Interaction Notes

Note 419
(with corrections)

April 1983

Bounding of Signal Levels at Terminations of a Multiconductor Transmission-Line Network

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Abstract

Starting from the norm concept for vectors and matrices, this note addresses the problem of bounding signal levels at terminations of a multiconductor transmission-line network. The overall network equation is formulated in terms of the combined voltage supervector (a special combination of the voltage and current vectors). Utilizing the scattering and propagation supermatrices for the waves on the transmission-line network and the combined voltage supervector for sources, the BLT equation is used to express the combined voltage supervectors and the voltage and current supervectors at the junctions. The upper and lower bounds for the combined voltage supervector, voltage supervector and current supervector are obtained in terms of the norms of the propagation and scattering supermatrices and the norm of the combined voltage source supervector. Various properties of the propagation and scattering supermatrices are discussed for two cases, namely, a uniform section of a multiconductor transmission line and a multiconductor transmission line with a branch. The expressions for upper and lower bounds for combined voltage supervectors and voltage and current supervectors are derived. Various norms of vectors, matrices, supervectors, and supermatrices are also discussed.

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PROLOGUE

SOCRATES: Phew! A considerable business still in front of us, Protarchus, and not exactly an easy one, I should say, to deal with now. It really looks as though I need fresh tactics. If my objective is to secure the second prize for reason I must have weapons different from those of my previous arguments, though possibly some may be the same. Is it to be, then?

PROTARCHUS: Yes, of course.

SOCRATES: Let us try to be very careful what starting point we take.

PROTARCHUS: Starting point?

SOCRATES: Of all that now exists in the universe let us make a twofold division, or rather, if you don't mind, a threefold.

PROTARCHUS: On what principle, may I ask?

SOCRATES: We might apply part of what we were saying a while ago.

PROTARCHUS: What part?

SOCRATES: We said, I fancy, that God had revealed two constituents of things, the unlimited and the limit.

PROTARCHUS: Certainly.

SOCRATES: Then let us take these as two of our classes, and as the third, something arising out of the mixture of them both, though I fear I'm a ridiculous sort of person with my sortings of things into classes and my enumerations.

PROTARCHUS: What are you making out, my good sir?

SOCRATES: It appears to me that I now need a fourth kind as well.

PROTARCHUS: Tell me what it is.

SOCRATES: Consider the cause of the mixing of these two things with each other, and put down that, please, as number four to be added to the other three.

PROTARCHUS: Are you sure you won't need a fifth to effect separation?

SOCRATES: Possibly, but not, I think, at the moment. But should the need arise, I expect you will forgive me if I go chasing after a fifth.

PROTARCHUS: Yes, to be sure.

from the dialogue Philebus, by Plato,
translated by R. Hackforth

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I. INTRODUCTION

In designing or analyzing the response of an electronic system to some kind of electromagnetic interference such as the nuclear electromagnetic pulse (EMP), one is overwhelmed by the complexity of the problem. There are too many individual components with an enormous number of interconnections. An example of such a complex system is a multiconductor cable network inside an aircraft.

When an aircraft is in an EMP environment, the cables inside the aircraft will be excited by an electromagnetic field which penetrates the aircraft body through a large number of penetrating conductors, small antennas, apertures, and diffusion through the skin of the aircraft. There are many parameters which determine the response of a multiconductor cable; these include polarization, angle of incidence, planarity and spectral contents of the incident field, number of points of entry (POEs), size, shape and location of POEs, physical properties of the transmission line and the surrounding medium, and the configuration of load impedances. These large number of variables together with the complexity of the multiconductor cables make it very difficult to get simple insights into how to control the system performance in an electromagnetic environment.

In evaluating the system vulnerability to EMP, it is often desirable to evaluate upper bounds on the problem rather than compute the full coupling and interaction evaluation for the cases of interest so as to determine the system survivability/vulnerability with high confidence. In most of the cases the latter may be effectively impossible due to system complexity and lack of complete and correct definition. To deal with this complexity one needs ways to identify and deal with a set of important variables which, if controlled, control the system performance. An approach to this problem has been developed (Ref. 1) which can be referred to as electromagnetic topology.

Having defined the electromagnetic topology and the related interaction sequence diagram (graph), one can write a general matrix equation (BLT equation, Ref. 1). The resulting supermatrix equation admits an approximate solution which shows the dependence of the system performance on system shielding parameters. One can also formulate a BLT equation for transmission-line networks within the system (Ref. 2). This equation shows the dependence of the cable network response on the induced sources, physical configuration of the cables in the network, and the load configurations. Certain approximate bounds for the termination voltages and currents can be obtained from norm concepts (Ref. 3).

In Reference 4, upper bounds were obtained for voltages and currents at terminations of a multiconductor transmission line excited by a single aperture, but bounds were not established for physical parameters of the line. For a moderately mismatched termination, the upper bound for the termination voltage was 10 times the actual maximum voltage.

In this paper, we establish upper and lower bounds on the voltages and currents at terminations of a multiconductor transmission-line network excited by an external electromagnetic field. The general matrix equation (BLT equation) is used as the basis for establishing upper and lower bounds on the termination voltages and currents. Upper and lower bounds on forward and backward traveling combined voltage waves are also established. These bounds are obtained in terms of upper bounds of several parameters, such as the source, load impedances, characteristic impedance of the line, etc. Upper bounds on these parameters are established for some special cases.

In Section II, the equations governing the response of a general multiconductor transmission-line network are discussed. In Section III, the upper and lower bounds for the combined voltages, voltages, and currents are obtained in terms of the induced sources, physical properties of the cable network, and the load configurations. The bounds on the ratio of the maximum pin current to the bundle current are also discussed. In Sections IV and V, bounds are obtained for two specal cases of a general multiconductor transmission-line network, namely, a uniform section of a multiconductor transmission line and a multiconductor transmission line with a branch. Procedures for obtaining bounds on parameters of the line such as the characteristic-impedance matrix, reflection-coefficient matrix, and the scattering matrix are discussed. Bounds for induced sources are also discussed for these two cases.

II. GENERAL MULTICONDUCTOR TRANSMISSION-LINE NETWORK EQUATIONS

In this section we will review the multiconductor transmission-line equations for a general network. The detailed derivation of these equations is discussed in Reference 2. These equations form a basis for the evaluation of upper bounds on voltages and currents at terminations of a multiconductor line network.

2.1 PROPAGATION ON A UNIFORM N-WIRE TRANSMISSION LINE

Let us first consider a single section of an N-wire transmission line. An N-wire transmission line is one that consists of N conductors and a reference conductor (or an equivalent one). Figure 2.1 shows per-unit-length equivalent circuit of the line with distributed sources. The equations governing the voltage and current propagation on an N-wire transmission line are the generalized multiconductor transmission-line equations:

$$\frac{d}{dz}(\tilde{I}_n(z,s)) = -(\tilde{Y}'_{n,m}(s)) \cdot (\tilde{V}_n(z,s)) + (\tilde{I}_n^{(s)'}(z,s)) \qquad (2.1)$$

$$\frac{d}{dz}(\tilde{V}_n(z.s)) = -(\tilde{Z}'_{n,m}(s)) \cdot (\tilde{I}_n(z,s)) + (\tilde{V}_n^{(s)'}(z,s)) \qquad (2.2)$$

where z = position along the line

$(\tilde{I}_n(z,s))$ = current vector at z

$(\tilde{V}_n(z,s))$ = voltage vector at z

$(\tilde{Y}'_{n,m}(s))$ = per-unit-length shunt admittance matrix

$(\tilde{Z}'_{n,m}(s))$ = per-unit-length series impedance matrix

$(\tilde{I}_n^{(s)'}(z,s))$ = per-unit-length shunt current source vector

$(\tilde{V}_n^{(s)'}(z,s))$ = per-unit-length series voltage source vector

It is noted that all vectors are of dimension N, and all matrices are N x N.

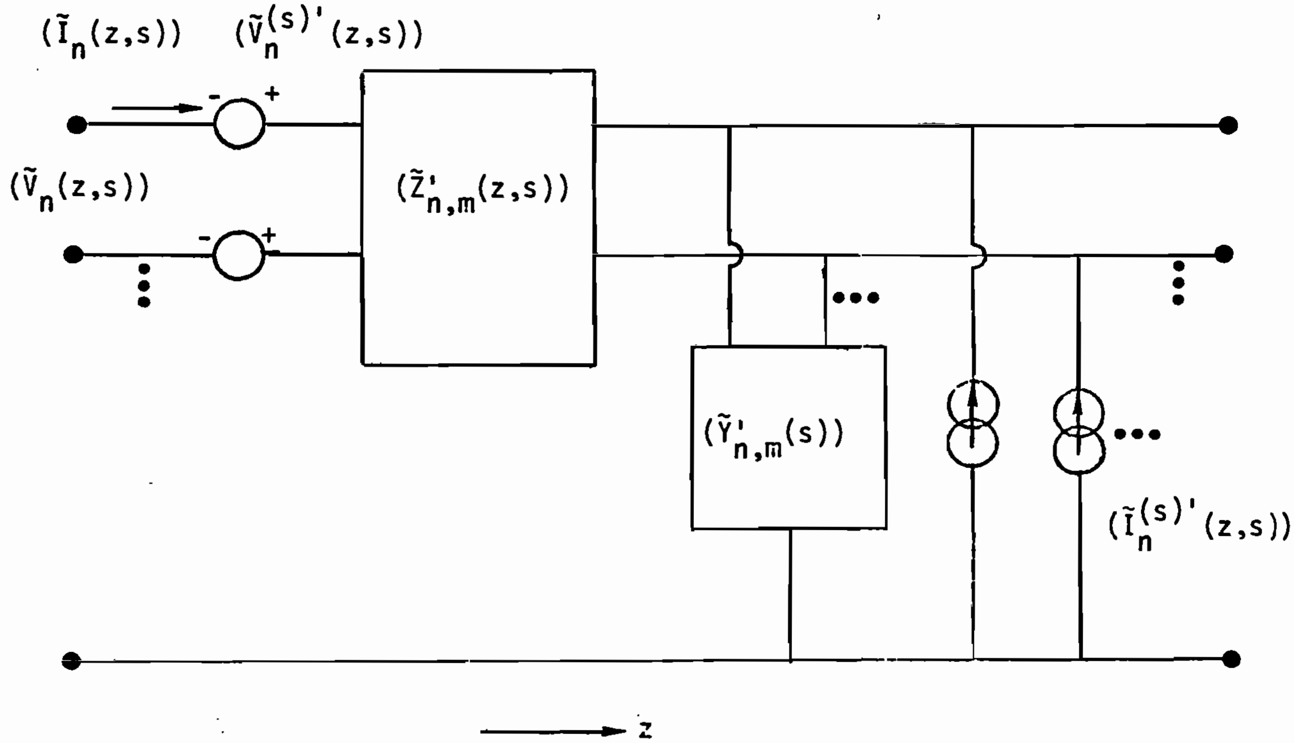

Figure 2.1. The per-unit-length model of a multi-conductor transmission line.

By algebraic manipulations of Equations 1 and 2 one can obtain an equation for combined voltages as follows (Ref. 2)

$$\left[(1_{n,m})\frac{d}{dz} + q(\tilde{Y}_{c_{n,m}}(s))\right] \cdot (\tilde{V}_n(z,s))_q = (\tilde{V}_n^{(s)'}(z,s))_q \qquad (3)$$

$$1_{n,m} = \begin{cases} 1 \text{ for } n = m \\ 0 \text{ for } n \neq m \end{cases} \qquad (4)$$

$q = \pm$ for forward and backward traveling combined N-vector waves, respectively

$$(\tilde{Y}_{c_{n,m}}(s)) = \left[(\tilde{Z}'_{n,m}(s)) \cdot (\tilde{Y}'_{n,m}(s))\right]^{1/2} \qquad (5)$$

$$(\tilde{V}_n(z,s))_q = (\tilde{V}_n(z,s)) + q(\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n(z,s))$$

$$(\tilde{V}_n^{(s)'}(z,s))_q = (\tilde{V}_n^{(s)'}(z,s)) + q(\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n^{(s)'}(z,s)) \qquad (6)$$

$$(\tilde{Z}_{c_{n,m}}(s)) = (\tilde{Y}_{c_{n,m}}(s))^{-1} \cdot (\tilde{Z}'_{n,m}(s)) \qquad (7)$$

$$(\tilde{Y}_{c_{n,m}}(s)) = (\tilde{Z}_{c_{n,m}}(s))^{-1} \qquad (8)$$

$(\tilde{Z}_{c_{n,m}}(s)) \equiv$ characteristic-impedance matrix

$(\tilde{Y}_{c_{n,m}}(s)) \equiv$ characteristic-admittance matrix

Substituting $q = +1$ and $q = -1$ in Equation 6, one can obtain the following relations

$$(\tilde{V}_n(z,s))_+ = (\tilde{V}_n(z,s)) + (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n(z,s)) \qquad (9)$$

$$(\tilde{V}_n(z,s))_- = (\tilde{V}_n(z,s)) - (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n(z,s)) \qquad (10)$$

$$(\tilde{V}_n^{(s)'}(z,s))_+ = (\tilde{V}_n^{(s)'}(z,s)) + (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n^{(s)}(z,s)) \quad (2.11)$$

$$(\tilde{V}_n^{(s)'}(z,s))_- = (\tilde{V}_n^{(s)'}(z,s)) - (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n^{(s)}(z,s)) \quad (2.12)$$

$(\tilde{V}_n(z,s))_+ \equiv$ forward traveling combined voltage vector or wave

$(\tilde{V}_n(z,s))_- \equiv$ backward traveling combined voltage vector or wave

From Equations 2.9 and 2.10 we can reconstruct voltage and current vectors in terms of forward and backward waves. These are given by the following relations

$$(\tilde{V}_n(z,s)) = \frac{1}{2}\left[(\tilde{V}_n(z,s))_+ + (\tilde{V}_n(z,s))_-\right] \quad (2.13)$$

$$(\tilde{I}_n(z,s)) = \frac{1}{2}(\tilde{Y}_{c_{n,m}}(s)) \cdot \left[(\tilde{V}_n(z,s))_+ - (\tilde{V}_n(z,s))_-\right] \quad (2.14)$$

From the above definitions we can obtain two sets of waves propagating in opposite directions along z. For all modes we have

$$\exp\left[-(\tilde{\gamma}_{c_{n,m}}(s))z\right] \quad + \text{ propagating}$$

$$\exp\left[(\tilde{\gamma}_{c_{n,m}}(s))z\right] \quad - \text{ propagating}$$

Equation 2.3 can be integrated to obtain a solution for the combined voltage vectors to give

$$(\tilde{V}_n(z,s))_q = \exp\left\{-q(\tilde{\gamma}_{c_{n,m}}(s))[z - z_o]\right\} \cdot (\tilde{V}_n(z_o,s))_q$$

$$+ \int_{z_o}^{z} \exp\left\{-q(\gamma_{c_{n,m}}(s))[z - z']\right\} \cdot (\tilde{V}_n^{(s)'}(z',s))_q dz' \quad (2.15)$$

For a + wave (i.e., a wave propagating in the +z direction), let us assume that $(\tilde{V}_n(0,s))_+$ is specified, then Equation 2.15 gives

$$(\tilde{V}_n(z,s))_+ = \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))z\right\} \cdot (\tilde{V}_n(0,s))_+ + \int_0^z \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))[z - z']\right\} \cdot (\tilde{V}_n^{(s)'}(z',s))_+ dz' \tag{2.16}$$

Similarly for a - wave with $(\tilde{V}_n(L,s))_-$ assumed specified, we have

$$(\tilde{V}_n(z,s))_- = \exp\left\{(\tilde{\gamma}_{c_{n,m}}(s))[z - L]\right\} \cdot (\tilde{V}_n(L,s))_- + \int_L^z \exp\left\{(\tilde{\gamma}_{c_{n,m}}(s))[z - z']\right\} \cdot (\tilde{V}_n^{(s)'}(z',s))_- dz' \tag{2.17}$$

These results illustrate that the + wave depends only on the left boundary condition and the - wave depends only on the right boundary condition in a very compact way.

2.2 TERMINATION CONDITION OF A SINGLE SECTION OF THE LINE (TUBE)

A transmission line is usually terminated at the two ends $z = 0$ and $z = L$. The termination could be a lumped impedance, a distributed network, open circuit or short circuit. If sources are included, these conditions can be represented by a generalized Thévenin equivalent network or a generalized Norton equivalent network.

Passive terminations can be specified as an impedance matrix $(\tilde{Z}_{T_{n,m}}(z,s))$ or an admittance matrix $(\tilde{Y}_{T_{n,m}}(z,s))$, where $z = 0$ or L. The terminating conditions can be specified by scattering matrices $(\tilde{S}_{n,m}(z,s))$, where $z = 0$ or L. Consider at $z = L$ (see Fig. 2.2); let the incoming waves be designated by a superscript - and the outgoing waves +. The scattering matrix is defined by

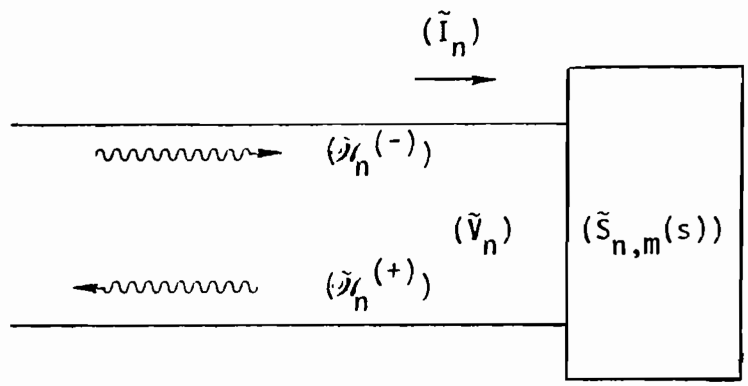

Figure 2.2. Incoming and outgoing wave at a junction

$$(\tilde{\mathscr{V}}^{(+)}(s)) = (\tilde{S}_{n,m}(z,s)) \cdot (\tilde{\mathscr{V}}_n^{(-)}(s)) \tag{18}$$

For the case illustrated in Figure 2, one observes that if this termination is taken as $z = L$, then

$$\begin{aligned}(\tilde{\mathscr{V}}_n^{(+)}(s)) &= (\tilde{V}_n(L,s))_- \\ (\tilde{\mathscr{V}}_n^{(-)}(s)) &= (\tilde{V}_n(L,s))_+\end{aligned} \tag{19}$$

And if the termination is taken as $z = 0$, then

$$\begin{aligned}(\tilde{\mathscr{V}}_n^{(+)}(s)) &= (\tilde{V}_n(0,s))_+ \\ (\tilde{\mathscr{V}}_n^{(-)}(s)) &= (\tilde{V}_n(0,s))_-\end{aligned} \tag{20}$$

One can then rewrite Equation 18 for $z = 0$ and $z = L$ as

$$(\tilde{V}_n(L,s))_- = (\tilde{S}_{n,m}(L,s)) \cdot (\tilde{V}_n(L,s))_+ \tag{21}$$

$$(\tilde{V}_n(0,s))_+ = (\tilde{S}_{n,m}(0,s)) \cdot (\tilde{V}_n(0,s))_- \tag{22}$$

which in this terminating case is the same as the definition of a reflection matrix and these are given by the following relations:

$$(\tilde{S}_{n,m}(L,s)) = \left[(\tilde{Z}_{T_{n,m}}(L,s)) + (\tilde{Z}_{c_{n,m}}(s))\right]^{-1} \cdot \left[(\tilde{Z}_{T_{n,m}}(L,s)) - (\tilde{Z}_{c_{n,m}}(s))\right] \tag{23}$$

$$(\tilde{S}_{n,m}(0,s)) = \left[(\tilde{Z}_{T_{n,m}}(0,s)) + (\tilde{Z}_{c_{n,m}}(s))\right]^{-1} \cdot \left[(\tilde{Z}_{T_{n,m}}(0,s)) - (\tilde{Z}_{c_{n,m}}(s))\right] \tag{24}$$

The scattering matrices in Equations 23 and 24 can also be represented in terms of the characteristic-admittance matrix and the load-admittance matrix as

$$(\tilde{S}_{n,m}(L,s)) = \left[(\tilde{Y}_{c_{n,m}}(s)) + (\tilde{Y}_{T_{n,m}}(L,s))\right]^{-1} \cdot \left[(\tilde{Y}_{c_{n,m}}(s)) - (\tilde{Y}_{T_{n,m}}(L,s))\right] \tag{2.25}$$

$$(\tilde{S}_{n,m}(0,s)) = \left[(\tilde{Y}_{c_{n,m}}(s)) + (\tilde{Y}_{T_{n,m}}(0,s))\right]^{-1} \cdot \left[(\tilde{Y}_{c_{n,m}}(s)) - (\tilde{Y}_{T_{n,m}}(0,s))\right] \tag{2.26}$$

Having defined the general transmission-line equations and termination conditions for a uniform multiconductor transmission line, we shall now consider multitube multiconductor transmission-line networks. Before deriving the BLT equation, we shall first discuss the scattering supermatrix for a general network.

2.3 SCATTERING SUPERMATRIX

The concept of scattering matrices introduced in the previous section for a terminated tube is extended here for junctions where more than one tube is connected. Collections and suitable ordering of scattering matrices at all junctions of the transmission-line network form a scattering supermatrix.

a. Junction scattering supermatrix

Consider the νth junction, J_ν, with tube ends denoted by $J_{\nu;r}$ with index r denoting the rth tube. Let this junction be characterized by an impedance matrix

$$(\tilde{Z}_{n,m}(s))_\nu = (\tilde{Y}_{n,m}(s))^{-1} \tag{2.27}$$

The junction scattering matrix is defined so that

$$(\tilde{V}_n(s))_{\nu,+} = (\tilde{S}_{n,m}(s))_\nu \cdot (\tilde{V}_n(s))_{\nu,-}$$

where the subscripts + and - refer to the aggregate of respectively outgoing and incoming waves on the various tubes in the form of combined voltage vectors.

In the supermatrix form partition according to waves on the r_ν tube ends connected to J_ν as

$$((\tilde{V}_n^{(0)}(s))_r)_\nu = (\tilde{Z}_{n,m}(s)_{r,r'})_\nu \odot (\tilde{I}_n^{(0)}(s))_\nu$$

$$((\tilde{Y}_{n,m}(s))_{r,r'})_\nu \equiv ((\tilde{Z}_{n,m}(s))_{r,r'})_\nu^{-1} \tag{2.28}$$

where

$$(\tilde{V}_n^{(0)}(s))_{r;\nu} \ , \ (\tilde{I}_n^{(0)}(s))_{r;\nu}$$

$$r = 1,2,\ldots,r_\nu$$

are the voltage and current vectors on the rth tube ends at J_ν with current convention into J_ν.

The tube associated with the rth tube end at J_ν has characteristic impedance and admittance matrices which can be put in supermatrix form for J_ν as

$$((\tilde{Z}_{c_{n,m}}(s))_{r,r'})_\nu \equiv \text{tube-end characteristic-impedance supermatrix for } J_\nu$$

$$((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_\nu \equiv \text{tube-end characteristic-admittance supermatrix for } J_\nu \tag{2.29}$$

where

$$(\tilde{Z}_{c_{n,m}}(s))_{r,r';\nu} \equiv \begin{cases} \text{characteristic-impedance matrix for rth tube end at } J_\nu \text{ for } r = r' \\ (0_{n,m}) \text{ for } r \neq r' \end{cases}$$

$$(\tilde{Y}_{c_{n,m}}(s))_{r,r';\nu} \equiv \begin{cases} \text{characteristic-admittance matrix for rth tube end at } J_\nu \text{ for } r = r' \\ (0_{n,m}) \text{ for } r \neq r' \end{cases}$$

$$(\tilde{Y}_{c_{n,m}}(s))_{r,r;\nu} = (\tilde{Z}_{c_{n,m}}(s))^{-1}_{r,r;\nu} \tag{2.30}$$

The impedance and admittance supermatrices for the tube ends at a given junction are block diagonal and may be represented in terms of the direct sum $\oplus$ as

$$((\tilde{Z}_{c_{n,m}}(s))_{r,r'})_\nu \equiv (\tilde{Z}_{c_{n,m}}(s))_{1,1;\nu} \oplus (\tilde{Z}_{c_{n,m}}(s))_{2,2;\nu} \oplus \cdots \oplus (\tilde{Z}_{c_{n,m}}(s))_{r_\nu,r_\nu;\nu}$$

$$\equiv \bigoplus_{\nu=1}^{r_\nu} (Z_{c_{n,m}}(s))_{r,r;\nu}$$

$$((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_\nu \equiv (\tilde{Y}_{c_{n,m}}(s))_{1,1;\nu} \oplus (\tilde{Y}_{c_{n,m}}(s))_{2,2;\nu} \oplus \cdots \oplus (\tilde{Y}_{c_{n,m}}(s))_{r_\nu,r_\nu;\nu}$$

$$\equiv \bigoplus_{\nu=1}^{r_\nu} (\tilde{Y}_{c_{n,m}}(s))_{r,r;\nu} \qquad (2.31)$$

The scattering supermatrix for J_ν is defined by

$$((\tilde{V}_n(s))_r)_{\nu,+} \equiv ((\tilde{S}_{n,m}(s))_{r,r'})_\nu : ((\tilde{V}_n(s))_r)_{\nu,-}$$

$$((\tilde{V}_n(s))_r)_{\nu,+} \equiv ((\tilde{V}_n^{(0)}(s))_r)_\nu - ((\tilde{Z}_{c_{n,m}}(s))_{r,r'})_\nu : ((\tilde{I}_n^{(0)}(s))_r)_\nu$$

$\equiv$ outgoing wave supervector at J_ν

$$((\tilde{V}_n(s))_r)_{\nu,-} \equiv ((\tilde{V}_n^{(0)}(s))_r)_\nu + ((\tilde{Z}_{c_{n,m}}(s))_{r,r'})_\nu : ((\tilde{I}_n^{(0)}(s))_r)_\nu$$

$\equiv$ incoming wave supervector at J_ν (2.32)

By solving Equations 2.28 and 2.32 we can obtain the junction scattering supermatrix as (Ref. 2)

$$((\tilde{S}_{n,m}(s))_{r,r'})_\nu = \left[((\tilde{Z}_{n,m}(s))_{r,r'})_\nu : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_\nu + ((1_{n,m})_{r,r'})_\nu\right]^{-1}$$

$$: \left[((\tilde{Z}_{n,m}(s))_{r,r'})_\nu : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_\nu - ((1_{n,m})_{r,r'})_\nu\right]$$

$$= \left[((1_{n,m})_{r,r'})_\nu + ((\tilde{Z}_{c_{n,m}}(s))_{r,r'})_\nu : ((\tilde{Y}_{n,m}(s))_{r,r'})_\nu\right]^{-1}$$

$$: \left[((1_{n,m})_{r,r'})_\nu - ((\tilde{Z}_{c_{n,m}}(s))_{r,r'})_\nu : ((\tilde{Y}_{n,m}(s))_{r,r'})_\nu\right] \qquad (2.33)$$

b. Scattering supermatrix

The proper ordering of all the junction scattering matrices into one large matrix forms the system (or network) scattering supermatrix $((\tilde{S}_{n,m}(s))_{u,v})$. This supermatrix is a collection of the junction scattering matrices, which themselves are collections of individual tube scattering matrices. The latter are matrices containing reflection and transmission coefficients of individual wires within the tubes.

