterior Resonance Ray Orbits

It is easy to appreciate that the corresponding eigenvalues in the exterior problem are complex. The imaginary part of the eigen-frequencies s_n associated with scalar transient f(t) are associated with the object circumference. Let f(t) be written as the inverse Laplace transform of F(s).

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, e^{st}\, ds$$

Let s_n be a simple pole of F(s) in the left half plane of the s domain, i.e.,

$$s_n = \sigma_n + i\,\omega_n \quad , \quad \sigma_n < 0.$$

The real part σ_n is physically associated with radiation damping and ray tube spreading upon reflection from the convex surface. The idea is demonstrated in Figure 4.

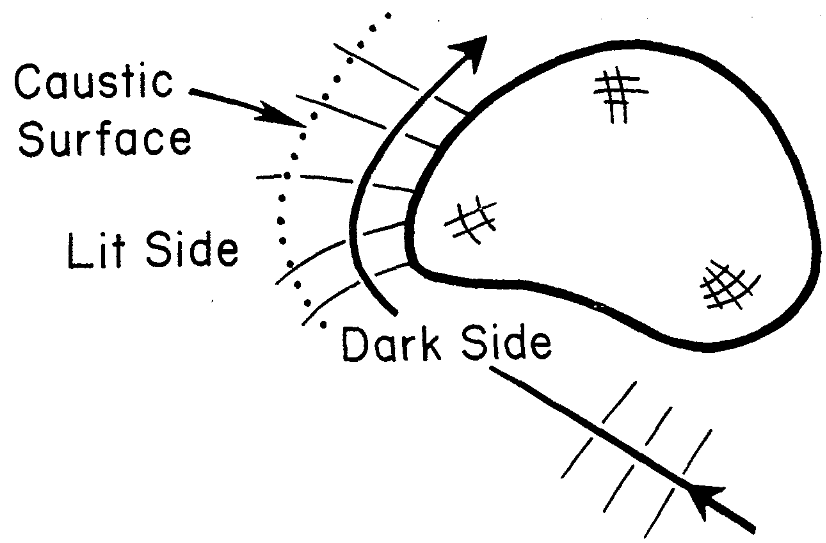


Figure 4. Heuristic Definition of Caustic

The caustic surface for the external problem can heuristically be defined as the locus where the local wave front tangential velocity exceeds that vacuum velocity c. The energy must, therefore, detach from the packet. This phenomenon is associated with the inertia of the electromagnetic field: there is a cause and effect relationship between local spatial curvature and electromagnetic energy density.

Such a relationship was obtained by Einstein in his theory of General Relativity in the form of a conservation law involving the covariant derivative of the electromagnetic energy momentum tensor $T^{\mu\nu}$ The mathematical statement is given by

$$\partial/\partial x_\nu \ (\sqrt{g}\ T^\nu{}_\lambda) - \frac{1}{2} \left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right) \sqrt{g}\ T^{\mu\nu} = 0 \tag{10}$$

where $g^{\mu\nu}$ is the space time metric tensor and $g = \det (g^{\mu\nu})$ [8,9].

It has been shown by Choudhary and Felsen [10] that geometric ray tracing in evanescent regions is further complicated by the non-congruence of the phase propagation paths of simple geometric theory and the power flow trajectories.

To obtain a geometric theory for the exterior eigenfrequencies, we postulate that the periodic damped motion be described by a suitable modification of the Bohr-Sommerfeld quantization rule for the action integral of the "Old Quantum Theory," [4,11,12].

$$\oint p_i \, dq_i = n_i h \quad , \quad i = 1,2,3\ldots, \ n_i = 1,2,3. \tag{11}$$

Here, p_i, q_i are the canonically conjugate momenta and coordinates as defined by the equations of motion in the Hamiltonian formalism. An important point is that for each independent momentum coordinate set (p_i, q_i) , $i = 1,2,\ldots$ there is a corresponding quantization or resonance condition. In the classical physics problem of determining the resonant frequencies of an electromagnetic cavity oscillator, Keller and Rubinow [7] showed that Equation (7) can be used to compute the associated eigenvalue spectrum.

