Interaction Notes

Note 381

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Static Electric and Magnetic Field Penetration of a Spherical Shield Through a Circular Aperture

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Abstract

Low-frequency electromagnetic penetration of a closed shielded region via an aperture in the shield is considered by investigating the canonical problems in which the shield is a perfectly conducting spherical shell, the aperture is circular, and the applied field is uniform. Each of these problems reduces to that of solving a set of dual series equations. The solutions of previously solved problems are presented as well as those of heretofore unsolved problems. The penetration of the shielded region is measured by the ratio of the field at the center of the sphere to the external applied uniform field. It has been previously shown that these ratios are the same for an applied magnetic field parallel to the symmetry axis and an applied electric field perpendicular to this axis; in this note it is shown that the ratios are the same for an applied electric field parallel to the axis when the shell is uncharged and for an applied magnetic field perpendicular to the axis. In addition, a new approach to the solution of certain class of dual series equations is found and exploited in the solution of two of the canonical problems.

I. Introduction

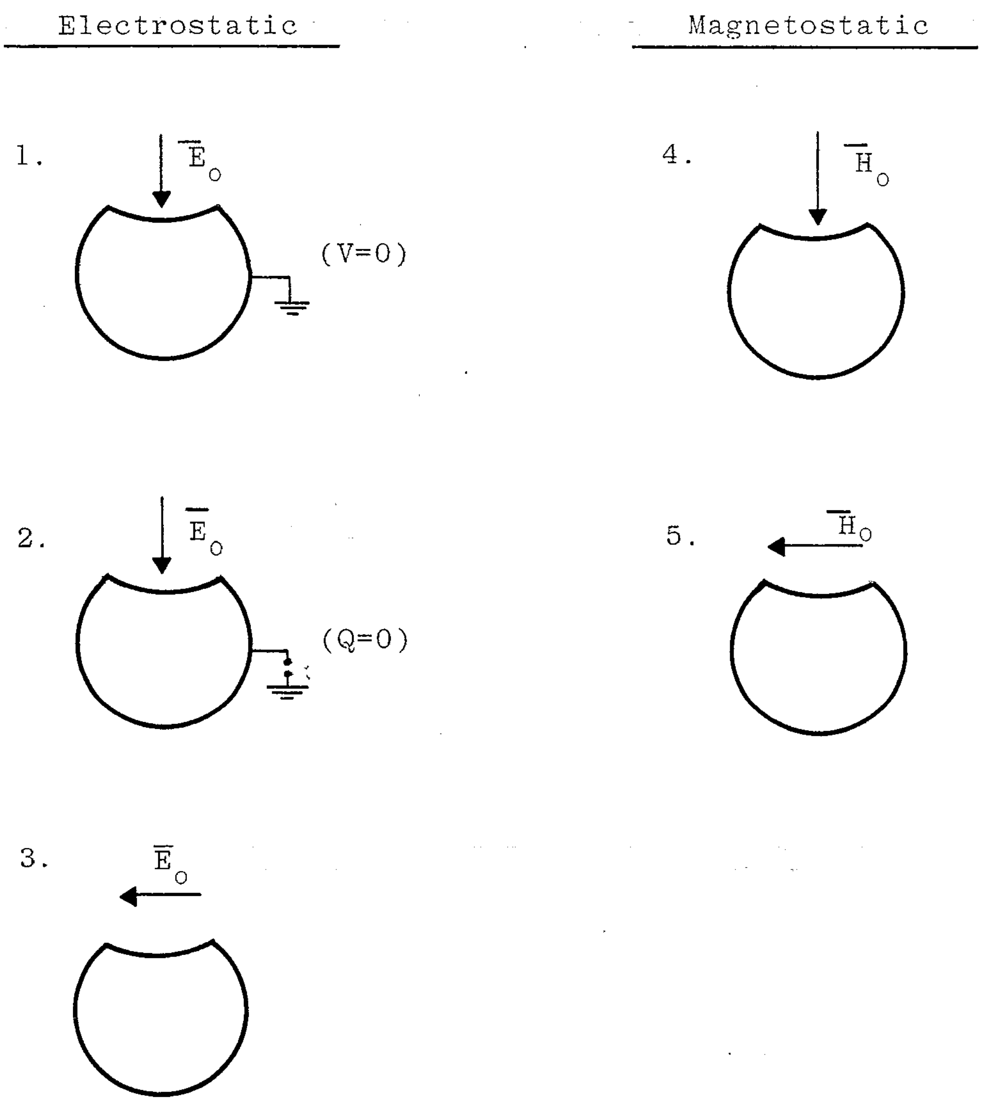
It is widely recognized that the most important penetrations of shielded regions by electromagnetic fields are those which occur through apertures and along conductors entering the shielded region. In this note we shall address the canonical problems of quasi-static electromagnetic aperture penetrations of a spherical conducting shield. The aperture is taken to be circular. This configuration is the simplest possible in a separable geometry which incorporates the two fundamental features of interest, viz.

1. finite volume of the shielded region
2. aperture penetration

There are five canonical problems in all. The five problems are listed in table 1: three of these are electrostatic problems, and the remaining two are magnetostatic. Some of these problems have been solved by other authors: in particular, the solution to the problem of the grounded spherical shell with an applied electric field parallel to the symmetry axis (problem 1) may be found in Sneddon [1]; and the problems of the spherical shell's interation with an applied electric field perpendicular to the symmetry axis (problem 3) and with an applied magnetic field parallel to the symmetry axis (problem 4) have been solved by de Logi [2].

It is our purpose in this note to review the solutions of the solved problems and to present the solutions of the heretofore unsolved problems, as well as to point out some relationships which exist among these problems. The electrostatic problems are formulated and solved in the next section, and the magnetostatic problems are addressed in section III. The equivalent aperture dipole moments are discussed in section IV. Section V concludes the note.

Table 1. The Canonical Shielding Problems



II. The Canonical Electrostatic Problems

The three electrostatic problems shown in table 1 all involve the interaction between an applied uniform electric field $\overline{E}_o$ and a spherical shell with circular aperture. The geometry of the problems is shown in figure 1. The conducting spherical shell, which is taken to be infinitesimally thin, occupies the surface $r = a$, $0 \le \theta < \alpha$ in the spherical coordinates (r, θ, ϕ). The angle of the opening, θ_o, is related to α by $\theta_o = \pi - \alpha$.