The wave-wave matrix $(W_{u,v})$ gives the structure of the scattering supermatrix since the scattering supermatrix is in general block sparse as

$$((\tilde{S}_{n,m}(s))_{u,v}) = ((0_{n,m})_{u,v}) \quad \text{for } W_{u,v} = 0 \tag{2.34}$$

We form the network elementary scattering matrices as

$$(\tilde{S}_{n,m}(s))_{u,v} \equiv \begin{cases} (\tilde{S}_{n,m}(s))_{r,r';\nu} & \text{for } \nu_1 = \nu_2 = \nu \text{ or } W_v \\ & \text{scattering into } W_u \text{ at } J_\nu \\ (0_{n,m}) = (0_{n,m})_{u,v} & \text{for } \nu_1 \neq \nu_2 \text{ or } W_v \text{ not} \\ & \text{scattering into } W_u \end{cases} \tag{2.35}$$

The wave-wave matrix is defined as

$$W_{u,v} = \begin{cases} 1 \text{ for } \nu_1 = \nu_2 = \nu \text{ and } W_v \text{ scattering into } W_u \text{ at } J_\nu \\ 0 \text{ for } \nu_1 \neq \nu_2 \text{ or } W_v \text{ not scattering into } W_u \end{cases} \tag{2.36}$$

The scattering supermatrix is $N_W \times N_W$ in terms of the u,v indices, i.e.,

$$u,v = 1,2,\ldots,N_W \tag{2.37}$$

where N_W is equal to twice the number of tubes. The elementary scattering matrices $(\tilde{S}_{n,m}(s))_{u,v}$ are $N_u \times N_v$, i.e.,

$$\begin{aligned} n &= 1,2,\ldots,N_u \\ m &= 1,2,\ldots,N_v \end{aligned} \tag{2.38}$$

where

N_u = number of conductors (not including reference) on the tube with uth wave (2.39)

and likewise for N_v.

As a special case, if there are no selftubes (with both ends connected to the same junction), then

$$W_{u,u} = 0 \quad \text{for } u = 1,2,\ldots,N_W \text{ for no selftubes}$$
$$(\tilde{S}_{n,m}(s))_{u,u} = (0_{n,m})_{u,u} \quad \text{for } n,m = 1,2,\ldots,N_u \text{ (square)} \tag{2.40}$$

2.4 DEFINITIONS OF SOME IMPORTANT SUPERMATRIX AND SUPERVECTOR QUANTITIES

This section takes the results for the combined voltages on a tube and separates them into wave variables for the network. The resulting equation for a general combined voltage wave W_u is used to relate the combined voltage waves at both ends of the tube with the sources along the tube. Each term is generalized to a form appropriate to the transmission-line network, i.e., supermatrices and supervectors, by aggregating the results for all W_u for $u = 1,2,\ldots N_W$.

Let us identify the two waves on the tube with the two waves of the transmission-line network, say W_u and W_v.

Consider the + wave; call this W_u and set the coordinate and dimension variable as

$$L_u \equiv L \equiv \text{length of path for } W_u$$
$$z_u \equiv z \equiv \text{wave coordinate for } W_u$$
$$0 \leq z_u \leq L_u \tag{2.41}$$
$$N_u \equiv N \equiv \text{number of conductors (less reference) on tube and dimension of vectors for } W_u$$

The wave and source conventions are then

$$(\tilde{V}_n(z_u,s))_u \equiv (\tilde{V}_n(z,s))_+ = (\tilde{V}_n(z_u,s)) + (\tilde{Z}_{c_{n,m}}(s))_u \cdot (\tilde{I}_n(z_u,s))$$

$$\equiv \text{combined voltage for } W_u$$

$$(\tilde{V}_n^{(s)'}(z_u,s))_u \equiv (\tilde{V}_n^{(s)'}(z,s))_+ = (\tilde{V}_n^{(s)'}(z_u,s)) + (\tilde{Z}_{c_{n,m}}(s))_u \cdot (\tilde{I}_n^{(s)'}(z_u,s))$$

$$\equiv \text{combined voltage source per unit length for } W_u$$

$$(\tilde{Z}_{c_{n,m}}(s))_u \equiv (\tilde{Y}_{c_{n,m}}(s))_u^{-1} \equiv \text{characteristic-impedance matrix for } W_u$$

$$(\tilde{\gamma}_{c_{n,m}}(s))_u \equiv (\tilde{\gamma}_{c_{n,m}}(s)) \equiv \text{propagation matrix for } W_u \tag{2.42}$$

The combined voltage vector for the wave W_u is given by

$$(\tilde{V}_n(z_u,s))_u = \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))_u z_u\right\} \cdot (\tilde{V}_n(0,s))_u + \int_0^{z_u} \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))_u\left[z_u - z_u'\right]\right\} \cdot (\tilde{V}_n^{(s)'}(z_u',s))_u dz_u' \tag{2.43}$$

Similarly, the combined voltage vector for the wave W_v can be defined. In Equation 2.43 we have the combined voltage at any z_u in terms of the value (boundary condition) at $z_u = 0$. Setting $z_u = L_u$, we introduce the boundary value there as giving

$$(\tilde{V}_n(L_u,s))_u = \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))_u L_u\right\} \cdot (\tilde{V}_n(0,s))_u + \int_0^{L_u} \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))_u\left[L_u - z_u'\right]\right\} \cdot (\tilde{V}_n^{(s)'}(z_u',s))_u dz_u' \tag{2.44}$$

This evidently relates $(\tilde{V}_n(0,s))_u$ which is an outgoing wave from the junction at $z_u = 0$, to $(\tilde{V}_n(L_u,s))_u$ which is an incoming wave to the junction at $z_u = L_u$.

As a matter of convention, let all the sources be considered as being present in the tubes instead of at the junctions. If one has a junction with an equivalent circuit containing sources, then the sources can be moved just across the terminals into the tube, a movement of zero distance.

a. Propagation characteristic supermatrix

Considering the various terms in Equation 2.44, let us first aggregate all the propagation terms not associated with the sources into a block diagonal propagation supermatrix as

$$((\tilde{\Gamma}_{n,m}(s))_{u,v})$$

$$\equiv \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))_1 L_1\right\} \oplus \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))_2 L_2\right\} \oplus \cdots \oplus \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))_{N_W} L_{N_W}\right\}$$

$$\equiv \bigoplus_{u=1}^{N_W} \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))_u L_u\right\} \tag{2.45}$$

$$\equiv \text{propagation supermatrix}$$

where the elementary matrices (blocks) are given by

$$(\tilde{\Gamma}_{n,m}(s))_{u,v} = \begin{cases} \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))_u L_u\right\} & \text{for } u = v \\ (0_{n,m}) & \text{for } u \neq v \end{cases}$$

$$= 1_{u,v} \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))_u L_u\right\} \tag{2.46}$$

b. Source supervector and combined voltage supervector

From Equation 2.44 let us define a source vector for W_u in traveling from $z_u = 0$ to $z_u = L_u$ as

$$(\tilde{V}_n^{(s)}(s))_u \equiv \int_0^{L_u} \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))_u\left[L_u - z_u'\right]\right\} \cdot (\tilde{V}_n^{(s)'}(z_u',s)_u dz_u' \tag{2.47}$$

The source supervector is then merely

$$((\tilde{V}_n^{(s)}(s))_u) = \left(\int_0^{L_u} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_u[L_u - z_u']\} \cdot (\tilde{V}_n^{(s)'}(z_u',s))_u dz_u' \right) \tag{2.48}$$

For completeness we have the aggregate of combined voltage vectors in Equation 2.43 as

$$\begin{aligned} ((\tilde{V}_n(0,s))_u) &\equiv \text{combined voltage supervector of outgoing waves at the junctions} \\ ((\tilde{V}_n(L_u,s))_u) &\equiv \text{combined voltage supervector of incoming waves at junctions} \end{aligned} \tag{2.49}$$

2.5 BLT EQUATION

Combining the results of the previous derivations we can write the BLT equation for the description of the transmission-line network. In Reference 2 the BLT equation was derived for the combined voltage waves leaving the junctions. Here, we shall derive the BLT equation for four variables, namely, combined voltage waves leaving the junctions, combined voltage waves entering the junctions, the total voltage vectors at the junctions, and the total current vectors at the junctions. We begin with the scattering supermatrix which relates the incoming waves to the outgoing waves as

$$(\tilde{V}_n(0,s))_u = ((\tilde{S}_{n,m}(s))_{u,v}) : ((\tilde{V}_n(L_u,s)_u) \tag{2.50}$$

Next, relate the incoming waves at the output ends of the tubes ($z_u = L_u$) to the same waves at the input end of the same tubes ($z_u = 0$), albeit at different junctions in general. Writing Equation 2.44 in supermatrix form we have

$$((\tilde{V}_n(L_u,s))_u) = ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{V}_n(0,s))_u) + ((\tilde{V}_n^{(s)}(s))_u) \tag{2.51}$$

Combining Equations 2.50 and 2.51 we have

$$((\tilde{V}_n(0,s))_u) = ((\tilde{S}_{n,m}(s))_{u,v}) : ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{V}_n(0,s))_u)$$

$$+ ((\tilde{S}_{n,m}(s))_{u,v}) : ((\tilde{V}_n^{(s)}(s))_u) \qquad (2.52)$$

That is rearranged by use of the supermatrix identity as

$$\left[((1_{n,m})_{u,v}) - ((\tilde{S}_{n,m}(s))_{u,v}) : ((\tilde{\Gamma}_{n,m}(s))_{u,v})\right] : ((\tilde{V}_n(0,s))_u)$$

$$= ((S_{n,m}(s))_{u,v}) : ((\tilde{V}_n^{(s)}(s))_u) \qquad (2.53)$$

This can be rearranged to obtain

$$((\tilde{V}_n(0,s))_u) = \left[((1_{n,m})_{u,v}) - ((\tilde{S}_{n,m}(s))_{u,v}) : ((\tilde{\Gamma}_{n,m}(s))_{u,v}\right]^{-1}$$

$$: ((\tilde{S}_{n,m}(s))_{u,v}) : ((\tilde{V}_n^{(s)}(s))_u) \qquad (2.54)$$

This is one form of the BLT equation with unknowns taken as the combined voltage waves leaving the junctions. Similarly, the BLT equation can be obtained with unknowns taken as the combined voltage waves entering the junctions. By rearranging Equation 2.51, we obtain

$$((\tilde{V}_n(0,s))_u) = ((\tilde{\Gamma}_{n,m}(s))_{u,v})^{-1} : ((\tilde{V}_n(L_u,s))_u) - ((\tilde{\Gamma}_{n,m}(s))_{u,v})^{-1} : ((\tilde{V}_n^{(s)}(s))_u) \qquad (2.55)$$

Combining Equations 2.50 and 2.54 we obtain

$$\left[((\tilde{\Gamma}_{n,m}(s))_{u,v})^{-1} - ((\tilde{S}_{n,m}(s))_{u,v})\right] : ((V_n(L_u,s))_u)$$

$$= ((\tilde{\Gamma}_{n,m}(s))_{u,v})^{-1} : ((\tilde{V}_n^{(s)}(s))_u) \qquad (2.56)$$

By rearranging Equation 2.56, we obtain

$$((\tilde{V}_n(L_u,s))_u) = \left[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})\right]^{-1} : ((\tilde{V}_n^{(s)}(s))_u) \tag{2.57}$$

This is another form of the BLT equation with unknown taken as the combined voltage waves entering the junctions. From Equations 2.54 and 2.57 we can derive the BLT equation in terms of the total voltage and total current supervectors. Note the order of multiplication of scattering and propagation supermatrices in Equations 2.54 and 2.57. We shall rearrange Equation 2.54 so that the order of multiplication of matrices is as that in Equation 2.57. Equation 2.54 can be rearranged to give

$$((\tilde{V}_n(0,s))_u) = ((\tilde{S}_{n,m}(s))_{u,v}) : \left[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})\right]^{-1} : ((\tilde{V}_n^{(s)}(s))_u) \tag{2.58}$$

From Equations 2.13 and 2.14, we can write the supervectors for voltages and currents at the junctions in terms of the combined voltage waves leaving and entering junctions as

$$((\tilde{V}_n^{(0)}(s))_u) = \frac{1}{2}\left[((\tilde{V}_n(0,s))_u) + ((P_{n,m})_{u,v}) : ((\tilde{V}_n(L_u,s))_u\right] \tag{2.59}$$

$$((\tilde{I}_n^{(0)}(s))_u) = \frac{1}{2}\ ((\tilde{Y}_{c_{n,m}}(s))_{u,v}) : \left[((\tilde{V}_n(0,s))_u) - ((P_{n,m})_{u,v}) : ((\tilde{V}_n(L_u,s))_u)\right] \tag{2.60}$$

where $\tilde{V}_n^{(0)}(s)$ and $\tilde{I}_n^{(0)}(s)$ are voltage and current on the nth conductor in the tube containing the uth wave at the junction from which the uth wave leaves.

In Equations 2.59 and 2.60 we have introduced a permulation supermatrix $((P_{n,m})_{u,v})$ in order to sum the appropriate outgoing and incoming waves at the junctions. The permulation supermatrix has blocks with the following properties:

$$(P_{n,m})_{u,v} = \begin{cases} (1_{n,m})_{u,v} & \text{if } W_u \text{ and } W_v \text{ are on the same tube and } u \neq v \text{ (noting that this is a square matrix)} \\ (0_{n,m})_{u,v} & \text{if } W_u \text{ and } W_v \text{ are not on the same tube or } u = v \end{cases}$$

and

$$u,v = 1,2,\ldots,N_W$$

$$n = 1,2,\ldots,N_u$$

$$m = 1,2,\ldots,N_v \tag{2.61}$$

Only one block matrix $(P_{n,m})_{u,v}$ is equal to $(1_{n,m})$ on any row or column with respect to indices u or v. Thus $((P_{n,m})_{u,v})$ is an orthogonal supermatrix.

Substituting Equations 2.57 and 2.58 into Equations 2.59 and 2.60 we obtain

$$((\tilde{V}_n^{(0)}(s))_u) = \frac{1}{2}\left[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v}\right] \\ \odot \left[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) \odot ((\tilde{S}_{n,m}(s))_{u,v}\right]^{-1} \odot ((\tilde{V}_n^{(s)}(s))_u) \tag{2.62}$$

$$((\tilde{I}_n^{(0)}(s))_u) = \frac{1}{2}((\tilde{Y}_{c_{n,m}}(s))_{u,v}) \odot \left[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})\right] \\ \odot \left[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) \odot ((\tilde{S}_{n,m}(s))_{u,v})\right]^{-1} \odot ((\tilde{V}_n^{(s)}(s))_u) \tag{2.63}$$

Equations 2.61 and 2.62 are two forms of the BLT equation in terms of the voltage and current supervectors at the junctions.

III. BOUNDS FOR SIGNALS ON A MULTICONDUCTOR CABLE NETWORK

Having derived the BLT equations for general multiconductor-line networks, we can now establish upper and lower bounds on combined voltages, voltages, and currents, using the norm concept discussed in Appendix A. The BLT equations give voltages, currents, and combined voltages at the junctions. From these one can find voltages and currents essentially everywhere, including at the junction terminals and at arbitrary positions on the tubes. However, we shall limit ourselves to the junctions for the purpose of establishing bounds.

3.1 BOUNDS ON COMBINED VOLTAGES, VOLTAGES, AND CURRENTS

Taking the norm of both sides of Equation 2.58 we get

$$\|((\tilde{V}_n(0,s))_u)\| = \|((\tilde{S}_{n,m}(s))_{u,v}) : [((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1} : ((\tilde{V}_n^{(s)}(s))_u)\| \tag{3.1}$$

Using Equation A6 in Equation 3.1, we obtain

$$\|((\tilde{V}_n(0,s))_u)\| \leq \|((\tilde{S}_{n,m}(s))_{u,v})\| \; \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v}]^{-1}\| : \|((\tilde{V}_n^{(s)}(s))_u\| \tag{3.2}$$

Rearranging Equation 2.58, we can write

$$[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})] : ((\tilde{S}_{n,m}(s))_{u,v})^{-1} : ((\tilde{V}_n(0,s))_u) = ((\tilde{V}_n^{(s)}(s))_u) \tag{3.3}$$

Taking the norm of both sides of Equation 3.3 and using Equation A6, we obtain

$$\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})] : ((S_{n,m}(s))_{u,v})^{-1}\| \cdot \|((\tilde{V}_n(0,s))_u)\| \geq \|((\tilde{V}_n^{(s)}(s))_u)\|$$

or

$$\|((\tilde{V}_n(0,s))_u)\| \geq \frac{\|((\tilde{V}_n^{(s)}(s))_u)\|}{\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})] : ((\tilde{S}_{n,m}(s))_{u,v})^{-1}\|} \tag{3.4}$$

Equations 3.2 and 3.4 give upper and lower bounds on the norm of the combined voltage supervector for all waves leaving junctions, in terms of the norms of other quantities, such as combined voltage source waves, scattering supermatrix, and propagation supermatrix.

Similarly, we can obtain upper and lower bounds on the norm of combined voltage supervector for waves entering junctions. Taking the norm of both sides of Equation 2.57 and using Equation A6, we get

$$\|((\tilde{V}_n(L_u))_u)\| \leq \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\| \cdot \|((\tilde{V}_n^{(s)}(s))_u)\| \tag{3.5}$$

and

$$\|((\tilde{V}_n(L_u,s))_u)\| \geq \frac{\|((\tilde{V}_n^{(s)}(s))_u)\|}{\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((S_{n,m}(s))_{u,v})]\|} \tag{3.6}$$

Similarly, from Equations 2.62 and 2.63, the upper and lower bounds on the norms of voltage and current supervectors are given by

$$\|((\tilde{V}_n^{(0)}(s))_u)\| \le \frac{1}{2} \|[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]\|$$

$$\cdot \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\| \|((\tilde{V}_n^{(s)}(s))_u)\| \quad (3.7)$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\| \le \frac{1}{2} \|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\| \; \|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]\|$$

$$\cdot \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\| \; \|((\tilde{V}_n^{(s)}(s))_u)\| \quad (3.8)$$

$$\|((\tilde{V}_n^{(0)}(s))_u)\| \ge \frac{1}{2} \|((\tilde{V}_n(s))_u)\| \Big/ \Big[\| [((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]$$

$$: [((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]^{-1}\|\Big] \quad (3.9)$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\| \ge \frac{1}{2} \|((\tilde{V}_n^{(s)}(s))_u)\| \Big/ \Big[\| [((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]$$

$$: [((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m}))_{u,v})]^{-1} : ((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|\Big] \quad (3.10)$$

Equations 3.7 and 3.8 give upper bounds on the voltages and currents, respectively, and Equations 3.9 and 3.10 give lower bounds on voltages and currents, respectively.

Before evaluating these upper and lower bounds, we shall illustrate what these bounds mean. The upper and lower bounds defined in Equations 3.2, 3.4, and 3.5 through 3.10 are upper and lower bounds on the norm of vectors. In Appendix A, 1, 2, and ∞ norms for vectors and matrices are defined. The above equations are valid for any norm as long as they are consistent on both sides of the equations. For the purpose of bounding signal levels, ∞ norm for vectors is most appropriate, for it gives the magnitude of the largest element of a vector. For a voltage or current vector at a junction or at any point along the line, the ∞ norm gives the magnitude of the maximum conductor voltage or current (pin voltage or current at terminations). Thus an upper and lower bound on the ∞ norm of a vector gives, respectively, an upper and lower bound on the magnitude of the largest element of the vector. The lower bound should not be confused with the magnitude of the smallest element of the vector.

Since the 2 norm of a matrix is obtained from the knowledge of its eigenvalues, it is possible to evaluate it from the characteristic properties of the matrix and, therefore, we shall use this norm for matrices in the evaluation of upper and lower bounds. Using ∞ and 2 norms and Equations A64 and A94 through A99, we can write upper and lower bounds for combined voltages, voltages, and currents as follows.

Using 2 norms on both sides of Equation 3.2 and substituting Equation A63, we get

$$\|((\tilde{V}_n(0,s))_u)\|_\infty \le \|((\tilde{S}_{n,m}(s))_{u,v})\|_2$$

$$\cdot \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2 \; \|((\tilde{V}_n^{(s)}(s))_u)\|_2 \tag{3.11}$$

Substituting Equation A65 into Equation 3.11 we get

$$\|((\tilde{V}_n(0,s))_u)\|_\infty \le \sqrt{N_s}\; \|((\tilde{S}_{n,m}(s))_{u,v})\|_2$$

$$\cdot \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2 \; \|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \tag{3.12}$$

Where N_s is the dimension (numbers of components) of the source supervector.

A lower bound on the combined voltages for all waves leaving junctions is obtained by using ∞ norms on both sides of Equation 3.4 as

$$\|((\tilde{V}_n(0,s))_u)\|_\infty \geq \|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \Big/ \Big[\|[((1_{n,m})_{u,v})-((\tilde{\Gamma}_{n,m}(s))_{u,v}):((\tilde{S}_{n,m}(s))_{u,v})]$$

$$:((S_{n,m}(s))_{u,v})^{-1}\|_\infty\Big] \qquad (3.13)$$

Substituting Equations A6 and A98 into Equation 3.13 we get

$$\|((\tilde{V}_n(0,s))_u)\|_\infty \geq \|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \Big/ \Big[\sqrt{N_s}\|[((1_{n,m})_{u,v})-((\tilde{\Gamma}_{n,m}(s))_{u,v}):((\tilde{S}_{n,m}(s))_{u,v})]\|_2$$

$$\cdot \|((\tilde{S}_{n,m}(s))_{u,v})^{-1}\|_2\Big] \qquad (3.14)$$

where N_s is the size of the supermatrices in the demoninator.

An upper bound on the combined voltages for all waves entering junctions is obtained from Equation 3.5 using Equation A63 as

$$\|((\tilde{V}_n(L_u,s))_u)\|_\infty \leq \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2$$

$$\cdot \|((\tilde{V}_n^{(s)}(s))_u)\|_2 \qquad (3.15)$$

Substituting Equation A65 into Equation 3.15 we get

$$\|((\tilde{V}_n(L_u,s))_u)\|_\infty \leq \sqrt{N_s}\ \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2$$

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \qquad (3.16)$$

Similarly, a lower bound on the combined voltages waves for all waves entering junctions obtained from Equation 3.6 using Equation A98 as

$$\|((\tilde{V}_n(L_u,s))_u)\|_\infty \geq \frac{\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]\|_\infty}$$

$$\geq \frac{\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{\sqrt{N_s}\,\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]\|_2} \tag{3.17}$$

Similarly, an upper and lower bound on the voltages and currents at the junctions is obtained from Equations 3.7 through 3.10 using Equations A6, A63, A65 and A98 as

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty \leq \frac{1}{2}\,\|[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]\|_2$$

$$\cdot\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2\,\|((\tilde{V}_n^{(s)}(s))_u)\|_2$$

$$\leq \frac{1}{2}\sqrt{N_s}\,\|[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]\|_2$$

$$\cdot\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2\,\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \tag{3.18}$$

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty \geq \frac{1}{2}\,\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \Big/ \Big[\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]$$

$$: [((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]^{-1}\|_\infty\Big]$$

$$\geq \frac{1}{2}\,\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \Big/ \Big[\sqrt{N_s}\,\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]\|_2$$

$$\cdot\,\|[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]^{-1}\|_2\Big] \tag{3.19}$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \le \frac{1}{2} \|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\|_2 \; \|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]\|_2$$

$$\cdot \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2 \|((\tilde{V}_n^{(s)}(s))_u)\|_2$$

$$\le \frac{1}{2}\sqrt{N_s} \; \|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\|_2 \|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]\|_2$$

$$\cdot \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2 \|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \tag{3.20}$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \ge \frac{1}{2} \|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \Big/ \Big[\| [((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]$$

$$: [((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]^{-1} \; : \; ((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|_\infty\Big]$$

$$\ge \frac{1}{2} \|((\tilde{V}_n^{(s)}(s))_u)\|_\infty$$

$$\Big/ \sqrt{N_s} \; \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]\|_2$$

$$\cdot \; \|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]^{-1}\|_2 \|((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|_2\Big] \tag{3.21}$$

So far we have derived relations for upper and lower bounds for combined voltages, voltages, and currents in terms of norms of other parameters, such as the propagation and scattering supermatrices, the characteristic impedance or admittance supermatrix, and the source supervector. Thus, to establish upper and lower bounds on combined voltages, voltages, and currents, we have to first establish bounds on the parameters. To establish bounds on the parameters and the sources for a general multiconductor cable network is very difficult. Furthermore, if such bounds could be established on parameters, the resulting bounds on the voltages and currents may be unrealistic. In order to get reasonable bounds we shall consider some special canonical configurations of a multiconductor cable network in Sections IV and V.

The simplest of these configurations is a uniform section of a multiconductor transmission line in a homogeneous medium terminated at both ends and excited by an external field or voltages and currents at terminations. To make the transmission line configuration more complex, we can add a branch to a uniform section of the line. The branched line will serve the purpose of illustrating the procedure for calculation of bounds for cable networks with junctions. These two configurations will be considered in Sections IV and V.

3.2 BOUNDS IN TERMS OF BULK CURRENT

In evaluating the EMP vulnerability of a system, the bounds which are of most interest are the bounds on pin currents in terms of the bulk current. The bulk current on a multiconductor transmission line is defined as the algebraic sum of all the wire currents at a given cross section. This concept of pin current bounding in terms of the bulk current has tremendous implications for aircraft testing. If such a bound can be established, then one only need to measure bulk currents on cables in an aircraft, thereby reducing the number of measurements by orders of magnitude. In this section we address the above problem.

Since the 1 norm of a vector is defined as the sum of the magnitudes of its components, and the bulk current is the algebraic sum of the wire current in a cable, then for current vector on a multiconductor line at a termination we have

$$\|(\tilde{I}_n^{(0)}(s))_{r;\nu}\|_1 \geq \left|\tilde{I}_B^{(0)}(s)_{r;\nu}\right| \tag{3.22}$$

where $(\tilde{I}_n^{(0)}(s))_{r;\nu}$ is the current vector for the rth tube at the νth junction, and $\tilde{I}_B^{(0)}(s)_{r;\nu}$ is the bulk current on the rth tube at the νth junction and is defined as $\tilde{I}_B^{(0)}(s)_{r;\nu} \equiv \sum_{n=1}^{N_u} \tilde{I}_n^{(0)}(s)_{r;\nu}$.

