Interaction Notes
Note 391

August 1979

EMP Study of a Proposed Underground Antenna (U)

Air Force Weapons Laboratory

Abstract

The frequency-dependent series impedance and shunt admittance of the time-domain transmission-line equations are evaluated in the frequency domain and represented in the time domain as solutions of differential equations. The use of central differencing in the solution is demonstrated and compared with the frequency-domain solution.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

IN 391

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ~~APPENDIX B~~ EMP STUDY OF A PROPOSED UNDERGROUND ANTENNA (U)		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER AMRC-R-193
7. AUTHOR(s) R. H. St. John, J. R. Schmelzer		8. CONTRACT OR GRANT NUMBER(s) F29601-79-C-0035 Program Plan Amendment 1.2.1
9. PERFORMING ORGANIZATION NAME AND ADDRESS MISSION RESEARCH CORPORATION, 1400 San Mateo Blvd., S. E., Suite A, Albuquerque, New Mexico 87108		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Weapons Laboratory, Kirtland Air Force Base, New Mexico 87117		12. REPORT DATE August 1979
		13. NUMBER OF PAGES
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

This project has been partially supported by the Defense Nuclear Agency (DNA) under:

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Transmission Line
Frequency Domain
Time Domain
Prony's Method
Differential Equation Representation
Three-Media Problem

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The frequency-dependent series impedance and shunt admittance of the time-domain transmission-line equations are evaluated in the frequency domain and represented in the time domain as solutions of differential equations. The use of central differencing in the solution is demonstrated and compared with the frequency-domain solution.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

CONTENTS

ILLUSTRATIONS

APPENDIX B

TIME-DOMAIN REPRESENTATION OF FREQUENCY-DEPENDENT PARAMETERS

1. THEORY

In this appendix, the theory necessary for the representation of frequency-dependent, transmission-line parameters in the time domain is discussed.

The general formulation of the telegrapher's equations with nonconstant R, L, G, and C in the time domain involves convolution integrals of various functions of time. Because the nonlinear, transmission-line analysis employs the standard differencing techniques in the time domain, the resultant convolution integrals require large amounts of computer time and storage. This results from the backstoring of time function values at each cell in the transmission line, plus the necessary computation of the convolution integral.

This computer resource expenditure can be reduced by expressing the convolution integrals as solutions of differential equations. With this expression, the backstoring is unnecessary. However, the solution to the differential equations is still required at each cell in the differencing of the transmission line equations.

The development of the differential equations starts with the frequency-domain telegrapher's equations,

$$\frac{d}{dx} V(x,s) = - Z(s)I(x,s) + E(x,s) \qquad \text{(B1a)}$$

$$\frac{d}{dx} I(x,s) = - Y(s)V(x,s) \qquad \text{(B1b)}$$

where V(x,s) and I(x,s) are the voltage and current as functions of the Laplace transform variable, s, and distance, x, along the transmission line, and E(x,s) is the distributed voltage source. The quantities of interest are Z(s) and Y(s), the series impedance and shunt admittance per unit length.

The convolution-integral, time-domain representations of these equations are

$$\frac{\partial}{\partial x} v(x,t) = - \int_0^t L^{-1}[Z(s)]\ (t-\tau)\ i(x,\tau)d\tau + e(x,t) \qquad \text{(B2a)}$$

$$\frac{\partial}{\partial x} i(x,t) = - \int_0^t L^{-1}[Y(s)]\ (t-\tau)\ v(x,\tau)d\tau \qquad \text{(B2b)}$$

where v(x,t), i(x,t), and e(x,t) are the inverse transforms of V, I and E, and L^{-1} is the inverse Laplace transform operator.

As is the technique of Reference 1, the inverse transforms of Z and Y are represented in the time domain as a sum of decaying exponentials via a Prony analysis. This representation allows the formulation of the differential equations to which the convolution integrals are solutions.

Let

$$L^{-1}\ [Z(s)]\ (t) = \sum_{n=1}^{N} a_n e^{s_n t} \qquad \text{(B3a)}$$

and

$$L^{-1}\ [Y(s)]\ (t) = \sum_{n=1}^{M} b_n e^{p_n t} \qquad \text{(B3b)}$$

1. Price, H., R. H. St. John and D. Merewether, Two-Port Representation of a Linear Transmission Line in the Time Domain, Mission Research Corporation, AMRC-R-170, January 1979.

With this form of the inverse transforms, Equation B2 is now

$$\frac{\partial}{\partial x} v(x,t) = - \sum_{n=1}^{N} \int_{o}^{t} a_n e^{s_n(t-\tau)} i(x,\tau)d\tau$$

$$+ e(x,t) \qquad \text{(B4a)}$$

A similar expression is obtained for Equation B2b,

$$\frac{\partial}{\partial x} i(x,t) = - \sum_{n=1}^{M} \int_{o}^{t} b_n e^{p_n(t-\tau)} v(x,\tau)d\tau$$

If we let

$$y_n(x,t) = \int_{o}^{t} a_n e^{s_n(t-\tau)} i(x,\tau)d\tau \qquad \text{(B5a)}$$

and

$$z_n(x,t) = \int_{o}^{t} b_n e^{p_n(t-\tau)} v(x,\tau)d\tau \qquad \text{(B5b)}$$

then the transmission line equations (B2a and B2b) become

$$\frac{\partial}{\partial x} v(x,t) = - \sum_{n=1}^{N} y_n(x,t) + e(x,t) \qquad \text{(B6)}$$

and similarly for Equation B2b. The differential equations can now be found from the expressions for $y_n(x,t)$ and $z_n(x,t)$ in Equation B5. Taking the derivatives of Equations B5 with respect to time, and obtains

$$\frac{\partial}{\partial t} y_n(x,t) = a_n i(x,t) + s_n \, y_n(x,t) \qquad \text{(B7)}$$

and a similar expression for $\frac{\partial}{\partial t} z_n(x,t)$. These results are compatible with those obtained from the state-variable procedure of Reference 1. The difference is that the present differential equation may be complex if the poles and residues obtained from the Prony representations of $L^{-1}[Z(s)]$ and L^{-1} $[Y(s)]$ are complex. This is not a problem since the complex poles are residues resulting from the Prony analysis are always in complex conjugate pairs, with the result that Σy_n and Σz_n are real. The telegrapher's equations (Equation 2) are now

$$\frac{\partial}{\partial x} v(x,t) = - \sum_{n=1}^{N} y_n(x,t) + e(x,t) \qquad \text{(B8a)}$$

$$\frac{\partial}{\partial t} y_n(x,t) = a_n i(x,t) + s_n y_n(x,t) \qquad \text{(B8b)}$$

$$\frac{\partial}{\partial t} i(x,t) = - \sum_{n=1}^{M} z_n(x,t) \qquad \text{(B8c)}$$

and

$$\frac{\partial}{\partial t} z_n(x,t) = b_n\, v(x,t) + p_n z_n(x,t) \qquad \text{(B8d)}$$

This derivation shows the most straightforward application of this method for the transmission-line equations. However, the transmission line equations in this form have problems when the numerical inverse transformations of Z(s) and Y(s) are to be made.