In problems 1 and 2, the applied electric field is $\overline{E}_o = E_{zo}\overline{a}_z$. We denote the potential of the shell by V_o and its total charge by Q_o. In problem 1, $V_o = 0$ (the shell is grounded); and in problem 2, $Q_o = 0$. The electric field $\overline{E}$ is given in terms of the electric scalar potential V by $\overline{E} = -\nabla V$, where $V = -E_{zo} r\cos\theta + V_1$ and

$$\nabla^2 V_1 = 0 \quad \text{off the shell}$$

$$\lim_{r\to\infty} rV_1 = \text{const} = 4\pi\varepsilon_o Q_o \tag{1}$$

$$V_1 = V_o + E_{zo} a\cos\theta \quad \text{on the shell}$$

Appropriate representations for V_1 in the regions $r \le a$ and $r \ge a$ are

$$r \le a : V_1(r,\theta) = E_{zo} a \sum_{n=0}^{\infty} a_n \left(\frac{r}{a}\right)^n P_n(\cos\theta)$$

$$r \ge a : V_1(r,\theta) = E_{zo} a \sum_{n=0}^{\infty} a_n \left(\frac{r}{a}\right)^{-n-1} P_n(\cos\theta) \tag{2}$$

The coefficients a_n are to be determined and $P_n(\cdot)$ denotes the Legendre polynomial of degree n. The potential represented by equations (2) is continuous at $r = a$ and has the prescribed behavior as $r\to\infty$. Now $V_1(a,\theta) = V_o + E_{zo} a\cos\theta$ on the shell $(0 \le \theta < \alpha)$ and $\partial V_1/\partial r$ must be continuous over the aperture $(\alpha < \theta \le \pi)$. Enforcing these conditions on the potential given above yields the dual series equations

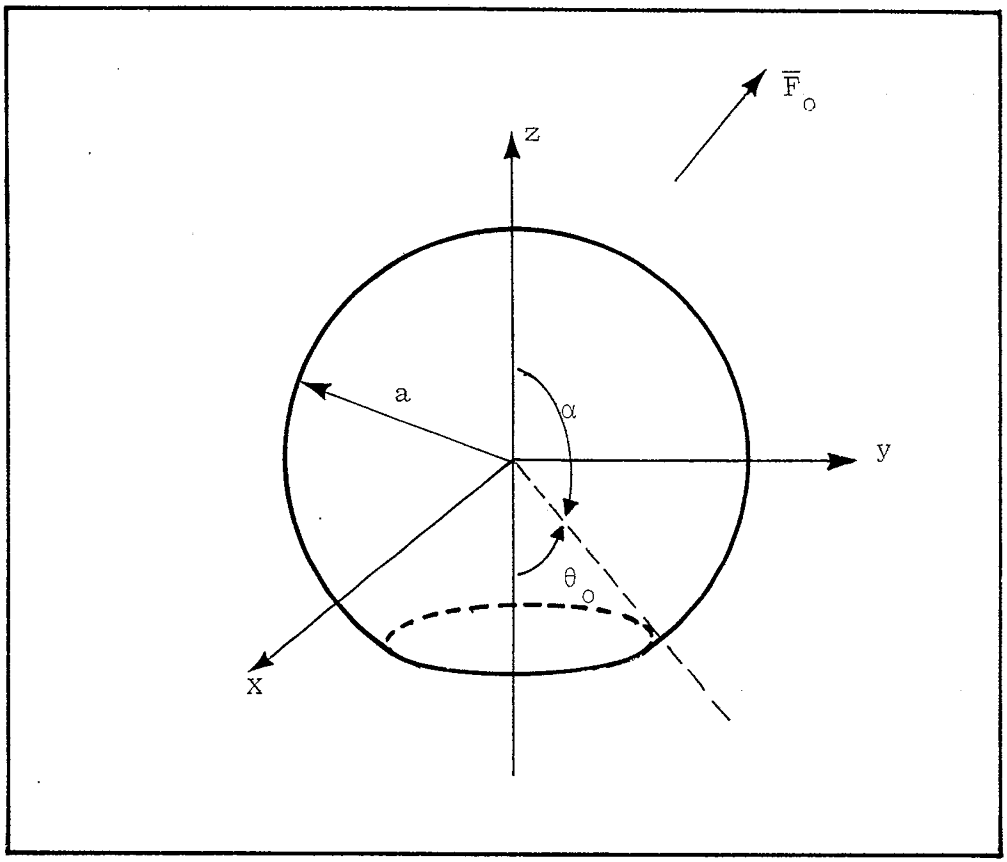


Figure 1. Geometry of the problems. $\overline{F}_o$ denotes the applied field: in problems 1 and 2, $\overline{F}_o = E_{zo}\overline{a}_z$; in problem 3, $\overline{F}_o = E_{xo}\overline{a}_x$; in problem 4, $\overline{F}_o = H_{zo}\overline{a}_z$; and in problem 5, $\overline{F}_o = H_{xo}\overline{a}_x$.

$$\sum_{n=0}^{\infty} a_n P_n(\cos\theta) = \frac{V_o}{E_{zo} a} + \cos\theta \qquad (0 \le \theta < \alpha)$$

$$\sum_{n=0}^{\infty} (2n + 1) a_n P_n(\cos\theta) = 0 \qquad (\alpha < \theta \le \pi) \tag{3}$$

Thus problems 1 and 2 reduce to that of solving the dual series equations (3): problem 1 has the additional requirement that $V_o = 0$; and problem 2 requires that $Q_o = 4\pi\varepsilon_o a^2 E_{zo} a_o = 0$, or that $a_o = 0$.

Equations 3 are a special case of the more general dual series equations

$$\sum_{n=m}^{\infty} {}_m a_n P_n^m(\cos\theta) = F(\theta) \qquad (0 \le \theta < \alpha)$$

$$\sum_{n=m}^{\infty} (2n + 1)\, {}_m a_n P_n^m(\cos\theta) = 0 \qquad (\alpha < \theta \le \pi) \tag{4}$$

in which $F(\theta)$ is a prescribed function of θ and $P_n^m(\cdot)$ denotes the associated Legendre function of degree n and order m. The solution of equations (4) can be written [1]

$${}_m a_n = \frac{2^{m+1/2}}{\pi} \frac{(n - m)!}{(n + m)!} \int_0^{\alpha} F_m^*(u) \left(\cos \frac{u}{2}\right)^{2m+1} \cdot \left(\frac{1}{\sin u} \frac{d}{du}\right)^m \left[\frac{\cos(n + 1/2)u}{\cos \frac{u}{2}}\right] du \tag{5}$$

where

$$F_m^*(u) = \frac{d}{du} \int_0^u \frac{\left(\tan \frac{\theta}{2}\right)^m F(\theta)\sin\theta d\theta}{\sqrt{\cos\theta - \cos u}} \tag{6}$$

In particular, if m = 0 and $F(\theta) = V_o/E_{zo}a + \cos\theta$, we find

$$a_n = \frac{V_o}{\pi E_{zo} a}\left[\frac{\sin n\alpha}{n} + \frac{\sin(n+1)\alpha}{n+1}\right]$$

$$+ \frac{1}{\pi}\left[\frac{\sin(n-1)\alpha}{n-1} + \frac{\sin(n+2)\alpha}{n+2}\right] \qquad (n \geq 0) \tag{7}$$

Now the coefficients for problem 1, $a_n^{(1)}$, are simply

$$a_n^{(1)} = \frac{1}{\pi}\left[\frac{\sin(n-1)\alpha}{n-1} + \frac{\sin(n+2)\alpha}{n+2}\right] \qquad (n \geq 0) \tag{8}$$

Setting $a_o = 0$ for problem 2 yields the relation

$$\frac{V_o}{\pi E_{zo} a}(\alpha + \sin\alpha) + \frac{1}{\pi}\left(\sin\alpha + \frac{1}{2}\sin 2\alpha\right) = 0 \tag{9}$$

so that the coefficients for problem 2, $a_n^{(2)}$, are

$$a_n^{(2)} = a_n^{(1)} - \frac{1}{\pi}\left(\frac{\sin\alpha + \frac{1}{2}\sin 2\alpha}{\alpha + \sin\alpha}\right) \cdot \left(\frac{\sin n\alpha}{n} + \frac{\sin(n+1)\alpha}{n+1}\right) \qquad (n \geq 0) \tag{10}$$

