Interaction Notes

Note 453

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Transient Response of an Infinite Wire in a Dissipative Medium

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Abstract

Currents on a buried wire due to both a localized voltage source and a normal incident wave from above an air-ground interface are calculated using a theory developed earlier. The diffusion limit is carried out to give the following: (1) universal curves for numerical applications, and (2) the effect of an air-ground interface.

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Section 1. Introduction and Summary

The determination of the transient response of an infinite wire in a dissipative medium is significant in numerous applications. Sunde[1] in his early investigation is most concerned with the attachment of a lightning stroke on a buried wire. He used a diffusion approximation to obtain the current propagation function to calculate the wire current as a function of distance from the transmission point. More recent applications are the calculation of the current on a buried wire due to an incident time domain wave or a localized voltage source. The incident electromagnetic field can be due to an indirect lightning source, a High Altitude Burst (HAB) and a Surface Burst (SB) Electromagnetic Pulse (NEMP). The buried wire can be a telephone line, or a power line, or a VLF antenna. The problem can also arise from the need to determine the transient antenna current on a long dipole buried in the earth or submerged in water.

To give a more specific example, consider the problem of a surface nuclear burst. When a nuclear device detonates low enough in altitude as to cause the fireball to attach electrically to a buried power line, the gamma ray knocks electrons off the atmospheric molecules and thus creates the Compton current. This current, in turn, generates the electric field along the power line. In general, the electric field falls off in distance away from the fireball; most of the voltage drop occurs approximately 200 meters from the fireball. Therefore, when calculating the induced line current at a distance greater than 200 meters, it is reasonable to assume an input voltage applied at a delta gap of an infinite wire. Furthermore, the air conductivity due to radiation drops off so quickly, it is negligible.

The above example typifies how the transient solution of an infinite wire in a dissipative medium can be useful. Recently a theory for the transient response of an infinite cylindrical antenna in a dissipative medium was developed.[2] This new theory gives a very accurate description of the transient response. Our principal objective here is to show the use of this theory in applications. In particular: (1) the accuracy of the conventional diffusion approximation is determined, (2) under the diffusion approximation the effect of the air-ground interface is included, and (3) universal curves for numerical applications are given. During the course of this investigation results obtained are

1. Clarification of an important discrepancy in the early-time solution for the wire current in free space (Appendix A).

2. The time-harmonic solutions for an infinite wire in a dissipative medium (Appendix B).

3. A simple integral formula for the wire current due to a double exponential input voltage (Section 2).

4. The effect of the air-ground interface under the diffusion approximation (Section 3).

5. A numerical example to treat a realistic case where the relative dielectric constant is a function of frequency (Section 4).

6. The current on a buried wire under a normally incident waveform above an air-ground interface (Section 5).

All physical quantities are given in the M.K.S. System; i.e., current in Amps and voltages in Volts.

Section 2. An Integral Formula for Obtaining the Wire Current Due to a Double Exponential Input Voltage

When the input voltage is

$$\mathscr{V}_o(t) = e^{-t/t_o} u(t) \qquad (1)$$

where u(t) is the unit step function, the method of superposition can be applied to calculate the corresponding current:

$$\mathscr{I}_o(z,t) = \int_{-\infty}^{\infty} \left[\left(\frac{\partial}{\partial t'} + \frac{\sigma}{\varepsilon} \right) I(z,t) \right] e^{-(t-t')/t_o} u(t - t')\, dt' \quad . \qquad (2)$$

Normalizing the variables in (2) to the unit of wire radius $t_n = ct/a$, $t_{on} = ct_o/a$, and $z_n = z/a$, and evaluating the resulting integral by parts, yields

$$\mathscr{I}_o = I(z,t) + \left(2\alpha - t_{on}^{-1}\right) \int_0^{t_n} I(z',t)\, e^{-(t_n - t_n')/t_{on}}\, dt_n' \quad . \qquad (3)$$

Notice I(z,t) is given by Reference 1, Equations (2) or (3) or (6). One special case is of great interest. Let $t_{on} = (2\alpha)^{-1}$, then $\mathscr{I}_o \equiv I(z,t)$. This is the wire current due to an input voltage of

$$\mathscr{V}_o(t) = e^{-2\alpha t_n} u(t) \quad . \qquad (4)$$

In free space, the input voltage reduces to a unit step function. This is the case discussed extensively in the literature.[3,4,5]

The input voltage, Equation (4), contains a discontinuity at $t = o$. This gives rise to the singularity in the current $I(z,t)$ at $ct - z = o$ (Appendix A). In applications, the input voltage does not contain a discontinuity. In the following, a construction of the solution is shown for this case. Begin with Equation (3), which is the current due to an input voltage of Equation (1). The first term on the right-hand side has been shown to be the response current to the voltage given by Equation (4). Therefore the second term on the right-hand side of Equation (3) is the current due to the input voltage of

$$\mathcal{V}(t) = \left(e^{-t/t_o} - e^{-\sigma t/\varepsilon}\right) u(t) = \left(e^{-t_n/t_{on}} - e^{-2\alpha t_n}\right) u(t_n) \quad . \qquad (5)$$