Writing Equation 3.22 for currents at all the junctions, we have

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_1 \geq \left| \sum_{r=1}^{N_u} \sum_{\nu=1}^{\nu} \tilde{I}_B^{(0)}(s)_{r;\nu} \right| \equiv |\tilde{I}_B| \tag{3.23}$$

where the right-hand side is the sum of all the bulk currents in all the tubes at all the junctions.

We shall now express the ∞ norm of the current supervector in terms of the total bulk current. This can be further decomposed in terms of bulk currents on tubes at various junctions for specific problems. Substituting Equation A63 into Equation 3.21 we obtain

$$\begin{aligned}\|((\tilde{I}_n^{(0)}(s))_u)\|_1 \geq \frac{1}{2} \|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \\ \Big/ \Big[\sqrt{N_s}\, \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]\|_2 \\ \cdot \|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]^{-1}\|_2 \|((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|_2\Big]\end{aligned} \tag{3.24}$$

and taking 1 norm of both sides of Equation 2.63 and then substituting Equations A6 and A94 into the result we get

$$\begin{aligned}\|((\tilde{I}_n^{(0)}(s))_u)\|_1 \leq \frac{1}{2} \sqrt{N_s}\, \|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\|_2 \|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]\|_2 \\ \cdot \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m})_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2 \|((\tilde{V}_n^{(s)}(s))_u)\|_1\end{aligned} \tag{3.25}$$

In Equations 3.24 and 3.25, 2 norms of matrices are used, since these can be computed from energy conservation. Dividing Equation 3.21 by Equation 3.25 and then substituting Equation 3.23 in the result, we obtain

$$\frac{\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty}{|\tilde{I}_B|}$$

$$\geq \|((\tilde{V}_n^{(s)}(s))_u\|_\infty / [N_s \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]\|_2$$

$$\cdot \|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]^{-1}\|_2 \; \|((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|_2$$

$$\cdot \; \|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\|_2 \; \|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]\|_2$$

$$\cdot \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2 \; \|((\tilde{V}_n^{(s)}(s))_u)\|_1]$$

(3.26)

Thus Equation 3.26 gives a lower bound on the ratio of the maximum pin current to the bulk current.

Similarly, dividing Equation 3.20 by Equation 3.24 we obtain

$$\frac{\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty}{\|((\tilde{I}_n^{(0)}(s))_u)\|_1}$$

$$\leq N_s \; \|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\|_2 \|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]\|_2$$

$$\cdot \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2 \|((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|_2$$

$$\cdot \|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]\|_2 \|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]^{-1}\|_2$$

(3.27a)

Also, from Equation A64

$$\frac{\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty}{\|((\tilde{I}_n^{(0)}(s))_u)\|_1} \leq 1 \qquad (3.27b)$$

In Equation 3.27 the upper bound is independent of sources. In Equations 3.26 and 3.27 the norms of matrices and their inverses occur in pairs. From Equation A56, the product of the norm of a matrix and the norm of its inverse is greater than or equal to 1.

From physical principles and Equation A65 the lower bound on the ratio of the maximum pin current to the bulk current is $1/N_s$. Thus, with the result in Equation A56 in mind, the lower bound in Equation 3.26 is not useful since it gives a lower bound which is less than $1/N_s$.

Equation 3.27 gives an upper bound on the ratio of the maximum pin current to the sum of the magnitudes of all the pin currents. Since we cannot substitute the 1 norm in the denominator with the bulk current, this bound is not very useful either. It is obvious from the above discussion that an upper bound on the ratio of the maximum pin current to the bulk current cannot be obtained analytically. However, it is seen easily that, in general, pin current is not bounded with respect to bulk current, since the bulk current in a cable can be zero while the individual pin currents are non-zero; for example, a two-wire cable excited in the differential mode has non-zero pin current and zero bulk current.

IV. BOUNDS FOR A UNIFORM SECTION OF A MULTICONDUCTOR TRANSMISSION LINE

In this section we shall consider a special case of a general multiconductor cable network, a uniform section of a multiconductor transmission line terminated at both ends. Two types of excitations will be considered. In the first type of excitation, the line is excited by an incident external field, and in the second type, the line is excited by voltage or current sources at the terminations.

Consider a multiconductor transmission line formed by N conductors plus a reference conductor or ground as shown in Figure 3. The line is assumed to be uniform along its length (z coordinate), but with arbitrary cross section. In general, the dielectric surrounding the line is inhomogeneous (e.g., cable made of insulated conductors having different geometries and dielectric materials).

The wave traveling in +z direction is denoted by wave W_1 or simply wave 1, and the wave traveling in -z direction as W_2 or wave 2, as shown in Figure 3. Then the combined voltage vectors for multiconductor transmission-line in Figure 3 are given by

$$((\tilde{V}_n(0,s))_u) = \begin{pmatrix} (\tilde{V}_n(0,s))_1 \\ (\tilde{V}_n(0,s))_2 \end{pmatrix} \tag{91}$$

$$((\tilde{V}_n(L_u,s))_u) = \begin{pmatrix} (\tilde{V}_n(L,s))_1 \\ (\tilde{V}_n(L,s))_2 \end{pmatrix} \tag{92}$$

where $(\tilde{V}_n(0,s))_1$ and $(\tilde{V}_n(0,s))_2$ are the waves leaving junctions at z = 0 and z = L, respectively, and $(\tilde{V}_n(L,s))_1$ and $(\tilde{V}_n(L,s))_2$ are the waves entering junctions at z = L and z = 0, respectively.

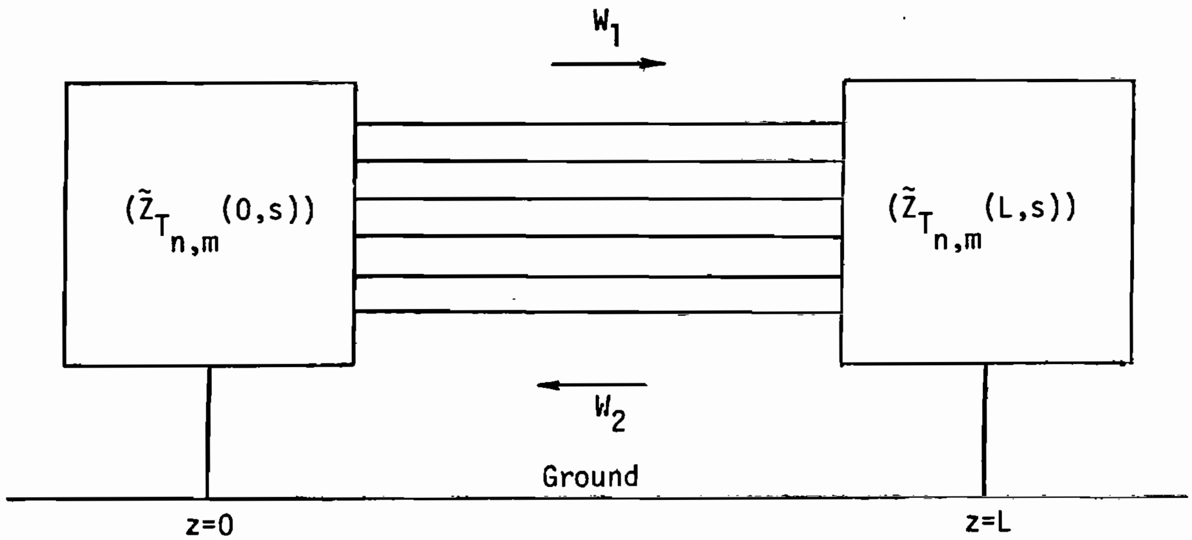

Figure 4.1. A multiconductor transmission line over a ground plane, terminated at both ends.

The waves leaving and entering junctions are related through scattering matrices as

$$(\tilde{V}_n(0,s))_1 = (\tilde{S}_{n,m}(s))_{1,2} \cdot (\tilde{V}_n(L,s))_2 \tag{4.3}$$

$$(\tilde{V}_n(0,s))_2 = (\tilde{S}_{n,m}(s))_{2,1} \cdot (\tilde{V}_n(L,s))_1 \tag{4.4}$$

where $(\tilde{S}_{n,m}(s))_{1,2}$ and $(\tilde{S}_{n,m}(s))_{2,1}$ are scattering matrices of junctions at $z = 0$ and $z = L$, respectively. The subscripts 1,2 and 2,1 indicate that the 2 wave is scattered into 1 wave and 1 wave is scattered into 2 wave, respectively. Combining Equations 4.3 and 4.4 and writing the scattering matrices in supermatrix form we have

$$\begin{pmatrix} (\tilde{V}_n(0,s))_1 \\ (\tilde{V}_n(0,s))_2 \end{pmatrix} = \begin{pmatrix} (0_{n,m})_{1,1} & (\tilde{S}_{n,m}(s))_{1,2} \\ (\tilde{S}_{n,m}(s))_{2,1} & (0_{n,m})_{2,2} \end{pmatrix} : \begin{pmatrix} (\tilde{V}_n(L,s))_1 \\ (\tilde{V}_n(L,s))_2 \end{pmatrix} \tag{4.5}$$

or

$$((\tilde{V}_n(0,s))_u) = ((\tilde{S}_{n,m}(s))_{u,v}) : ((\tilde{V}_n(L,s))_u) \tag{4.6}$$

where

$$((\tilde{S}_{n,m}(s))_{u,v}) = \begin{pmatrix} (0_{n,m}) & (\tilde{S}_{n,m}(s))_{1,2} \\ (\tilde{S}_{n,m}(s))_{2,1} & (0_{n,m}) \end{pmatrix} \tag{4.7}$$

$\equiv$ scattering supermatrix

$u = 1,2$

$v = 1,2$

$n = m = 1,2,\ldots,N$

$(\tilde{S}_{n,m}(0,s))_{1,2}$ and $(\tilde{S}_{n,m}(L,s))_{2,1}$ are the reflection coefficient matrices at $z = 0$ and $z = L$, respectively, and are given by Equations 2.23 through 2.26.

From Equation 2.43, we can write the relation between the 1 wave at $z = L$ in terms of the 1 wave at $z = 0$ as

$$(\tilde{V}_n(L,s))_1 = \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))L\} \cdot (\tilde{V}_n(0,s))_1 + \int_0^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))[L - z']\} \cdot (\tilde{V}_n^{(s)'}(z',s))_1 dz' \tag{4.8}$$

Note that

$$z_u = \begin{cases} z & \text{for } u = 1 \\ L-z & \text{for } u = 2 \end{cases}$$

Similarly, the 2 wave at $z = L$ can be expressed in terms of the 2 wave at $z = 0$

$$(\tilde{V}_n(L,s))_2 = \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))L\} \cdot (\tilde{V}_n(0,s))_2 - \int_0^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))z''\} \cdot (\tilde{V}_n^{(s)'}(z'',s))_2 dz'' \tag{4.9}$$

where $z'' = L - z'$. Combining Equations 4.8 and 4.9 we get

$$\begin{pmatrix} (\tilde{V}_n(L,s))_1 \\ (\tilde{V}_n(L,s))_2 \end{pmatrix} = \begin{pmatrix} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))L\} & (0_{n,m}) \\ (0_{n,m}) & \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))L\} \end{pmatrix} : \begin{pmatrix} (\tilde{V}_n(0,s))_1 \\ (\tilde{V}_n(0,s))_2 \end{pmatrix} + \begin{pmatrix} \int_0^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))[L - z']\} \cdot (\tilde{V}_n^{(s)'}(z',s))_1 dz' \\ -\int_0^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))z''\} \cdot (\tilde{V}_n^{(s)'}(z'',s))_2 dz'' \end{pmatrix} \tag{4.10}$$

We can write Equation 4.10 in supermatrix notation as

$$((\tilde{V}_n(L,s))_u) = ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{V}_n(0,s))_u) + ((\tilde{V}_n^{(s)}(s))_u) \tag{4.11}$$

where

$$((\tilde{\Gamma}_{n,m}(s))_{u,v}) = \begin{pmatrix} (\tilde{\Gamma}_{n,m}(s))_{1,1} & (0_{n,m}) \\ (0_{n,m}) & (\tilde{\Gamma}_{n,m}(s))_{2,2} \end{pmatrix} \tag{4.12}$$

$$(\tilde{\Gamma}_{n,m}(s))_{1,1} = (\tilde{\Gamma}_{n,m}(s))_{2,2} = (\tilde{\Gamma}_{n,m}(s)) = \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))L\} \tag{4.13}$$

$$((\tilde{V}_n^{(s)}(s))_u) = \begin{pmatrix} \int_o^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))[L - z']\} \cdot (\tilde{V}_n^{(s)'}(z',s))_1 dz' \\ -\int_o^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))z''\} \cdot (\tilde{V}_n^{(s)'}(z',s))_2 dz'' \end{pmatrix} \tag{4.14}$$

Substituting Equations 2.11 and 2.12 into Equation 4.14, we get the source supervector for the two waves in terms of the voltage and current source vectors as

$$((\tilde{V}_n^{(s)}(s))_u)$$
$$= \begin{pmatrix} \int_o^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))[L-z']\} \cdot [(\tilde{V}_n^{(s)'}(z',s)) + (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n^{(s)'}(z',s))]dz' \\ -\int_o^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))z''\} \cdot [(\tilde{V}_n^{(s)'}(z'',s)) - (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n^{(s)'}(z'',s))]dz'' \end{pmatrix} \tag{4.15}$$

where $(\tilde{I}_n^{(s)'}(z',s))$ is now taken positive in W_1 (or +z) direction.

4.1 NORM OF THE SCATTERING SUPERMATRIX

The scattering supermatrix for a uniform section of a multiconductor transmission line is given by Equation 4.7, and has its diagonal block matrices as null matrices and off-diagonal block matrices as the reflection coefficient matrices at z = 0 and z = L. From Equation 4.7, we can write

$$((\tilde{S}_{n,m}(s))_{u,v})^{\dagger} \cdot ((\tilde{S}_{n,m}(s))_{u,v})$$

$$= \begin{pmatrix} (\tilde{S}_{n,m}(s))_{2,1}^{\dagger} \cdot (\tilde{S}_{n,m}(s))_{2,1} & (0_{n,m}) \\ (0_{n,m}) & (\tilde{S}_{n,m}(s))_{1,2}^{\dagger} \cdot (\tilde{S}_{n,m}(s))_{1,2} \end{pmatrix}$$

$$\equiv \left[(\tilde{S}_{n,m}(s))_{2,1}^{\dagger} \cdot (\tilde{S}_{n,m}(s))_{2,1}\right] \oplus \left[(\tilde{S}_{n,m}(s))_{1,2}^{\dagger} \cdot (\tilde{S}_{n,m}(s))_{1,2}\right] \tag{4.16}$$

The supermatrix in Equation 4.16 is block diagonal and, therefore, its eigenvalues are the eigenvalues of its block matrices. From Equations 4.16 and A73 we have

$$\|((\tilde{S}_{n,m}(s))_{u,v})\|_2 = \left[\lambda_{max}\begin{Bmatrix} (\tilde{S}_{n,m}(s))_{2,1}^{\dagger} \cdot (\tilde{S}_{n,m}(s))_{2,1} \\ (\tilde{S}_{n,m}(s))_{1,2}^{\dagger} \cdot (\tilde{S}_{n,m}(s))_{1,2} \end{Bmatrix}\right]^{1/2}$$

$$= \max_{\substack{u=1,v=2 \\ u=2,v=1}} \|(\tilde{S}_{n,m}(s))_{u,v}\|_2 \tag{4.17}$$

Thus the 2 norm of the scattering supermatrix of a uniform section of a multiconductor line is the larger of the 2 norms of the scattering matrices at the terminations.

For passive terminations, we can establish an upper bound on the 2 norm of the scattering supermatrix. If $(\tilde{Y}_{c_{n,m}}(s))$ is a real, diagonal matrix with equal diagonal elements, i.e., the lines are decoupled and have the same characteristic admittances, then the 2 norms of the scattering matrices (reflection matrices) $(\tilde{S}_{n,m}(s))_{1,2}$ and $(\tilde{S}_{n,m}(s))_{2,1}$ satisfy the inequality

$$\|(\tilde{S}_{n,m}(s))\|_2 \leq 1 \qquad \text{for } s = j\omega \tag{4.18}$$

The proof of Equation 4.18 is illustrated in Appendix B.

Similarly, we can establish a lower bound on the norm of the inverse of the scattering supermatrix for passive terminations, under the above assumptions.

From Equation 4.7 we can write the inverse of the scattering supermatrix as

$$((\tilde{S}_{n,m}(s))_{u,v})^{-1} = \begin{pmatrix} (0_{n,m}) & (\tilde{S}_{n,m}(s))^{-1}_{1,2} \\ (\tilde{S}_{n,m}(s))^{-1}_{2,1} & (0_{n,m}) \end{pmatrix} \qquad (4.19)$$

Since $(\tilde{S}_{n,m}(s))_{1,2}$ and $(\tilde{S}_{n,m}(s))_{2,1}$ are square matrices for a uniform section of a line. Then from Equation 4.19

$$((\tilde{S}_{n,m}(s))_{u,v})^{-1\dagger} : ((\tilde{S}_{n,m}(s))_{u,v})^{-1}$$

$$= \begin{pmatrix} (\tilde{S}_{n,m}(s))^{-1\dagger}_{2,1} \cdot (\tilde{S}_{n,m}(s))^{-1}_{2,1} & (0_{n,m}) \\ (0_{n,m}) & (\tilde{S}_{n,m}(s))^{-1\dagger}_{1,2} \cdot (\tilde{S}_{n,m}(s))^{-1}_{1,2} \end{pmatrix}$$

$$\equiv \left[(\tilde{S}_{n,m}(s))^{-1\dagger}_{2,1} \cdot (\tilde{S}_{n,m}(s))^{-1}_{2,1}\right] \oplus \left[(\tilde{S}_{n,m}(s))^{-1\dagger}_{1,2} \cdot (\tilde{S}_{n,m}(s))^{-1}_{1,2}\right] \qquad (4.20)$$

From Equations 4.20 and A73 we have

$$\|((\tilde{S}_{n,m}(s))_{u,v})^{-1}\|_2 = \left[\lambda_{max}\begin{Bmatrix} (\tilde{S}_{n,m}(s))^{-1\dagger}_{2,1} \cdot (\tilde{S}_{n,m}(s))^{-1}_{2,1} \\ (\tilde{S}_{n,m}(s))^{-1\dagger}_{1,2} \cdot (\tilde{S}_{n,m}(s))^{-1}_{1,2} \end{Bmatrix}\right]^{1/2}$$

$$= \max_{\substack{u=1,v=2 \\ u=2,v=1}} \|(\tilde{S}_{n,m}(s))^{-1}_{u,v}\|_2 \qquad (4.21)$$

The 2 norm of the inverse of the scattering supermatrix is greater than or equal to one for $s = j\omega$ (see Appendix B).

4.2 NORM OF THE PROPAGATION SUPERMATRIX

The propagation supermatrix for a uniform section of a multiconductor transmission line in Equation 4.12 is block diagonal, with block matrices equal to the propagation matrix of the line given in Equation 4.13. From Equation A73, we can write the 2 norm of the scattering supermatrix as

$$\|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 = \|(\tilde{\Gamma}_{n,m}(s))\|_2$$

$$= \|\exp\{-(\tilde{\gamma}_{c_{n,m}}(s))L\}\|_2 \qquad (4.22)$$

Thus the 2 norm of the propagation supermatrix is equal to the 2 norm of the propagation matrix $(\tilde{\Gamma}_{n,m}(s))$ of the line.

The propagation matrix $(\tilde{\Gamma}_{n,m}(s))$ is a complex, nonsymmetric matrix in general. The calculation of the eigenvalues of the propagation matrix requires knowledge of the propagation modes, eigenvalues, and eigenvectors of the characteristic propagation matrix $(\tilde{\gamma}_{c_{n,m}}(s))$. Since it is difficult to find eigenvectors of the propagation matrix without the complete knowledge of the matrix itself, for the purpose of establishing bounds we shall limit our investigation to a homogeneous medium case. For a multiconductor transmission line surrounded by a homogeneous medium, the characteristic propagation matrix is diagonal with equal elements since all the modes propagate with the same speed. The diagonal elements of the characteristic propagation matrix for a homogeneous passive case are given by

$$\tilde{\gamma}_{c_{n,n}}(s) = \tilde{\alpha}(s) + j\tilde{\beta}(s) \qquad \text{for } s = j\omega \qquad (4.23)$$

$$\tilde{\alpha}(s) \geq 0$$

where α and β are the attenuation and phase constants.

From Equation 4.23, diagonal terms of the matrix $(\tilde{\Gamma}_{n,m}(s))$ are given by

$$\tilde{\Gamma}_{n,n}(s) = \exp\{-[\tilde{\alpha}(s) + j\tilde{\beta}(s)]L\}$$

$$= \exp\{-\tilde{\alpha}(s)L\} \exp\{-j\tilde{\beta}(s)L\} \qquad \text{for } s = j\omega \qquad (4.24)$$

From Equation 4.24, the 2 norm of the matrix $(\tilde{\Gamma}_{n,m}(s))$ can be written as

$$\|(\tilde{\Gamma}_{n,m}(s))\|_2 = \|(\tilde{\Gamma}_{n,m}(s))\| = |\exp\{-\alpha(s)L\} \exp\{-j\beta(s)L\}|$$

$$= \exp\{-\alpha(s)L\} \qquad \text{for } s = j\omega \qquad (4.25)$$

since the magnitude of the second exponential term is equal to one.

From Equation 4.25 we can conclude that

$$\|(\tilde{\Gamma}_{n,m}(s))\|_2 \leq 1 \qquad \text{for } s = j\omega \qquad (4.26)$$

and hence

$$\|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \leq 1 \qquad \text{for } s = j\omega \qquad (4.27)$$

4.3 NORM OF THE SOURCE SUPERVECTOR

The source supervector is given by Equation 4.15 and, using Equation A62, its norm can be expressed as

$$\|((\tilde{V}_n^{(s)}(s))_u)\|$$

$$= \left\| \begin{pmatrix} \left\| \int_o^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))[L - z']\} \cdot [(\tilde{V}_n^{(s)'}(z',s)) + (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n^{(s)'}(z',s))]dz' \right\| \\ \left\| -\int_o^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))z''\} \cdot [(\tilde{V}_n^{(s)'}(z'',s)) - (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n^{(s)'}(z'',s))]dz'' \right\| \end{pmatrix} \right\|$$

(4.28)

Using Equations A91 and A6, the norm of the vectors in Equation 4.28 can be expressed as

$$\left\| \int_0^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))[L-z']\} \cdot [\tilde{V}_n^{(s)'}(z',s) + (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n(z',s))]dz' \right\|$$

$$\leq \int_0^L \|\exp\{-(\tilde{\gamma}_{c_{n,m}}(s))[L-z']\}\| \, \|[(\tilde{V}_n^{(s)'}(z',s)) + (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n(z',s))]\| dz'$$

$$\leq \int_0^L \|\exp\{-(\tilde{\gamma}_{c_{n,m}}(s))[L-z']\}\| \, [\|\tilde{V}_n^{(s)'}(z',s)\| + \|(\tilde{Z}_{c_{n,m}}(s))\| \, \|(\tilde{I}_n(z',s))\|]dz'$$

(4.29)

$$\left\| -\int_0^L \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))z''\} \cdot [(\tilde{V}_n^{(s)'}(z'',s)) - (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n^{(s)'}(z'',s))]dz'' \right\|$$

$$\leq \int_0^L \|\exp\{-(\tilde{\gamma}_{c_{n,m}}(s))z''\}\| \, \|[(\tilde{V}_n^{(s)'}(z'',s)) - (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n^{(s)'}(z'',s))]\| dz''$$

$$\leq \int_0^L \|\exp\{-(\tilde{\gamma}_{c_{n,m}}(s))z''\| \, [\|(\tilde{V}_n^{(s)'}(z'',s))\| + \|(\tilde{Z}_{c_{n,m}}(s))\| \, \|(\tilde{I}_n^{(s)'} z'',s))\|]dz''$$

(4.30)

For a homogeneous medium, from Equation 4.25 the norm of the propagation matrix is bounded by (for $s = j\omega$)

$$\|\exp\{-(\tilde{\gamma}_{c_{n,m}}(s))[L - z']\| \leq 1 \qquad (4.31)$$

$$\|\exp\{-(\tilde{\gamma}_{c_{n,m}}(s))z''\| \leq 1 \qquad (4.32)$$

Substituting Equations 4.29 through 4.32 into Equation 4.28 we get

$$\|((\tilde{V}_n^{(s)}(s))_u)\| \leq \left\| \begin{pmatrix} \int_0^L [\|(\tilde{V}_n^{(s)'}(z',s))\| + \|(\tilde{Z}_{c_{n,m}}(s))\| \, \|(\tilde{I}_n^{(s)'}(z',s))\|]dz' \\ \int_0^L [\|(\tilde{V}_n^{(s)'}(z'',s))\| + \|(\tilde{Z}_{c_{n,m}}(s))\| \, \|(\tilde{I}_n^{(s)'}(z'',s))\|]dz'' \end{pmatrix} \right\|$$

(4.33)

Note that the two integrals are equal so that we only need one and the norms can be expressed in terms of the norm of this one as

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_1 \leq 2\left[\int_0^L [\|(\tilde{V}_n^{(s)\prime}(z',s))\|_1 + \|(\tilde{Z}_{c_{n,m}}(s))\|_1 \; \|(\tilde{I}_n^{(s)\prime}(z',s))\|_1]dz'\right] \tag{4.34a}$$

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_2 \leq \sqrt{2}\left[\int_0^L [\|(\tilde{V}_n^{(s)\prime}(z',s)\|_2 + \|(\tilde{Z}_{c_{n,m}}(s))\|_2 \; \|(\tilde{I}_n^{(s)\prime}(z',s))\|_2]dz'\right] \tag{4.34b}$$

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \leq \int_0^L [\|(\tilde{V}_n^{(s)\prime}(z',s))\|_\infty + \|(\tilde{Z}_{c_{n,m}}(s))\|_\infty \; \|(\tilde{I}_n^{(s)\prime}(z',s))\|_\infty]dz' \tag{4.34c}$$

If the per-unit-length voltage and current source vectors along the line can be expressed using delta functions as

$$(\tilde{V}_n^{(s)\prime}(z',s)) = \sum_{\sigma=1}^{\sigma_{max}} (\tilde{V}_n^{(s)}(s))_\sigma \delta(z'-\zeta_\sigma) \; , \; (\tilde{I}_n^{(s)\prime}(z',s)) = \sum_{\sigma=1}^{\sigma_{max}} (\tilde{I}_n^{(s)}(s))_\sigma \delta(z'-\zeta_\sigma)$$

where $\sigma = 1,2,\ldots,\sigma_{max}$, then Equations 4.34a through c can be written as

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_1 \leq 2 \sum_{\sigma=1}^{\sigma_{max}} [\;\|(\tilde{V}_n^{(s)}\;(s))_\sigma\|_1 + \|(\tilde{Z}_{c_{n,m}}(s))\|_1 \|(\tilde{I}_n^{(s)}\;(s))_\sigma\|_1] \tag{4.34d}$$