For a cable buried in an infinite lossy medium, the propagation constant, h, is the solution of a transcendental equation (Ref. 2). With h known, the characteristic impedance, Z_c, of the line can be found as shown in Reference 3. From the propagation constant and characteristic impedance, the series impedance and shunt admittance are defined as

2. Stratton, J. A., Electromagnetic Theory, McGraw-Hill, 1941, p. 547.

3. Hill, J. R. and M. R. Wilson, Buried Cable Transmission Line Parameters: A Comparison of Two Theoretical Models, Mission Research Corporation, AMRC-N-5, March 1973.

$$Z(s) = ih(s)Z_c(s) \tag{B9a}$$

$$Y(s) = ih(s)/Z_c(s) \tag{B9b}$$

Figures B1 and B2 show the real and imaginary parts of Z(s) as functions of frequency for a typical buried cable (Y(s) displays similar behavior). The next step is to find the inverse transform of a function with this frequency behavior.

To demonstrate this, let us examine the simple L-R series circuit shown in Figure B3.

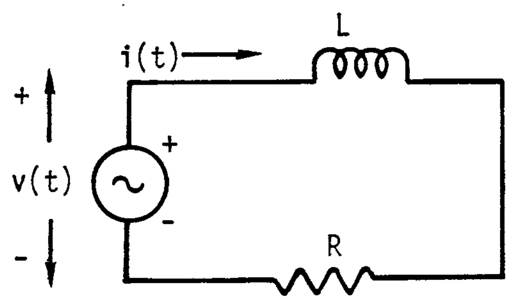

Figure B3. Simple L-R series circuit.

The time domain relation between the current and voltage is

$$v(t) = Ri(t) + L \frac{d}{dt} i(t) \tag{B10a}$$

The Laplace transform of this equation is

$$V(s) = (R+sL)\ I(s) \tag{B10b}$$

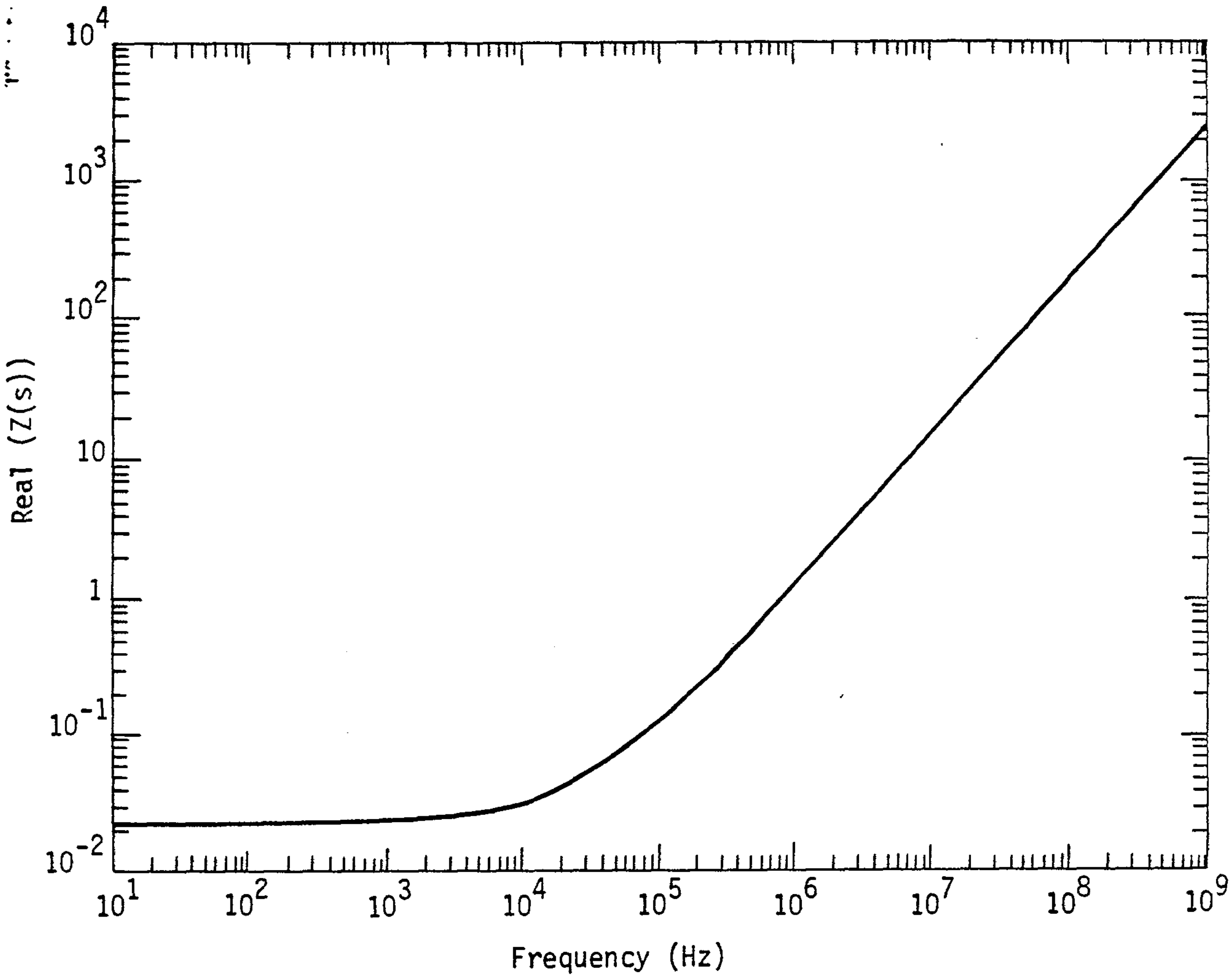

Figure B1. Real part of the series impedance, Z(s), as a function of frequency.