The corresponding current response is

$$\mathcal{I}(z,t) = \left(\frac{\sigma}{\varepsilon} - t_o^{-1}\right)\int_o^t I(z,t')\, e^{-(t-t')/t_o}\, dt'$$

$$= \left(2\alpha - t_{on}^{-1}\right)\int_o^\tau e^{-\left(t_n-\sqrt{\tau'^2+z_n^2}\right)/t_{on}}\, I(z,t')\, \frac{\tau' d\tau'}{\sqrt{\tau'^2 + z_n^2}} \qquad (6)$$

where $\quad t_n' = \sqrt{\tau'^2 + z_n^2} \quad .$

The wavefront singularity of $I(z,t')$ is shown to be of $1/\tau'$ in Appendix A. Since it is multiplied by τ' in the integrand, it contributes negligibly to the integral. Therefore, Reference 2 Equation (6) can be used for determining the numerical values of the wire current.

To obtain the current response for a double exponential input voltage, e.g.,

$$\mathcal{V}(t) = \left(e^{-t/t_f} - e^{-t/t_r}\right) u(t) \quad , \tag{7}$$

write Equation (7) as

$$\mathcal{V}(t) = \left(e^{-t/t_f} - e^{-\sigma t/\varepsilon}\right) u(t) - \left(e^{-t/t_r} - e^{-\sigma t/\varepsilon}\right) u(t) \quad . \tag{8}$$

As a result, the corresponding current response can be constructed easily from Equation (6). A simple computer program using Equation (6) can be written to generate the wire current for any input voltage at the gap. However, the diffusion limiting case of Equation (6) plays an important role in applications. Universal curves for obtaining the current waveforms for this important case will be given.

The diffusion limit of Reference 2 Equation (6) for $I(z,t)$ yields accurate numerical results when two conditions are met. The first is that $I_o(\alpha\tau)$ be replaced by its leading asymptotic term. When the argument $\alpha\tau = 5$, the asymptote deviates less than 1 percent from $I_O(\alpha\tau)$. The second condition is that $t >> z/c$, when $t = 3\ z/c$, $\left(t - z^2/2c^2t\right)$ deviates less than 0.5 percent from $\sqrt{t^2 - z^2/c^2}$. The arctangent function can also be replaced by its argument with minimum errors introduced. The diffusion approximation obtained is

$$I(z,t) \sim 2\varepsilon\left(\frac{\pi}{\mu\sigma t}\right)^{1/2} \frac{e^{-z^2/2\delta^2}}{2\ell n\ \delta/a + \ell n\ 2-\gamma} \tag{9}$$

where $\delta = \sqrt{2t/\sigma\mu}$ (10)

and γ = Euler's constant = 0.57721 ... .

Next, evaluate Equation (6) with I(z,t) approximated by Equation (9):

$$\mathscr{I}(z,t) \sim \left(\sigma/\varepsilon - t_o^{-1}\right) 2\varepsilon(\pi/\mu\sigma)^{1/2} \int_o^t \frac{e^{-\frac{z^2\sigma\mu}{4(t-t')} - t'/t_o} dt'}{(t-t')^{1/2}\left\{\ell n\left[\frac{2(t-t')}{\sigma\mu a^2}\right] + \ell n\ 2-\gamma\right\}} \quad (11)$$

The second factor in the denominator of the above integrand is the impedance factor that is slowly varying and can be taken out of the integral. Thus,

$$\mathscr{I}(z,t) \sim \frac{\left(\sigma/\varepsilon - t_o^{-1}\right) 4\varepsilon\left(\frac{\pi t_o}{\mu\sigma}\right)^{1/2}}{2\ell n\left(\frac{\delta}{a}\right) + \ell n\ 2-\gamma}\, g(z,t)$$

$$= Z(t_o,t)\ g(z,t) \quad (12)$$

where $g(z,t) = e^{-t/t_o} \int_0^{\sqrt{t/t_o}} e^{-A/u^2+u^2}\, du$, (13)

$$A = \frac{z^2\sigma\mu}{4t_o} = \frac{z^2}{4(ct_o)(ct_{re\ell})} \left(= \frac{z_n^2}{2t_{on}\alpha^{-1}}\right) ,$$

$\tau_{rel} = \varepsilon/\sigma$ = the relaxation time of the medium

$$Z(t_o,t) = \frac{4\sqrt{\pi}(t_o - t_{rel})}{\zeta_o(t_o t_{rel})^{1/2}\left[2\ln\frac{\delta}{a} + \ln 2-\gamma\right]}$$

Notice $Z(t_o,t)$ can be explained as the admittance factor, and $g(z,t)$ is a normalized admittance function. When $t_o >> t_{rel}$, $Z(t_o,t)$ is approximately proportional to $\sqrt{t_o}$.

Notice that the last expression for A is in a normalized notation defined in Reference 2, and A is a function of t_o. In Figure 1, $g(z,t)$ is shown as a function of t/t_o for $A = 10^{-4}$, 10^{-2}, 10^{-1}, 1, 10, and 100. Similar curves were given by Sunde[1] when he considered the propagation function for the surge current in a buried wire. Our excitation is an input voltage at the delta gap. Another important difference is that the input voltage considered is Equation (5), instead of the more conventional $e^{-t/t_o}u(t)$. This stretches the applicability of the diffusion approximation by removing the discontinuity in the input voltage.