We want to show that analogous independent closed curves Γ_p , $p = 1,2,3\ldots$ can also be determined and that the independent momentum-coordinate defined action integrals for the exterior problem determine the eigenvalue spectrum.

The Bohr-Sommerfeld rule was developed for the determination of electron orbits and associated energy eigenvalues of atomic physics. However, the wave nature of particles is manifest according to de Broglie's relation that a particle with momentum p has associated wave properties with wavelength λ [6]. This relation is

$$\lambda = h/p \quad \text{or} \quad k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h} \tag{12}$$

Thus, the wavenumber momentum relation corresponding to (1) is

$$\oint_{\Gamma_i} k_i \, dq_i = 2\pi n_i \qquad \begin{array}{l} i = 1,2,3\ldots \\ n_i = 0,1,2,3\ldots \end{array} \tag{13}$$

We postulate that (13) with proper interpretation determines asymptotically the complex resonant frequencies of an exterior scatterer. In the original development of the Old Quantum Theory, the correspondence principle was used to bridge the gap between classical and quantum mechanical domains. Then, the classical limit is approached as the integer n_i in Equation (11) becomes large. Similarly, the asymptotic optical limit for electromagnetic analogue in Equation (13) is approached when the corresponding n_i becomes large. To apply these concepts we first develop the relevant scattering problem.

Our boundary value problem is as follows. We wish to solve the scalar wave Equation (2) for $\psi(\bar{x})$ in the region exterior to the surface S_o as shown in Figure 3. The boundary conditions on $\psi(\bar{x})$ is an impedance condition of the form

$$\left(\frac{\partial\psi}{\partial\eta} + q\psi\right)\Big|_{\bar{x}\varepsilon S_o} = 0 \tag{14}$$

where $q = -i\,\omega\,\varepsilon_o\,z_o\,\Delta$, $z_o = 120\,\pi\,\Omega$, and

Δ = normalized surface impedance at grazing incidence

The same type of asymptotic solution described in Equations (3) through (7) are again used. The equation corresponding to (13) for the exterior ray orbits over an impedance surface is

$$k_o \oint_{\Gamma_i} \bar{\nabla} S \cdot d\bar{l} = 2\pi \left(n_i - \frac{m_i}{4} - \phi_i\right) \qquad i = 1,2,3 \ldots \tag{15}$$

The interpretation of Equation (15) is the same as (13) where now the phase ϕ_i is given by

$$\phi_i = \frac{1}{2\pi} \arg(\Gamma), \qquad \Gamma = \frac{\Delta-1}{\Delta+1} \tag{16}$$

where Δ is the normalized outward pointing surface impedance introduced in Equation (14). Thus

$$\arg(\Gamma) = \mathrm{Im}(\ln(\Gamma))$$

and the paths Γ_i are yet to be determined. The caustic in the exterior problem, as shown in Figure 3, separates the dark from the light side. The dark side is closer to the surface S_o; outside the caustic the energy radiates away from the surface. The caustic or turning point then separates the evanescent and radiating regions. The fact that the phase changes $\pi/2$ radians at the caustic can be seen from Equation (16) if the appropriate normalized impedance $\Delta = i$ is used.

As described by Howard [13], this caustic phase advance is central to the bending loss associated with open waveguides. In addition to the bending losses and the ohmic losses in the scattering object, there is a third loss mechanism in exterior problems. This is the ray spreading caused by reflection off the convex surface S_o. One way to account for this ray tube divergence is to place the amplitude function $A(\bar{x})$ into the exponent and then replace $\bar{\nabla} S$ in Equation (15) by

$$\nabla S - i/k_o \, \nabla \ln A \tag{17}$$

The dependence of the amplitude function A near a surface such as S_o has been carried out by Kouyoumjian and Pathak [14,15].