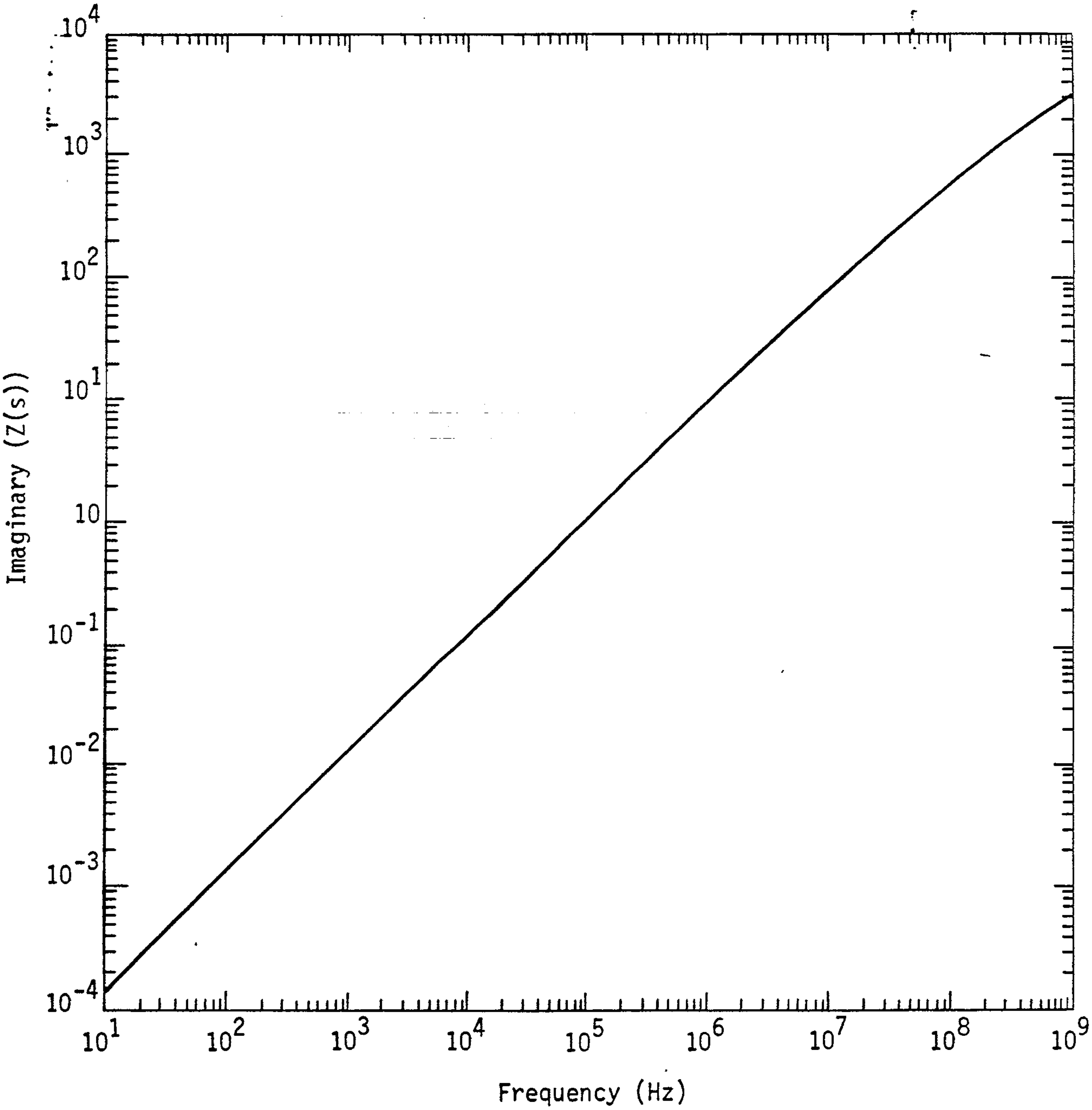

Figure B2. Imaginary part of the series impedance, Z(s), as a function of frequency.

By comparison with Equation B1a, the series impedance is $Z(s) = R+sL$. The imaginary part of $Z(s)$ for the simple L-R circuit displays the behavior shown in Figure B3, but the real part is constant, unlike the real part of the actual series impedance of Figure B2. This form of the actual series impedance makes it difficult to separate the behavior which results in the impulse, doublet, and inverse transformable time domain aspects.

The intractability of this problem can be surmounted by the realization that all that is needed is a relationship between the current and voltage, not necessarily the one implied by Equations B1a and B1b. To clarify this, a comparison using Equation B10b can be made. Written in the form

$$I(s) = \frac{1}{R+sL} V(s) \tag{B11}$$

the inverse transform of this admittance has no frequency-domain behavior which prevents the immediate numerical (or analytical in this case) inverse transformation into the time domain

$$i(t) = \frac{1}{L} \int_0^t e^{-\frac{R}{L}(t-\tau)} v(\tau) d\tau \tag{B12}$$

This same process can be applied to Equation B1a by setting

$$\frac{d}{dx} V(x,s) = - U(x,s) + E(x,s) \tag{B13a}$$

where

$$U(x,s) = Z(s)\, I(x,s) \tag{B13b}$$

or

$$I(x,s) = \frac{1}{Z(s)} U(x,s) \tag{B13c}$$

Figures B4 and B5 show the real end imaginary parts of $1/Z(s)$; these have no high frequency behavior which indicates impulses or doublets in the time domain and can therefore be numerically inverse transformed. The frequency behavior of $1/Y(s)$ has similar real and imaginary parts.