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_2 \leq \sqrt{2} \sum_{\sigma=1}^{\sigma_{max}} [\;\|(\tilde{V}_n^{(s)}\;(s))_\sigma\|_2 + \|(\tilde{Z}_{c_{n,m}}(s))\|_2 \|(\tilde{I}_n^{(s)}\;(s))_\sigma\|_2] \tag{4.34e}$$

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \leq \sum_{\sigma=1}^{\sigma_{max}} [\;\|(\tilde{V}_n^{(s)}\;(s))_\sigma\|_\infty + \|(\tilde{Z}_{c_{n,m}}(s))\|_\infty \|(\tilde{I}_n^{(s)}\;(s))_\sigma\|_\infty] \tag{4.34f}$$

The above equations (4.34a through 4.34f) express the norms of the source supervector in terms of the norms of the per-unit-length voltage and current source vectors on the line and the norms of the characteristic-impedance matrix of the line. The expressions for the norms of the source supervector can be simplified for the following three special cases.

a. Sources are delta functions; that is, the sources exist only at a point along the line (localized sources). In this case the 1, 2, and ∞ norms of the source supervector in Equation 4.34 reduce to

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_1 \leq 2\,\|(\tilde{V}_n^{(s)}(s))\|_1 + 2\,\|(\tilde{Z}_{c_{n,m}}(s))\|_1\,\|(\tilde{I}_n^{(s)}(s))\|_1 \qquad (4.35a)$$

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_2 \leq \sqrt{2}\,\|(\tilde{V}_n^{(s)}(s))\|_2 + \sqrt{2}\,\|(\tilde{Z}_{c_{n,m}}(s))\|_2\,\|(\tilde{I}_n^{(s)}(s))\|_2 \qquad (4.35b)$$

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \leq \|(\tilde{V}_n^{(s)}(s))\|_\infty + \|(\tilde{Z}_{c_{n,m}}(s))\|_\infty\,\|(\tilde{I}_n^{(s)}(s))\|_\infty \qquad (4.35c)$$

b. Sources are uniform along the line. In this case the 1, 2, and ∞ norms of the source supervector in Equation 4.34 reduce to

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_1 \leq 2L\,\|(\tilde{V}_n^{(s)'}(s))\|_1 + 2L\,\|(\tilde{Z}_{c_{n,m}}(s))\|_1\,\|(\tilde{I}_n^{(s)'}(s))\|_1 \qquad (4.36a)$$

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_2 \leq \sqrt{2}\,L\,\|(\tilde{V}_n^{(s)'}(s))\|_2 + \sqrt{2}\,L\,\|(\tilde{Z}_{c_{n,m}}(s))\|_2\,\|(\tilde{I}_n^{(s)'}(s))\|_2 \qquad (4.36b)$$

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \leq L\,\|(\tilde{V}_n^{(s)'}(s))\|_\infty + L\,\|(\tilde{Z}_{c_{n,m}}(s))\|_\infty\,\|(\tilde{I}_n^{(s)'}(s))\|_\infty \qquad (4.36c)$$

c. Sources are sort of uniform; that is, the variation of per-unit-length sources along the line is small. In this case, it is appropriate to use the maximum so that we can write the 1, 2, and ∞ norms of the source supervector as

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_1 \le 2L[\ \|(\tilde{V}_n^{(s)'}(z,s))\|_1 + \|(\tilde{Z}_{c_{n,m}}(s))\|_1\ \|(\tilde{I}_n^{(s)'}(s))\|_1]_{max} \tag{4.37a}$$

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_2 \le \sqrt{2}\ L[\ \|(\tilde{V}_n^{(s)'}(z,s))\|_2 + \|(\tilde{Z}_{c_{n,m}}(s))\|_2\ \|(\tilde{I}_n^{(s)'}(s))\|_2]_{max} \tag{4.37b}$$

$$\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \le L[\ \|(\tilde{V}_n^{(s)'}(z,s))\|_\infty + \|(\tilde{Z}_{c_{n,m}}(s))\|_\infty\ \|(\tilde{I}_n^{(s)'}(s))\|_\infty]_{max} \tag{4.37c}$$

4.4 NORM OF THE MATRIX $[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}$

From Equations 4.18 and 4.27 the 2 norms of the scattering and propagation matrices are less than or equal to one, and, hence, for $s = j\omega$

$$\|((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})\|_2 \le \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2 \le 1 \tag{4.38}$$

Then from Equation A48 we can write

$$\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2 \le \frac{1}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\ \|((\tilde{S}_{n,m}(s))_{u,v})\|_2} \tag{4.39}$$

for $s = j\omega$

Note that since the product of the norms in the denominator of Equation 4.39 is less than or equal to one, we cannot use this upper bound for calculating upper bound for the norm on the left-hand side of Equation 4.39, for it gives an infinitely large bound which is not useful. In order to get a finite bound in Equation 4.39, tighter bounds for the scattering and the propagation matrices are required. For a homogeneous medium, the norm of the propagation matrix is given by Equation 4.25 as

$$\|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 = e^{-\alpha(s)L} \qquad \text{for } s = j\omega \tag{4.40}$$

for a lossless case Equation 4.40 reduces to (for $s = j\omega$)

$$\|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 = 1 \tag{4.41}$$

Hence, for a lossless case, from Equation 4.39, we can write

$$\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1}\|_2 \leq \frac{1}{1 - \|((\tilde{S}_{n,m}(s))_{u,v})\|_2}$$

$$\text{for } s = j\omega \tag{4.42}$$

The norm of the scattering supermatrix for a uniform section of a multiconductor transmission line can be calculated from the knowledge of the termination impedances which we shall consider next.

4.5 NORM OF THE SCATTERING MATRICES AT TERMINATIONS

The scattering or reflection coefficient matrices $(\tilde{S}_{n,m}(s))_{1,2}$ and $(\tilde{S}_{n,m}(s))_{2,1}$ at the terminations of a uniform multiconductor line are given by Equations 2.25 and 2.26 in terms of the characteristic-admittance and termination admittance matrices as

$$(\tilde{S}_{n,m}(s))_{1,2} = [(\tilde{Y}_{c_{n,m}}(s)) + (\tilde{Y}_{T_{n,m}}(0,s))]^{-1} \cdot [(\tilde{Y}_{c_{n,m}}(s)) - (\tilde{Y}_{T_{n,m}}(0,s))] \tag{4.43}$$

$$(\tilde{S}_{n,m}(s))_{2,1} = [(\tilde{Y}_{c_{n,m}}(s)) + (\tilde{Y}_{T_{n,m}}(L,s))]^{-1} \cdot [(\tilde{Y}_{c_{n,m}}(s)) - (\tilde{Y}_{T_{n,m}}(L,s))] \tag{4.44}$$

where $(\tilde{Y}_{T_{n,m}}(0,s))$ and $(\tilde{Y}_{T_{n,m}}(L,s))$ are the termination-admittance matrices at $z = 0$ and $z = L$, respectively. These are related to the termination-impedance matrices by the following relations:

$$(\tilde{Y}_{T_{n,m}}(0,s)) = (\tilde{Z}_{T_{n,m}}(0,s))^{-1} \tag{4.45}$$

$$(\tilde{Y}_{T_{n,m}}(L,s)) = (\tilde{Z}_{T_{n,m}}(L,s))^{-1} \tag{4.46}$$

From Equations 4.43 and 4.44, using Equation A6, we can write

$$\|(\tilde{S}_{n,m}(s))_{1,2}\| \leq \|[(\tilde{Y}_{c_{n,m}}(s)) + (\tilde{Y}_{T_{n,m}}(0,s))]^{-1}\| \cdot \|[(\tilde{Y}_{c_{n,m}}(s)) - (\tilde{Y}_{T_{n,m}}(0,s))]\| \qquad (4.47)$$

$$\|(\tilde{S}_{n,m}(s))_{2,1}\| \leq \|[(\tilde{Y}_{c_{n,m}}(s)) + (\tilde{Y}_{T_{n,m}}(L,s))]^{-1}\| \cdot \|[(\tilde{Y}_{c_{n,m}}(s)) - (\tilde{Y}_{T_{n,m}}(L,s))]\| \qquad (4.48)$$

For a short- or open-circuit termination (all termination impedances are zero or infinity) the scattering matrices in Equations 4.43 and 4.44 are equal to, respectively, - or + the identity matrix. And since the eigenvalues of the identity matrix are all equal to one, the norm of the reflection-coefficient matrices is exactly equal to one for short-circuit or open-circuit termination and, therefore, we will exclude these two cases and assume that the termination impedances are finite and non-zero.

An estimation of upper bounds for norms of scattering matrices in Equations 4.47 and 4.48 is quite difficult without a complete knowledge of the characteristic-admittance and termination-admittance matrices. However, things can be simplified somewhat if we assume that the termination-admittance matrices are real and diagonal; the real, diagonal matrix implies resistive diagonal loads, that is, there are no loads between conductors and each conductor is terminated to ground in a resistive load. This is not a severe assumption since in practice diagonal loads are very common for electronic systems connected by multiconductor cables. Further, we assume that the medium is lossless, or the losses are small so that the characteristic-admittance matrix is real.

For diagonal resistive loads, we have

$$(\tilde{Y}_{T_{n,m}}(0,s)) = (G_{T_{n,m}}(0))$$
$$(\tilde{Y}_{T_{n,m}}(L,s)) = (G_{T_{n,m}}(L)) \tag{4.49}$$

where

$$\left.\begin{array}{l} G_{T_{n,m}}(0) = 0 \\ G_{T_{n,m}}(L) = 0 \end{array}\right\} \quad \text{if } n \neq m$$

and for n = m

$G_{T_{n,n}}(0)$ = the conductance between nth and ground conductors at z = 0

$G_{T_{n,n}}(L)$ = the conductance between nth and ground conductors at z = L

For a lossless case, the characteristic-admittance matrix is independent of frequency and can be written as

$$(\tilde{Y}_{c_{n,m}}(s)) = (\tilde{Y}_{c_{n,m}}) \tag{4.50}$$

Substituting Equations 4.49 and 4.50 into Equations 4.47 and 4.48, we get

$$\|(\tilde{S}_{n,m})_{1,2}\|_2 \leq \|[(\tilde{Y}_{c_{n,m}}) + (G_{T_{n,m}}(0))]^{-1}\|_2 \; \|[(\tilde{Y}_{c_{n,m}}) - (G_{T_{n,m}}(0))]\|_2 \tag{4.51}$$

$$\|(\tilde{S}_{n,m})_{2,1}\|_2 \leq \|[(\tilde{Y}_{c_{n,m}}) + (G_{T_{n,m}}(L))]^{-1}\|_2 \; \|[(\tilde{Y}_{c_{n,m}}) - (G_{T_{n,m}}(L))]\|_2 \tag{4.52}$$

Note that if the line is terminated in its characteristic admittance, the scattering matrix is a null matrix and its norm is zero. Since the termination-admittance matrices are diagonal, their norms are simply equal to the largest element, i.e.,

$$\|(G_{T_{n,n}}(^{0}_{L}))\|_2 = \max G_{T_{n,n}}(^{0}_{L}) \tag{4.53}$$

Thus the 2 norm of the termination-admittance matrix is equal to the reciprocal of the smallest value of the terminating resistor. The characteristic-admittance matrix is a diagonally dominant, real symmetric matrix (Ref. 13). The diagonally dominant property is defined as (Ref. 6)

$$|\tilde{Y}_{c_{n,n}}| \geq \sum_{\substack{m \\ n \neq m}} |\tilde{Y}_{c_{n,m}}| \qquad \text{for all } n \tag{4.54}$$

Since $(G_{T_{n,m}})$ is positive and diagonal, the matrix sum $[(\tilde{Y}_{c_{n,m}}) + (G_{T_{n,m}})]$ is also diagonally dominant. Then from Equation A37 we can write

$$\| [(\tilde{Y}_{c_{n,m}}) + (G_{T_{n,m}} \binom{0}{L})]^{-1} \|_2 \leq \frac{1}{\min_n \{ |\tilde{Y}_{c_{n,n}} + (G_{T_{n,n}} \binom{0}{L})| - \sum_{\substack{m=1 \\ m \neq n}}^{N} |\tilde{Y}_{c_{n,m}} + (G_{T_{n,m}} \binom{0}{L})| \}} \tag{4.55}$$

An upper bound for the characteristic-admittance matrix can be obtained using Equation A38 as

$$\begin{aligned} \|(\tilde{Y}_{c_{n,m}})\|_2 &\leq \sum_n |\tilde{Y}_{c_{n,n}}| \\ &\leq N \max_{n,m} |\tilde{Y}_{c_{n,m}}| \end{aligned} \tag{4.56}$$

For a homogeneous case, the characteristic-admittance matrix can be obtained from the per-unit-length inductance matrix using the relation

$$(\tilde{Y}_{c_{n,m}}) = \frac{1}{v} (L_{n,m})^{-1} \tag{4.57}$$

where v is the speed of propagation on the transmission line. The self and mutual terms of the inductance matrix for a multiconductor line can be estimated approximately using the following relations (Ref. 9).

$$L_{n,n} = 0.2\ \ell n[4\ H_n/d_n]\ \mu H/m$$
$$L_{n,m} = 0.2\ \ell n[B_{n,m}/D_{n,m}]\ \mu H/m \tag{4.58}$$

The parameters in Equation 4.58 are defined as

d the diameter of the conductor

H the distance from a conductor to ground plane

D the distance between two conductors (between centers)

B the distance from the conductor to the image of a second

The relations in Equation 4.58 are valid if the distances between conductors are greater than or equal to 5 times the radius of conductors.

Similarly, using the procedure described above, we can calculate an upper bound for the inverse of the reflection coefficient matrices. From Equations 4.43 and 4.44, for diagonal, resistive loads and a lossless case, we can write

$$(\tilde{S}_{n,m}(s))^{-1}_{1,2} = [(\tilde{Y}_{c_{n,m}}) - (G_{T_{n,m}}(0))]^{-1} \cdot [(\tilde{Y}_{c_{n,m}}) + (G_{T_{n,m}}(0))] \tag{4.59}$$

$$(\tilde{S}_{n,m}(s))^{-1}_{2,1} = [(\tilde{Y}_{c_{n,m}}) - (G_{T_{n,m}}(L))]^{-1} \cdot [(\tilde{Y}_{c_{n,m}}) + (G_{T_{n,m}}(L))] \tag{4.60}$$

Using Equation A6 in Equations 4.59 and 4.60, we can write

$$\|(\tilde{S}_{n,m}(s))^{-1}_{1,2}\|_2 \leq \|[(\tilde{Y}_{c_{n,m}}) - (G_{T_{n,m}}(0))]^{-1}\|_2 \cdot \|[(\tilde{Y}_{c_{n,m}}) + (G_{T_{n,m}}(0))]\|_2 \tag{4.61}$$

$$\|(\tilde{S}_{n,m}(s))^{-1}_{2,1}\|_2 \leq \|[(\tilde{Y}_{c_{n,m}}) - (G_{T_{n,m}}(L))]^{-1}\|_2 \cdot \|[(\tilde{Y}_{c_{n,m}}) + (G_{T_{n,m}}(0))]\|_2 \tag{4.62}$$

The norms in Equations 4.61 and 4.62 can be evaluated for diagonal loads using the relations for the norms of the characteristic-admittance matrix, the load admittance matrix and the matrix $[(\tilde{Y}_{c_{n,m}}) - (G_{T_{n,m}})]^{-1}$.

Having defined norms of the scattering and propagation supermatrices and the source supervector, we can now calculate upper and lower bounds for combined voltage waves, voltages, and currents using the relations derived in Section III.

4.6 BOUNDS FOR COMBINED VOLTAGES, VOLTAGES, AND CURRENTS

Substituting Equation 4.39 into 3.12, we obtain an upper bound for the combined voltage waves leaving junctions (terminations for a uniform section of line) as

$$\|((\tilde{V}_n(0,s))_u)\|_\infty \le \frac{\sqrt{N_s}\,\|((\tilde{S}_{n,m}(s))_{u,v})\|_2\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\|((\tilde{S}_{n,m}(s))_{u,v})\|_2} \tag{4.63}$$

where $((\tilde{V}_n(0,s))_u)$, $((\tilde{V}_n^{(s)}(s))_u)$, $((\tilde{\Gamma}_{n,m}(s))_{u,v})$ and $((\tilde{S}_{n,m}(s))_{u,v})$ are given by Equations 4.1, 4.14, 4.12, and 4.7, respectively. Note that N_s is the dimension of the source supervector and is equal to 2N, where N is the number of conductors in the transmission line.

The ∞ norm of the source supervector is given by Equation 4.34 and the 2 norm of the propagation supermatrix is given by Equation 4.25. The calculation of the norm of the scattering supermatrix was discussed in Section 4.5. Note that for a lossless case, the 2 norm of the propagation supermatrix is exactly equal to one (for $s = j\omega$), and use of the inequality (Equation 4.18) in Equation 4.63 gives an infinitely large bound for the combined voltage waves leaving the termination, which is not useful. Therefore, the knowledge of a tighter upper bound on the norm of the scattering supermatrix is essential to obtain a practical bound, and this can be obtained by using relations discussed in Section 4.5. A lower bound for the combined voltage waves leaving terminations is given by Equation 3.14 as

$$\|((\tilde{V}_n(0,s))_u)\|_\infty$$

$$\geq \frac{\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{\sqrt{N_s}\,\|[((1_{n,m})_{u,v}) - ((\tilde{\Gamma}_{n,m}(s))_{u,v}) : ((\tilde{S}_{n,m}(s))_{u,v})]\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})^{-1}\|_2} \tag{4.64}$$

An upper bound forthe norm of the inverse of the scattering supermatrix can be obtained from Equations 4.61 and 4.62. Note that, in this case, N_s is the order of the supermatrices in the denominator and is equal to 2N.

Using Equation A6 in 4.64, we get

$$\|((\tilde{V}_n(0,s))_u)\|_\infty$$

$$\geq \frac{\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{\sqrt{N_s}[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]\|((S_{n,m}(s))_{u,v})^{-1}\|_2} \tag{4.65}$$

Substituting Equations 4.39 and A6 into Equation 3.16, we obtain an upper bound for the combined voltage waves arriving at the junctions as

$$\|((\tilde{V}_n(L_u,s))_u)\|_\infty \leq \frac{\sqrt{N_s}\;\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2} \tag{4.66}$$

where $((\tilde{V}_n(L_u,s))_u$ is given by Equation 4.2 and N_s = 2N.

A lower bound for the combined voltage waves arriving at the junctions is obtained from Equation 3.17 using Equation A6 as

$$\|((\tilde{V}_n(L_u,s))_u)\|_\infty \geq \frac{\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{\sqrt{N_s}[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]} \tag{4.67}$$

Similarly, substitution of Equation 4.39 into Equations 3.18 and 3.20 gives an upper bound for voltages and currents at the junctions as

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty$$

$$\leq \frac{1}{2}\ \frac{\sqrt{N_s}\,[\|((\tilde{S}_{n,m}(s))_{u,v})\|_2 + \|((P_{n,m})_{u,v})\|_2]\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\|((\tilde{S}_{n,m}(s))_{u,v})\|_2} \qquad (4.68)$$

$$\|(\tilde{I}_n^{(0)}(s))_u\|_\infty$$

$$\leq \frac{1}{2}\ \frac{\sqrt{N_s}\ \|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\|_2[\|((\tilde{S}_{n,m}(s))_{u,v})\| + \|((P_{n,m})_{u,v})\|]\ \|((\tilde{V}_n(s))_u)\|_\infty}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\|((\tilde{S}_{n,m}(s))_{u,v})\|_2} \qquad (4.69)$$

and a lower bound for voltages and currents is obtained from Equations 3.19 and 3.21 using Equation A6 as

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty$$

$$\geq \frac{\frac{1}{2}\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{\sqrt{N_s}[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\|((\tilde{S}_{n,m}(s))_{u,v})\|_2]\|[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]^{-1}\|_2} \qquad (4.70)$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \geq \frac{1}{2}\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty \Big/ \Big[\sqrt{N_s}\ [1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\|((\tilde{S}_{n,m}(s))_{u,v})\|_2] \cdot \|[((\tilde{S}_{n,m}(s))_{u,v}) - ((\tilde{P}_{n,m})_{u,v})]^{-1}\|_2\|((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|_2\Big] \qquad (4.71)$$

Since the permutation supermatrix $((P_{n,m})_{u,v})$ is an orthogonal supermatrix, we have

$$\|((P_{n,m})_{u,v})\| = \|((P_{n,m})_{u,v})^{-1}\| = 1 \qquad (4.72)$$

where $((P_{n,m})_{u,v})$ is given by the definition in Equation 2.61 as

$$((P_{n,m})_{u,v}) = \begin{pmatrix} (0_{n,m})_{1,1} & (1_{n,m})_{1,2} \\ (1_{n,m})_{2,1} & (0_{n,m})_{2,2} \end{pmatrix} \tag{4.73}$$

Also,

$$[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]^{-1}$$

$$= [((1_{n,m})_{u,v}) + ((P_{n,m})_{u,v})^{-1} : ((\tilde{S}_{n,m}(s))_{u,v})]^{-1} : ((P_{n,m})_{u,v})^{-1} \tag{4.74}$$

and

$$\|((P_{n,m})_{u,v})^{-1} : ((\tilde{S}_{n,m}(s))_{u,v})\|_2 \le \|((\tilde{P}_{n,m})_{u,v})^{-1}\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2 \tag{4.75}$$

Substituting Equations 4.72 and 4.18 into Equation 4.75 we get

$$\|((P_{n,m})_{u,v})^{-1} : ((\tilde{S}_{n,m}(s))_{u,v})\|_2 \le 1 \qquad \text{for } s = j\omega \tag{4.76}$$

Using Equations A47, 4.72, 4.76, and A6 in Equation 4.74, we obtain

$$\|[((\tilde{S}_{n,m}(s))_{u,v}) + ((P_{n,m})_{u,v})]^{-1}\|_2 \le \frac{1}{1 - \|((P_{n,m})_{u,v})^{-1} : ((S_{n,m}(s))_{u,v})\|_2} \qquad \text{for } s = j\omega \tag{4.77}$$

Similarly, we can write

$$\|[((\tilde{S}_{n,m}(s))_{u,v}) - ((P_{n,m})_{u,v})]^{-1}\|_2 \le \frac{1}{1 - \|((P_{n,m})_{u,v})^{-1} : ((\tilde{S}_{n,m}(s))_{u,v})\|_2} \qquad \text{for } s = j\omega \tag{4.78}$$

Substituting Equation 4.72 into Equations 4.68 and 4.69, and Equations 4.77 and 4.78 into Equations 4.70 and 4.71, we obtain upper and lower bounds for voltages and currents at the terminations, which are given (for $s = j\omega$) by

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty \le \frac{1}{2}\,\frac{\sqrt{N_s}[1 + \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\|((\tilde{S}_{n,m}(s))_{u,v})\|_2} \tag{4.79}$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty$$

$$\le \frac{1}{2}\,\frac{\sqrt{N_s}\,\|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\|_2\,[1 + \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\|((\tilde{S}_{n,m}(s))_{u,v})\|_2} \tag{4.80}$$

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty$$

$$\ge \frac{1}{2}\,\frac{\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty[1 - \|((P_{n,m})_{u,v})^{-1} : ((S_{n,m}(s))_{u,v})\|_2]}{\sqrt{N_s}[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\|((\tilde{S}_{n,m}(s))_{u,v})\|_2]} \tag{4.81}$$

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty$$

$$\ge \frac{1}{2}\,\frac{\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty[1 - \|((P_{n,m})_{u,v})^{-1} : ((\tilde{S}_{n,m}(s))_{u,v})\|_2]}{\sqrt{N_s}\,[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2\|((\tilde{S}_{n,m}(s))_{u,v})\|_2]\|((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|_2} \tag{4.82}$$

From Equation A11 we can write

$$\|((P_{n,m})_{u,v})^{-1} : ((\tilde{S}_{n,m}(s))_{u,v})\|_2$$

$$= [\lambda_{max}\{((\tilde{S}_{n,m}(s))_{u,v})^{\dagger} : ((P_{n,m})_{u,v}) : ((P_{n,m})_{u,v})^{-1} : ((\tilde{S}_{n,m}(s))_{u,v})\}]^{\frac{1}{2}}$$

$$= [\lambda_{max}\{((\tilde{S}_{n,m}(s))_{u,v})^{\dagger} : ((\tilde{S}_{n,m}(s))_{u,v})\}]^{\frac{1}{2}}$$

$$= \|((\tilde{S}_{n,m}(s))_{u,v})\|_2 \tag{4.83}$$

Substituting Equation 4.83 into Equations 4.81 and 4.82 we get

$$\|((\tilde{V}_n(0,s))_u)\|_\infty \geq \frac{1}{2} \frac{\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty [1 - \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]}{\sqrt{N_s}[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \ \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]} \tag{4.84}$$

$$\|((\tilde{I}_n(0,s))_u)\|_\infty \geq \frac{1}{2} \frac{\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty [1 - \|((\tilde{S}_{n,m}(s))_{u,v})\|]_2}{\sqrt{N_s}[1+\|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2] \|((\tilde{Z}_{c_{n,m}}(s))_{u,v})\|_2} \tag{4.85}$$

The characteristic-impedance supermatrix for the uniform section of a line in Equation 4.85 is given by the relation

$$((\tilde{Z}_{c_{n,m}}(s))_{u,v}) = \begin{pmatrix} (\tilde{Z}_{c_{n,m}}(s))_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (\tilde{Z}_{c_{n,m}}(s))_{2,2} \end{pmatrix} \tag{4.86}$$

and the characteristic-admittance supermatrix is given by the relation

$$((\tilde{Y}_{c_{n,m}}(s))_{u,v}) = ((\tilde{Z}_{c_{n,m}}(s))_{u,v})^{-1}$$

$$= \begin{pmatrix} (\tilde{Y}_{c_{n,m}}(s))_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (\tilde{Y}_{c_{n,m}}(s))_{2,2} \end{pmatrix} \tag{4.87}$$

where $(\tilde{Z}_{c_{n,m}}(s))$ and $(\tilde{Y}_{c_{n,m}}(s))$ are the characteristic-impedance and admittance matrices of the line.

Thus a lower and an upper bound for the combined voltage waves leaving junctions, the combined voltage waves entering junctions, voltages at the junctions and currents at the junctions can be calculated using Equations 4.63, 4.65; 4,66, 4.67; 4.79, 4.84; and 4.80, 4.85, respectively.

V. BOUNDS FOR A MULTICONDUCTOR TRANSMISSION LINE WITH A BRANCH

Having defined upper and lower bounds for voltages and currents at terminations of a uniform section of a multiconductor transmission line, we now turn our attention to a somewhat more complicated transmission-line network, a multiconductor line with a branch (T-network). All the branches of the T-network are terminated at their respective ends. Two types of excitations will be considered. In the first type of excitation, the network is excited by an incident external field, and in the second type, the line is excited by voltage or current sources at the terminations.