APPLICATION TO A SPHERE

The exterior resonances of a sphere have been thoroughly investigated beginning with Thompson's 1884 treatment [16]. A more modern discussion can be found in Stratton's book [17]. Numerical results have been presented by Martinez et al [18]. In these references it is shown that the complex resonances for the sphere are associated with the complex zeros ρ_{nm} of the spherical outward radiating Hankel function $h_n^{(1)}$. In the case of perfect conductivity the mode equations are

H Type Modes (Horizontal Polarization, $q \to \infty$)

$$h_n^{(1)}(\rho_{nm}) = 0 \tag{18}$$

E Type Modes (Vertical Polarization, $q \to 0$).

$$(\rho h_n^{(1)}(\rho))' = 0 \tag{19}$$

The poles s_{nm} in the Lapace s domain which corresponds to the complex resonant frequencies are given by

$$s_{nm} = -i \frac{\rho_{nm}}{a\sqrt{\mu\varepsilon}}, \; m = 1,2,3 \ldots \tag{20}$$

where a is the sphere radius.

Because the spherical Hankel function of order n is a polynomial of order n multiplied by an exponential function, the natural frequencies for a sphere are particularly easy to compute.

The geometric theory as represented by the generalized Bohr-Sommerfeld-Wilson condition (15) should asymptotically produce the complex frequencies as defined by Equations (18), (19) and (20).

For the general case of an arbitrarily shaped convex scattering body, a major difficulty in applying Equation (15) is in the determination of the fundamental paths Γ_i. It is anticipated that the required geodesic paths will be best computed using local geodesic coordinates on the surface of S_o as defined, for example by Struik [19] or O'Neill [20]. These local solutions then are integrated over the global geodesic paths to determine the resonant frequencies.

For a sphere, the local solution can be trivially translated into the global solution. A coordinate independent general way to approach the problem is to compute the ray divergence contribution to the loss term in Equation (17) through the geometrical formulation of reflection from curved surfaces as developed for example by Kouyoumjian and Pathak [14].

Keller has carried out this approach for asymptotic solutions to the Schrodinger equation [4]. He found that the amplitude contribution to the phase (as determined by Equation (17) here, for example) gives rise to half-integer quantum numbers.

At this stage of progress in our work, a heuristic approach to the geometric computation of the exterior resonances of a metallic sphere is given.

The convex scattering surface, as shown in Figure 3, can be defined mathematically by an equation of the form

$$f(x,y,z) = 0 \tag{21}$$

In a neighborhood of the point (0,0,0) it is possible to represent f in the form

$$f = z + 1/2\ (\alpha x^2 + 2\beta xy + \gamma y^2) \tag{22}$$

where the convexity of the surface and fixing the orientation of positive z to the convex side of f requires that

$$\alpha > 0,\ \beta > 0,\ \alpha\gamma - \beta^2 > 0 \tag{23}$$

The coefficients α, β, γ are given by

$$\alpha = -\left.\frac{\alpha^2 z}{\alpha x^2}\right|_{x = y = 0},\ \beta = -\left.\frac{\partial^2 z}{\partial x \partial y}\right|_{x = y = 0},\ \gamma = \left.\frac{-\partial^2 z}{\partial y^2}\right|_{x = y = 0} \tag{24}$$

so that representation (22) is just a McLaurin series through second order terms about the point $x = y = 0$ of the surface (21). At the point (0,0,0) on the surface the normal curvature in the x direction is $a = 1/\alpha$ and the analogous y quantity is $b = 1/\beta$.

To model a convex body with two principal radii of curvature we choose the system defined by the arc length formula

$ds^2 = \rho^2\, d\theta^2 + b^2\, d\phi^2 + d\rho^2$.