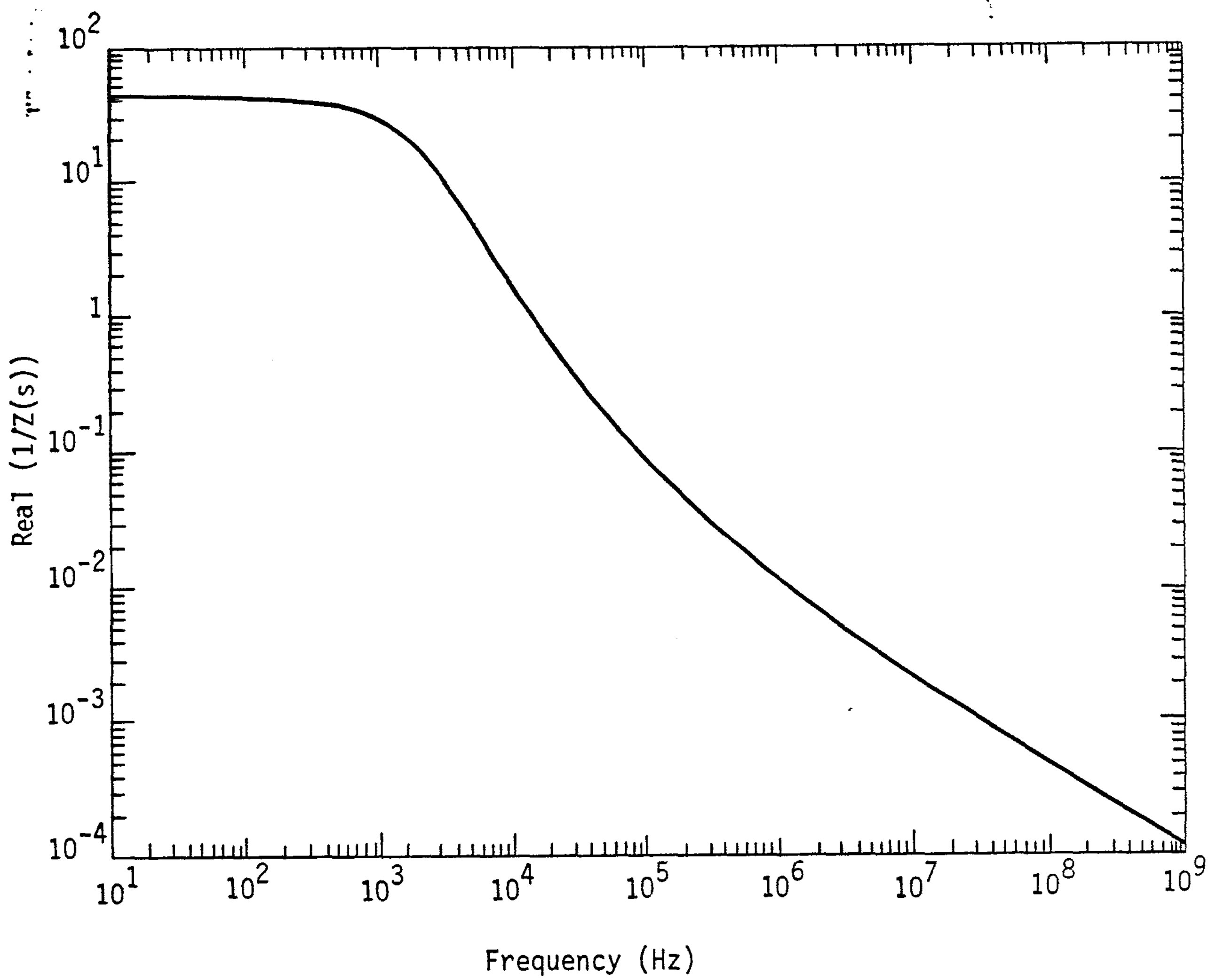

Figure B4. Real part of the inverse of the series impedance as a function of frequency.

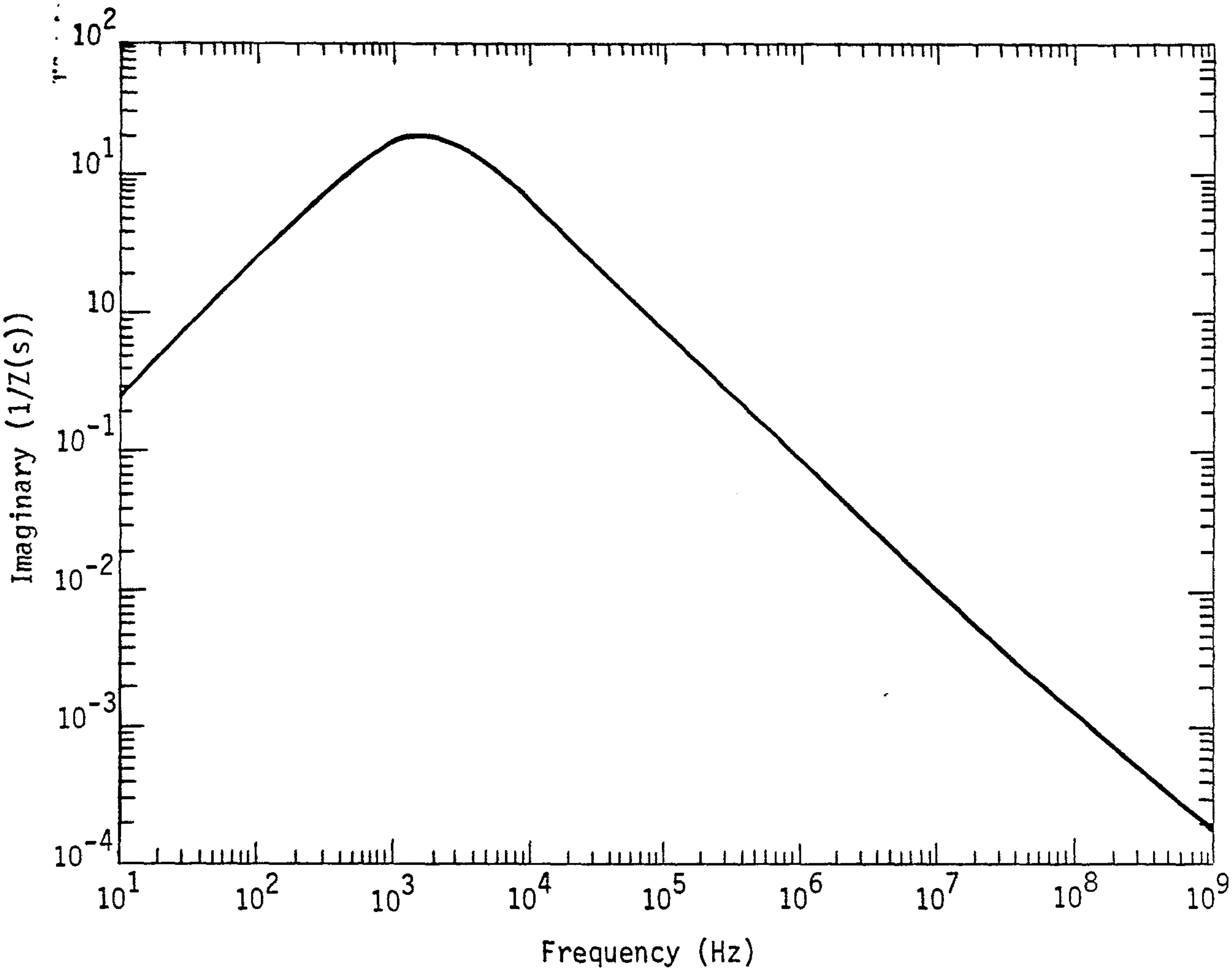

Figure B5. Imaginary part of the inverse of the series impedance as a function of frequency.

The utilization of this form of the series impedance and shunt admittance requires a form of the telegrapher's equations which is different from that shown in Equation B8.

Examining the voltage equation (Equation B13a), the inverse transform is

$$\frac{\partial}{\partial x} v(x,t) = - u(x,t) + e(x,t) \tag{B14}$$

The voltage term u(x,t) now appears in the differential equation resulting from the inverse transform of Equation B13c,

$$i(x,t) = \int_0^t L^{-1} [1/Z(s)] (t-\tau) u(\tau)d\tau \tag{B15}$$

If L^{-1} [1/Z(s)] (t) is represented as the sum of decaying exponentials as was done with L^{-1} [Z(s)] in Equation B3a, one obtains, by an analagous derivation,

$$i(x,t) = \sum_{n-1}^{N} y_n(x,t) \tag{B16a}$$

where $y_n(x,t)$ is now related to u(x,t) by

$$\frac{\partial}{\partial t} y_n(x,t) = a_n u(x,t) + s_n y_n(x,t) \tag{B16b}$$

By solving for the unknown u(x,t), the telegrapher's equations now have the form

$$i(x,t) = \sum_{n=1}^{N} y_n(x,t) \tag{B17a}$$

$$\frac{\partial}{\partial t} y_n(x,t) = a_n [e(x,t) - \frac{\partial}{\partial x} v(x,t)] + s_n y_n(x,t) \tag{B17b}$$

and by a similar analysis involving L^{-1} [1/Y(s)],

$$v(x,t) = \sum_{n=1}^{M} z_n(x,t) \tag{B17c}$$

$$\frac{\partial}{\partial t} z_n(x,t) = -b_n \frac{\partial}{\partial x} i(x,t) + p_n z_n(x,t) \tag{B17d}$$

2. RELATIONSHIP TO ELEMENTARY CIRCUITS

The series R-L circuit of Figure B3 gives considerable insight into the physical interpretation of the Prony poles and residues of L^{-1} [1/Z(s)] of Equation B15. The current-voltage relationship for the series R-L circuit is shown in Equation B12. In this case the Prony analysis would give one pole and one residue of $a = 1/L$ and $s = - R/L$. From this, one may expect that the dominant pole and residue obtained from L^{-1} [1/Z(s)] for the actual buried cable would be related to a constant L and R of the simple circuit in Figure B3.