In order to determine the accuracy of Figure 1 in applications, it is necessary to investigate the numerical errors introduced by the various approximations. Extensive numerical investigations have been made on this. The parameters chosen for this discussion are: for σ = 0.01 S/m, and ϵ_r = 40, $t_o = 2t_e$, $5t_e$, $10t_e$, and $40t_e$. Comparison of various approximations are shown in Figures 2a-c through 5a-c. In Part a of these figures, Equation (12) is plotted by using $g(z,t)$ given in Figure 1, which is a diffusion approximation with time-domain impedance taken out of the integrand. In Part b we plotted Equation (11), which is a diffusion

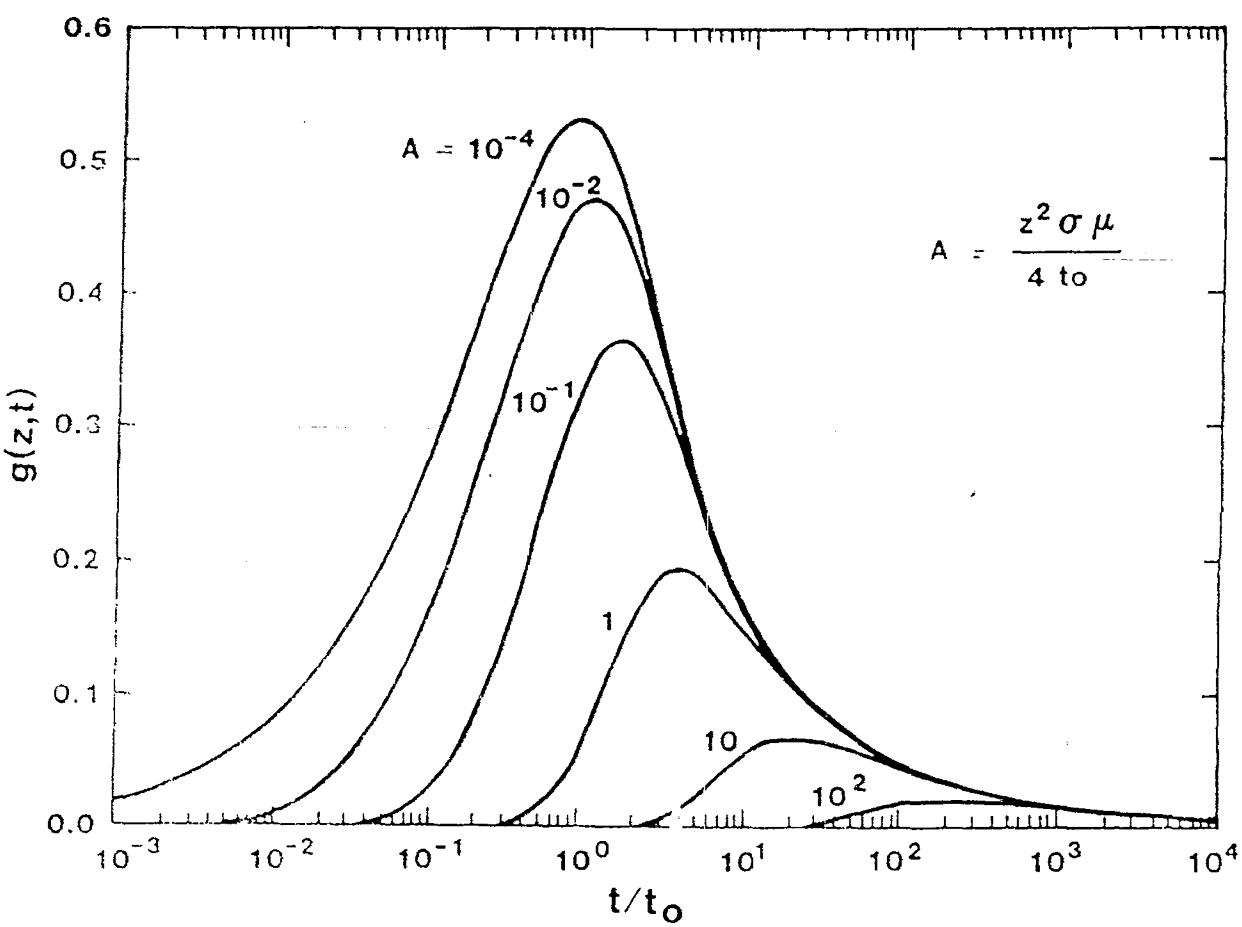


Figure 1. Universal Curves for Wire Currents Due to a Localized Voltage Source. These curves are obtained by a diffusion approximation.