Consider a multiconductor line T-network as shown in Figure 5.1. The network topology involves three sections of uniform multiconductor transmission lines (tubes), and four junctions denoted by 1, 2, 3, and 4. The three tubes of the network meet at junction 2. The transmission lines are terminated at their respective ends. Let the number of conductors in tubes 1, 2, and 3 be n_1, n_2, and n_3, respectively, and their lengths be denoted by ℓ_1, ℓ_2, and ℓ_3, respectively. The medium surrounding the network is assumed to be homogeneous. It is assumed that the junction 2 is to be of zero length, and there is no direct coupling between branches. The forward and backward traveling waves on tube 1, tube 2, and tube 3 are denoted by W_1 and W_2, W_3 and W_4, and W_5 and W_6, respectively. The combined voltage vectors at different junctions for various tubes are defined as:

$$
\text{Tube 1}\quad
\begin{cases}
\text{Junction 1} & \begin{cases} V_n(0,s))_1 & \text{wave leaving the junction} \\ V_n(L_2,s))_2 & \text{wave arriving at the junction} \end{cases} \\
\text{Junction 2} & \begin{cases} \tilde{V}_n(0,s))_2 & \text{wave leaving the junction} \\ \tilde{V}_n(L_1,s))_1 & \text{wave arriving at the junction} \end{cases}
\end{cases}
\qquad (5.1)
$$

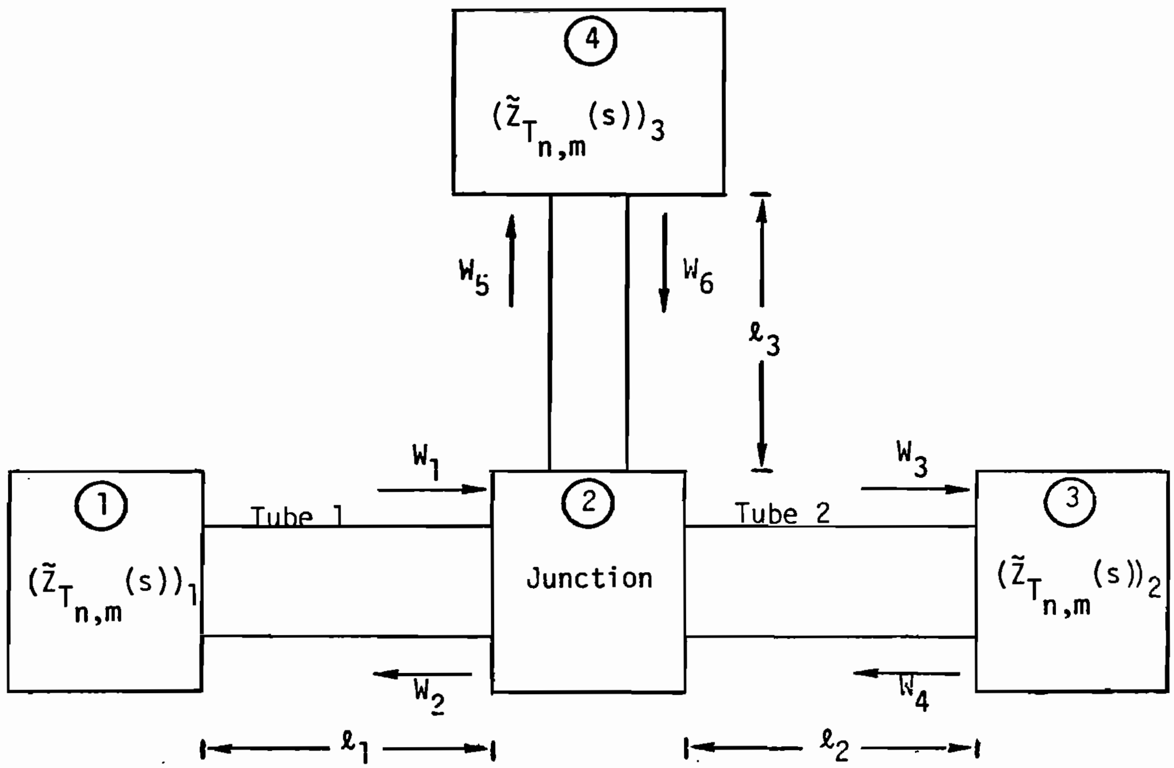

Figure 5.1. A multiconductor transmission line with a branch

$$\text{Tube 2}\quad \begin{cases} \text{Junction 2} \begin{cases} (\tilde{V}_n(0,s))_3 & \text{wave leaving the junction} \\ (\tilde{V}_n(L_4,s))_4 & \text{wave arriving at the junction} \end{cases} \\ \text{Junction 3} \begin{cases} (\tilde{V}_n(0,s))_4 & \text{wave leaving the junction} \\ (\tilde{V}_n(L_3,s))_3 & \text{wave arriving at the junction} \end{cases} \end{cases} \tag{179}$$

$$\text{Tube 3}\quad \begin{cases} \text{Junction 2} \begin{cases} (\tilde{V}_n(0,s))_5 & \text{wave leaving the junction} \\ (\tilde{V}_n(L_6,s))_6 & \text{wave arriving at the junction} \end{cases} \\ \text{Junction 4} \begin{cases} (\tilde{V}_n(0,s))_6 & \text{wave leaving the junction} \\ (\tilde{V}_n(L_5,s))_5 & \text{wave arriving at the junction} \end{cases} \end{cases} \tag{180}$$

where $L_1 = L_2 = \ell_1$, $L_3 = L_4 = \ell_2$, and $L_5 = L_6 = \ell_3$. Having defined the combined voltage waves for different tubes, we can now define the propagation supermatrix, scattering supermatrix, and the source supermatrix for the network.

1. PROPAGATION SUPERMATRIX

For tube 1, the waves leaving and entering junctions are related through the propagation supermatrix as

$$\begin{pmatrix} (\tilde{V}_n(L_1,s))_1 \\ (\tilde{V}_n(L_2,s))_2 \end{pmatrix} = \begin{pmatrix} (\tilde{\Gamma}_{n,m}(s))_{1,1} & (0_{n,m}) \\ (0_{n,m}) & (\tilde{\Gamma}_{n,m}(s))_{2,2} \end{pmatrix} : \begin{pmatrix} (\tilde{V}_n(0,s))_1 \\ (\tilde{V}_n(0,s))_2 \end{pmatrix}$$

$$+ \begin{pmatrix} \int_0^{L_1} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_1[L_1-z']\} \cdot (\tilde{V}_n^{(s)'}(z',s))_1 dz' \\ -\int_0^{L_2} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_2 z''\} \cdot (\tilde{V}_n^{(s)'}(z'',s))_2 dz'' \end{pmatrix} \tag{181}$$

where $z'' = L_2 - z'$

where

$$(\tilde{\Gamma}_{n,m}(s))_{1,1} = (\tilde{\Gamma}_{n,m}(s))_{2,2} = \exp\{-(\tilde{\gamma}_{1c_{n,m}}(s))\ell_1\}$$

$$(\tilde{\gamma}_{c_{n,m}}(s))_1 = (\tilde{\gamma}_{c_{n,m}}(s))_2 = (\tilde{\gamma}_{1c_{n,m}}(s))$$

$$(\tilde{\gamma}_{1c_{n,m}}(s)) \equiv \text{characteristic-propagation matrix for tube 1}$$

Similarly, we can write relations between waves leaving and entering junctions for tubes 2 and 3 as

$$\begin{pmatrix} (\tilde{V}_n(L_3,s))_3 \\ (\tilde{V}_n(L_4,s))_4 \end{pmatrix} = \begin{pmatrix} (\tilde{\Gamma}_{n,m}(s))_{3,3} & (0_{n,m}) \\ (0_{n,m}) & (\tilde{\Gamma}_{n,m}(s))_{4,4} \end{pmatrix} : \begin{pmatrix} (\tilde{V}_n(0,s))_3 \\ (\tilde{V}_n(0,s))_4 \end{pmatrix}$$

$$+ \begin{pmatrix} \int_0^{L_3} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_3[L_3-z']\} \cdot (\tilde{V}_n^{(s)'}(z',s))_3 dz' \\ -\int_0^{L_4} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_4 z''\} \cdot (\tilde{V}_n^{(s)'}(z'',s))_4 dz'' \end{pmatrix} \quad (5.5)$$

where

$$(\tilde{\Gamma}_{n,m}(s))_{3,3} = (\tilde{\Gamma}_{n,m}(s))_{4,4} = \exp\{-(\tilde{\gamma}_{2c_{n,m}}(s))\ell_2\}$$

$$(\tilde{\gamma}_{c_{n,m}}(s))_3 = (\tilde{\gamma}_{c_{n,m}}(s))_4 = (\tilde{\gamma}_{2c_{n,m}}(s))$$

$$(\tilde{\gamma}_{2c_{n,m}}(s)) \equiv \text{characteristic-propagation matrix for tube 2}$$

$$z'' = L_4 - z'$$

and

$$\begin{pmatrix}(\tilde{V}_n(L_5,s))_5 \\ (\tilde{V}_n(L_6,s))_6\end{pmatrix} = \begin{pmatrix}(\tilde{\Gamma}_{n,m}(s))_{5,5} & (0_{n,m}) \\ (0_{n,m}) & (\tilde{\Gamma}_{n,m}(s))_{6,6}\end{pmatrix} : \begin{pmatrix}(\tilde{V}_n(0,s))_5 \\ (\tilde{V}_n(0,s))_6\end{pmatrix}$$

$$+ \begin{pmatrix}\int_0^{L_5} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_5[L_5-z']\} \cdot (\tilde{V}_n^{(s)'}(z',s))_5 dz' \\ -\int_0^{L_6} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_6 z''\} \cdot (\tilde{V}_n^{(s)'}(z'',s))_6 dz''\end{pmatrix} \quad (5.6)$$

where

$$(\tilde{\Gamma}_{n,m}(s))_{5,5} = (\tilde{\Gamma}_{n,m}(s))_{6,6} = \exp\{-(\tilde{\gamma}_{3c_{n,m}}(s))\ell_3\}$$

$$(\tilde{\gamma}_{c_{n,m}}(s))_5 = (\tilde{\gamma}_{c_{n,m}}(s))_6 = (\tilde{\gamma}_{3c_{n,m}}(s))$$

$(\tilde{\gamma}_{3c_{n,m}}(s)) \equiv$ characteristic-propagation matrix for tube 3

$$z'' = L_6 - z'$$

Note that $(\tilde{\Gamma}_{n,m}(s))_{1,1}$, $(\tilde{\Gamma}_{n,m}(s))_{3,3}$ and $(\tilde{\Gamma}_{n,m}(s))_{5,5}$ are $\eta_1 \times \eta_1$, $\eta_2 \times \eta_2$ and $\eta_3 \times \eta_3$ matrices, respectively, and $(\tilde{\gamma}_{1c_{n,m}}(s))$, $(\tilde{\gamma}_{2c_{n,m}}(s))$, and $(\tilde{\gamma}_{3c_{n,m}}(s))$ are $\eta_1 \times \eta_1$, $\eta_2 \times \eta_2$ and $\eta_3 \times \eta_3$ matrices, respectively.

The network propagation supermatrix $((\tilde{\Gamma}_{n,m}(s))_{u,v})$ and the network source supervector can be obtained by combining the results above in the following manner:

$$\begin{pmatrix} (\tilde{V}_n(L_1,s))_1 \\ (\tilde{V}_n(L_2,s))_2 \\ (\tilde{V}_n(L_3,s))_3 \\ (\tilde{V}_n(L_4,s))_4 \\ (\tilde{V}_n(L_5,s))_5 \\ (\tilde{V}_n(L_6,s))_6 \end{pmatrix}$$

$$= \begin{pmatrix} (\tilde{\Gamma}_{n,m}(s))_{1,1} & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) \\ (0_{n,m}) & (\tilde{\Gamma}_{n,m}(s))_{2,2} & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) \\ (0_{n,m}) & (0_{n,m}) & (\tilde{\Gamma}_{n,m}(s))_{3,3} & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) \\ (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (\tilde{\Gamma}_{n,m}(s))_{4,4} & (0_{n,m}) & (0_{n,m}) \\ (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (\tilde{\Gamma}_{n,m}(s))_{5,5} & (0_{n,m}) \\ (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (\tilde{\Gamma}_{n,m}(s))_{6,6} \end{pmatrix}$$

$$\vdots \begin{pmatrix} (\tilde{V}_n(0,s))_1 \\ (\tilde{V}_n(0,s))_2 \\ (\tilde{V}_n(0,s))_3 \\ (\tilde{V}_n(0,s))_4 \\ (\tilde{V}_n(0,s))_5 \\ (\tilde{V}_n(0,s))_6 \end{pmatrix} + \begin{pmatrix} (\tilde{V}_n^{(s)}(s))_1 \\ (\tilde{V}_n^{(s)}(s))_2 \\ (\tilde{V}_n^{(s)}(s))_3 \\ (\tilde{V}_n^{(s)}(s))_4 \\ (\tilde{V}_n^{(s)}(s))_5 \\ (\tilde{V}_n^{(s)}(s))_6 \end{pmatrix} \tag{5.7}$$

Equation 5.7 can be written in supermatrix notation as

$$((\tilde{V}_n(L_u,s))_u) = ((\tilde{\Gamma}_{n,m}(s))_{u,v}) \vdots (\tilde{V}_n(0,s))_u) + ((\tilde{V}_n^{(s)}(s))_u) \tag{5.8}$$

where

$$((\tilde{\Gamma}_{n,m}(s))_{u,v}) = (\tilde{\Gamma}_{n,m}(s))_{1,1} \oplus (\tilde{\Gamma}_{n,m}(s))_{2,2} \oplus (\tilde{\Gamma}_{n,m}(s))_{3,3}$$

$$\oplus (\tilde{\Gamma}_{n,m}(s))_{4,4} \oplus (\tilde{\Gamma}_{n,m}(s))_{5,5} \oplus (\tilde{\Gamma}_{n,m}(s))_{6,6} \quad (5.9)$$

$\equiv$ propagation supermatrix

and

$$((\tilde{V}_n^{(s)}(s))_u) = \begin{pmatrix} \int_0^{L_1} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_1[L_1 - z']\} \cdot (\tilde{V}_n^{(s)'}(z',s))_1 dz' \\ -\int_0^{L_2} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_2 z''\} \cdot (\tilde{V}_n^{(s)'}(z'',s))_2 dz'' \\ \int_0^{L_3} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_3[L_3 - z']\} \cdot (\tilde{V}_n^{(s)'}(z',s))_3 dz' \\ -\int_0^{L_4} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_4 z''\} \cdot (\tilde{V}_n^{(s)'}(z'',s))_4 dz'' \\ \int_0^{L_5} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_5[L_5 - z']\} \cdot (\tilde{V}_n^{(s)'}(z',s))_5 dz' \\ -\int_0^{L_6} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_6 z''\} \cdot (\tilde{V}_n^{(s)'}(z'',s))_6 dz'' \end{pmatrix}$$

$\equiv$ source supervector

$z'' = L_u - z'$ for $u = 2,4,6$ (5.10)

5.2 SCATTERING SUPERMATRIX

For convenience in referencing junctions, they are assigned numbers 1, 2, 3, and 4, as shown in Figure 5.1. For junction 1 where tube 1 is terminated in the impedance $(\tilde{Z}_{T_{n,m}}(s))_1$, the incoming and outgoing waves are related by the following relation

$$(\tilde{V}_n(0,s))_1 = (\tilde{S}_{n,m}(s))_{1,2} \cdot (\tilde{V}_n(L_1,s))_2 \quad (5.11)$$

From Equation 2.26, the scattering matrix $(\tilde{S}_{n,m}(s))_{1,2}$ is given by

$$(\tilde{S}_{n,m}(s))_{1,2} = [(\tilde{Y}_{c_{n,m}}(s))_1 + (\tilde{Y}_{T_{n,m}}(s))_1]^{-1} \cdot [(\tilde{Y}_{c_{n,m}}(s))_1 - (\tilde{Y}_{T_{n,m}}(s))_1] \tag{5.12}$$

where

$$(\tilde{Y}_{T_{n,m}}(s))_1 = (\tilde{Z}_{T_{n,m}}(s))_1^{-1}$$

and

$$(\tilde{Y}_{c_{n,m}}(s))_1 \equiv \text{characteristic-admittance matrix of tube 1}$$

Similarly, we can write relationships between outgoing and incoming waves at junction 3 and 4 as

$$(\tilde{V}_n(0,s))_4 = (\tilde{S}_{n,m}(s))_{4,3} \cdot (\tilde{V}_n(L_3,s))_3 \tag{5.13}$$

$$(\tilde{V}_n(0,s))_6 = (\tilde{S}_{n,m}(s))_{6,5} \cdot (\tilde{V}_n(L_5,s))_5 \tag{5.14}$$

$$(\tilde{S}_{n,m}(s))_{4,3} = [(\tilde{Y}_{c_{n,m}}(s))_2 + (\tilde{Y}_{T_{n,m}}(s))_2]^{-1} \cdot [(\tilde{Y}_{c_{n,m}}(s))_2 - (\tilde{Y}_{T_{n,m}}(s))_2] \tag{5.15}$$

$$(\tilde{S}_{n,m}(s))_{6,5} = [(\tilde{Y}_{c_{n,m}}(s))_3 + (\tilde{Y}_{T_{n,m}}(s))_3]^{-1} \cdot [(\tilde{Y}_{c_{n,m}}(s))_3 - (\tilde{Y}_{T_{n,m}}(s))_3] \tag{5.16}$$

$$(\tilde{Y}_{T_{n,m}}(s))_2 = (\tilde{Z}_{T_{n,m}}(s))_2^{-1}$$

$$(\tilde{Y}_{T_{n,m}}(s))_3 = (\tilde{Z}_{T_{n,m}}(s))_3^{-1}$$

$$(\tilde{Y}_{c_{n,m}}(s))_2 \equiv \text{characteristic-admittance matrix of tube 2}$$

$$(\tilde{Y}_{c_{n,m}}(s))_3 \equiv \text{characteristic-admittance matrix of tube 3}$$

The outgoing and incoming waves at junction 2 are related in the following manner:

$$\begin{pmatrix} (\tilde{V}_n(0,s))_2 \\ (\tilde{V}_n(0,s))_3 \\ (\tilde{V}_n(0,s))_5 \end{pmatrix} = ((\tilde{S}_{n,m}(s))_{u,v})_2 : \begin{pmatrix} (\tilde{V}_n(L_1,s))_1 \\ (\tilde{V}_n(L_4,s))_4 \\ (\tilde{V}_n(L_6,s))_6 \end{pmatrix} \quad (5.17)$$

where $((\tilde{S}_{n,m}(s))_{u,v})_2$ is the scattering supermatrix of the junction 2.

It is assumed here that junction 2 contains wires only which are interconnected; that is, there are no impedances involved at the junction 2. The procedure for calculating junction scattering supermatrices in general are discussed in References 10 and 11. Here, we shall illustrate the procedure for the case shown in Figure 5.1. Further, the junction is considered lossless, i.e., all the energy incident at the junction is reflected and/or transmitted.

At a junction where there are several tubes interconnected to one another, the Kirchhoff's current law and the Kirchhoff's voltage law have to be enforced.

Kirchhoff's current law states that the sum of the currents flowing into a node is zero. For the case where n_1th wire of tube 1 is connected to the n_2th wire of tube 2, and to the n_3th wire of tube 3, and these wires are not connected to any other wires at this junction, we have

$$(I_{n_1}^{(0)}(s))_{r,1} + (I_{n_2}^{(0)}(s))_{r,2} + (I_{n_3}^{(0)}(s))_{r,3} = 0 \quad (5.18)$$

Equation 5.18 can be put into supermatrix form, i.e.,

$$(\overset{\text{tube 1}}{0\ 0\ldots1\ldots0} \,\vdots\, \overset{\text{tube 2}}{0\ldots1\ldots0} \,\vdots\, \overset{\text{tube 3}}{0\ 0\ldots1\ldots1} \,\vdots) : \begin{pmatrix} (\tilde{I}_n^{(0)}(s))_{r,1} \\ (\tilde{I}_n^{(0)}(s))_{r,2} \\ (\tilde{I}_n^{(0)}(s))_{r,3} \end{pmatrix} = ((0_n)_r) \quad (5.19)$$

where $(\tilde{I}_n^{(0)}(s))_{r,1}$, $(\tilde{I}_n^{(0)}(s))_{r,2}$, and $(\tilde{I}_n^{(0)}(s))_{r,3}$ are current vectors at the junction associated with tubes 1, 2, and 3, respectively.

In Equation 5.19, all elements in the left vector are zero, unless they correspond to the conductors which are connected at the node. For N_c connections at the junction, there are N_c equations similar to Equation 5.19 and we can define the junction connection supermatrix $((C_{I_{n,m}})_{a,b})$ so that

$$((C_{I_{n,m}})_{a,b}) : \begin{pmatrix} (\tilde{I}_n^{(0)}(s))_{r,1} \\ (\tilde{I}_n^{(0)}(s))_{r,2} \\ (\tilde{I}_n^{(0)}(s))_{r,3} \end{pmatrix} = ((0_n)_a) \tag{5.20}$$

where $((C_{I_{n,m}})_{a,b})$ is an $N_c \times M_j$ supermatrix, and M_j is the total number of conductors entering the junction. In this case, $M_j = n_1 + n_2 + n_3$.

Kirchhoff's voltage law requires all voltages associated with each conductor to be the same at the same node. Thus for the above example, we have

$$\begin{aligned} \tilde{V}_{n_1}^{(0)}(s)_{r,1} - \tilde{V}_{n_2}^{(0)}(s)_{r,2} &= 0 \\ \tilde{V}_{n_1}^{(0)}(s)_{r,1} - \tilde{V}_{n_3}^{(0)}(s)_{r,3} &= 0 \end{aligned} \tag{5.21}$$

If there are M conductors being connected to the same node, there are M-1 equations in Equation 5.21. Equation 5.21 can also be written in supermatrix form as

$$\left(\begin{array}{c|c|c} 0\ 0\ldots1\ldots0 & 0\ldots-1\ldots.0 & 0\ 0\ldots\ldots0\ldots0 \\ \hline 0\ 0\ldots0\ldots0 & 0\ldots\ 1\ldots0\ldots & 0\ 0\ldots-1\ldots0\ldots0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ 0\ 0\ldots0\ldots0 & 0\ 1\ldots0\ldots0 & 0\ 0\ldots-1\ldots.0 \end{array}\right) : \begin{pmatrix} (\tilde{V}_n^{(0)}(s))_{r,1} \\ (\tilde{V}_n^{(0)}(s))_{r,2} \\ (\tilde{V}_n^{(0)}(s))_{r,3} \end{pmatrix} = ((0_n)_a) \tag{5.22}$$

where $(\tilde{V}_n^{(0)}(s))_{r,1}$, $(\tilde{V}_n^{(0)}(s))_{r,2}$, and $(\tilde{V}_n^{(0)}(s))_{r,3}$ are voltage vectors at the junctions associated with tubes 1, 2, and 3, respectively. Here, each row contains one 1 and one -1, and all other values are zero. Note that the subscripts 1, 2, and 3 on voltage and current vectors denote tube numbers, not waves.

For N_c connections there are $M_j - N_c$ equations. Let us denote the corresponding supermatrix as $((C_{V_{n,m}})_{a,b})$

$$((C_{V_{n,m}})_{a,b}) : \begin{pmatrix} (\tilde{V}_n^{(0)}(s))_{r,1} \\ (\tilde{V}_n^{(0)}(s))_{r,2} \\ (\tilde{V}_n^{(0)}(s))_{r,3} \end{pmatrix} = ((0_n)_r) \tag{5.23}$$

At the junction, the total voltage and current are related to the incident and reflected voltage waves as

$$((\tilde{V}_n^{(0)}(s))_r) = \frac{1}{2} [((\tilde{V}_n(s))_r)_+ + ((\tilde{V}_n(s))_r)_-] \tag{5.24}$$

$$((\tilde{I}_n^{(0)}(s))_r) = \frac{1}{2} ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 \cdot [((\tilde{V}_n(s))_r)_+ - ((\tilde{V}_n(s))_r)_-] \tag{5.25}$$

where $(\tilde{V}_n(s))_{r;+}$ and $(\tilde{V}_n(s))_{r;-}$ are outgoing and incoming waves on the rth tube in the form of combined voltage vectors at the junction, and $((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2$ is the characteristic-admittance matrix of the junction and is given by Equation 2.31 as

$$((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 = \bigoplus_{r=1}^{3} (\tilde{Y}_{c_{n,m}}(s))_{r,r;2} \tag{5.26}$$

where $(\tilde{Y}_{c_{n,m}}(s))_{r,r';\nu}$ is the characteristic-admittance matrix of the rth tube, at the junction 2.

Using Equations 5.20 and 5.25, we get

$$((C_{I_{n,m}})_{a,b}) : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 : ((\tilde{V}_n(s))_r)_+$$

$$= ((C_{I_{n,m}})_{a,b}) : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 : ((\tilde{V}_n(s))_r)_- \tag{5.27a}$$

Premultiply Equation 5.27a by a normalizing nonsingular impedance supermatrix $((\tilde{Z}_{n,m}(s))_{a\ ,b\ })$

$$((\tilde{Z}_{n,m}(s))_{a,b}) : ((C_{I_{n,m}})_{a,b}) : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 : ((\tilde{V}_n(s))_r)_+$$

$$= ((\tilde{Z}_{n,m}(s))_{a,b}) : ((C_{I_{n,m}})_{a,b}) : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'}) : ((\tilde{V}_n(s))_r)_- \tag{5.27b}$$

$$a\ ,b\ = 1,2,\ldots,N_c$$

and, similarly, from Equations 5.23 and 5.24 we get

$$(C_{V_{n,m}}) : ((\tilde{V}_n(s))_r)_+ = -(C_{V_{n,m}}) : ((\tilde{V}_n(s))_r)_- \tag{5.28}$$

Note that $((C_{I_{n,m}})_{a,b})$ and $((C_{V_{n,m}})_{a,b})$ are supermatrices of size $N_c \times M_j$ and $(M_j - N_c) \times M_j$, while $(\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2$ is of size $M_j \times M_j$. The vectors $((\tilde{V}_n(s))_r)_+$ and $((\tilde{V}_n(s))_r)_-$ are of size M_j.