A similar analysis of a parallel R-C circuit gives analogous results for the dominant term in the Prony analysis of L^{-1} [1/Y(s)]: $b = 1/C$ and $p = - G/C$ where G is 1/R in the simple R-C circuit.

Depending on the relative importance of the first few terms in the expansion of L^{-1} [1/Z(s)] or L^{-1} [1/Y(s)], it may be possible to represent the transmission line with constant parameters. If the energy of the second term obtained by the Prony analysis is three or four orders of magnitude less than the energy of the first term, an adequate representation of the transmission line may be made with constant R, L, G, and C.

In the case where there is one dominant term in the expansions, this procedure gives the best constant R, L, G, and C which can be used. Of course, when this is true, the differencing of the telegrapher's equations can be achieved by either the method discussed here (Equations B8) or by the differencing of the time-domain representation of Equations B1.

3. COMPARISON TO THE FREQUENCY DOMAIN SOLUTION

To test the accuracy of the new representation of the telegrapher's equations, several comparisons were made between the predictions of the frequency-domain solution and the time-domain solution. All comparisons were made with sources driving the cable such that the response was linear. Under this condition, the frequency-domain prediction is assumed to be "correct".

The transmission line to be analyzed consisted of a 46 meter long, single, insulated conductor buried in an infinite soil medium with the parameters shown in Table 1. The termination impedances are 50 ohms on the "source" end of the cable and 10^6 ohms on the "load" end.

TABLE 1

CABLE AND SOIL PROPERTIES

	Conductor	Dielectric	Soil
Radius (r)	3.937×10^{-3} meters	1.270×10^{-3} meters	---
Permittivity (ε)	ε_0 farads/meter	$2.3 \times \varepsilon_0$ farads/meter	frequency dependent[4]
Permeability (μ)	μ_0 henries/meter	μ_0 henries/meter	μ_0 henries/meter
Conductivity (σ)	5.8×10^7 mhos/meter	0.0	frequency dependent[4]

The solution of the three media problem with the parameters listed in Table 1 resulted in the quantities L^{-1} [1/Z] (Fig. B6) and L^{-1} [1/Y] as functions of time. A Prony analysis on these functions resulted in the decaying exponential representations

$$L^{-1}\left[\frac{1}{Z}\right] \approx 5.32 \times 10^5 \exp(-1.197 \times 10^4 t) + 1.12 \times 10^5 \exp(-2.022 \times 10^6 t) \quad \text{(B18a)}$$

$$L^{-1}\left[\frac{1}{Y}\right] \approx 9.24 \times 10^9 \exp(-1.073 \times 10^3 t) + 1.69 \times 10^9 \exp(-4.502 \times 10^6 t) \quad \text{(B18b)}$$

The coding of Equations B17 was done as shown in the next part of Section II. Two comparisons were made between this formulation of the telegrapher's equations and the frequency-domain solution. The first employed a source, e(t), at the "source" end of the line with the following form

4. Longmire, C. L. and K. S. Smith, A Universal Impedance for Soils, Mission Research Corporation, MRC-N-214, October 1975.

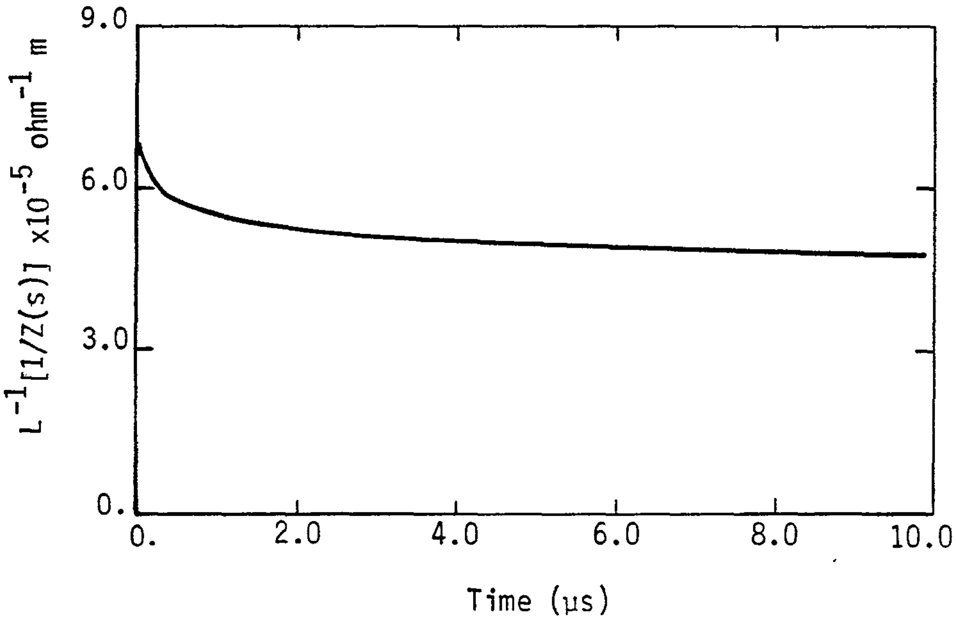

Figure B6. Time-domain representation of $L^{-1}[1/Z(s)]$.

$$e(t) = 1.0 - e^{-\alpha t} \tag{B19}$$

where

$$\alpha = 10^{-6}.$$

Figures B7 and B8 show the voltage and current given by the two approaches at a distance of 26 meters from the source end of the cable. Comparisons of results at the ends of the cable resulted in similar correlations between the two techniques.

The second comparison involved a distributed source of the form

$$e(t,x) = A(e^{-\alpha t} - e^{-\beta t})\, e^{-Gx} \tag{B20}$$

where $A = 10^4$ volts/meter, $\alpha = 4.0 \times 10^{-6}$ sec^{-1}, $\beta = 4.76 \times 10^{-8}$ sec^{-1}, $G = 10^{-3}$ m^{-1}, and t is the retarded time. This distributed source represents a field traveling at the speed of light from the source to the load end of the cable.

Figure B9 shows the voltage across the source impedance predicted by two techniques. The discrepancy between the two responses results from inconsistencies inadvertently introduced into the analysis. These can be avoided in the future but, because of time limitations, must remain present in the calculations of this report.

The problem arises from the inherent time resolution of time-domain and frequency-domain analyses. The time step, Δt, employed in the time-domain calculations results from the Courant stability condition, $\Delta t \leq \Delta x/2v$ where Δx is the cell size and v is the velocity of propagation of a signal along the cable. The time resolution employed in the Prony analysis of L^{-1} [1/Z] is not as good as that implied by Δt of the time-domain analysis; thus the reflections of the pulses are not modeled as well in the frequency-domain analysis <u>and</u> the poles and residues employed by the differential equations in the time-domain analysis do not give a true representation of the voltages and currents. As can be seen in the comparisons (Figure B9), the discrepancy is not major and can be ignored.