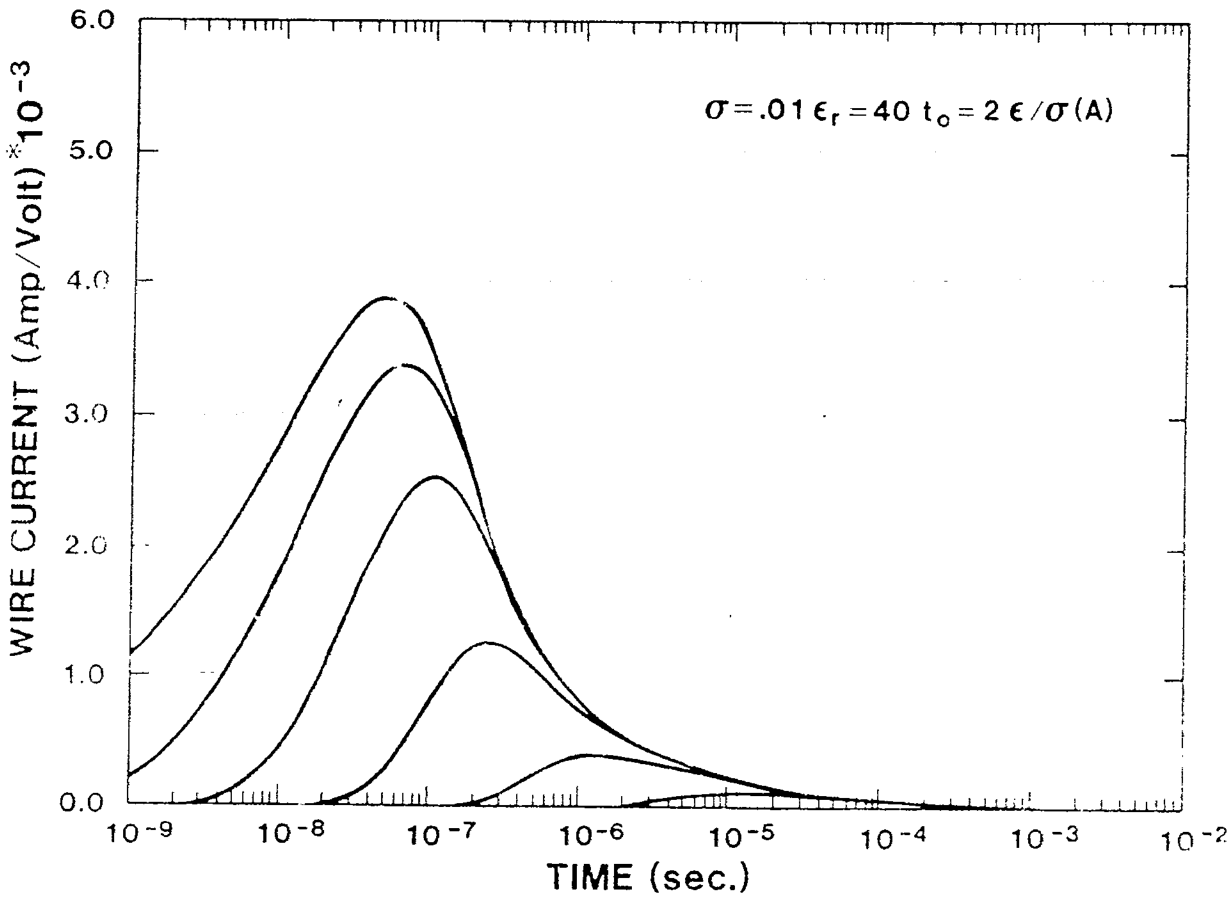


Figure 2a. Wire Currents for σ = 0.01 S/m, and ϵ_r = 40, t_o = $2t_e$

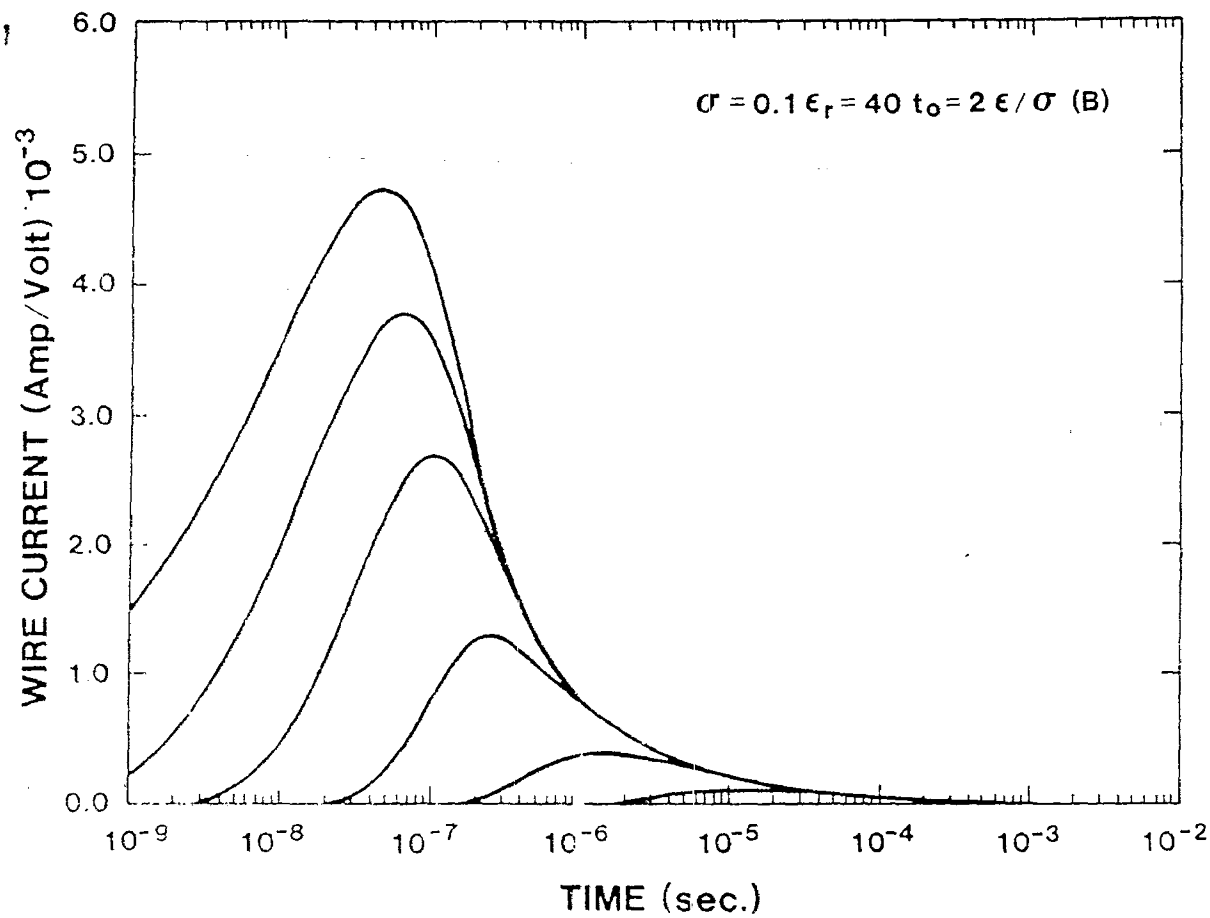