A convenient measure of the field penetration to the interior of the shielded region is the ratio of the electric field at the center of the spherical shell to the applied uniform field E_{zo}. This ratio is simply

$$\frac{E_z(0)}{E_{zo}} = 1 - a_1 \tag{11}$$

Expressing $E_z(0)/E_{zo}$ in terms of θ_o, the half-angle of the aperture opening, yields the following results:

grounded shell:

$$\left.\frac{E_z(0)}{E_{zo}}\right|_{V_o=0} = \frac{1}{\pi}\left(\theta_o - \frac{1}{3}\sin 3\theta_o\right) \tag{12}$$

uncharged shell:

$$\left.\frac{E_z(0)}{E_{zo}}\right|_{Q_o=0} = \frac{1}{\pi}\left[\theta_o - \frac{1}{3}\sin 3\theta_o + \frac{\left(\sin\theta_o - \frac{1}{2}\sin 2\theta_o\right)^2}{\pi - \theta_o + \sin\theta_o}\right] \tag{13}$$

These ratios are plotted as functions of θ_o in figure 2.

In problem 3, the applied electrostatic field is $\overline{E}_o = E_{xo}\overline{a}_x$ and the sphere is uncharged. Thus $V = V_1 - E_{xo} r\sin\theta\cos\phi$, and appropriate representations for the potential V_1 in the regions $r \le a$ and $r \ge a$ are

$$r \le a : V_1(r,\theta,\phi) = E_{xo} a\cos\phi \sum_{n=1}^{\infty} a_n^{(3)} \left(\frac{r}{a}\right)^n P_n^1(\cos\theta)$$

$$r \ge a : V_1(r,\theta,\phi) = E_{xo} a\cos\phi \sum_{n=1}^{\infty} a_n^{(3)} \left(\frac{r}{a}\right)^{-n-1} P_n^1(\cos\theta) \tag{14}$$

By symmetry, the potential of the shell must be zero, so that $V_1 = E_{xo} a\sin\theta\cos\phi$ on the shell. Enforcing this condition and requiring that $\partial V_1/\partial r$ be continuous over the aperture leads to the dual series equations

$$\sum_{n=1}^{\infty} a_n^{(3)} P_n^1(\cos\theta) = \sin\theta \qquad (0 \le \theta < \alpha)$$

$$\sum_{n=1}^{\infty} (2n+1) a_n^{(3)} P_n^1(\cos\theta) = 0 \qquad (\alpha < \theta \le \pi) \tag{15}$$

whose solution, from equations (4)-(6), is

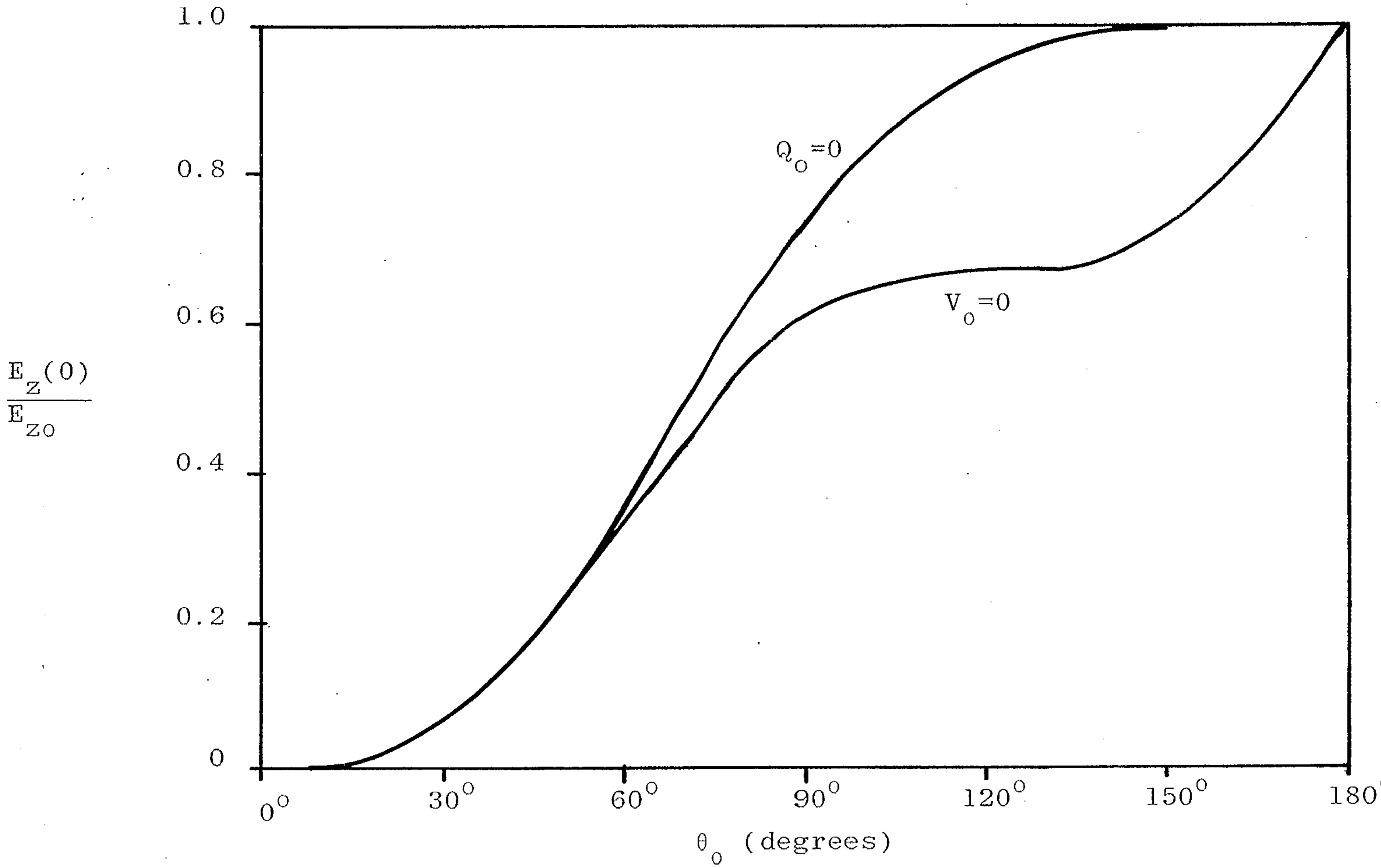


Figure 2. $E_z(0)/E_{zo}$ vs. θ_o for $V_o=0$ (problem 1) and $Q_o=0$ (problem 2)

$$a_n^{(3)} = \frac{-1}{\pi n(n + 1)}\left[\frac{n + 1}{n - 1} \sin(n - 1)\alpha + \sin n\alpha \right.$$
$$\left. - \sin(n + 1)\alpha - \frac{n}{n + 2} \sin(n + 2)\alpha\right] \qquad (n \geq 1) \tag{16}$$

The ratio $E_x(0)/E_{xo} = 1 + a_1^{(3)}$, expressed in terms of the aperture angle θ_o, is

$$\frac{E_x(0)}{E_{xo}} = \frac{1}{\pi}\left(\theta_o - \frac{1}{2} \sin\theta_o - \frac{1}{2} \sin 2\theta_o + \frac{1}{6} \sin 3\theta_o\right) \tag{17}$$

This ratio is plotted as a function of θ_o in figure 3. As we expect, $E_x(0)/E_{xo}$ is always less than $E_z(0)/E_{zo}$ when the sphere is uncharged.