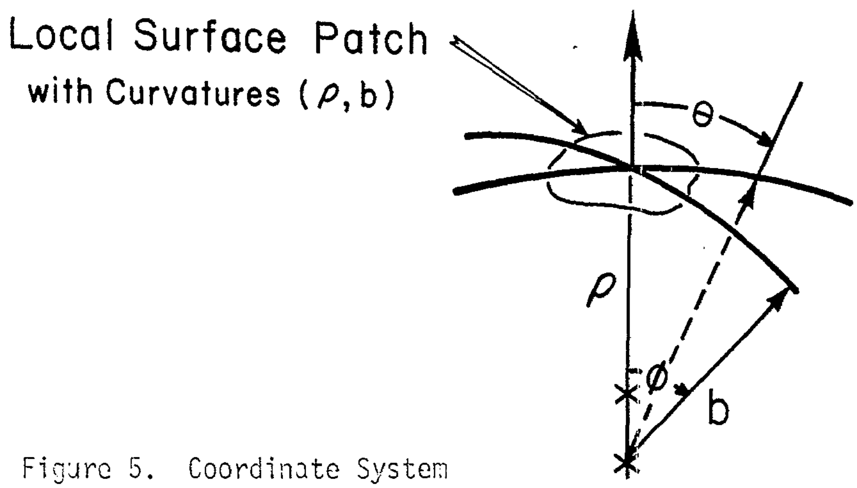


Figure 5. Coordinate System

The Laplacian operator is determined to be

$$\nabla^2\psi = \frac{1}{\rho b}\left[\frac{\partial}{\partial\theta}\left(\frac{b}{\rho}\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{\rho}{b}\frac{\partial\psi}{\partial\phi}\right) + \frac{\partial}{\partial\rho}\left(\rho b\frac{\partial\psi}{\partial\rho}\right)\right] \tag{25}$$

Consider ray solutions propagating in the direction of increasing θ. We transform out the transverse ϕ dependence. Thus, let

$$\psi(\rho,\theta,\phi) = \int_{-\infty}^{\infty} \hat{\psi}(\rho,\theta,t)\, e^{ik\phi t}\, dt \tag{26}$$

The wave equation

$$(\nabla^2 + k^2)\,\psi = 0$$

becomes upon using (25) and (26),

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\hat{\psi}}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2}+k^2(1-t^2/b^2)\,\hat{\psi}=0 \tag{27}$$

Assume a product solution of the form

$$\hat{\psi}(\rho,\theta,t)=R(\rho)\;e^{i\gamma ka\theta} \tag{28}$$

where a is the local radius of curvature in the ray direction, and γ is a normalized propagation constant to be determined. The ordinary differential equation for $R(\rho)$ is thus

$$\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\left((k\rho)^2-k^2t^2\left(\frac{\rho}{b}\right)^2-\gamma^2(ka)^2\right)R(\rho)=0 \tag{29}$$

Let $x=k\rho\left(1-\left(\frac{t^2}{b}\right)\right)^{1/2}$, $\nu=\gamma ka$, $u(x)=R\,(\rho)$

Then, the equation for $u(x)$ is

$$x\,(x\,u'(x))'+(x^2-\nu^2)\;u(x)=0 \tag{30}$$

In equation (30), make the change of variable

$$y=\ln\,(x/x_a)\;,\quad x_a=x\,\Big|_{\rho=a}=ka\alpha\;,$$

$$f(y)=u(x)\qquad,\qquad\alpha=(1-(t/b)^2)^{1/2}$$

The resulting differential equation for $f(y)$ is

$$\frac{d^2f(y)}{dy^2}+\Omega^2n^2(y)\;f(y)=0 \tag{31}$$

where

$$n^2(y)=\alpha^2\;e^{2y}-\gamma^2\qquad,\;\Omega=ka$$

The radial Bohr-Sommerfeld resonance condition for equation (31), which corresponds to a ray trip up and back between s_o and the caustic in Figure 3, becomes

$$2\;\Omega\int_o^{y_o}n(y)dy=-\;\pi/2 \tag{32}$$

Here we have chosen $m_i = 1$ in Equation (15) and y_o is the turning point of Equation (31), i.e., $n(y_o) = 0$. Also in Equation (32), ϕ_i as defined by Equation (16) is zero. This is the case for vertical polarized E field on a perfectly conducting sphere. In Equation (32), we make the change of variable

$$\tau = e^{-(y_o - y)} \text{ and define } z = \frac{\alpha}{\gamma} .$$

Then Equation (32) becomes

$$2i\Omega\gamma \int_z^1 \frac{(1-\tau^2)^{1/2}}{\tau} d\tau = -\pi/2 . \tag{33}$$