Figure 2b. Wire Currents for σ = 0.01 S/m, and ε_r = 40, t_o = $2t_e$

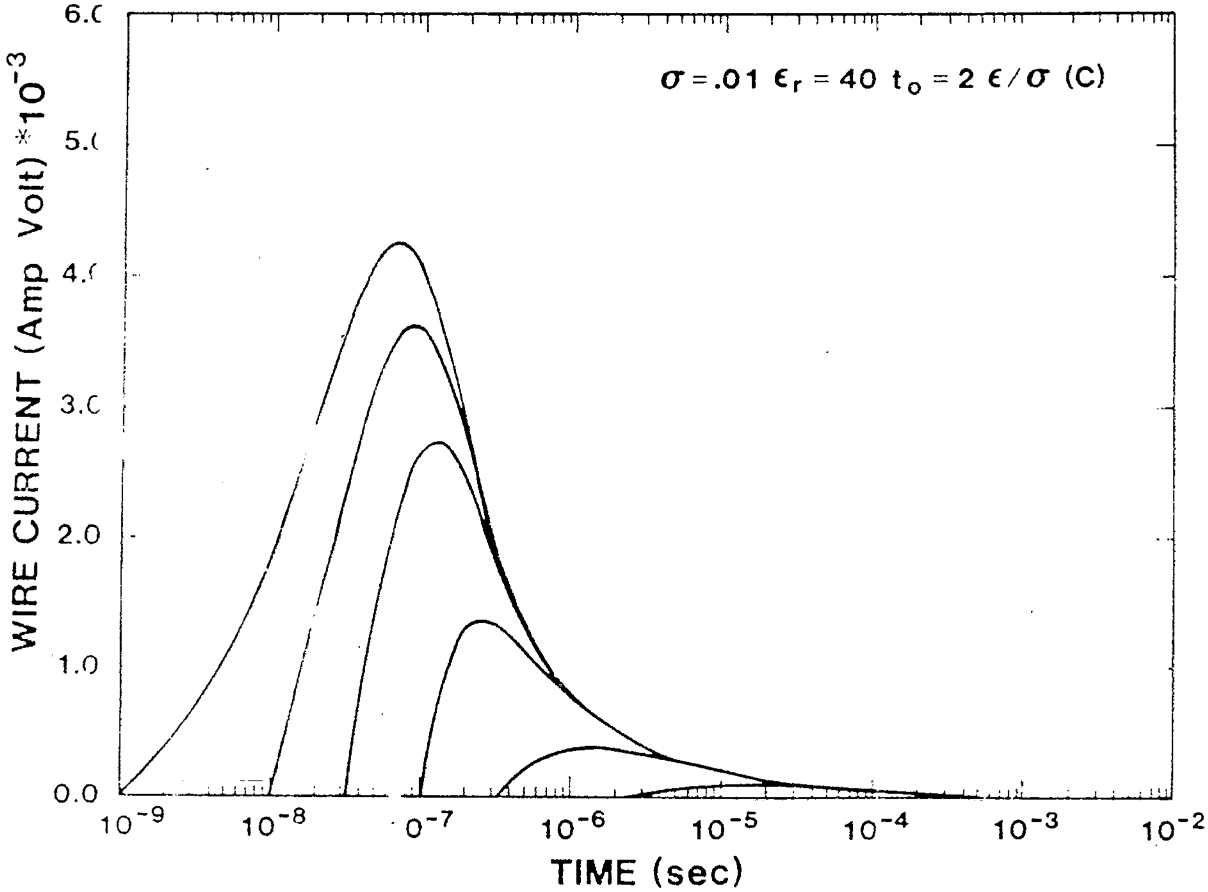


Figure 2c.—Wire Currents for σ = 0.01 S/m, and ϵ_r = 40, $t_o = 2t_e$