This completes our treatment of the three canonical electrostatic problems. We now turn to the two canonical magnetostatic problems.

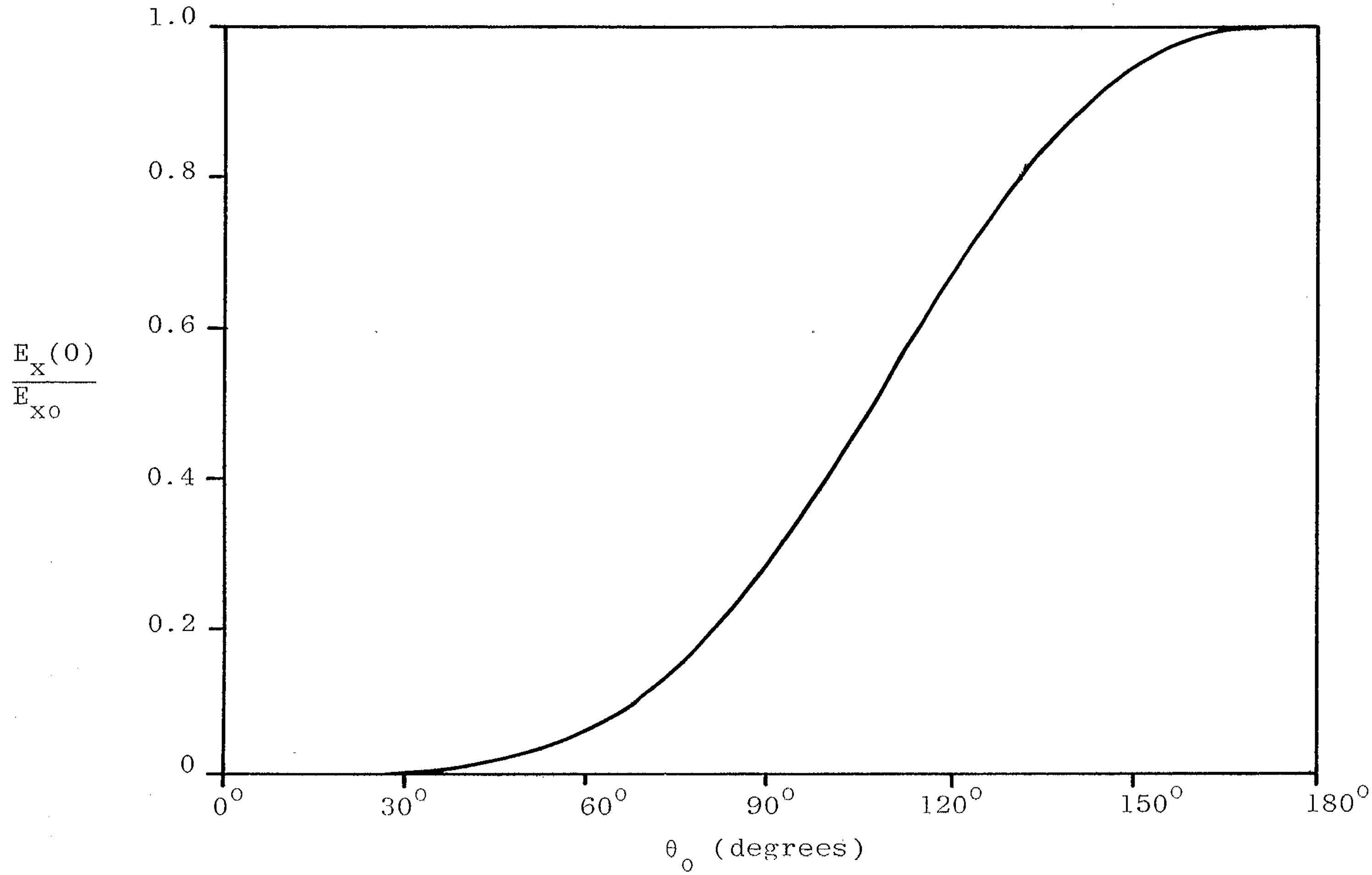


Figure 3. $E_x(0)/E_{xo}$ vs. θ_o (problem 3)

III. The Canonical Magnetostatic Problems

The two magnetostatic problems shown in table 1 both involve the interaction between an applied uniform magnetostatic field and a conducting spherical shell with a circular aperture. The geometry of these problems is identical to that shown in figure 1; only the applied field is different.

In problem 4, the applied magnetic field is $\overline{H}_o = H_{zo}\overline{a}_z$. By virtue of the azimuthal symmetry of the problem, it is convenient to write the magnetic field in terms of the magnetic vector potential $\overline{A} = A_\phi \overline{a}_\phi$ as $\overline{H} = \nabla \times A_\phi \overline{a}_\phi$, in which $A_\phi = \left(H_{zo} r/2\right) \cdot \sin\theta - A_{\phi 1}$, where $A_{\phi 1}$ is independent of ϕ and

$$\nabla^2 A_{\phi 1} - \frac{A_{\phi 1}}{r^2 \sin^2\theta} = 0 \qquad \text{off the shell}$$

$$\lim_{r\to\infty} rA_{\phi 1} = 0 \tag{18}$$

$$A_{\phi 1} = \frac{H_{zo}a}{2}\sin\theta \qquad \text{on the shell*}$$

Appropriate representations for $A_{\phi 1}$ in the regions $r \leq a$ and $r \geq a$ are

$$r \leq a : A_{\phi 1}(r,\theta) = \frac{1}{2} H_{zo} a \sum_{n=1}^{\infty} a_n^{(4)} \left(\frac{r}{a}\right)^n P_n^1(\cos\theta)$$

$$r \geq a : A_{\phi 1}(r,\theta) = \frac{1}{2} H_{zo} a \sum_{n=1}^{\infty} a_n^{(4)} \left(\frac{r}{a}\right)^{-n-1} P_n^1(\cos\theta) \tag{19}$$

Setting $A_{\phi 1}(a,\theta) = \left(H_{zo} a/2\right)\sin\theta$ on the shell $(0 \leq \theta < \alpha)$ and forcing $\partial/\partial r(rA_{\phi 1})$ to be continuous over the aperture $(r = a, \alpha < \theta \leq \pi)$ yields the dual series equations

* The solution to problem 4 via the magnetic scalar potential is described in the Appendix.

$$\sum_{n=1}^{\infty} a_n^{(4)} P_n^1(\cos\theta) = \sin\theta \qquad (0 \le \theta < \alpha)$$

$$\sum_{n=1}^{\infty} (2n + 1)a_n^{(4)} P_n^1(\cos\theta) = 0 \qquad (\alpha < \theta \le \pi) \tag{20}$$

These equations are identical to equations (15). Thus

$$a_n^{(4)} = a_n^{(3)} = \frac{-1}{\pi n(n+1)}\left[\frac{n+1}{n-1}\sin(n-1)\alpha + \sin n\alpha - \sin(n+1)\alpha - \frac{n}{n+2}\sin(n+2)\alpha\right] \qquad (n \ge 1) \tag{21}$$

and the ratio $H_z(0)/H_{zo}$ is simply

$$\frac{H_z(0)}{H_{zo}} = \frac{E_x(0)}{E_{xo}} = \frac{1}{\pi}\left[\theta_o - \frac{1}{2}\sin\theta_o - \frac{1}{2}\sin 2\theta_o + \frac{1}{6}\sin 3\theta_o\right] \tag{22}$$

This ratio has been plotted in figure 3.