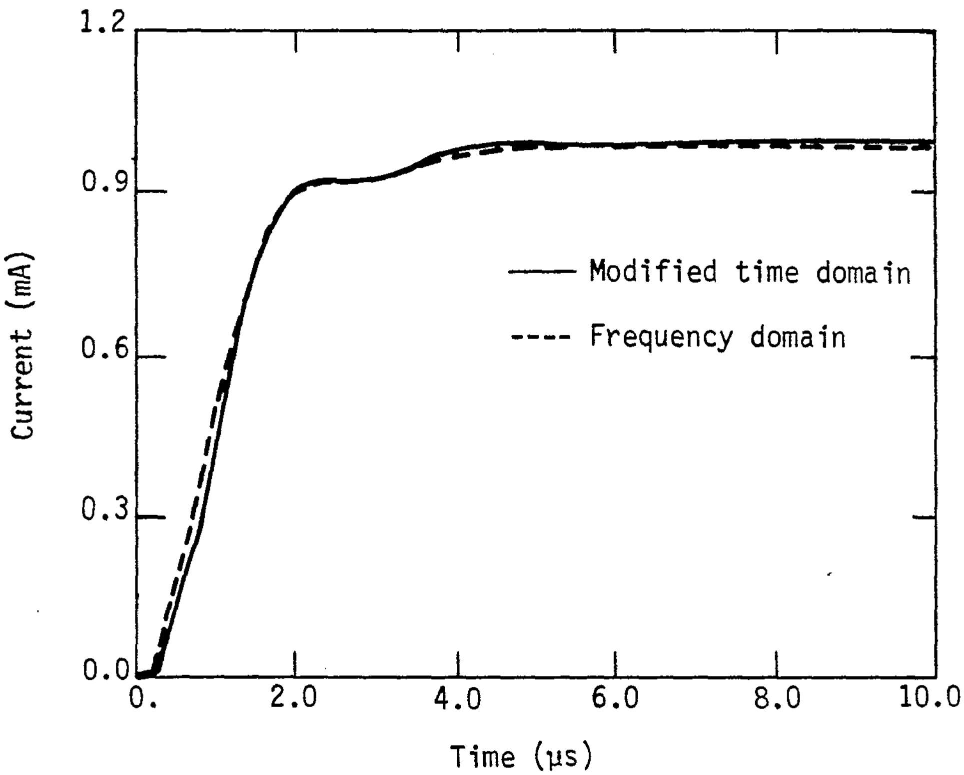

Figure B7. Voltage at the center of the cable - predicted by the modified time-domain and frequency domain approaches (source is equation B19).

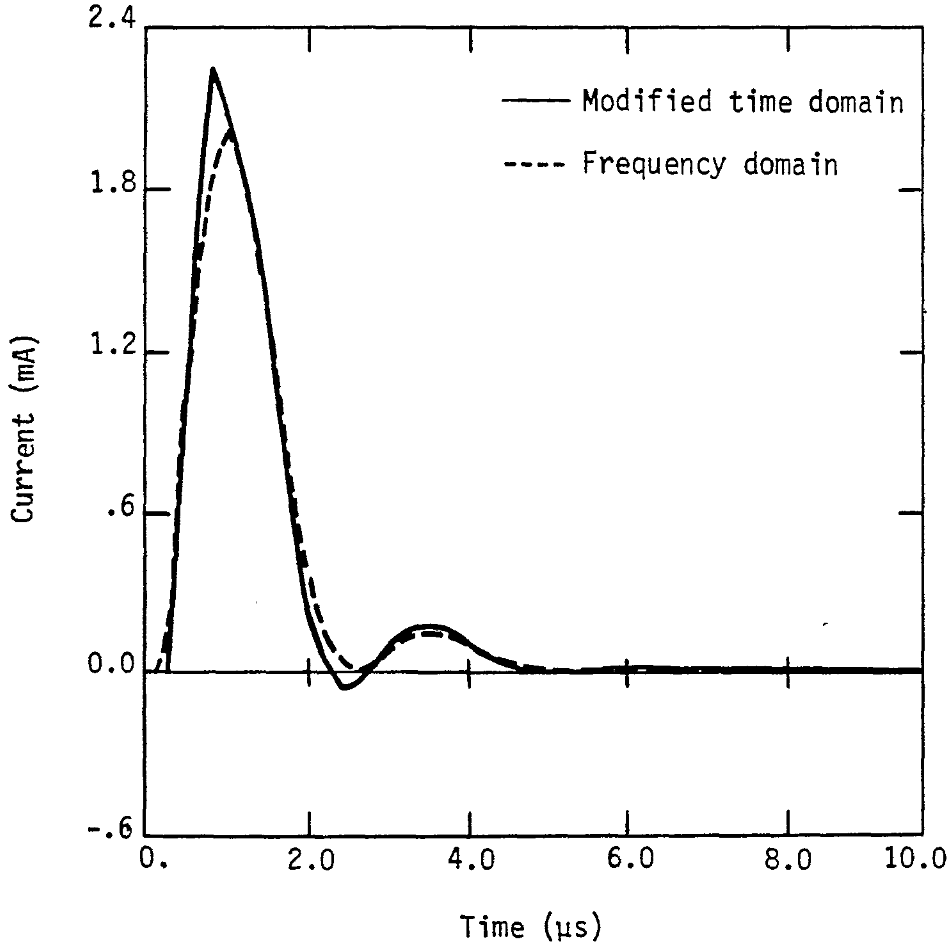

Figure B8. Current at the center of the cable - predicted by the modified time-domain and frequency domain approaches (source is equation B19).