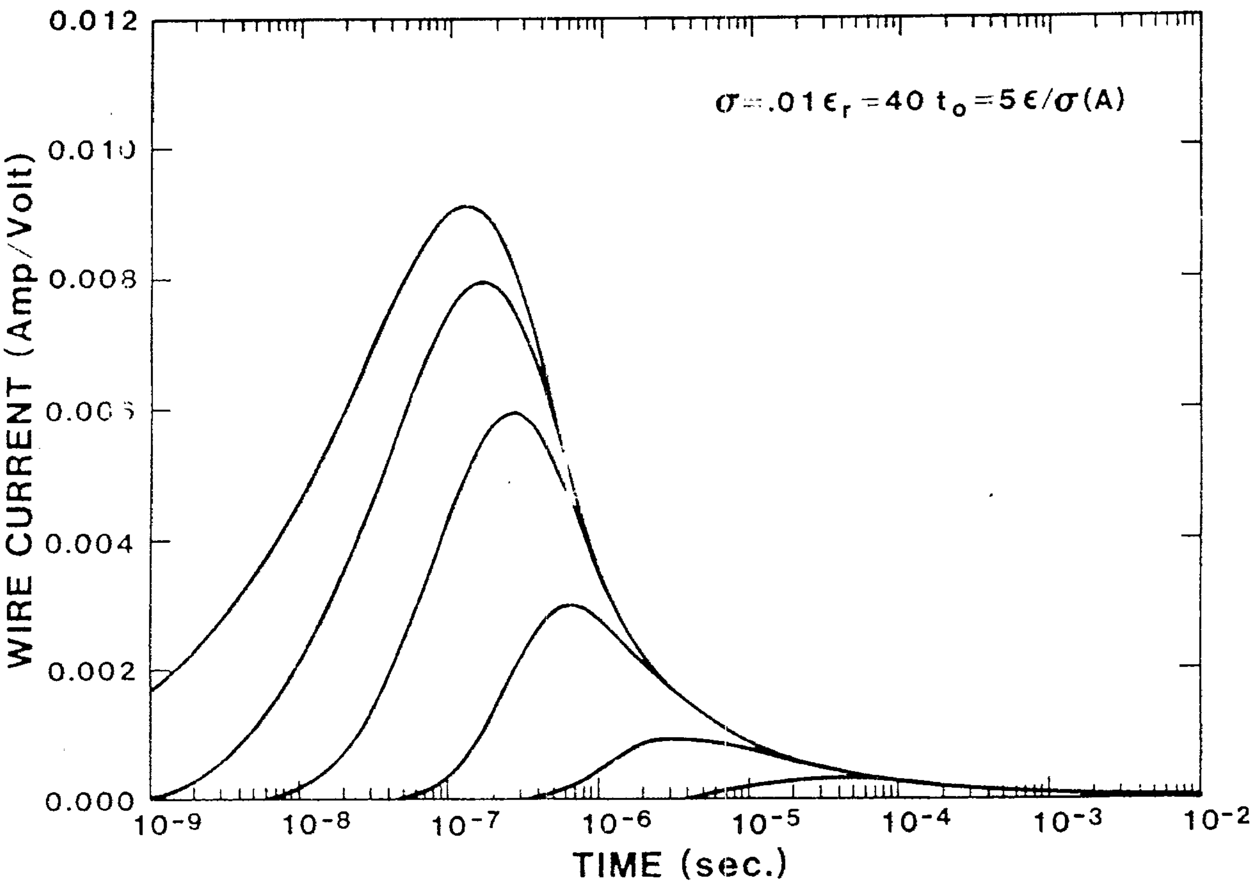


Figure 3a. Wire Currents for σ = 0.01 S/m, and ε_r = 40, t_o = $5t_e$

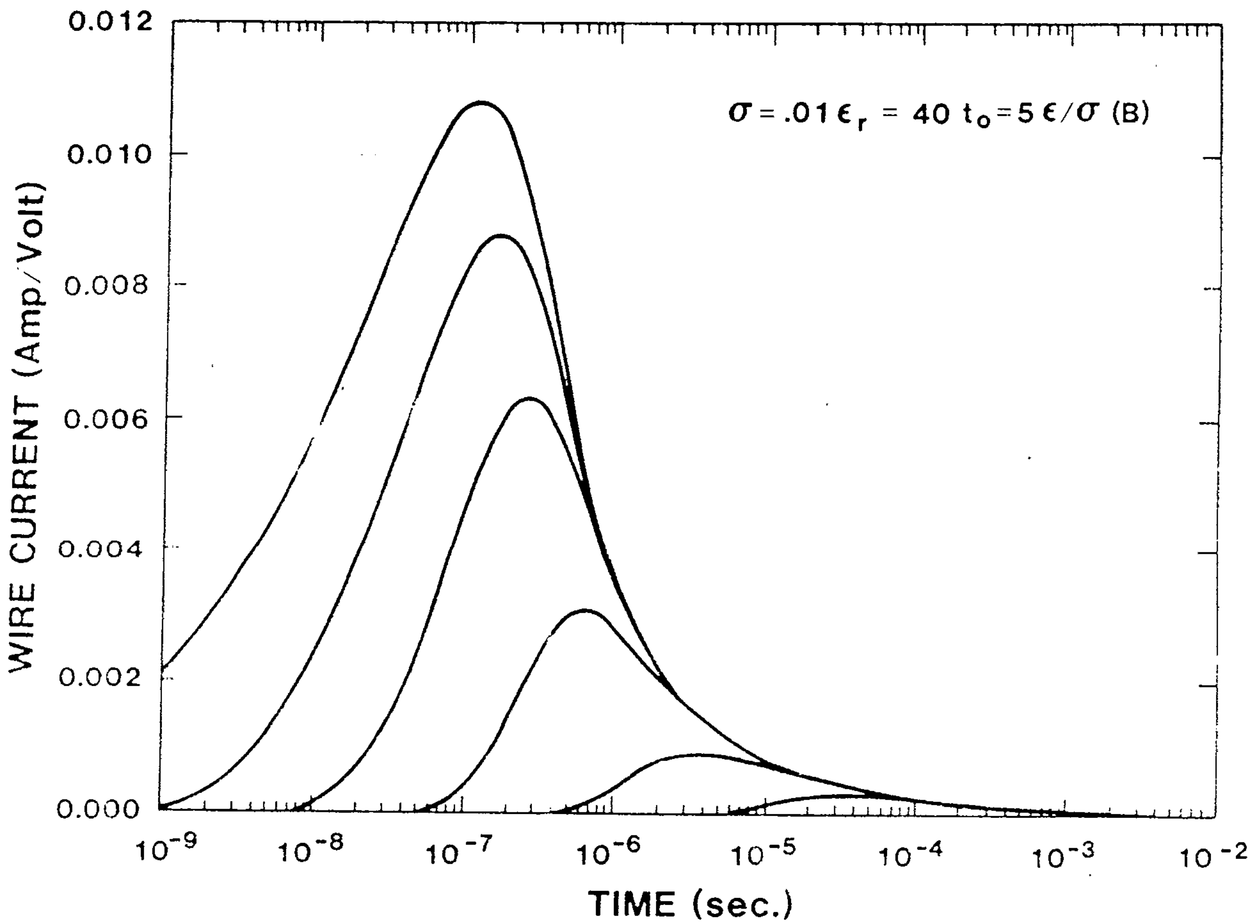