In problem 5, the applied magnetostatic field is $\overline{H}_o = H_{xo}\overline{a}_x$. The absence of azimuthal symmetry forbids the use of the magnetic vector potential; instead, we write $\overline{H} = -\nabla V_m$, where V_m is the magnetic scalar potential. We have $V_m = -H_{xo} r\sin\theta\cos\phi + V_{m1}$, where

$$\nabla^2 V_{m1} = 0 \qquad \text{off the shell}$$

$$\lim_{r\to\infty} rV_{m1} = 0 \tag{23}$$

$$\frac{\partial V_{m1}}{\partial r} = H_{xo}\sin\theta\cos\phi \qquad \text{on the shell}$$

Appropriate representations for V_{m1} in the regions $r < a$ and $r > a$ are

$$r < a : V_{m1}(r,\theta,\phi) = H_{xo} a\cos\phi \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r}{a}\right)^{n} a_n^{(5)} P_n^1(\cos\theta)$$

$$r > a : V_{m1}(r,\theta,\phi) = -H_{xo} a\cos\phi \sum_{n=1}^{\infty} \frac{1}{n+1} a_n^{(5)} \left(\frac{r}{a}\right)^{-n-1} P_n^1(\cos\theta) \qquad (24)$$

These representations guarantee that $\partial V_{m1}/\partial r$ will be continuous at $r = a$. Setting $\partial V_{m1}/\partial r = H_{xo}\sin\theta\cos\phi$ on the shell ($r = a$, $0 \leq \theta < \alpha$) and forcing V_{m1} to be continuous through the aperture ($r = a$, $\alpha < \theta \leq \pi$) yields the dual series equations

$$\sum_{n=1}^{\infty} a_n^{(5)} P_n^1(\cos\theta) = \sin\theta \qquad (0 \leq \theta < \alpha)$$

$$\sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} a_n^{(5)} P_n^1(\cos\theta) = 0 \qquad (\alpha < \theta \leq \pi) \qquad (25)$$

It will be noted that these dual series equations are of a form different from those seen in the previous problems. However, we observe that the first of equations (25) can be written as an equivalent "serio-differential" equation:

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - \frac{1}{\sin^2\theta}\right] \sum_{n=1}^{\infty} \frac{a_n^{(5)}}{n(n+1)} P_n^1(\cos\theta)$$

$$= -\sin\theta \qquad (0 \leq \theta < \alpha) \qquad (26)$$

Solving this differential equation, we find that

$$\sum_{n=1}^{\infty} \frac{a_n^{(5)}}{n(n+1)} P_n^1(\cos\theta) = \frac{1}{2}\sin\theta + A\tan\frac{\theta}{2}$$

$$+ B\cot\frac{\theta}{2} \qquad (0 \leq \theta < \alpha) \qquad (27)$$

in which A and B are constants to be determined. The functions $\tan\frac{\theta}{2}$, $\cot\frac{\theta}{2}$ are the solutions of the homogeneous equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{df}{d\theta}\right) - \frac{f}{\sin^2\theta} = 0 \tag{28}$$

Now defining

$$\hat{a}_n^{(5)} = \frac{a_n^{(5)}}{n(n+1)} \tag{29}$$

we obtain the dual series equations

$$\begin{aligned} \sum_{n=1}^{\infty} \hat{a}_n^{(5)} P_n^1(\cos\theta) &= \frac{1}{2}\sin\theta + A\tan\frac{\theta}{2} + B\cot\frac{\theta}{2} \quad (0 \le \theta < \alpha) \\ \sum_{n=1}^{\infty} (2n+1)\hat{a}_n^{(5)} P_n^1(\cos\theta) &= 0 \quad (\alpha < \theta \le \pi) \end{aligned} \tag{30}$$

which are of the form previously considered.

Since we have no reason to suspect the existence of singular behavior at $\theta = 0$, we set the constant B equal to zero. Then, by using equations (4)-(6) and (29), we find

$$\begin{aligned} a_n^{(5)} = \frac{1}{\pi}\frac{\cos\left(n+\frac{1}{2}\right)\alpha}{\cos\frac{\alpha}{2}}&\left[A(\alpha + \sin\alpha) + \left(\sin\alpha + \frac{1}{2}\sin 2\alpha\right)\right] \\ &- \frac{1}{\pi}\left[\frac{\sin(n-1)\alpha}{n-1} + \frac{\sin(n+2)\alpha}{n+2}\right] \\ &- \frac{A}{\pi}\left[\frac{\sin n\alpha}{n} + \frac{\sin(n+1)\alpha}{n+1}\right] \quad (n \ge 1) \end{aligned} \tag{31}$$

in which the constant A remains to be determined.

The determination of the constant A follows from consideration of the first of the original dual series equations (25). Specifically, the sum

$$\sum_{n=1}^{\infty} a_n^{(5)} P_n^1(\cos\theta)$$

must exist, and it must equal $\sin\theta$ over the interval $0 \le \theta < \alpha$. By virtue of the manipulations used to convert the first of equations (25) into equation (27), it is clear that if this series converges, it will converge to $\sin\theta$ in the interval $0 \le \theta < \alpha$. Now when $n \to \infty$ and $\pi > \theta > 0$ [3],

$$P_n^1(\cos\theta) \sim \left(\frac{2n}{\pi\sin\theta}\right)^{1/2} \cos\left[\left(n + \frac{1}{2}\right)\theta + \frac{\pi}{4}\right] \tag{32}$$

from which it is obvious that the series in question will converge only if the factor multiplying $\cos(n + 1/2)\alpha$ in equation (31) vanishes, i.e., if

$$A = \frac{-\left(\sin\alpha + \frac{1}{2}\sin 2\alpha\right)}{\alpha + \sin\alpha} \tag{33}$$

Thus

$$\begin{aligned} a_n^{(5)} &= \frac{-1}{\pi}\left[\frac{\sin(n-1)\alpha}{n-1} + \frac{\sin(n+2)\alpha}{n+2}\right] \\ &+ \frac{1}{\pi}\left[\frac{\sin\alpha + \frac{1}{2}\sin 2\alpha}{\alpha + \sin\alpha}\right]\left[\frac{\sin n\alpha}{n} + \frac{\sin(n+1)\alpha}{n+1}\right] \quad (n \ge 1) \end{aligned} \tag{34}$$

which is equal to $-a_n^{(2)}$. The ratio $H_x(0)/H_{xo}$ is therefore

$$\begin{aligned} \frac{H_x(0)}{H_{xo}} = \left.\frac{E_z(0)}{E_{zo}}\right|_{Q_o=0} &= \frac{1}{\pi}\left[\theta_o - \frac{1}{3}\sin 3\theta_o \right. \\ &\left. + \frac{\left(\sin\theta_o - \frac{1}{2}\sin 2\theta_o\right)^2}{\pi - \theta_o + \sin\theta_o}\right] \end{aligned} \tag{35}$$

This ratio has been plotted in figure 2.