Olver has made use of this integral extensively in his work on special functions [21,22]. He defines the implicit relationship $z(\zeta)$ as

$$2/3\ \zeta^{3/2} = -\int_1^z \left(\frac{1 - t^2}{t}\right)^{1/2} dt \tag{34}$$

In our application ζ is given and z is to be computed. This is a complex valued transcendental equation in which it is quite useful to introduce an intermediate change of variable

$$z = \operatorname{sech} \sigma \tag{35}$$

since then it can be shown that

$$2/3\ \zeta^{3/2} = \sigma - \tanh \sigma \tag{36}$$

Details can be found in Appendix A.

Equation (33) then is the radial resonance condition. For the sphere, the angular resonance condition from Equation (15) and our assumed angular dependence as given in relation (28) is

$$2\pi\gamma\Omega \;=\; 2\pi n \qquad n = 1,2,3,\; \Omega = k_o a \tag{37}$$

since the ray orbits are obviously great circles. (All great circles on a sphere are geodesics.) Combining Equations (33), (34) and (37) yields the mode equation

$$\frac{4in}{3}\zeta^{3/2} = -\pi/2 \quad (\mathrm{mod}\ 2\pi). \tag{38}$$

This is a phase requirement so that $\zeta(z)$ as defined by Equation (38) is multivalued. Thus, from Equation (38) it follows

$$2/3\,\zeta^{3/2} = -\frac{i\pi}{4n}(4m-1) \quad , \quad m = 1,2,3\ldots \tag{39}$$

On the other hand, the uniform asymptotic formula for the roots of the spherical Hankel function $h_n^{(1)}(z)$ are given by [23].

$$n^{2/3}\,\zeta\,(z_{nm})\,e^{i2\pi/3} \;=\; a_m \tag{40}$$

where a_m is the m^{th} root of the Airy function Ai [23]. The zeros of Ai are along the negative real axis. The lead term of the asymptotic expansion for the zeros of Ai is (see [23], p 450)

$$a_m \simeq a_m^{(0)} \;=\; -\,(3\pi\,(4m-1)/8)^{2/3}\;,\quad m = 1,2,3\;\ldots \tag{41}$$

If expression (41) is substituted into (40), the result is identical to the mode equation (39). Thus, we have shown that the modified Bohr-Sommerfeld-Wilson quantum conditions when applied to the exterior resonances of a sphere result in a mode equation which is identical to the uniform asymptotic formulas for the complex zeros of the Hankel functions as given by Abramovitz and Stegun [23] or Olver [21].

To give an indication of the accuracy of the method, a table of the exact (Z_{nk}) and asymptotic ($\tilde{Z}_{nk}$ as computed using Equation (39) and results of Appendix A) complex zeros of $h_4^{(1)}(z)$ is given.

Table 2. Exterior Resonance Comparison of Exact (Z_{nk}) and Asymptotic ($\tilde{Z}_{nk}$) Zeros of $h_4^{(1)}$ (z).

k (n = 4)	Z_{nk}		$\tilde{Z}_{nk}$	
1	2.6574,	-2.1038	2.6539,	-2.1035
2	0.8672,	-2.8962	0.8628,	-2.8950
3	-0.8672,	-2.8962	-0.8741,	-2.8927
4	-2.6574,	-2.1038	-2.6740,	-2.0892

Again, as in the interior resonance comparison of Table 1, the geometric results are surprisingly accurate.

Discussions and Recommendations

In conclusion, we have shown that a modification of the Bohr-Sommerfeld-Wilson quantum condition can predict exterior resonance frequencies. The method is geometric and hence applies to non-separable geometrics. It has the potential to order the pole trajectory plots of SEM into radial and angular families. The grouping of eigenvalues in SEM has been discussed from an eigenmode point of view by Baum [24].