Figure 3b. Wire Currents for σ = 0.01 S/m, and ε_r = 40, t_o = $5t_e$

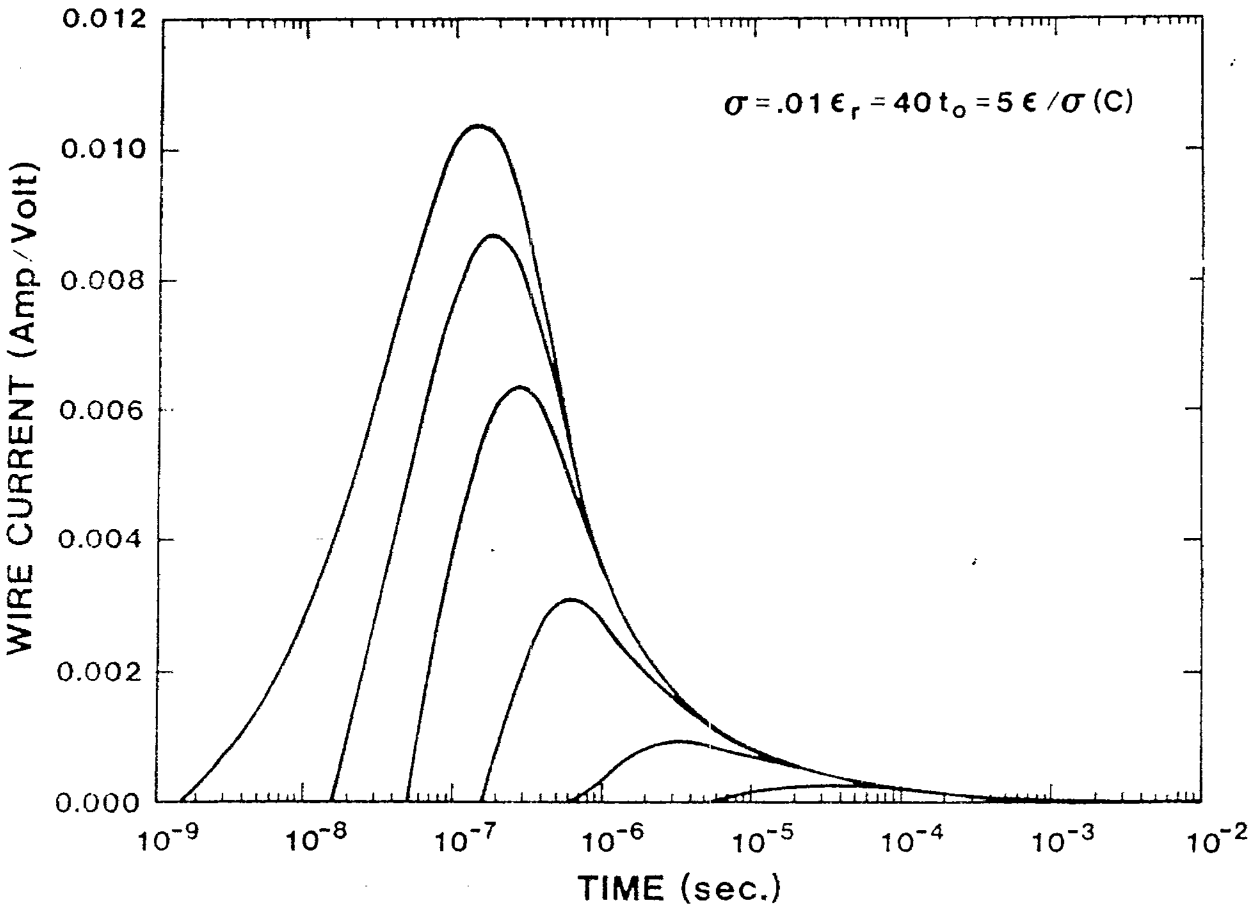


Figure 3c. Wire Currents for σ = 0.01 