Combining Equations 5.27 and 5.28, we get

$$\begin{pmatrix} -((C_{V_{n,m}})_{a,b}) \\ ((\tilde{Z}_{n,m}(s))_{a,b}) : ((C_{I_{n,m}})_{a,b}) : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 \end{pmatrix} : ((\tilde{V}_n(s))_r)_+$$

$$= \begin{pmatrix} ((C_{V_{n,m}})_{a,b}) \\ ((\tilde{Z}_{n,m}(s))_{a,b}) : ((C_{I_{n,m}})_{a,b}) : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 \end{pmatrix} : ((\tilde{V}_n(s))_r)_- \tag{5.29}$$

or

$$((\tilde{V}_n(s))_r)_+ = \begin{pmatrix} -((C_{V_{n,m}})_{a,b}) \\ ((\tilde{Z}_{n,m}(s))_{a,b}) : ((C_{I_{n,m}})_{a,b}) : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 \end{pmatrix}^{-1}$$

$$: \begin{pmatrix} ((C_{V_{n,m}})_{a,b}) \\ ((\tilde{Z}_{n,m}(s))_{a,b}) : ((C_{I_{n,m}})_{a,b}) : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 \end{pmatrix} : ((\tilde{V}_n(s))_r)_- \tag{5.30}$$

From Equation 5.30, the scattering supermatrix for the junction 2 is

$$((\tilde{S}_{n,m}(s))_{r,r'})_2 = \begin{pmatrix} -((C_{V_{n,m}})_{a,b}) \\ ((\tilde{Z}_{n,m}(s))_{a,b}) : ((C_{I_{n,m}})_{a,b}) : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 \end{pmatrix}^{-1}$$

$$: \begin{pmatrix} ((C_{V_{n,m}})_{a,b}) \\ ((\tilde{Z}_{n,m}(s))_{a,b}) : ((C_{I_{n,m}})_{a,b}) : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2 \end{pmatrix} \tag{5.31}$$

Note that the normalizing supermatrix $((\tilde{Z}_{n,m}(s))_{a,b})$ makes the two supermatrices in Equation 5.31 unitless and well conditioned. Without the supermatrix $((\tilde{Z}_{n,m}(s))_{a,b})$ the elements of matrix $((C_{I_{n,m}})_{a,b}) : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_2$ will be small compared to the elements of $((C_{V_{n,m}})_{a,b})$.

For the network of Figure 5.1, the outgoing and incoming combined voltage waves at the junction 2 are

$$((\tilde{V}_n(s))_r)_+ = \begin{pmatrix} (\tilde{V}_n(0,s))_2 \\ (\tilde{V}_n(0,s))_3 \\ (\tilde{V}_n(0,s))_5 \end{pmatrix} \tag{5.32}$$

$$((\tilde{V}_n(s))_r)_- = \begin{pmatrix} (\tilde{V}_n(L_1,s))_1 \\ (\tilde{V}_n(L_4,s))_4 \\ (\tilde{V}_n(L_6,s))_6 \end{pmatrix} \tag{5.33}$$

The scattering supermatrix in Equation 5.31 is of the order $M_j \times M_j$.

For convenience in properly ordering variables in the scattering supermatrix for the network, let us write the scattering supermatrix for junction 2 in terms of its block matrices; then using Equations 5.32 and 5.33 in Equation 5.30 we get,

$$\begin{pmatrix} (\tilde{V}_n(0,s))_2 \\ (\tilde{V}_n(0,s))_3 \\ (\tilde{V}_n(0,s))_5 \end{pmatrix} = \begin{pmatrix} (\tilde{S}_{n,m}(s))_{2,1} & (\tilde{S}_{n,m}(s))_{2,4} & (\tilde{S}_{n,m}(s))_{2,6} \\ (\tilde{S}_{n,m}(s))_{3,1} & (\tilde{S}_{n,m}(s))_{3,4} & (\tilde{S}_{n,m}(s))_{3,6} \\ (\tilde{S}_{n,m}(s))_{5,1} & (\tilde{S}_{n,m}(s))_{5,4} & (\tilde{S}_{n,m}(s))_{5,6} \end{pmatrix} \odot \begin{pmatrix} (\tilde{V}_n(L_1,s))_1 \\ (\tilde{V}_n(L_4,s))_4 \\ (\tilde{V}_n(L_6,s))_6 \end{pmatrix} \tag{5.34}$$

Combining Equations 5.11, 5.13, 5.14, and 5.34, and rearranging the junction scattering matrices so that the ordering of the components of the incident and reflected waves is the same as in the propagation supermatrix equation, we get

$$\begin{pmatrix} (\tilde{V}_n(0,s))_1 \\ (\tilde{V}_n(0,s))_2 \\ (\tilde{V}_n(0,s))_3 \\ (\tilde{V}_n(0,s))_4 \\ (\tilde{V}_n(0,s))_5 \\ (\tilde{V}_n(0,s))_6 \end{pmatrix} = \begin{pmatrix} (0_{n,m}) & (\tilde{S}_{n,m}(s))_{1,2} & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) \\ (\tilde{S}_{n,m}(s))_{2,1} & (0_{n,m}) & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{2,4} & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{2,6} \\ (\tilde{S}_{n,m}(s))_{3,1} & (0_{n,m}) & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{3,4} & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{3,6} \\ (0_{n,m}) & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{4,3} & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) \\ (\tilde{S}_{n,m}(s))_{5,1} & (0_{n,m}) & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{5,4} & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{5,6} \\ (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{6,5} & (0_{n,m}) \end{pmatrix}$$

$$: \begin{pmatrix} (\tilde{V}_n(L_1,s)_1 \\ (\tilde{V}_n(L_2,s)_2 \\ (\tilde{V}_n(L_3,s)_3 \\ (\tilde{V}_n(L_4,s)_4 \\ (\tilde{V}_n(L_5,s)_5 \\ (\tilde{V}_n(L_6,s)_6 \end{pmatrix} \tag{5.35}$$

From Equation 5.35, the scattering supermatrix of the network is

$$((\tilde{S}_{n,m}(s))_{u,v})$$

$$= \begin{pmatrix} (0_{n,m}) & (\tilde{S}_{n,m}(s))_{1,2} & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) \\ (\tilde{S}_{n,m}(s))_{2,1} & (0_{n,m}) & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{2,4} & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{2,6} \\ (\tilde{S}_{n,m}(s))_{3,1} & (0_{n,m}) & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{3,4} & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{3,6} \\ (0_{n,m}) & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{4,3} & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) \\ (\tilde{S}_{n,m}(s))_{5,1} & (0_{n,m}) & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{5,4} & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{5,6} \\ (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (\tilde{S}_{n,m}(s))_{6,5} & (0_{n,m}) \end{pmatrix} \tag{5.36}$$

The size of the matrix in Equation 5.36 is $2N \times 2N$, where $N = \eta_1 + \eta_2 + \eta_3$. Note that all the block matrices on the diagonal are null matrices.

5.3 NORM OF THE PROPAGATION SUPERMATRIX

The propagation supermatrix given by Equation 5.9 is block-diagonal, with block matrices equal to the propagation matrices of the various uniform sections of the line (tubes). From Equations A73 and 5.9 we can write the 2 norm of the scattering supermatrix as

$$\|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 = \max_r \|(\tilde{\Gamma}_{n,m}(s))_{r,r}\|_2 \tag{5.37}$$

where r is the tube number ($r = 1,2,3$).

The 2 norm of the propagation matrix of a uniform section of a multiconductor line was discussed in Section 4.2. For a homogeneous medium surrounding the multiconductor cable network, from Equation 4.25, we have (for $s = j\omega$)

$$\|(\tilde{\Gamma}_{n,m}(s))_{r,r}\|_2 = \exp\{-\alpha_r(s)\ell_r\} \leq 1 \tag{5.38}$$

Substitution of Equation 5.38 into 5.37 yields

$$\|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 = \max_r[\exp\{-\alpha_r(s)\ell_r\}] \leq 1 \tag{5.39}$$

where the subscript r represents the tube number.

5.4 NORM OF THE SCATTERING SUPERMATRIX

For passive terminations and at all junctions, if all the tubes have decoupled lines with equal characteristic admittances the 2 norm of the scattering supermatrix satisfies the inequality

$$\|((\tilde{S}_{n,m}(s))_{u,v})\|_2 \leq 1 \qquad (\text{for } s = j\omega) \tag{5.40}$$

This is due to the fact that reflected power from all junctions is always less than or equal to incident power for physically realizable systems (power conservation). The following derivation illustrates the proof for Equation 5.40.

The power-conservation condition can be expressed for lossless tubes (see Appendix B) as

$$((\tilde{V}_n(L_u,s))_u) : ((\tilde{I}_n(L_u,s))_u)^* \leq ((\tilde{V}_n(0,s))_u) : ((\tilde{I}_n(0,s))_u)^* \tag{5.41}$$

where $((\tilde{I}_n(0,s))_u)$ and $((\tilde{I}_n(L_u,s))_u)$ are the combined current supervectors for waves leaving and entering junctions, respectively.

The combined current vectors are related to the combined voltage vectors in the following manner:

$$((\tilde{V}_n(0,s))_u) = ((\tilde{Z}_{c_{n,m}}(s))_{u,v}) : ((\tilde{I}_n(0,s))_u) \tag{5.42}$$

$$((\tilde{V}_n(L,s))_u) = ((\tilde{Z}_{c_{n,m}}(s))_{u,v}) : ((\tilde{I}_n(L,s))_u) \tag{5.43}$$

$$((\tilde{I}_n(0,s))_u) = ((\tilde{Y}_{c_{n,m}}(s))_{u,v}) : ((\tilde{V}_n(0,s))_u) \tag{5.44}$$

$$((\tilde{I}_n(L,s))_u) = ((\tilde{Y}_{c_{n,m}}(s))_{u,v}) : ((\tilde{V}_n(L,s))_u) \tag{5.45}$$

where

$$((\tilde{Z}_{c_{n,m}}(s))_{u,v}) = ((\tilde{Y}_{c_{n,m}}(s))_{u,v})^{-1}$$

$$\equiv \text{characteristic-impedance matrix of the network} \qquad (5.46)$$

The characteristic-impedance matrix for the network is given by

$$((\tilde{Z}_{c_{n,m}}(s))_{u,v}) =$$

$$\begin{pmatrix} (\tilde{Z}_{c_{n,m}}(s))_{1,1} & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) \\ (0_{n,m}) & (\tilde{Z}_{c_{n,m}}(s))_{2,2} & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) \\ (0_{n,m}) & (0_{n,m}) & (\tilde{Z}_{c_{n,m}}(s))_{3,3} & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) \\ (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (\tilde{Z}_{c_{n,m}}(s))_{4,4} & (0_{n,m}) & (0_{n,m}) \\ (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (\tilde{Z}_{c_{n,m}}(s))_{5,5} & (0_{n,m}) \\ (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (0_{n,m}) & (\tilde{Z}_{c_{n,m}}(s))_{6,6} \end{pmatrix}$$

(5.46)

Equation 5.46 can be written as

$$((\tilde{Z}_{c_{n,m}}(s))_{u,v}) = \bigoplus_{u=1}^{6} (\tilde{Z}_{c_{n,m}}(s))_{u,u} \qquad (5.47)$$

where $(\tilde{Z}_{c_{n,m}}(s))_{u,u}$ is the characteristic-impedance matrix of the tube associated with the uth wave. If all the branches at the junction are identical and the characteristic-admittance matrices of the branches are real, diagonal matrices with equal diagonal elements, i.e., the lines in the branches are decoupled and have the same characteristic admittances, then substitution of Equations 5.44 and 5.45 into 5.41 yields

$$((\tilde{V}_n(L_u,s))_u) : ((\tilde{V}_n(L_u,s))_u)^* \leq ((\tilde{V}_n(0,s))_u) : ((\tilde{V}_n(0,s))_u)^*$$

or

$$((\tilde{V}_n(L_u,s))_u)^* : ((\tilde{V}_n(L_u,s))_u) \leq ((\tilde{V}_n(0,s))_u)^* : ((\tilde{V}_n(0,s))_u) \qquad (5.48)$$

Equation 5.48 is similar to Equation B5 and, following the procedure of Appendix B, we can easily prove that

$$\|((\tilde{S}_{n,m}(s))_{u,v})\|_2 \leq 1 \qquad (\text{for } s = j\omega) \qquad (5.49)$$

and

$$\|((\tilde{S}_{n,m}(s))_{u,v})^{-1}\|_2 \geq 1 \qquad (\text{for } s = j\omega) \qquad (5.50)$$

a. Norm of the junction scattering supermatrix

The 2 norm of the junction scattering supermatrix given in Equation 5.34 is less than or equal to 1. This can easily be proven by following the procedure described above for the network scattering supermatrix. The 2 norm of the junction scattering supermatrix of a lossless junction is exactly equal to one. Further, the junction scattering supermatrix of a lossless junction is unitary (Ref. 12). The proof of these properties is illustrated in Appendix C.

b. Norm of the scattering supermatrix in terms of its block matrices

An upper and lower bound for the 2 norm of the scattering supermatrix can be obtained in terms of the 2 norms of its elementary block matrices using the relation (Eq. A102) in Appendix A. From Equation A102 the 2 norm of the scattering supermatrix is bounded by the following relation:

$$\frac{1}{\sqrt{N_S N_V}} \max_{u,v} \|((\tilde{S}_{n,m})_{u,v})\|_2 \leq ((\tilde{S}_{n,m}(s))_{u,v})\|_2 \leq \sqrt{N_S} \max_{v} \sum_{u=1}^{N} \sqrt{N_V} \|(\tilde{S}_{n,m})_{u,v})\|_2 \qquad (5.51)$$

Note that the block matrices in Equation 5.36 are of two kinds: 1) the reflection coefficient matrices at the terminations, 2) partitioned block matrices of the junction scattering supermatrix. An upper bound for the reflection coefficient matrices can be obtained from the knowledge of the termination impedance and the characteristic-impedance matrices of the tubes, using the relations in Section 4.5. The junction scattering supermatrix is obtained from the knowledge of interconnections at the junction.

In Equation 5.51 we observe that due to the presence of factors N_s and N, the upper and lower bounds for the 2 norm of the scattering supermatrix may be very loose and may not be very practical, since the upper bound for the 2 norm of the scattering supermatrix is one.

5.5 NORM OF THE SOURCE SUPERVECTOR

The source supervector is given by Equation 5.10, and using Equation A62 its norm can be expressed as

$$\|((\tilde{V}_n^{(s)}(s))_u)\| =$$

$$\left\| \begin{pmatrix} \left\| \int_0^{L_1} \exp\{-(\tilde{\gamma}_{1c_{n,m}}(s))[L_1 - z']\} \cdot [(\tilde{V}_{1n}^{(s)'}(z',s)) + (\tilde{Z}_{c_{n,m}}(s))_{1,1} \cdot (\tilde{I}_{1n}^{(s)'}(z',s))] dz' \right\| \\ \left\| - \int_0^{L_2} \exp\{-(\tilde{\gamma}_{1c_{n,m}}(s)) z''\} \cdot [(\tilde{V}_{1n}^{(s)'}(z'',s) - (\tilde{Z}_{c_{n,m}}(s))_{1,1} \cdot (\tilde{I}_{1n}^{(s)'}(z'',s))] dz'' \right\| \\ \left\| \int_c^{L_3} \exp\{-(\tilde{\gamma}_{2c_{n,m}}(s))[L_3 - z']\} \cdot [(\tilde{V}_{2n}^{(s)'}(z',s) + (\tilde{Z}_{c_{n,m}}(s))_{2,2} \cdot (\tilde{I}_{2n}^{(s)'}(z',s))] dz' \right\| \\ \left\| - \int_0^{L_4} \exp\{-(\tilde{\gamma}_{2c_{n,m}}(s)) z''\} \cdot [(\tilde{V}_{2n}^{(s)'}(z'',s)) - (\tilde{Z}_{c_{n,m}}(s))_{2,2} \cdot (\tilde{I}_{2n}^{(s)'}(z'',s))] dz'' \right\| \\ \left\| \int_0^{L_5} \exp\{-(\tilde{\gamma}_{3c_{n,m}}(s))[L_5 - z']\} \cdot [(\tilde{V}_{3n}^{(s)'}(z',s)) + (\tilde{Z}_{c_{n,m}}(s))_{3,3} \cdot (\tilde{I}_{3n}^{(s)'}(z',s))] dz' \right\| \\ \left\| - \int_0^{L_6} \exp\{-(\tilde{\gamma}_{3c_{n,m}}(s)) z''\} \cdot [(\tilde{V}_{3n}^{(s)'}(z'',s)) - (\tilde{Z}_{c_{n,m}}(s))_{3,3} \cdot (\tilde{I}_{3n}^{(s)'}(z'',s))] dz'' \right\| \end{pmatrix} \right\|$$

$$z'' = L_u - z' \quad \text{for } u = 2,4,6 \qquad (5.52)$$

where $(\tilde{V}_{rn}^{(s)'}(z',s))$ and $(\tilde{I}_{rn}^{(s)'}(z',s))$ are the per-unit-length voltage and current source vectors, respectively, on the rth tube.

For a homogeneous medium, from Equations 4.31 and 4.32, the norms of the exponential matrices in Equation 5.52 are less than or equal to one. Following the procedure used in the derivation of Equation 4.33, an upper bound for the norm of the source supervector is obtained as (for $s = j\omega$)

$$\|((\tilde{V}_n^{(s)}(s))_u)\|$$

$$\leq \left\| \begin{pmatrix} \int_0^{L_1} [\|(\tilde{V}_{1n}^{(s)'}(z',s))\| + \|(\tilde{Z}_{c_{n,m}}(s))_{1,1}\| \; \|(\tilde{I}_{1n}^{(s)'}(z',s))\|]dz' \\ \int_0^{L_2} [\|(\tilde{V}_{1n}^{(s)'}(z'',s))\| + \|(\tilde{Z}_{c_{n,m}}(s))_{1,1}\| \; \|(\tilde{I}_{1n}^{(s)'}(z'',s))\|]dz'' \\ \int_0^{L_3} [\|(\tilde{V}_{2n}^{(s)'}(z',s))\| + \|(\tilde{Z}_{c_{n,m}}(s))_{2,2}\| \; \|(\tilde{I}_{2n}^{(s)'}(z',s))\|]dz' \\ \int_0^{L_4} [\|(\tilde{V}_{2n}^{(s)'}(z'',s))\| + \|(\tilde{Z}_{c_{n,m}}(s))_{2,2}\| \; \|(\tilde{I}_{2n}^{(s)'}(z'',s))\|]dz'' \\ \int_0^{L_5} [\|(\tilde{V}_{3n}^{(s)'}(z',s))\| + \|(\tilde{Z}_{c_{n,m}}(s))_{3,3}\| \; \|(\tilde{I}_{3n}^{(s)'}(z',s))\|]dz' \\ \int_0^{L_6} [\|(\tilde{V}_{3n}^{(s)'}(z'',s))\| + \|(\tilde{Z}_{c_{n,m}}(s))_{3,3}\| \; \|(\tilde{I}_{3n}^{(s)'}(z'',s))\|]dz'' \end{pmatrix} \right\| \tag{5.53}$$

If the per-unit-length voltage and current source vectors along the tubes can be expressed as delta functions as

$$(\tilde{V}_{r,n}^{(s)'}(z',s)) = \sum_{\sigma_r=1}^{\sigma_{r_{max}}} (\tilde{V}_{r,n}^{(s)}(s))_{\sigma_r} \delta(z' - \zeta_{\sigma_r})$$

$$(\tilde{I}_{r,n}^{(s)'}(z',s)) = \sum_{\sigma_r=1}^{\sigma_{r_{max}}} (\tilde{I}_{r,n}^{(s)}(s))_{\sigma_r} \delta(z' - \zeta_{\sigma_r})$$

where

$$\sigma_r = 1,2,\ldots,\sigma_{r_{max}}$$

$$r = 1,2,3$$

then Equation 5.53 can be written as

$$\|((\tilde{V}_n^{(s)}(s))_u\|$$

$$\leq \begin{pmatrix} \sum_{\sigma_1=1}^{\sigma_{1max}} [\|(\tilde{V}_{1n}^{(s)}(s))\sigma_1\| + \|(\tilde{Z}_{c_{n,m}}(s))_{1,1}\| \|(\tilde{I}_{1n}^{(s)}(s))\sigma_1\|] \\ \sum_{\sigma_1=1}^{\sigma_{1max}} [\|(\tilde{V}_{1n}^{(s)}(s))\sigma_1\| + \|(\tilde{Z}_{c_{n,m}}(s))_{1,1}\| \|(\tilde{I}_{1n}^{(s)}(s))\sigma_1\|] \\ \sum_{\sigma_2=1}^{\sigma_{2max}} [\|(\tilde{V}_{2n}^{(s)}(s))\sigma_2\| + \|(\tilde{Z}_{c_{n,m}}(s))_{2,2}\| \|(\tilde{I}_{2n}^{(s)}(s))\sigma_2\|] \\ \sum_{\sigma_2=1}^{\sigma_{2max}} [\|(\tilde{V}_{2n}^{(s)}(s))\sigma_2\| + \|(\tilde{Z}_{c_{n,m}}(s))_{2,2}\| \|(\tilde{I}_{2n}^{(s)}(s))\sigma_2\|] \\ \sum_{\sigma_3=1}^{\sigma_{3max}} [\|(\tilde{V}_{3n}^{(s)}(s))\sigma_3\| + \|(\tilde{Z}_{c_{n,m}}(s))_{3,3}\| \|(\tilde{I}_{3n}^{(s)}(s))\sigma_3\|] \\ \sum_{\sigma_3=1}^{\sigma_{3max}} [\|(\tilde{V}_{3n}^{(s)}(s))\sigma_3\| + \|(\tilde{Z}_{c_{n,m}}(s))_{3,3}\| \|(\tilde{I}_{3n}^{(s)}(s))\sigma_3\|] \end{pmatrix} \tag{5.54}$$

Equation 5.54 can be simplified for the following three special cases:

a. Sources are delta functions; that is, the sources exist only at a point along the tubes (localized sources). In this case Equation 5.54 reduces to

$$\|((\tilde{V}_n^{(s)}(s))_u)\|$$

$$\leq \left\| \begin{pmatrix} \|(\tilde{V}_{1n}^{(s)}(z',s))\| + \|(\tilde{Z}_{c_{n,m}}(s))_{1,1}\| \; \|(\tilde{I}_{1n}^{(s)}(z',s))\| \\ \|(\tilde{V}_{1n}^{(s)}(z'',s))\| + \|(\tilde{Z}_{c_{n,m}}(s))_{1,1}\| \; \|(\tilde{I}_{1n}^{(s)}(z'',s))\| \\ \|(\tilde{V}_{2n}^{(s)}(z',s))\| + \|(\tilde{Z}_{c_{n,m}}(s))_{2,2}\| \; \|(\tilde{I}_{2n}^{(s)}(z',s))\| \\ \|(\tilde{V}_{2n}^{(s)}(z'',s))\| + \|(\tilde{Z}_{c_{n,m}}(s))_{2,2}\| \; \|(\tilde{I}_{2n}^{(s)}(z'',s))\| \\ \|(\tilde{V}_{3n}^{(s)}(z',s))\| + \|(\tilde{Z}_{c_{n,m}}(s))_{3,3}\| \; \|(\tilde{I}_{3n}^{(s)}(z',s))\| \\ \|(\tilde{V}_{3n}^{(s)}(z'',s))\| + \|(\tilde{Z}_{c_{n,m}}(s))_{3,3}\| \; \|(\tilde{I}_{3n}^{(s)}(z'',s))\| \end{pmatrix} \right\| \tag{5.55}$$

b. Sources are uniform along the line. In this case Equation 5.54 reduces to

$$\|((\tilde{V}_n^{(s)}(s))_u)\|$$

$$\leq \left\| \begin{pmatrix} \|(\tilde{V}_{1n}^{(s)'}(s))\| \, \ell_1 + \|(\tilde{Z}_{c_{n,m}}(s))_{1,1}\| \; \|(\tilde{I}_{1n}^{(s)'}(s))\| \, \ell_1 \\ \|(\tilde{V}_{1n}^{(s)'}(s))\| \, \ell_1 + \|(\tilde{Z}_{c_{n,m}}(s))_{1,1}\| \; \|(\tilde{I}_{1n}^{(s)'}(s))\| \, \ell_1 \\ \|(\tilde{V}_{2n}^{(s)'}(s))\| \, \ell_2 + \|(\tilde{Z}_{c_{n,m}}(s))_{2,2}\| \; \|(\tilde{I}_{2n}^{(s)'}(s))\| \, \ell_2 \\ \|(\tilde{V}_{2n}^{(s)'}(s))\| \, \ell_2 + \|(\tilde{Z}_{c_{n,m}}(s))_{2,2}\| \; \|(\tilde{I}_{2n}^{(s)'}(s))\| \, \ell_2 \\ \|(\tilde{V}_{3n}^{(s)'}(s))\| \, \ell_3 + \|(\tilde{Z}_{c_{n,m}}(s))_{3,3}\| \; \|(\tilde{I}_{3n}^{(s)'}(s))\| \, \ell_3 \\ \|(\tilde{V}_{3n}^{(s)'}(s))\| \, \ell_3 + \|(\tilde{Z}_{c_{n,m}}(s))_{3,3}\| \; \|(\tilde{I}_{3n}^{(s)'}(s))\| \, \ell_3 \end{pmatrix} \right\| \tag{5.56}$$

c. Sources are sort of uniform, that is, the variation of per-unit-length sources along the tubes is small. In this case, Equation 5.54 reduces to

$$\|((\tilde{V}_n^{(s)}(s))_u)\|$$

$$\leq \left\| \begin{pmatrix} [\|(\tilde{V}_{1n}^{(s)'}(z,s))\|\,\ell_1 + \|(\tilde{Z}_{c_{n,m}}(s))_{1,1}\|\,\|(\tilde{I}_{1n}^{(s)'}(z,s))\|\,\ell_1]_{max} \\ [\|(\tilde{V}_{1n}^{(s)'}(z,s))\|\,\ell_1 + \|(\tilde{Z}_{c_{n,m}}(s))_{1,1}\|\,\|(\tilde{I}_{1n}^{(s)'}(z,s))\|\,\ell_1]_{max} \\ [\|(\tilde{V}_{2n}^{(s)'}(z,s))\|\,\ell_2 + \|(\tilde{Z}_{c_{n,m}}(s))_{2,2}\|\,\|(\tilde{I}_{2n}^{(s)'}(z,s))\|\,\ell_2]_{max} \\ [\|(\tilde{V}_{2n}^{(s)'}(z,s))\|\,\ell_2 + \|(\tilde{Z}_{c_{n,m}}(s))_{2,2}\|\,\|(\tilde{I}_{2n}^{(s)'}(z,s))\|\,\ell_2]_{max} \\ [\|(\tilde{V}_{3n}^{(s)'}(z,s))\|\,\ell_3 + \|(\tilde{Z}_{c_{n,m}}(s))_{3,3}\|\,\|(\tilde{I}_{3n}^{(s)'}(z,s))\|\,\ell_3]_{max} \\ [\|(\tilde{V}_{3n}^{(s)'}(z,s))\|\,\ell_3 + \|(\tilde{Z}_{c_{n,m}}(s))_{3,3}\|\,\|(\tilde{I}_{3n}^{(s)'}(z,s))\|\,\ell_3]_{max} \end{pmatrix} \right\| \qquad (5.57)$$

Thus an upper bound for the source supervector can be calculated from Equations 5.53 through 5.57 in terms of the norms of per-unit-length voltage and current source vectors on the various tubes and the characteristic-impedance matrices of the various tubes.