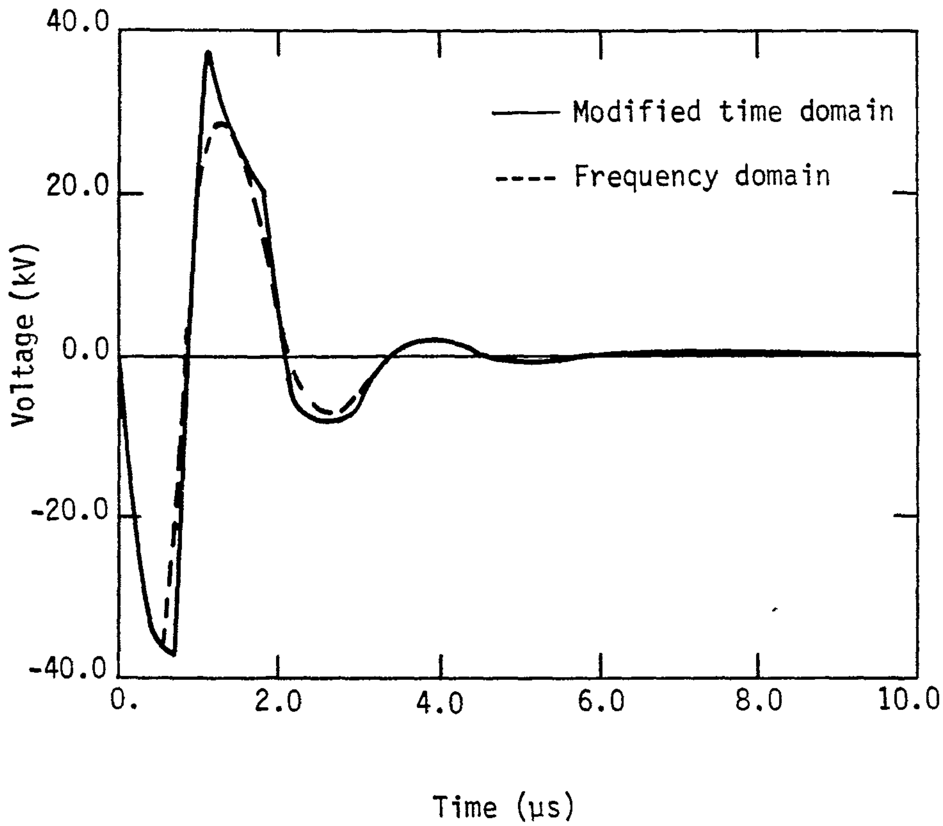

Figure B9. Voltage across the 50 ohm source impedance - predicted by the modified time-domain and frequency-domain approaches (source is equation B20).

The last comparison used the distributed source of Equation B19 and compares the time-domain coding modified by the differential equation approach and a time-domain code employing constant R, L, G, and C transmission-line parameters. One case of constant R, L, G, and C chosen was the dominant-pole-and residue parameters discussed previously. The other case used R, L, G, and C parameters derived without benefit of the frequency-domain analysis. If one expects a response near a certain frequency, a common derivation of the constant R, L, G, and C parameters is

$$R = \left[\pi(2r_c\delta_c - \delta_c^{\,2})\ \sigma_c\right]^{-1} \tag{B21a}$$

$$L = \frac{\mu_o}{2\pi} \ln(\delta_s/r_c) \tag{B21b}$$

$$C = 2\pi\varepsilon_s/\ln(\delta_s/r_c) \tag{B21c}$$

$$G = \frac{\sigma_s}{\varepsilon_s} C \tag{B21d}$$

where $\delta = \sqrt{2/\omega\mu\sigma}$, the skin depth in the soil (δ_s) or the conductor (δ_c), r_c and σ_c are the radius and conductivity of the conductor, and μ_o and ε_s are the permeability and permittivity of the soil.

The three voltage responses shown in Figure B10 represent predictions by (1) the differential-equation-modified, time-domain code, (2) a straight time-domain code employing the constant R, L, G, and C derived from the dominant poles and residues as discussed earlier and (3) the time-domain code employing the constant R, L, G, and C of Equations B21.

Reasonable agreement is found between the time-domain predictions employing some benefit of frequency-domain analysis. The code using the constant, R, L, G, and C of Equations B21 disagrees mainly because of the large shunt admittance given by Equation B21d.

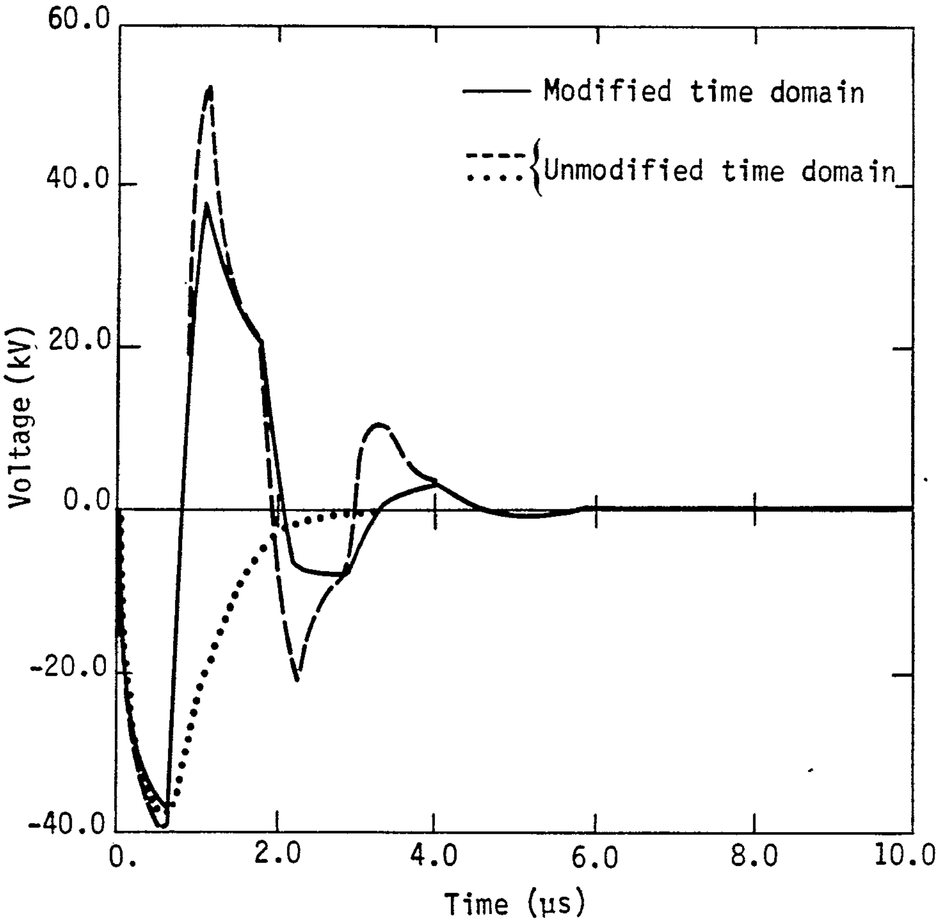

Figure B10. Voltage across the 50 ohm source impedance predicted by three different time domain approaches. (—— the modified time-domain code, ---- the unmodified time-domain code employing the constant R, L, G, and C derived from the dominant pole and residue, ... the unmodified time-domain code employing constant R, L, G, and C derived without the benefit of frequency-domain analysis.)

4. DIFFERENCING OF THE TELEGRAPHER'S EQUATIONS

The equations to be differenced implicitly contain the differential-equation representation of the frequency-dependent variables. In the case of the regular telegrapher's equations there are two equations involved. The modified equations not only contain two equations to be differenced (Equations B17b and B17d) but two auxilliary equations (Equations B17a and B17c) which represent summations to obtain the current and voltage.