We have plotted the field ratios for the five problems together as functions of θ_o in figure 4. These ratios are also given in table 2 for $0^\circ \le \theta_o \le 90^\circ$, and their small-argument approximations are given below:

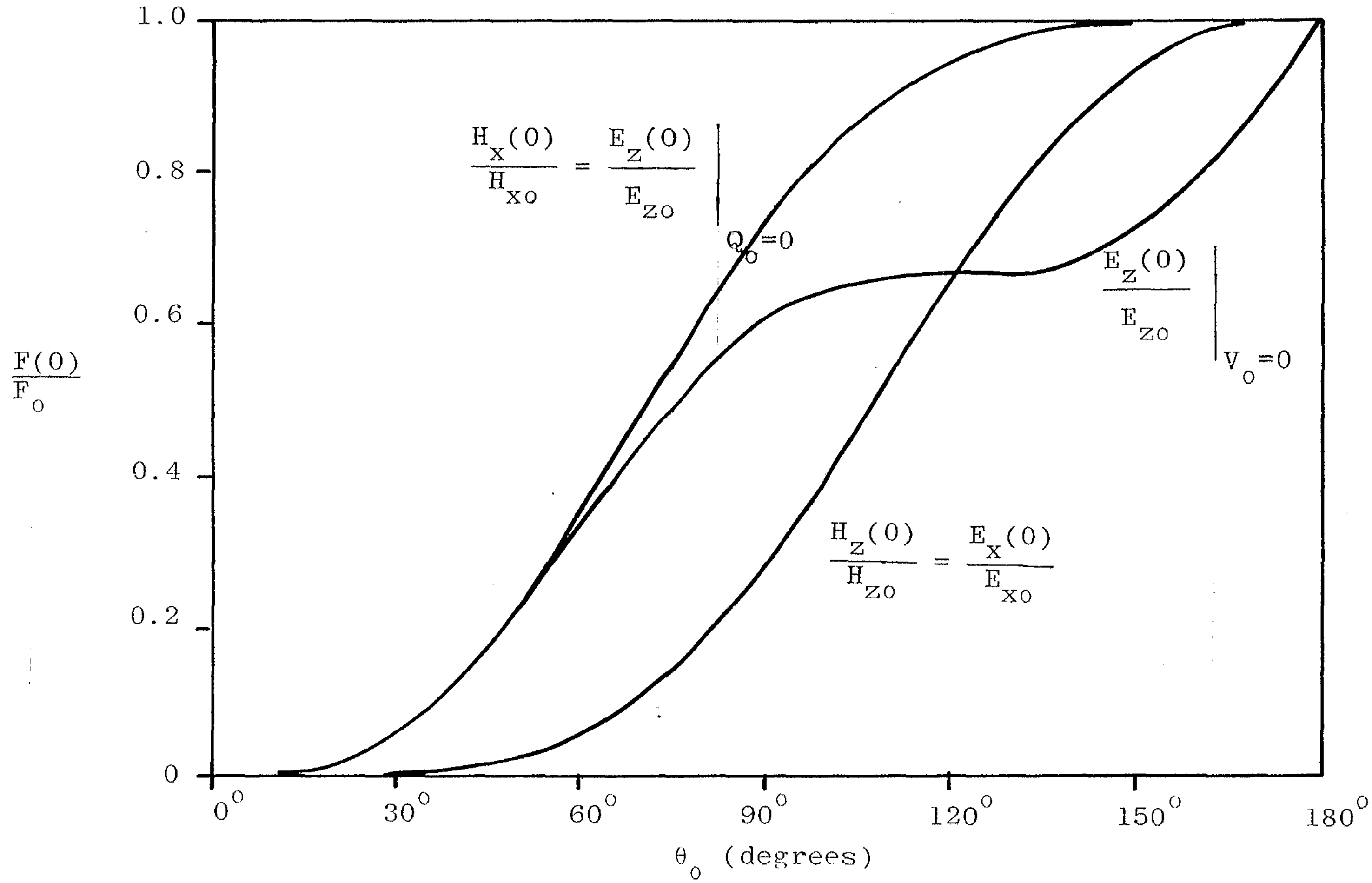


Figure 4. Field ratios $F(0)/F_o$ vs. θ_o for problems 1-5

Table 2. Field Ratios versus θ_o ($0^\circ \leq \theta_o \leq 90^\circ$)

θ_o	$\left.\frac{E_z(0)}{E_{zo}}\right\|_{Q_o=0} = \frac{H_x(0)}{H_{xo}}$	$\left.\frac{E_z(0)}{E_{zo}}\right\|_{V_o=0}$	$\frac{E_x(0)}{E_{xo}} = \frac{H_z(0)}{H_{zo}}$
0°	0.0000	0.0000	0.0000
5°	0.0003	0.0003	0.0000
10°	0.0025	0.0025	0.0000
15°	0.0083	0.0083	0.0001
20°	0.0192	0.0192	0.0003
25°	0.0364	0.0364	0.0010
30°	0.0610	0.0606	0.0023
35°	0.0931	0.0920	0.0048
40°	0.1327	0.1303	0.0091
45°	0.1794	0.1750	0.0158
50°	0.2326	0.2247	0.0256
55°	0.2910	0.2781	0.0394
60°	0.3535	0.3333	0.0577
65°	0.4185	0.3886	0.0812
70°	0.4845	0.4419	0.1105
75°	0.5500	0.4920	0.1458
80°	0.6135	0.5363	0.1873
85°	0.6739	0.5747	0.2348
90°	0.7299	0.6061	0.2878

$$\left.\frac{E_z(0)}{E_{zo}}\right|_{Q_o=0} = \frac{H_x(0)}{H_{xo}} \simeq \left.\frac{E_z(0)}{E_{zo}}\right|_{V_o=0} \simeq \frac{3\theta_o^3}{2\pi} \quad (\theta_o \to 0)$$

$$\frac{E_x(0)}{E_{xo}} = \frac{H_z(0)}{H_{zo}} \simeq \frac{\theta_o^5}{5\pi} \quad (\theta_o \to 0) \tag{36}$$

IV. Aperture Dipole Moments

It is of interest to calculate the dipole moments of the circular aperture for problems 1, 2, and 5 and to consider the effect of the surface curvature on these dipole moments. We use the "imaged" electric and magnetic dipole moments defined by

$$\bar{p}_{ai} = \frac{\varepsilon_o}{2} \int_A \bar{r} \times \bar{n} \times \bar{E}_a \, ds$$
$$\bar{m}_{ai} = - \int_A \bar{r} H_n \, ds \tag{37}$$

where $\bar{r}$ denotes the position vector, $\bar{n}$ is the unit vector normal to the aperture, $\bar{E}_a$ is the electric field in the aperture, and H_n denotes the component of magnetic field normal to the aperture.