5.6 BOUNDS FOR COMBINED VOLTAGES, VOLTAGES, AND CURRENTS

In Sections 5.3 and 5.4, we established that the norms of the scattering and propagation supermatrices are less than or equal to one. Following the procedure used in Section 4.6 for the derivation of upper and lower bounds for combined voltages, voltages, and currents, we can write similar relations for the present network.

The upper and lower bounds for the combined voltages, voltages, and currents for a multiconductor transmission line with a branch (Fig. 5.1) are given by the following relations (for $s = j\omega$):

a. Combined voltage vector for waves leaving junctions

$$\|((\tilde{V}_n(0,s))_u)\|_\infty \le \frac{\sqrt{N_s}\,\|((\tilde{S}_{n,m}(s))_{u,v})\|_2 \|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2} \tag{5.58}$$

$$\|((\tilde{V}_n(0,s))_u)\|_\infty \ge \frac{\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{\sqrt{N_s}[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2] \|((\tilde{S}_{n,m}(s))_{u,v})^{-1}\|_2} \tag{5.59}$$

b. Combined voltage vector for waves entering junctions

$$\|((\tilde{V}_n(L_u,s))_u)\|_\infty \le \frac{\sqrt{N_s}\,\|((\tilde{V}_n(s))_u)\|_\infty}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2} \tag{5.60}$$

$$\|((\tilde{V}_n(L_u,s))_u)\|_\infty \ge \frac{\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{\sqrt{N_s}[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]} \tag{5.61}$$

c. Voltage vector at junctions

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty \le \frac{1}{2}\,\frac{\sqrt{N_s}[1 + \|((\tilde{S}_{n,m}(s))_{u,v})\|_2] \|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2} \tag{5.62}$$

$$\|((\tilde{V}_n^{(0)}(s))_u)\|_\infty \geq \frac{1}{2} \frac{\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty [1 - \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]}{\sqrt{N_s}[1 + \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \ \|((\tilde{S}_{n,m}(s))_{u,v}\|_2]}$$

(5.63)

d. Current vector at the junctions

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \leq \frac{1}{2} \frac{\sqrt{N_s}\ \|((\tilde{Y}_{c_{n,m}}(s))_{u,v})\|_2 [1+ \|((\tilde{S}_{n,m}(s))_{u,v})\|_2] \|((\tilde{V}_n^{(s)}(s))_u)\|_\infty}{1 - \|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2}$$

(5.64)

$$\|((\tilde{I}_n^{(0)}(s))_u)\|_\infty \geq \frac{1}{2} \frac{\|((\tilde{V}_n^{(s)}(s))_u)\|_\infty [1 - \|((\tilde{S}_{n,m}(s))_{u,v})\|_2]}{\sqrt{N_s}[1+\|((\tilde{\Gamma}_{n,m}(s))_{u,v})\|_2 \|((\tilde{S}_{n,m}(s))_{u,v})\|_2] \|((Z_{c_{n,m}}(s))_{u,v})\|_2}$$

(5.65)

VI. CONCLUSIONS

This note has developed a formulation for the computation of upper and lower bounds on signals at terminations of a multiconductor cable network. The BLT equation expresses the characteristics of a multiconductor transmission line network in a single supermatrix notation. The upper and lower bounds on signals are obtained by using norms of vectors, matrices, supervectors, and supermatrices. Various norms and their properties for vectors, matrices, supervectors, and supermatrices are discussed.

Having developed the general formulation for the computation of upper and lower bounds on signals at terminations of a general multiconductor cable network, two special cases are considered: 1) a uniform section of a multiconductor transmission line and 2) a multiconductor transmission line with a branch. For these two cases scattering and propagation supermatrices are derived and their properties are discussed. The norm of the scattering supermatrix can be estimated for passive terminations. Expressions for upper and lower bounds on signals for these two cases are derived.

A natural extension of this work will be to compare these bounds with exact calculations for a number of canonical configurations when the parameters of the cable network, sources, and load configurations are varied. This will establish the tightness of these bounds. If practically acceptable bounds result from the algorithm presented in this note, it will have tremendous implications on testing of electronic systems for EMP survivability/vulnerability.

APPENDIX A

NORMS OF VECTORS AND MATRICES

In this appendix we will review norms of vectors, supervectors, matrices, and supermatrices. Of special interest are the norms of vectors and matrices which are needed to establish lower and upper bounds on the combined voltage waves and the voltages and currents in the BLT equations, derived in Section II.

A.1 VECTOR NORMS

The norm of a vector (a_n) is denoted by $\|(a_n)\|$ and it satisfies the following properties (Refs. 3,5)

$$\|(a_n)\| \geq 0 \text{ with } \|(a_n)\| = 0 \text{ iff } (a_n) = (0_n)$$

$$\|\alpha(a_n)\| = |\alpha| \, \|(a_n)\|$$

$$\|(a_n) + (b_n)\| \leq \|(a_n)\| + \|(b_n)\| \qquad \text{(A1)}$$

$\|(a_n)\|$ depends continuously on (a_n)

where $(a_n),(b_n)$ are N-component complex vectors

α is a complex number

$|\alpha| \equiv$ magnitude of α

A common type of vector norm is referred to as the p norm defined by

$$\|(a_n)\|_p \equiv \left\{ \sum_{n=1}^{N} |a_n|^p \right\}^{1/p} \quad \text{for any } p \geq 1 \qquad \text{(A2)}$$

This has important special cases

$$\|(a_n)\|_1 \equiv \sum_{n=1}^{N} |a_n|$$

$$\|(a_n)\|_2 = \left\{ \sum_{n=1}^{N} |a_n|^2 \right\}^{1/2} \equiv \{(a_n) \cdot (a_n)^*\}^{1/2} \equiv |(a_n)| \tag{A3}$$

$$\|(a_n)\|_\infty \equiv \max_{1 \le n \le N} |a_n|$$

The 2 norm is then the euclidean norm or magnitude. The ∞ norm or maximum norm represents the magnitude of the maximum component of the vector. The 1 norm represents the sum of the magnitudes of the components of the vector.

From Equation A3, we can write

$$\|(a_n)\|_1 \ge \|(a_n)\|_2 \ge \|(a_n)\|_\infty$$

or, in general

$$\|(a_n)\|_p \ge \|(a_n)\|_q \qquad p \le q \tag{A4}$$

From Equation A3, we can also write the following relations between, 1, 2, and ∞ norms

$$\|(a_n)\|_1 \le N \|(a_n)\|_\infty$$

$$\|(a_n)\|_2 \le \sqrt{N} \|(a_n)\|_\infty \tag{A5}$$

$$\|(a_n)\|_1 \le \sqrt{N} \|(a_n)\|_2$$

A.2 MATRIX NORMS

Norms can also be defined for matrices. The norm of a matrix $(A_{n,m})$ is denoted by $\|(A_{n,m})\|$ and satisfies the following properties:

$$\|(A_{n,m})\| \ge 0 \text{ with } \|(A_{n,m})\| = 0 \text{ iff } (A_{n,m}) = (0_{n,m})$$

$$\|\alpha(A_{n,m})\| = |\alpha| \, \|(A_{n,m})\|$$

$$\|(A_{n,m}) + (B_{n,m})\| \leq \|(A_{n,m})\| + \|(B_{n,m})\|$$

$$\|(A_{n,m}) \cdot (B_{n,m})\| \leq \|(A_{n,m})\| \, \|(B_{n,m})\| \tag{A6}$$

For the above relations to be meaningful, we must have matrices of compatible order (Ref. 2).

It follows from Equation A6 that if $(A_{n,m})$ is a square matrix, we have

$$\|(A_{n,m})^q\| \leq \|(A_{n,m})\|^q$$

$$q \equiv \text{positive integer} \tag{A7}$$

A common way of constructing matrix norms uses the role of matrices in relating vectors via dot multiplication as in

$$(b_n) = (A_{n,m}) \cdot (X_n)$$

$$(A_{n,m}) \equiv N \times M \text{ complex matrix}$$

$$(X_n) \equiv M\text{-component complex vector} \tag{A8}$$

$$(b_n) \equiv N\text{-component complex vector}$$

If we define a matrix norm via

$$\|(A_{n,m})\| = \sup_{(X_n)\neq(0_n)} \frac{\|(A_{n,m}) \cdot (X_n)\|}{\|(X_n)\|} \tag{A9}$$

$$\sup \equiv \text{supremum} \equiv \text{least upper bound}$$

which makes the matrix norm a least upper bound over all (X_n) in Equation A8. The matrix norm in Equation A8 is referred to as an associated matrix norm and can be thought of as a minimum norm consistent with the chosen vector norm. We shall use only associated norms in the rest of the discussion.

For 1 and ∞ vector norms, the corresponding associated matrix norms are given respectively by (Refs. 3,5).

$$\|(A_{n,m})\|_1 = \max_{1\leq m\leq M} \sum_{n=1}^{N} |A_{n,m}| \equiv \text{maximum column magnitude sum}$$

$$\|(A_{n,m})\|_\infty = \max_{1\leq n\leq N} \sum_{m=1}^{M} |A_{n,m}| \equiv \text{maximum row magnitude sum} \tag{A10}$$

These results apply to general complex $N \times M$ matrices.

Corresponding to the vector 2 norm, the associated matrix norm is given by

$$\|(A_{n,m})\|_2 = [\lambda_{max}((A_{n,m})^\dagger \cdot (A_{n,m}))]^{1/2} \tag{A11}$$

where $\dagger$ represents conjugate transpose. Note that all the eigenvalues of $(A_{n,m})^\dagger \cdot (A_{n,m})$ are non-negative since this is a positive semidefinite matrix.

For general complex square $(N \times N)$ matrices we can define a spectral radius as

$$\rho((B_{n,m})) \equiv \text{spectral radius of } (B_{n,m})$$

$$\rho((B_{n,m})) = |\lambda((B_{n,m}))|_{max} \tag{A12}$$

where $|\lambda|_{max}$ is defined as an eigenvalue of $(B_{n,m})$ with maximum magnitude.

Having defined matrix norms, we shall now derive relations between different matrix norms.

A.3 SPECTRAL RADIUS AND ASSOCIATED MATRIX NORMS

For general complex square matrices we have (Ref. 5)

$$\|(A_{n,m})\| \geq \rho((A_{n,m})) = |\lambda((A_{n,m}))|_{max} \tag{A13}$$

so that the spectral radius is a lower bound for all associated matrix norms (for square matrices).

From Equations A13 and A11 we have

$$|\lambda((A_{n,m}))|_{max} \leq \lambda_{max}\{((A_{n,m})^{\dagger} \cdot (A_{n,m}))\}^{1/2} \tag{A14}$$

If $(A_{n,m})$ is real symmetric, then we have

$$\rho((A_{n,m})) = \|(A_{n,m})\|_2 \tag{A15}$$

and the eigenvalues of $(A_{n,m})$ are all real, since $(A_{n,m})$ is real symmetric.

A.4 RELATIONS BETWEEN 1, 2, AND ∞ ASSOCIATED MATRIX NORMS

For an $N \times M$ matrix we can write:

a. 1 and 2 norms--From Equation A9 we define the 1 norm of a matrix as

$$\|(A_{n,m})\|_1 = \sup_{(X_n)} \frac{\|(A_{n,m}) \cdot (X_n)\|_1}{\|(X_n)\|_1} \tag{A16}$$

From Equation A16 we have

$$\|(A_{n,m})\|_1 \leq \frac{\|(A_{n,m}) \cdot (X_n)\|_1}{(X_n)_1}$$

Substituting Equation A5 into Equation A16 we get

$$\|(A_{n,m})\|_1 \leq \sqrt{M}\,\frac{\|(A_{n,m}) \cdot (X_n)\|_2}{\|(X_n)\|_1}$$

$$\leq \sqrt{M}\,\frac{\|(A_{n,m})\|_2\,\|(X_n)\|_2}{\|(X_n)\|_1}$$

$$\leq \sqrt{M}\,\|(A_{n,m})\|_2 \tag{A17}$$

Similarly, from Equation A9 we define the 2 norm of a matrix as

$$\|(A_{n,m})\|_2 = \sup_{(X_n)} \frac{\|(A_{n,m}) \cdot (X_n)\|_2}{\|(X_n)\|_2} \tag{A18}$$

From Equation A18 we have

$$\|(A_{n,m})\|_2 \leq \frac{\|(A_{n,m}) \cdot (X_n)\|_2}{\|(X_n)\|_2} \tag{A19}$$

Substituting Equation A5 into Equation A19 we get

$$\begin{aligned} \|(A_{n,m})\|_2 &\leq \frac{\|(A_{n,m}) \cdot (X_n)\|_1}{\|(X_n)\|_2} \\ &\leq \frac{\|(A_{n,m})\|_1 \|(X_n)\|_1}{\|(X_n)\|_2} \\ &\leq \sqrt{M}\ \|(A_{n,m})\|_1 \end{aligned} \tag{A20}$$

b. <u>1 and ∞ norms</u>--From Equation A9 we define the ∞ norm of a matrix as

$$\|(A_{n,m})\|_\infty = \sup_{(X_n)} \frac{\|(A_{n,m}) \cdot (X_n)\|_\infty}{\|(X_n)\|_\infty} \tag{A21}$$

From Equation A21 we have

$$\|(A_{n,m})\|_\infty \leq \frac{\|(A_{n,m}) \cdot (X_n)\|_\infty}{\|(X_n)\|_\infty} \tag{A22}$$

Substituting Equation A5 into Equation A22 we get

$$\begin{aligned} \|(A_{n,m})\|_\infty &\leq \frac{\|(A_{n,m}) \cdot (X_n)\|_1}{\|(X_n)\|_\infty} \\ &\leq \frac{\|(A_{n,m})\|_1 \|(X_n)\|_1}{\|(X_n)\|_\infty} \\ &\leq M\ \|(A_{n,m})\|_1 \end{aligned} \tag{A23}$$

Similarly, from Equation A16 we have

$$\|(A_{n,m})\|_1 \le \frac{\|(A_{n,m}) \cdot (X_n)\|_1}{\|(X_n)\|_1} \tag{A24}$$

Substituting Equation A5 into Equation A24 we get

$$\|(A_{n,m})\|_1 \le M\frac{\|(A_{n,m}) \cdot (X_n)\|_\infty}{\|(X_n)\|_1}$$

$$\le M\frac{\|(A_{n,m})\|_\infty \|(X_n)\|_\infty}{\|(X_n)\|_1}$$

$$\le M\|(A_{n,m})\|_\infty \tag{A25}$$

c. <u>2 and ∞ norms</u>--From Equation A19 we have

$$\|(A_{n,m})\|_2 \le \frac{\|(A_{n,m}) \cdot (X_n)\|_2}{\|(X_n)\|_2} \tag{A26}$$

Substituting Equation A5 into Equation A26 we get

$$\|(A_{n,m})\|_2 \le \sqrt{M}\,\frac{\|(A_{n,m}) \cdot (X_n)\|_\infty}{\|(X_n)\|_2}$$

$$\le \sqrt{M}\,\frac{\|(A_{n,m})\|_\infty \|(X_n)\|_\infty}{\|(X_n)\|_2}$$

$$\le \sqrt{M}\,\|(A_{n,m})\|_\infty \tag{A27}$$

Similarly, Equation A22 gives

$$\|(A_{n,m})\|_\infty \le \frac{\|(A_{n,m}) \cdot (X_n)\|_\infty}{\|(X_n)\|_\infty} \tag{A28}$$

Substituting Equation A5 into Equation A28 we get

$$\|(A_{n,m})\|_\infty \le \frac{\|(A_{n,m}) \cdot (X_n)\|_2}{\|(X_n)\|_\infty}$$

$$\le \frac{\|(A_{n,m})\|_2 \, \|(X_n)\|_2}{\|(X_n)\|_\infty}$$

$$\le \sqrt{M}\|(A_{n,m})\|_2 \tag{A29}$$

We can now summarize the relations between 1, 2, and ∞ matrix norms as:

$$\frac{1}{\sqrt{M}} \|(A_{n,m})\|_2 \le \|(A_{n,m})\|_1 \le \sqrt{M}\,\|(A_{n,m})\|_2 \tag{A30}$$

$$\frac{1}{M}\|(A_{n,m})\|_\infty \le \|(A_{n,m})\|_1 \le M\,\|(A_{n,m})\|_\infty \tag{A31}$$

$$\frac{1}{\sqrt{M}} \|(A_{n,m})\|_1 \le \|(A_{n,m})\|_2 \le \sqrt{M}\,\|(A_{n,m})\|_1 \tag{A32}$$

$$\frac{1}{\sqrt{M}} \|(A_{n,m})\|_\infty \le \|(A_{n,m})\|_2 \le \sqrt{M}\,\|(A_{n,m})\|_\infty \tag{A33}$$

$$\frac{1}{\sqrt{M}} \|(A_{n,m})\|_2 \le \|(A_{n,m})\|_\infty \le \sqrt{M}\,\|(A_{n,m})\|_2 \tag{A34}$$

$$\frac{1}{M} \|(A_{n,m})\|_1 \le \|(A_{n,m})\|_\infty \le M\,\|(A_{n,m})\|_1 \tag{A35}$$

where M is the number of columns of $(A_{n,m})$.

A.5 BOUNDS ON THE NORM OF SQUARE MATRICES

The spectral radius of a square matrix $(A_{n,m})$ is bounded by (Ref. 5)

$$\rho((A_{n,m})) \le \max_n \sum_{m=1}^{N} |A_{n,m}| \equiv \|(A_{n,m})\|_\infty \tag{A36}$$

and the spectral radius of $(A_{n,m})^{-1}$ is such that

$$\frac{1}{\rho((A_{n,m})^{-1}} \ge \min_n \left(|A_{n,n}| - \sum_{\substack{m=1 \\ m\ne n}}^{N} |A_{n,m}|\right)$$

or

$$\rho((A_{n,m})^{-1} \leq \frac{1}{\min\limits_{n} (|A_{n,m}| - \sum\limits_{\substack{m=1 \\ m\neq n}}^{N} |A_{n,m}|)} \tag{A37}$$

Equation A37 gives a bound for the inverse of a square matrix. The norm of a square matrix is also bounded by the following inequality (Ref. 6)

$$\max_{n,m} |A_{n,m}| \leq \|(A_{n,m})\| \leq N \max_{n,m} |A_{n,m}| \tag{A38}$$

A.6 NORM OF DIAGONAL MATRICES

Very often, in dealing with electronic systems, one encounters matrices which are diagonal. The norms of diagonal matrices are relatively simple to evaluate. For a diagonal matrix, the associated norm is defined as

$$\|(A_{n,m})\| = \sup_{(X_n)} \frac{\|(A_{n,m}) \cdot (X_n)\|}{\|(X_n)\|}$$

$$= \sup_{(X_n)} \frac{\|(A_{n,n}X_n)\|}{\|(X_n)\|} \tag{A39}$$

From Equation A39 we observe that for any p norm of the matrix we have

$$\|(A_{n,m})\|_p = \sup_{(X_n)} \frac{\|(A_{n,n}X_n)\|_p}{\|(X_n)\|_p} \tag{A40}$$

From Equation A40 we observe that

$$\max \frac{\|(A_{n,n}X_n)\|_p}{\|(X_n)\|_p} = \max_n |A_{n,n}| = \max_{n,m} |A_{n,m}| \tag{A41}$$

Hence,

$$\|(A_{n,m})\|_p = \max_{n,m} |A_{n,m}| \tag{A42}$$

Also from the definitions of 1, 2, and ∞ norms for matrices in Equations A10 and A11, for diagonal matrices we have

$$\|(A_{n,m})\|_1 = \|(A_{n,m})\|_2 = \|(A_{n,m})\|_\infty = \max_n |A_{n,n}| \qquad \text{(A43)}$$

A.7 NORMS OF $[(1_{n,m}) + (A_{n,m})]^{-1}$ and $[(1_{n,m}) - (A_{n,m})]^{-1}$

If $\|(A_{n,m})\| < 1$, then we have (Ref. 5)

$$\|[(1_{n,m}) + (A_{n,m})]^{-1}\| \leq \frac{1}{1 - \|(A_{n,m})\|} \qquad \text{(A44)}$$

To prove Equation A44, let $(B_{n,m}) = [(1_{n,m}) + (A_{n,m})]^{-1}$. Then,

$$(1_{n,m}) = [(1_{n,m}) + (A_{n,m})] \cdot (B_{n,m})$$

or

$$(B_{n,m}) = (1_{n,m}) - (A_{n,m}) \cdot (B_{n,m}) \qquad \text{(A45)}$$

Taking norms of both sides and using Equation A6, we get

$$\|(B_{n,m})\| \leq \|(1_{n,m})\| + \|(A_{n,m})\| \; \|(B_{n,m})\| \qquad \text{(A46)}$$

Noting that $\|(1_{n,m})\| = 1$, from Equation A46 we get

$$\|(B_{n,m})\| \leq \frac{1}{1 - \|(A_{n,m})\|} \qquad \text{if } \|(A_{n,m})\| < 1$$

or

$$\|[(1_{n,m}) + (A_{n,m})]^{-1}\| \leq \frac{1}{1 - \|(A_{n,m})\|} \qquad \text{(A47)}$$

In Equation A47 if we replace $(A_{n,m})$ by $-(A_{n,m})$, we get

$$\|[(1_{n,m}) - (A_{n,m})]^{-1}\| \leq \frac{1}{1 - \|(A_{n,m})\|} \qquad \text{if } \|(A_{n,m})\| < 1 \qquad \text{(A48)}$$

Note that in Equation A48 we have used $\|-(A_{n,m})\| = \|(A_{n,m})\|$.

Corollary 1

Let $(C_{n,m}) = e^{j\delta} (A_{n,m})$

Then $$\| [(1_{n,m}) \pm (C_{n,m})]^{-1} \| \leq \frac{1}{1 - \|(A_{n,m})\|} \tag{A49}$$

Since from Equation A6

$$\|e^{j\delta} (A_{n,m})\| = |e^{j\delta}| \, \|(A_{n,m})\|$$

$$= \|(A_{n,m})\| \tag{A50}$$

Corollary 2

If $(A_{n,m})$ is such that $(A_{n,m})^{-1}$ exists and $\|(A_{n,m})^{-1}\| < 1$, then

$$[(1_{n,m}) + (A_{n,m})]^{-1} = (A_{n,m})^{-1} \cdot [(1_{n,m}) + (A_{n,m})^{-1}]^{-1} \tag{A51}$$

Taking norms of both sides and applying Equations A6 and A47, we get.

$$\| [(1_{n,m}) + (A_{n,m})]^{-1} \| \leq \frac{\|(A_{n,m})^{-1}\|}{1 - \|(A_{n,m})^{-1}\|} \tag{A52}$$

A.8 CONDITION NUMBER OF A MATRIX

The quantity $\|(A_{n,m})\| \, \|(A_{n,m})^{-1}\|$ is defined as the condition number of $(A_{n,m})$ and is denoted as $K((A_{n,m}))$ (Ref. 5). These numbers, defined for various matrix norms give a measure of the condition of $(A_{n,m})$ and are always greater than or equal to 1. This can be seen easily from the following:

$$K((A_{n,m})) = \|(A_{n,m})\| \, \|(A_{n,m})^{-1}\| \tag{A53}$$

From the property Equation A6, we have

$$\|(A_{n,m}) \cdot (B_{n,m})\| \leq \|(A_{n,m})\| \, \|(B_{n,m})\| \tag{A54}$$

Let $$(B_{n,m}) = (A_{n,m})^{-1} \tag{A55}$$

Then from Equation A54 we have

$$\|(A_{n,m})\| \; \|(A_{n,m})^{-1}\| \geq \|(1_{n,m})\|$$

and since $\|(1_{n,m})\| = 1$

we have,

$$\|(A_{n,m})\| \; \|(A_{n,m})^{-1}\| \geq 1 \tag{A56}$$

Equation A56 is valid for any associated matrix norm.

A.9 NORMS OF SUPERVECTORS

In Section II, we introduced supervectors or divectors whose components are vectors and are defined in the form

$$((a_n)_u) \tag{A57}$$

with elementary vectors as

$$\begin{aligned} &(a_n)_u \\ &n = 1,2,\ldots,N_u \\ &u = 1,2,\ldots,N \end{aligned} \tag{A58}$$

The elements of supervectors are designated as

$$a_{n;u} \tag{A59}$$

From the definition of vector norms as defined in Equations A2 and A3, the p norm, 1 norm, 2 norm, and ∞ norm of a supervector can be defined in terms of its elements as

$$\|((a_n)_u)\|_p = \left\{ \sum_{u=1}^{N} \sum_{n=1}^{N_u} |a_{n;u}|^p \right\}^{1/p} \qquad p \geq 1$$

$$\|((a_n)_u)\|_1 = \sum_{u=1}^{N} \sum_{n=1}^{N_u} |a_{n;u}| \tag{A60}$$

$$\|((a_n)_u)\|_2 = \left\{ \sum_{u=1}^{N} \sum_{n=1}^{N_u} |a_{n;u}|^2 \right\}^{\frac{1}{2}}$$

$$\|((a_n)_u)\|_\infty = \max_{\substack{1 \leq n \leq N_u \\ 1 \leq u \leq N}} |a_{n;u}|$$

Note that the norms in Equation A60 satisfy properties of Equation A1.

The p norm of a supervector can be expressed in terms of the norms of its elementary vectors as

$$\|((a_n)_u)\|_p = \|(\|(a_n)_u\|_p)\|_p$$

$$= \left\{ \sum_{u=1}^{N} \|(a_n)_u\|^p \right\}^{1/p} \tag{A61}$$

That is, the p norm of a supervector is equal to the p norm of a vector whose elements are the norms of the elementary vectors of the supervector.