Using a central-differencing scheme where n and j are the time and position indices, it can be seen from Figure B11 that Equation B17b becomes

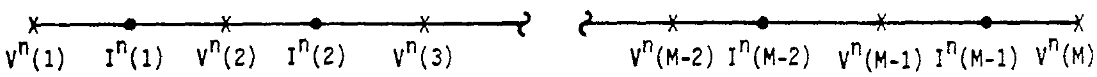

Figure B11. Relative placement of currents and voltages for differencing of the telegrapher's equations.

$$\frac{1}{\Delta t}\left[y_i^{n+1}(j) - y_i^n(j)\right] = \frac{1}{2} s_i \left[y_i^{n+1}(j) + y_i^n(j)\right]$$

$$+ a_i \left\{e^{n+1}(j) - \frac{1}{\Delta x}\left[V^n(j+1) - V^n(j)\right]\right\} \qquad \text{(B22a)}$$

or

$$y_i^{n+1}(j)\left(\frac{1}{\Delta t} - \frac{s_i}{2}\right) = y_i^n(j)\left(\frac{1}{\Delta t} + \frac{s_i}{2}\right)$$

$$+ a_i \left\{e^{n+1}(j) - \frac{1}{\Delta x}\left[V^n(j+1) - V^n(j)\right]\right\} \qquad \text{(B22b)}$$

by a similar process the z's of Equation B17d are found to be

$$z_i^{n+1}(j)\left(\frac{1}{\Delta t} - \frac{u_i}{2}\right) = z_i^n(j)\left(\frac{1}{\Delta t} + \frac{p_i}{2}\right)$$

$$- \frac{b_i}{\Delta x}\left[I^n(j) - I^n(j-1)\right] \qquad \text{(B23)}$$

After the y's and z's have been found, the currents and voltages are the sums shown in Equation B17a and B17c. The arbitrary placement of the voltage nodes of the ends of the transmission line results in the calculation of Equation B22b before Equation B23. This places the currents one-half time stop ahead of the voltages. The relationship of the discrete voltage and current representation to the continuous representation is therefore

$$I^n\ (j) = i\left[\left(j - \frac{1}{2}\right) \Delta x,\ (n-1)\ \Delta t\right] \tag{B24a}$$

$$V^n\ (j) = v\ \left[(j-1)\ \Delta x,\ \left(n - \frac{1}{2}\right) \Delta t\right] \tag{B24b}$$

$$e^n\ (j) = e\left[\left(j - \frac{1}{2}\right) \Delta x,\ (n-1)\ \Delta t\right] \tag{B24c}$$

where e(x,t) is the voltage driving term in Equation B6.

The boundary conditions of the ends of the transmission line determine the end voltages. At the source end

$$V^n\ (1) = -\ i\left[0,\ \left(n - \frac{1}{2}\right) \Delta t\right] R_s + V_s\left[\left(n - \frac{1}{2}\right) \Delta t\right] \tag{B25a}$$

where R_s and V_s are the source resistance and voltage and the current is defined to be positive flowing from the source to the load end of the cable.

At the load end of the transmission line, only a resistance, R_L, is assumed so

$$V^n\ (M) = i\left[\ell,\ \left(n - \frac{1}{2}\right) \Delta t\right] R_L \tag{B25b}$$

where ℓ is the length of the line.

Using central differencing, Equation B25a becomes

$$i\ \left[0,\ \left(n - \frac{1}{2}\right) \Delta t\right] = \frac{1}{R_s} \left\{V_s \left[\left(n - \frac{1}{2}\right) \Delta t\right] - \frac{1}{2}\left[V^{n+1}(1) + V^n(1)\right]\right\} \tag{B26}$$

Because $V^n(1)$ is a sum of z's (Equation B17c), the boundary condition for $Z_i^{n+1}(1)$ is found from Equation B23.

$$z_i^{n+1}(1)\left(\frac{1}{\Delta t} - \frac{p_i}{2} + \frac{b_i}{R_s \Delta x}\right) + \frac{b_i}{R_s \Delta x} \sum_{\ell \neq i}^{N} z_\ell^{n+1}(1)$$

$$= z_i(1)\left(\frac{1}{\Delta t} + \frac{p_i}{2} - \frac{b_i}{R_s \Delta x}\right) - \frac{2b_i}{\Delta x}\left[I^n(1) - \frac{1}{R_s} V_s + \frac{1}{2R_s} \sum_{\ell \neq i}^{N} z_\ell^n(1)\right] \qquad \text{(B27a)}$$

This results in a set of linear equations for the unknown $z_i^{n+1}(1)$ at the source boundary,

$$\begin{pmatrix} A_1 & D_1 & D_1 & \cdots & D_1 \\ D_2 & A_2 & D_2 & \cdots & D_2 \\ \vdots & & & & \vdots \\ D_N & \cdots & & & A_N \end{pmatrix} \begin{pmatrix} z_1^{n+1}(1) \\ z_2^{n+1}(1) \\ \vdots \\ z_N^{n+1}(1) \end{pmatrix} = \begin{pmatrix} z_1^n(1)B_1 - C_1 - D_1 \sum_{\ell \neq 1} z_\ell^n(1) \\ z_2^n(1)B_2 - C_2 - D_2 \sum_{\ell \neq 2} z_\ell^n(1) \\ \vdots \\ z_N(1)B_N - C_N - D_N \sum_{\ell \neq N} z_\ell^n(1) \end{pmatrix} \qquad \text{(B28a)}$$

Where N is the number of poles in the Prony approximation, and

$$A_i = \frac{1}{\Delta t} - \frac{p_i}{2} + \frac{b_i}{R_s \Delta x} \qquad \text{(B28b)}$$

$$B_i = \frac{1}{\Delta t} + \frac{p_i}{2} - \frac{b_i}{R_s \Delta x} \qquad \text{(B28c)}$$

$$C_i = \frac{2b_i}{\Delta x}\left[I^n(1) - \frac{1}{R_s} V_s\right] \qquad \text{(B28d)}$$

$$D_i = \frac{b_i}{R_s \Delta x} \qquad \text{(B28e)}$$

A similar matrix equation is found for the unknown $z_i^{n+1}(M)$ at the load end of the transmission line,

$$\begin{pmatrix} A_1 & D_1 & D_1 & \cdots & D_N \\ D_2 & A_2 & D_2 & \cdots & D_N \\ \vdots & & & & \vdots \\ D_N & \cdots & & \cdots & A_N \end{pmatrix} \begin{pmatrix} z_1^{n+1}(M) \\ z_2^{n+1}(M) \\ \vdots \\ z_N^{n+1}(M) \end{pmatrix} = \begin{pmatrix} z_1^n(M)\, B_1 + C_1 - D_1 \sum_{\ell \neq 1} z_\ell^n(M) \\ z_2^n(M)\, B_2 + C_2 - D_2 \sum_{\ell \neq 2} z_\ell^n(M) \\ \vdots \\ z_N^n(M)\, B_N + C_N - D_N \sum_{\ell \neq N} z_\ell^n(M) \end{pmatrix} \qquad \text{(B29a)}$$

where

$$A_i = \frac{1}{\Delta t} - \frac{p_i}{2} + \frac{b_i}{R_L \Delta x}$$

$$B_i = \frac{1}{\Delta t} + \frac{p_i}{2} - \frac{b_i}{R_L \Delta x}$$

$$C_i = \frac{2b_i}{\Delta x} \; I^n \; (M-1)$$

$$D_i = \frac{b_i}{R_L \Delta x}$$