It is easy to show that

$$\bar{p}_{ai} = \frac{4}{3} \pi a^3 \varepsilon_o E_{zo} \, (1-a_1^{(1,2)}) \bar{a}_z \tag{38}$$

for problems 1 and 2 and that

$$\bar{m}_{ai} = - \frac{4}{3} \pi a^3 \, H_{xo} \, (1+a_1^{(5)}) \bar{a}_x \tag{39}$$

for problem 5. In the limit $\theta_o \to 0$,

$$p_{aiz} \to 2\varepsilon_o E_{zo} \, (a\theta_o)^3$$
$$-m_{aix} \to 2H_{xo} \, (a\theta_o)^3 \tag{40}$$

Since the short-circuit electric and magnetic fields at the aperture center are respectively $3E_{zo}\bar{a}_z$ and $\frac{3}{2}H_{xo}\bar{a}_x$, we find that the "imaged" aperture polarizabilities α_{ei} and α_{mi} are, in the limit,

$$\alpha_{ei} = \frac{2}{3}(a\theta_o)^3$$
$$\alpha_{mi} = \frac{4}{3}(a\theta_o)^3 \qquad (\theta_o \to 0) \tag{41}$$

The quantity $a\theta_o$ is, of course, the aperture radius in this limit.

To see the behavior of the electric and magnetic dipole moments as functions of the radius of curvature of the surface, we define the functions

$$\xi^{(1,2)}(\theta) = \frac{p_{aiz}}{2\varepsilon_o E_{zo}(a\theta_o)^3} = \frac{2\pi}{3\theta_o^{\ 3}}\left(1-a_1^{(1,2)}\right)$$

$$\eta(\theta) = \frac{m_{aix}}{2H_{xo}(a\theta_o)^3} = \frac{2\pi}{3\theta_o^{\ 3}}\left(1 + a_1^{(5)}\right) \tag{42}$$

Now the function $\xi^{(2)}(\theta)$ (uncharged shell) is identical to $\eta(\theta)$. Curves of $\xi^{(1)}(\theta)$ (grounded shell) and $\eta(\theta)$ are shown in figure 5, and the small-argument approximations to these functions are given below:

$$\xi^{(1)}(\theta_o) \approx \xi^{(2)}(\theta_o) = \eta(\theta_o) \approx 1 - \frac{9}{20}\theta_o^{\ 2} \tag{43}$$

$\xi^{(1)}(\theta_o)$ and $\eta(\theta_o)$ are also given in table 3 for $0^o \leq \theta_o \leq 90^o$. As we expect, a decrease in radius of curvature (i.e., an increase in θ_o) yields a decrease in the equivalent dipole moment of the aperture. A similar result for a cylindrical geometry has been derived by Latham [4].

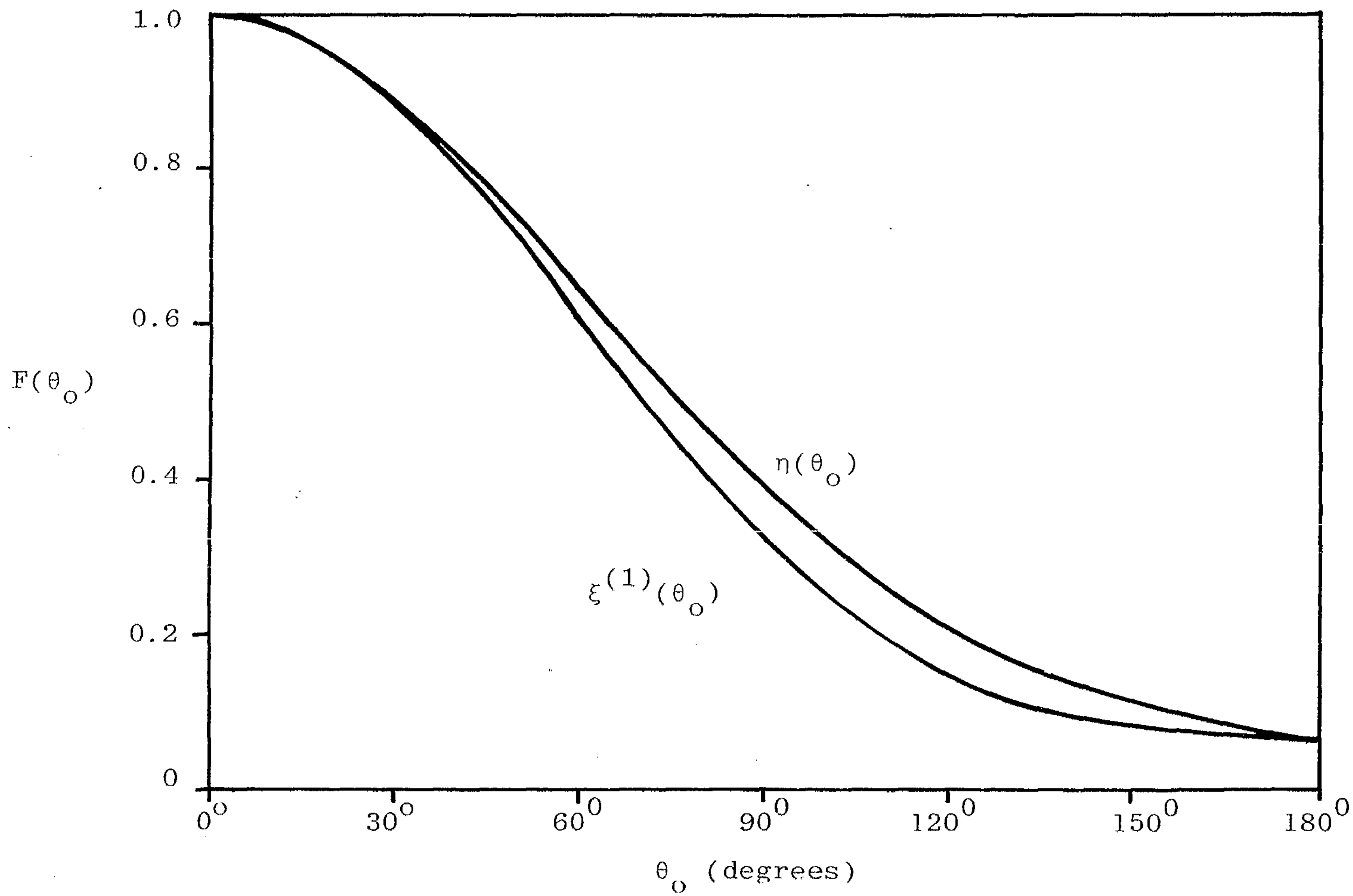


Figure 5. $\xi^{(1)}(\theta_o)$ and $\eta(\theta_o)$ vs. θ_o

Table 3. $\xi^{(1)}(\theta_o)$ and $\eta(\theta_o)$ versus θ_o ($0^o \le \theta_o \le 90^o$)