It is important to realize that the poles we have determined do not correspond to creeping waves. Thus, if s_{nk} is the position of the pole in the s plane, then far from the surface of the sphere the modes have the dependence

$$e^{s_{nk}(t - r/c)} \quad , \; s_{nk} \; = \; \frac{j \; c \; Z^*_{nk}}{a} \; , \; Re(s_{nk}) \leq 0 \qquad (42)$$

where we are using the traditional electrical engineering convention $e^{j\omega t}$ and $s \leftrightarrow j\omega$, and the complex conjugate Z^*_{nk} appearing in formula (42) is used to agree with this convention. The exponential form of (42) has the well known "exponential catastrophe" behavior as $r \to \infty$ [25]. The wave amplitude increases in the radial direction. This, as is well known, is accounted for in setting up the proper excitation conditions [9]. Creeping waves on the other hand obey radiation conditions at infinity. The relationship between creeping waves and ray orbits as developed herein remains to be determined.

An interesting question that arises is "how do the pole plots of s_{nk} move in the s plane when the object geometry changes continuously?" The interpretation of the resonance conditions (13) and (15) in terms of the residue theorem tells us that the pole positions will move little and their number will be conserved unless an additional singularity of the field is created by the geometry deformation. A deeper understanding of this analytic function method should be pursued. The significance of topological invariants of ("compact orientable geometric") surfaces such as the Euler-Poincaré characteristic [20] also needs to be investigated.

An immediate logical extension of this theory is to properly define the independent closed paths Γ_i for more general surfaces. The theory should then be checked against the known results for the prolate and oblate spheroid.

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APPENDIX A

The solution to the transcendental equation

$$2/3\,\zeta^{3/2} = -\int_1^z \frac{(1-t^2)^{1/2}}{t}\,dt \qquad (A.1)$$

is now given. Olver has determined that "where branches are defined to take their principal values when $z\varepsilon(0,1)$ and $\zeta\varepsilon(0,\infty)$ and are continuous elsewhere" (p. 421 of [19]).

The integral is doable:

$$2/3\,\zeta^{3/2} = \ln\left(\frac{1+\sqrt{1-z^2}}{z}\right) - \sqrt{1-z^2} \qquad (A.2)$$

To aid in the solution of this implicit function (i.e., given ζ find z) let us make the change of variable

$$z = \operatorname{sech}\sigma \text{ then } 2/3\,\zeta^{3/2} = \sigma - \tanh\sigma \qquad (A.3)$$

Thus, given ω where

$$\omega = \sigma - \tanh\sigma$$

σ is determined.

Let $\omega = u + iv$, $\sigma = \alpha + i\beta$, $\tanh\sigma = \tau + i\mu$

where $\tau = \sinh 2\alpha/\cosh(2\alpha) + \cos(2\beta))$

$\mu = \sin 2\beta/(\cosh(2\alpha) + \cos(2\beta))$

Notice then $z = \dfrac{\cosh\alpha\,\sin\beta - i\,\sinh\alpha\,\cos\beta}{\sinh^2\alpha + \sin^2\beta}$

so that

$$u = \alpha - \frac{\sinh 2\alpha}{\cosh 2\alpha + \cos 2\beta} \qquad \text{(a)}$$

$$v = \beta - \frac{\sin 2\beta}{\cosh 2\alpha + \cos 2\beta} \qquad \text{(b)} \qquad \text{(A.4)}$$

Thus, in (A.4) given u and v, we must solve for (α,β) simultaneously.

$$\cosh 2\alpha + \cos 2\beta = \frac{\sinh 2\alpha}{\alpha - u}, \quad \cosh 2\alpha + \cos 2\beta = \frac{\sin 2\beta}{\beta - v}$$

Therefore, $\sinh 2\alpha = \frac{\alpha-u}{\beta-v} \sin 2\beta$ (A.5)

Procedure:

Given (u,v) take approximate β value and solve for corresponding α in (A.5). Then, take this (α,β) pair and substitute them into (A.4a). Vary β with fixed α until (A.4a) is satisfied. Take this new value of β and substitute it into (A.5) to obtain an updated value of α. Stop iteration when $|z_n - z_{n-1}| < \varepsilon$ where ε is preset tolerance.