From Equation A61, the 1, 2, and ∞ norms of a supervector in terms of the norms of its elementary vectors are given by

$$\|((a_n)_u)\|_1 = \sum_{u=1}^{N} \|(a_n)_u\|_1$$

$$\|((a_n)_u)\|_2 = \left\{ \sum_{u=1}^{N} \|(a_n)_u\|_2^2 \right\}^{\frac{1}{2}} \tag{A62}$$

$$\|((a_n)_u)\|_\infty = \max_{1 \leq u \leq N} \|(a_n)_u\|_\infty$$

From Equation A62 we can write the following property for supervector (same as Eq. A4 for vectors)

$$\|((a_n)_u)\|_1 \geq \|((a_n)_u)\|_2 \geq \|((a_n)_u)\|_\infty \tag{A63}$$

or, in general

$$\|((a_n)_u)\|_p \geq \|((a_n)_u)\|_q \qquad p \leq q \tag{A64}$$

Similar to properties of Equation A5 for vectors, we can write the following relations for supervectors from Equation A63 as

$$\begin{aligned} \|((a_n)_u)\|_1 &\leq N_s \|((a_n)_u)\|_\infty \\ \|((a_n)_u)\|_2 &\leq \sqrt{N_s} \|((a_n)_u)\|_\infty \\ \|((a_n)_u)\|_1 &\leq \sqrt{N_s} \|((a_n)_u)\|_2 \end{aligned} \tag{A65}$$

where

$$N_s = \sum_{u=1}^{N} N_u. \tag{A66}$$

A.10 NORMS OF BLOCK-DIAGONAL SUPERMATRICES

Block-diagonal supermatrices were introduced in Section II. A block-diagonal supermatrix is defined as

$$((A_{n,m})_{u,u}) = \begin{pmatrix} (A_{n,m})_{1,1} & & & \bigcirc \\ & (A_{n,m})_{2,2} & & \\ & & \ddots & \\ \bigcirc & & & (A_{n,m})_{N,N} \end{pmatrix} \tag{A67}$$

where $(A_{n,m})_{u,u}$ are square matrices of size $N_u \times N_u$. The block-diagonal supermatrix in Equation A67 may be represented in terms of the direct sum $\oplus$ as

$$((A_{n,m})_{u,u}) \equiv (A_{n,m})_{1,1} \oplus (A_{n,m})_{2,2} \oplus \cdots \oplus (A_{n,m})_{N,N}$$

$$\equiv \bigoplus_{u=1}^{N} (A_{n,m})_{u,u} \qquad \text{(A68)}$$

Since $((A_{n,m})_{u,u})$ is block diagonal, its 1 and ∞ norms are given by

$$\| ((A_{n,m})_{u,u}) \|_{\substack{1\\ \infty}} = \max_{1 \leq u \leq N} \| (A_{n,m})_{u,u} \|_{\substack{1\\ \infty}} \qquad \text{(A69)}$$

The 2 norm of $((A_{n,m})_{u,u})$ is given by

$$\| ((A_{n,m})_{u,u}) \|_2 = [\lambda_{max}\{((A_{n,m})_{u,u})^{\dagger} : ((A_{n,m})_{u,u})\}]^{\frac{1}{2}} \qquad \text{(A70)}$$

Since

$$((A_{n,m})_{u,u})^{\dagger} : ((A_{n,m})_{u,u}) = \bigoplus_{u=1}^{N} (A_{n,m})^{\dagger}_{u,u} \cdot (A_{n,m})_{u,u} \qquad \text{(A71)}$$

and eigenvalues of

$$((A_{n,m})_{u,u})^{\dagger} : ((A_{n,m})_{u,u}) = \text{eigenvalues of } \{(A_{n,m})^{\dagger}_{u,u} \cdot (A_{n,m})_{u,u}\}$$

$$1 \leq u \leq N \qquad \text{(A72)}$$

Then from Equations A70 and A72, the 2 norm of $((A_{n,m})_{u,u})$ is given by

$$\| ((A_{n,m})_{u,u}) \|_2 = \max_u [\lambda_{max}\{(A_{n,m})^{\dagger}_{u,u} \cdot (A_{n,m})_{u,u}\}]^{\frac{1}{2}}$$

$$= \max_u \| (A_{n,m})_{u,u} \|_2$$

$$u = 1,2,\ldots,N \qquad \text{(A73)}$$

Thus the 2 norm of a block-diagonal supermatrix is simply the maximum 2 norm of its block matrices on the diagonal.

A.11 NORM OF AN EXPONENTIAL FUNCTION OF A SQUARE MATRIX

If the power series

$$f(z) = \sum_{k=0}^{\infty} c_k z^k \tag{A74}$$

in a complex variable z converges everywhere, then the matrix power series

$$\sum_{k=0}^{\infty} c_k (A_{n,m})^k \tag{A75}$$

in an $N \times N$ matrix $(A_{n,m})$ converges absolutely (Ref. 7).

In the scalar case e^z is defined by

$$e^z = 1 + z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \tag{A76}$$

Since the power series

$$\sum_{k=0}^{\infty} \frac{1}{k!} z^k \tag{A77}$$

converges everywhere, the matrix power series

$$\sum_{k=0}^{\infty} \frac{1}{k!} (A_{n,m})^k \tag{A78}$$

converges absolutely for any square matrix $(A_{n,m})$. The exponential function of a matrix can thus be defined for every square matrix $(A_{n,m})$ by

$$\begin{aligned} e^{(A_{n,m})} &= (1_{n,m}) + (A_{n,m}) + \frac{1}{2!} (A_{n,m})^2 + \frac{1}{3!} (A_{n,m})^3 + \cdots \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} (A_{n,m})^k \end{aligned} \tag{A79}$$

Using Equation A6 in Equation A79 we can write

$$\begin{aligned} \| e^{(A_{n,m})} \| &\leq \| (1_{n,m}) \| + \| (A_{n,m}) \| + \frac{1}{2!} \| (A_{n,m})^2 \| + \frac{1}{3!} \| (A_{n,m})^3 \| + \cdots \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \| (A_{n,m})^k \| \end{aligned} \tag{A80}$$

Substituting Equation A7 into Equation A80 we get

$$\|e^{(A_{n,m})}\| \le \|(1_{n,m})\| + \|(A_{n,m})\| + \frac{1}{2!}\|(A_{n,m})\|^2 + \frac{1}{3!}\|(A_{n,m})\|^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\|(A_{n,m})\|^k = e^{\|(A_{n,m})\|} \qquad \text{(A81)}$$

Therefore,

$$\|e^{(A_{n,m})}\| \le e^{\|(A_{n,m})\|} \qquad \text{(A82)}$$

Similarly, for an exponential function of $(A_{n,m})t$ we can write

$$\|e^{(A_{n,m})t}\| \le e^{\|(A_{n,m})t\|}$$

$$= e^{\|(A_{n,m})\|\,|t|} \qquad \text{for all finite } t \qquad \text{(A83)}$$

Note that, in general,

$$e^{\{(A_{n,m})+(B_{n,m})\}t} \ne e^{(A_{n,m})t} \cdot e^{(B_{n,m})t} \qquad \text{(A84)}$$

unless $(A_{n,m})$ and $(B_{n,m})$ commute, that is,

$$(A_{n,m}) \cdot (B_{n,m}) = (B_{n,m}) \cdot (A_{n,m}) \qquad \text{(A85)}$$

From the above discussion we can conclude that if a function of a square matrix $(A_{n,m})$ can be expressed as a convergent infinite series as

$$f((A_{n,m})) = \sum_{k=0}^{\infty} c_k (A_{n,m})^k \qquad \text{(A86)}$$

then

$$\|f((A_{n,m}))\| \le f_o(\|(A_{n,m})\|) \equiv \sum_{k=0}^{\infty} |c_k|\,\|(A_{n,m})\|^k \qquad \text{(A87)}$$

A.12 NORM OF FUNCTIONS INVOLVING INTEGRALS

Consider a vector expressed as an integral as

$$(a_n(z)) = \int_{z_o}^{z} (A_{n,m}(z')) \cdot (b_n(z'))\, dz' \qquad \text{(A88)}$$

Taking norm of both sides of Equation A88 we get

$$\|(a_n(z))\| = \left\| \int_{z_o}^{z} (A_{n,m}(z')) \cdot (b_n(z'))\, dz' \right\| \qquad \text{(A89)}$$

The norm of the integral in Equation A89 satisfies the following inequality (Ref. 8)

$$\left\|\int_{z_o}^{z} (A_{n,m}(z')) \cdot (b_n(z'))\, dz'\right\| \le \int_{z_o}^{z} \|(A_{n,m}(z'))\|\ \|(b_n(z'))\|\ dz' \qquad \text{(A90)}$$

Substituting Equation A90 into Equation A89 we get

$$\|(a_n(z))\| \le \int_{z_o}^{z} \|(A_{n,m}(z'))\|\ \|(b_n(z'))\|\ dz' \qquad \text{(A91)}$$

Equation A91 is an important relation which is very useful for many physical problems which often involve relations of the type in Equation A88.

A.13 NORMS OF SUPERMATRICES

Norms of supermatrices can be expressed in terms of norms of their block matrices. The 1 and ∞ norms of a supermatrix can be expressed in terms of 1 and ∞ norms of its block matrices in the following manner:

$$\max_{u,v} \|(A_{n,m})_{u,v}\|_1 \le \|((A_{n,m})_{u,v})\|_1 \le \max_{1\le v\le M} \sum_{u=1}^{N} \|(A_{n,m})_{u,v}\|_1 \qquad \text{(A92)}$$

$$\max_{u,v} \|(A_{n,m})_{u,v}\|_\infty \le \|((A_{n,m})_{u,v})\|_\infty \le \max_{1\le u\le N} \sum_{v=1}^{M} \|(A_{n,m})_{u,v}\|_\infty \qquad \text{(A93)}$$

where $(A_{n,m})_{u,v}$ is an elementary block matrix $(N_u \times N_v)$ of $((A_{n,m})_{u,v})$. $(A_{n,m})_{u,v}$ in general is rectangular.

The relations between 1, 2, and ∞ norms of supermatrices, similar to Equations A30 through A35, can be obtained by following the procedure in Section A.4, and the resulting relations are:

$$\frac{1}{\sqrt{N_s}} \| ((A_{n,m})_{u,v}) \|_2 \leq \| ((A_{n,m})_{u,v}) \|_1 \leq \sqrt{N_s} \| ((A_{n,m})_{u,v}) \|_2 \qquad (A94)$$

$$\frac{1}{N_s} \| ((A_{n,m})_{u,v}) \|_\infty \leq \| ((A_{n,m})_{u,v}) \|_1 \leq N_s \| ((A_{n,m})_{u,v}) \|_\infty \qquad (A95)$$

$$\frac{1}{\sqrt{N_s}} \| ((A_{n,m})_{u,v}) \|_1 \leq \| ((A_{n,m})_{u,v}) \|_2 \leq \sqrt{N_s} \| ((A_{n,m})_{u,v}) \|_1 \qquad (A96)$$

$$\frac{1}{\sqrt{N_s}} \| ((A_{n,m})_{u,v}) \|_\infty \leq \| ((A_{n,m})_{u,v}) \|_2 \leq \sqrt{N_s} \| ((A_{n,m})_{u,v}) \|_\infty \qquad (A97)$$

$$\frac{1}{\sqrt{N_s}} \| ((A_{n,m})_{u,v}) \|_2 \leq \| ((A_{n,m})_{u,v}) \|_\infty \leq \sqrt{N_s} \| ((A_{n,m})_{u,v}) \|_2 \qquad (A98)$$

$$\frac{1}{N_s} \| ((A_{n,m})_{u,v}) \|_1 \leq \| ((A_{n,m})_{u,v}) \|_\infty \leq N_s \| ((A_{n,m})_{u,v}) \|_1 \qquad (A99)$$

where N_s is the number of columns in the supermatrix, which is equal to the size of the supermatrix for the rectangular case.

From Equations A92 and A96 we get

$$\frac{1}{\sqrt{N_s}} \max_{u,v} \sum_{u=1}^{N} \| (A_{n,m})_{u,v} \|_1 \leq \| ((A_{n,m})_{u,v}) \|_2 \leq \sqrt{N_s} \max_{v} \sum_{u=1}^{N} \| (A_{n,m})_{u,v} \|_1 \qquad (A100)$$

Similarly, from Equations A93 and A97 we get

$$\frac{1}{\sqrt{N_s}} \max_{u,v} \sum_{v=1}^{M} \| (A_{n,m})_{u,v} \|_\infty \leq \| ((A_{n,m})_{u,v}) \|_2 \leq \sqrt{N_s} \max_{u} \sum_{v=1}^{M} \| (A_{n,m})_{u,v} \|_\infty \qquad (A101)$$

Substituting Equation A30 into Equation A100 and Equation A34 into A101, respectively, we get

$$\frac{1}{\sqrt{N_s}} \max_{u,v} \frac{1}{\sqrt{N_v}} \|(A_{n,m})_{u,v}\|_2 \leq \|((A_{n,m})_{u,v})\|_2 \leq \sqrt{N_s} \max_{v} \sum_{u=1}^{N} \sqrt{N_v} \|(A_{n,m})_{u,v}\|_2 \qquad \text{(A102)}$$

$$\frac{1}{\sqrt{N_s}} \max_{u,v} \frac{1}{\sqrt{N_v}} \|(A_{n,m})_{u,v}\|_2 \leq \|((A_{n,m})_{u,v})\|_2 \leq \sqrt{N_s} \max_{v} \sum_{v=1}^{M} \sqrt{N_v} \|(A_{n,m})_{u,v}\|_2 \qquad \text{(A103)}$$

where N is the number of columns in the u,v block matrix $(A_{n,m})_{u,v}$.

Equations A102 and A103 give the 2 norm of a supermatrix in terms of the 2 norms of its block matrices, and Equations 92 and 93 give the 1 and ∞ norms of a supermatrix in terms of 1 and ∞ norms of its block matrices, respectively.

APPENDIX B

TWO NORM OF THE SCATTERING MATRIX AT A TERMINATION OF A UNIFORM MULTICONDUCTOR TRANSMISSION LINE

From the power conservation, the reflected power from a passive termination is always less than or equal to the incident power for physically realizable systems. The power-conservation condition can be expressed in terms of the combined voltage vectors for waves leaving and entering the termination (for $s = j\omega$) as

$$\mathrm{Re}[(\tilde{V}_n(s))_+ \cdot (\tilde{I}_n(s))_+^*] \leq \mathrm{Re}[(\tilde{V}_n(s))_- \cdot (\tilde{I}_n(s))_-^*] \tag{B1}$$

where $(\tilde{V}_n(s))_+$ and $(\tilde{V}_n(s))_-$ are combined voltage vectors for waves leaving and entering the termination, respectively, and $(\tilde{I}_n(s))_+$ and $(\tilde{I}_n(s))_-$ are combined current vectors for waves leaving and entering the termination (junction). The $*$ represents a complex conjugate. Currents are positive into the junction.

Equation B1 can be rearranged to give

$$\mathrm{Re}[(\tilde{I}_n(s))_+^* \cdot (\tilde{V}_n(s))_+] \leq \mathrm{Re}[(\tilde{I}_n(s))_-^* \cdot (\tilde{V}_n(s))_-] \tag{B2}$$

The combined voltage and current vectors are related through the characteristic-admittance matrix of the transmission line as

$$(\tilde{I}_n(s))_+ = (\tilde{Y}_{c_{n,m}}(s)) \cdot (\tilde{V}_n(s))_+ \tag{B3}$$

$$(\tilde{I}_n(s))_- = (\tilde{Y}_{c_{n,m}}(s)) \cdot (\tilde{V}_n(s))_- \tag{B4}$$

If $(\tilde{Y}_{c_{n,m}}(s))$ is a real, diagonal matrix with equal diagonal elements, i.e., the lines are decoupled and have same characteristic admittances, then substitution of Equations B3 and B4 into B2 yields (the general case will be discussed in a future paper)

$$(\tilde{V}_n(s))_+^* \cdot (\tilde{V}_n(s))_+ \leq (\tilde{V}_n(s))_-^* \cdot (\tilde{V}_n(s))_-$$

or

$$(\tilde{V}_n(s))_+ \cdot (\tilde{V}_n(s))_+^* \leq (\tilde{V}_n(s))_- \cdot (\tilde{V}_n(s))_-^* \tag{B5}$$

Note that a minus sign would result on the right-hand side of Equation B5 after substitution but, since we are interested in magnitude on both sides, it has been dropped.

The combined voltage vectors for waves leaving and entering the termination (junction) are related through the scattering matrix $(\tilde{S}_{n,m}(s))$ of the termination as (for $s = j\omega$)

$$(\tilde{V}_n(s))_+ = (\tilde{S}_{n,m}(s)) \cdot (\tilde{V}_n(s))_- = (\tilde{V}_n(s))_- \cdot (\tilde{S}_{n,m}(s))^T \quad \text{(B6)}$$

Substituting Equation B6 into B5 we obtain

$$(\tilde{V}_n(s))_- \cdot (\tilde{S}_{n,m}(s))^T \cdot (\tilde{S}_{n,m}(s))^* \cdot (\tilde{V}_n(s))_-^* \leq (\tilde{V}_n(s))_- \cdot (\tilde{V}_n(s))_-^* \quad \text{(B7)}$$

For any eigenvector $(\tilde{X}_n(s))$ of matrix $(\tilde{S}_{n,m}(s))^T \cdot (\tilde{S}_{n,m}(s))^*$ with eigenvalues λ_n, one obtains

$$(\tilde{X}_n(s)) \cdot (\tilde{S}_{n,m}(s))^T \cdot (\tilde{S}_{n,m}(s))^* \cdot (\tilde{X}_n(s))^* = \lambda_n(\tilde{X}_n(s)) \cdot (\tilde{X}_n(s))^* \quad \text{(B8)}$$

But according to Equation B7

$$(\tilde{X}_n(s)) \cdot (\tilde{S}_{n,m}(s))^T \cdot (\tilde{S}_{n,m}(s))^* \cdot (\tilde{X}_n(s))^* \leq (\tilde{X}_n(s)) \cdot (\tilde{X}_n(s))^* \quad \text{(B9)}$$

Therefore,

$$\lambda_n(\tilde{X}_n(s)) \cdot (\tilde{X}_n(s))^* \leq (\tilde{X}_n(s)) \cdot (\tilde{X}_n(s))^* \quad \text{(B10)}$$

or

$$\lambda_n \leq 1 \quad \text{(B11)}$$

also $\lambda_n \geq 0$ since $(\tilde{S}_{n,m}(s))^T \cdot (\tilde{S}_{n,m}(s))^*$ is hermitian, positive semidefinite. Since Equation B11 is true for any eigenvalue, we have

$$\lambda_{max}\{(\tilde{S}_{n,m}(s))^T \cdot (\tilde{S}_{n,m}(s))^*\} \leq 1$$

or

$$\lambda_{max}\{(\tilde{S}_{n,m}(s))^{\dagger} \cdot (\tilde{S}_{n,m}(s))\}^{\frac{1}{2}} \leq 1 \tag{B12}$$

Hence

$$\|(\tilde{S}_{n,m}(s))\|_2 \leq 1 \tag{B13}$$

Following the procedure used above, we can show that the 2 norm of the inverse of the scattering matrix is greater than or equal to one.

From Equation B6, we can write

$$(\tilde{V}_n(s)) = (\tilde{S}_{n,m}(s))^{-1} \cdot (\tilde{V}_n(s))_+ = (\tilde{V}_n(s))_+ \cdot (\tilde{S}_{n,m}(s))^{-1T} \tag{B14}$$

Substituting Equation B14 into Equation B5 we get

$$(\tilde{V}_n(s))_+ \cdot (\tilde{V}_n(s))_+^* \leq (\tilde{V}_n(s))_+ \cdot (\tilde{S}_{n,m}(s))^{-1T} \cdot (\tilde{S}_{n,m}(s))^{-1*} \cdot (\tilde{V}_n(s))_+^* \tag{B15}$$

For any eigenvector $(\tilde{Y}_n(s))$ of matrix $(\tilde{S}_{n,m}(s))^{-1T} \cdot (\tilde{S}_{n,m}(s))^{-1*}$ with eigenvalues μ_n, one obtains

$$(\tilde{Y}_n(s)) \cdot (\tilde{S}_{n,m}(s))^{-1T} \cdot (\tilde{S}_{n,m}(s))^{-1*} \cdot (\tilde{Y}_n(s))^* = \mu_n(\tilde{Y}_n(s)) \cdot (\tilde{Y}_n(s))^* \tag{B16}$$

From Equations B15 and B16 we obtain

$$\mu_n(\tilde{Y}_n(s)) \cdot (\tilde{Y}_n(s))^* \geq (\tilde{Y}_n(s)) \cdot (\tilde{Y}_n(s))^*$$

or

$$\mu_n \geq 1 \tag{B17}$$

Since Equation B17 is true for any eigenvalue, we have

$$\lambda_{max}\{(\tilde{S}_{n,m}(s))^{-1\dagger} \cdot (\tilde{S}_{n,m}(s))\}^{\frac{1}{2}} \geq 1$$

or

$$\|(\tilde{S}_{n,m}(s))^{-1}\|_2 \geq 1 \tag{B18}$$

APPENDIX C
TWO NORM OF THE LOSSLESS JUNCTION SCATTERING SUPERMATRIX

From the power conservation condition, the reflected power is equal to the incident power for a lossless junction (a junction with interconnection of wires only). For a junction with identical branches, whose characteristic-admittance matrices are real, diagonal matrices with equal diagonal elements, i.e., the lines in the branches are decoupled and have same characteristic admittances, the power conservation condition can be expressed as (for $s = j\omega$)

$$((\tilde{V}_n(L_u,s))_u)_\nu : ((\tilde{V}_n(L_u,s))_u)_\nu^* = ((\tilde{V}_n(0,s))_u)_\nu : ((\tilde{V}_n(0,s))_u)_\nu^* \quad \text{(C1)}$$

where the subscript ν is for the νth junction.

The combined voltage supervector for waves leaving and entering the junction is related through the scattering supermatrix of the junction as

$$((\tilde{V}_n(0,s))_u)_\nu = ((\tilde{S}_{n,m}(s))_{u,v})_\nu : ((\tilde{V}_n(L_u,s))_u)_\nu \quad \text{(C2)}$$

where $((\tilde{S}_{n,m}(s))_{u,v})_\nu$ is the scattering supermatrix of the νth junction. Substituting Equation C2 into Equation C1 we get

$$((\tilde{V}_n(L_u,s))_u)_\nu : ((\tilde{V}_n(L_u,s))_u)_\nu^* =$$

$$((\tilde{V}_n(L_u,s))_u)_\nu : ((\tilde{S}_{n,m}(s))_{u,v})_\nu^T : ((\tilde{S}_{n,m}(s))_{u,v})_\nu^* : ((\tilde{V}_n(L_u,u))_u)_\nu^*$$

or

$$((\tilde{V}_n(L_u,s))_u)_\nu : [((1_{n,m})_{u,v})_\nu - ((\tilde{S}_{n,m}(s))_{u,v})_\nu^T : ((\tilde{S}_{n,m}(s))_{u,v})_\nu^*]$$

$$: ((\tilde{V}_n(L_u,s))_u)_\nu^* = ((0_n)_u) \quad \text{(C3)}$$

This equation can hold only if

$$((\tilde{S}_{n,m}(s))_{u,v})_\nu^T : ((\tilde{S}_{n,m}(s))_{u,v})_\nu^* = ((1_{n,m})_{u,v})_\nu$$

or

$$((\tilde{S}_{n,m}(s))_{u,v})^T_\nu = ((\tilde{S}_{n,m}(s))_{u,v})^{*-1}_\nu$$

or

$$((\tilde{S}_{n,m}(s))_{u,v})^\dagger_\nu = ((\tilde{S}_{n,m}(s))_{u,v})^{-1}_\nu \tag{C4}$$

Since $((\tilde{V}_n(L_u,s))_u)$ is not zero.

The result in Equation C4 is the definition of a unitary matrix. From Equation C4 we can write

$$((\tilde{S}_{n,m}(s))_{u,v})^\dagger_\nu : ((\tilde{S}_{n,m}(s))_{u,v})_\nu = ((1_{n,m})_{u,v}) \tag{C5}$$

Since all the eigenvalues of the identity matrix are equal to one, from the definition of the 2 norm of a matrix (Eq. A11) we obtain

$$\|((\tilde{S}_{n,m}(s))_{u,v})_\nu\|_2 = 1 \tag{C6}$$

REFERENCES

1. Baum, C. E., "Electromagnetic Topology: A Formal Approach to the Analysis and Design of Complex Electronic Systems," AFWL EMP Interaction Note 400, Air Force Weapons Laboratory, Kirtland AFB, NM, Spetember 1980.

2. Baum, C. E., T. K. Liu, and F. M. Tesche, "On the Analysis of General Multiconductor Line Networks," AFWL EMP Interaction Note 350, Air Force Weapons Laboratory, Kirtland AFB, NM, November 1978.

3. Baum, C. E., "Norms and Eigenvector Norms," Mathematics Note 63, Air Force Weapons Laboratory, Kirtland AFB, NM, November 1979.

4. Davis, W. A., and M. K. Sistanizadeh, "Bounding Signal Levels at Terminations of a Multiconductor Line Behind an Aperture," Air Force Weapons Laboratory Interaction Note 412, Kirtland AFB, NM, June 1981.

5. Ralston, A., and P. Rabinowitz, A First Course in Numerical Analysis, Second Edition, McGraw Hill, 1978.

6. Forsythe, G., and C. B. Moler, Computer Solution of Linear Algebriac Systems, Prentice Hall, 1967.

7. Ogata, K., State Space Analysis of Control Systems, Prentice-Hall, Inc., 1967.

8. Brauer, F., and J. A. Nohel, Qualitative Theory of Ordinary Differential Equations, W. A. Benjamin, Inc., NY, 1969.

9. Paul, C. R., "Applications of Multiconductor Transmission Line Theory to the Prediction of Cable Coupling, Volume I," Final Technical Report, RADC-TR-76-101, Griffiss AFB, Rome Air Development Center, Rome, NY, April 1976. Also published as AFWL EMP Interaction Note 370.

10. Baum, C. E., T. K. Liu, F. M. Tesche, and S. K. Chang, "Numerical Results for Multiconductor Transmission-Line Networks," AFWL EMP Interaction Note 322, Air Force Weapons Laboratory, Kirtland AFB, NM, September 1977.

11. Agrawal, A. K., H. M. Fowles, L. D. Scott, and S. H. Gurbaxani, "Time-Domain Analysis of Multiconductor Transmission Lines with Branches in Inhomogeneous Media," Air Force Weapons Laboratory Interaction Note 333, Kirtland AFB, NM, February 1978.

12. Collin, R. E., Foundations for Microwave Engineering, McGraw Hill, 1966.

13. Agrawal, A. K., H. M. Fowles, and L. D. Scott, "Experimental Characterization of Multiconductor Transmission Lines in Inhomogeneous Media Using Time-Domain Techniques, Air Force Weapons Laboratory Interaction Note 332, Kirtland AFB, NM, February 1978.

EPILOGUE

"Can you do Addition?" the White Queen asked. What's one and one and one and one and one and one and one and one and one and one?

"I don't know," said Alice. "I lost count."

"She can't do Addition," the Red Queen interrupted. "Can you do Subtraction? Take nine from eight."

"Nine from eight. I can't, you know," Alice replied very readily, "but--"

"She can't do Subtraction," said the White Queen. "Can you do Division? Divide a loaf by a knife--what's the answer to that?"

"I suppose--" Alice was beginning, but the Red Queen answered for her. "Bread and Butter, of course. Try another Subtraction sum. Take a bone from a dog; what remains?"

Alice considered. "The bone wouldn't remain, of course, if I took it--and the dog wouldn't remain; it would come to bite me--and I'm sure _I_ shouldn't remain!"

"Then you think nothing would remain?" said the Red Queen.

"I think that's the answer."

"Wrong, as usual," said the Red Queen. "The dog's temper would remain."

"But I don't see how--"

"Why, look here!" the Red Queen cried. "The dog would lose its temper, wouldn't it?"

"Perhaps it would," Alice replied cautiously.

"Then if the dog went away, its temper would remain!" the Queen exclaimed triumphantly.

Alice said, as gravely as she could, "They might go different ways." But she couldn't help thinking to herself, "What dreadful nonsense we _are_ talking!"

from Through the Looking Glass,
by Lewis Carroll

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