θ_o	$\xi^{(1)}(\theta_o)$	$\xi^{(2)}(\theta_o) = \eta(\theta_o)$
0^o	1.0000	1.0000
5^o	0.9966	0.9966
10^o	0.9864	0.9867
15^o	0.9696	0.9705
20^o	0.9466	0.9487
25^o	0.9177	0.9218
30^o	0.8836	0.8903
35^o	0.8408	0.8513
40^o	0.8022	0.8166
45^o	0.7564	0.7757
50^o	0.7082	0.7330
55^o	0.6585	0.6891
60^o	0.6079	0.6447
65^o	0.5524	0.5959
70^o	0.4978	0.5478
75^o	0.4591	0.5136
80^o	0.4127	0.4721
85^o	0.3687	0.4323
90^o	0.3275	0.3944

V. Discussion and Concluding Remarks

The solutions of the five boundary value problems considered in this note can all be expressed in terms of the solutions of certain dual series equations. It is interesting to note that although electric scalar potential, magnetic scalar potential, and magnetic vector potential formulations have been employed, there are in fact only two distinct sets of coefficients a_n. It will be recalled that $a_n^{(1)}$ and $a_n^{(2)}$ are special cases of a more general a_n given in equation (7) and that $a_n^{(5)} = -a_n^{(2)}$. Furthermore, $a_n^{(3)} = a_n^{(4)}$, and (see the Appendix) $2b_n = -n(n+1)a_n^{(3)}$. While these equalities of certain of the expansion coefficients may be merely fortuitous, one suspects that a more general conclusion may be drawn. For example, the result $a_n^{(3)} = a_n^{(4)}$ could have been predicted directly from the fact that $V_1 \sec\phi$ and $A_{\phi 1}$ satisfy the same partial differential equation and also satisfy similar boundary conditions. Furthermore, such a relationship is known for bodies of revolution in general [5]. There appears to be no comparable general relationship from which the equality $a_n^{(5)} = -a_n^{(2)}$ could have been deduced.

A contribution of this note is the solution procedure for the dual series equations (25). Dual series equations of that form do not appear to have been solved previously, and the conversion of one of them into an equivalent "serio-differential" equation in order to produce an alternate form containing arbitrary constants appears to be new. The procedure is obviously capable of generalization, and this topic will be dealt with in a future Mathematics Note.

References

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[2] Lee, K.S.H., ed., EMP Interaction: Principles, Techniques, and Reference Data, AFWL-TR-79-403, Kirtland Air Force Base, New Mexico, December 1979, pp. 536-542.

[3] Abramowitz, M. and Stegun, I.A., eds., Handbook of Mathematical Functions, Dover Publications, New York, 1965, p. 336.

[4] Latham, R.W., "Small Holes in Cable Shields," Interaction Notes, Note 118, September 1972. See also [2], pp. 514-515.

[5] Taylor, T.T., "Magnetic Polarizability of a Short Right Circular Cylinder," J. Res. NBS 64B4, 1960, p. 199.

Appendix

The Solution of Problem 4 via Scalar Potentials

We use $\overline{H} = -\nabla V_m$, where $V_m = -H_{zo} r\cos\theta + V_{m1}$ and

$$
\begin{aligned}
&\nabla^2 V_{m1} = 0 && \text{off the shell} \\
&\lim_{r\to\infty} rV_{m1} = 0 && \\
&\frac{\partial V_{m1}}{\partial r} = H_{zo}\cos\theta && \text{on the shell}
\end{aligned}
\qquad (A1)
$$

Appropriate representations for V_{m1} in the two regions of the problem are

$$
\begin{aligned}
r < a :\ V_{m1}(r,\theta) &= H_{zo} a \sum_{n=0}^{\infty} \frac{1}{n}\left(\frac{r}{a}\right)^{n} b_n P_n(\cos\theta) \\
r > a :\ V_{m2}(r,\theta) &= -H_{zo} a \sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{r}{a}\right)^{-n-1} b_n P_n(\cos\theta)
\end{aligned}
\qquad (A2)
$$

where $b_o = 0$. Imposing the conditions that $\partial V_{m1}/\partial r = H_{zo}\cos\theta$ on the shell and that V_{m1} be continuous over the aperture yields the dual series equations

$$
\begin{aligned}
&\sum_{n=0}^{\infty} b_n P_n(\cos\theta) = \cos\theta && (0 \le \theta < \alpha) \\
&\sum_{n=0}^{\infty} \frac{2n+1}{n(n+1)} b_n P_n(\cos\theta) = 0 && (\alpha < \theta \le \pi)
\end{aligned}
\qquad (A3)
$$

Applying the operator

$$
\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)\right]^{-1}
$$

to the first of these equations yields

$$\sum_{n=0}^{\infty} \frac{b_n}{n(n+1)} P_n(\cos\theta) = \frac{1}{2}\cos\theta + A + B \ln\left(\tan\frac{\theta}{2}\right) \quad (0 \leq \theta < \alpha)$$

$$\sum_{n=0}^{\infty} \frac{2n+1}{n(n+1)} b_n P_n(\cos\theta) = 0 \quad (\alpha < \theta \leq \pi) \tag{A4}$$

Now defining

$$\hat{b}_n = \frac{b_n}{n(n+1)} \tag{A5}$$

and setting B = 0, we obtain as dual series equations

$$\sum_{n=0}^{\infty} \hat{b}_n P_n(\cos\theta) = \frac{1}{2}\cos\theta + A \quad (0 \leq \theta < \alpha)$$

$$\sum_{n=0}^{\infty} (2n+1)\hat{b}_n P_n(\cos\theta) = 0 \quad (\alpha < \theta \leq \pi) \tag{A6}$$

These dual series equations are of the form given in equation (4). Solving for b_n, we obtain

$$b_n = n(n+1)\left\{\frac{A}{\pi}\left[\frac{\sin n\alpha}{n} + \frac{\sin(n+1)\alpha}{n+1}\right] + \frac{1}{2\pi}\left[\frac{\sin(n-1)\alpha}{(n-1)} + \frac{\sin(n+2)\alpha}{n+2}\right]\right\} \tag{A7}$$

and we note that $b_o = 0$ as required.

In order to determine A, we note that the series $\sum_{n=0}^{\infty} b_n P_n(\cos\theta)$ must be convergent and must represent the function $\cos\theta$ over the interval $0 \leq \theta < \alpha$. Now,

$$P_n(\cos\theta) \sim \left(\frac{2}{n\pi\sin\theta}\right)^{1/2} \cos\left[\left(n + \frac{1}{2}\right)\theta - \frac{\pi}{4}\right] \tag{A8}$$

so that for the series to converge,

$$\lim_{n\to\infty} b_n < \infty \tag{A9}$$

It is not difficult to show that the value of A for which this condition holds independent of α is

$$A = \frac{1}{2} - \cos\alpha \tag{A10}$$

so that

$$b_n = \frac{1}{2\pi}\left[\sin n\alpha - \sin(n + 1)\alpha + \frac{n + 1}{n - 1}\sin(n - 1)\alpha - \frac{n}{n + 2}\sin(n + 2)\alpha\right] = \frac{-n(n + 1)}{2} a_n^{(3)} \tag{A11}$$