S/m, and ε_r = 40, t_o = $5t_e$

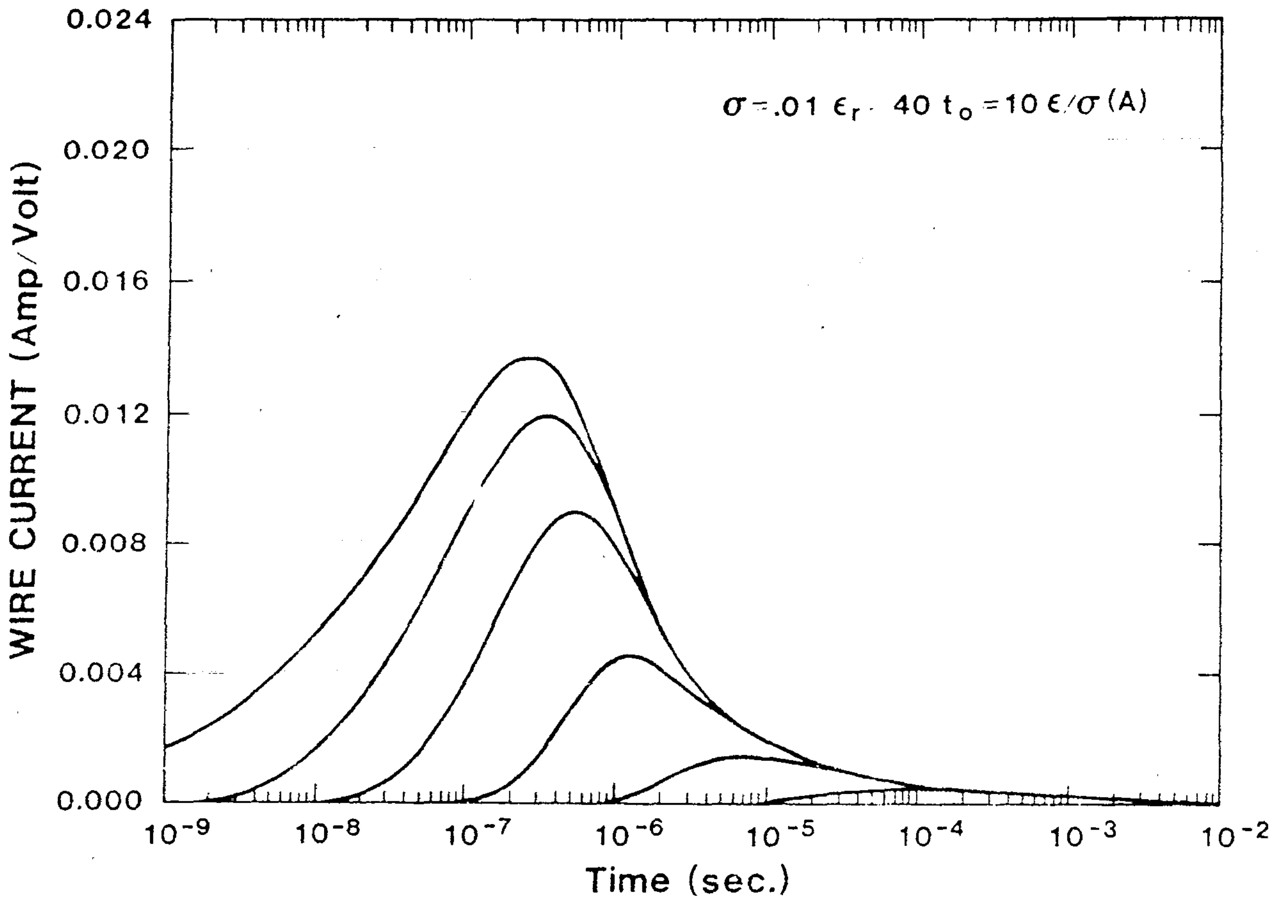


Figure 4a. Wire Currents for $\sigma = 0.01$ S/m, and $\epsilon_r = 40$, $t_o = 10t_e$

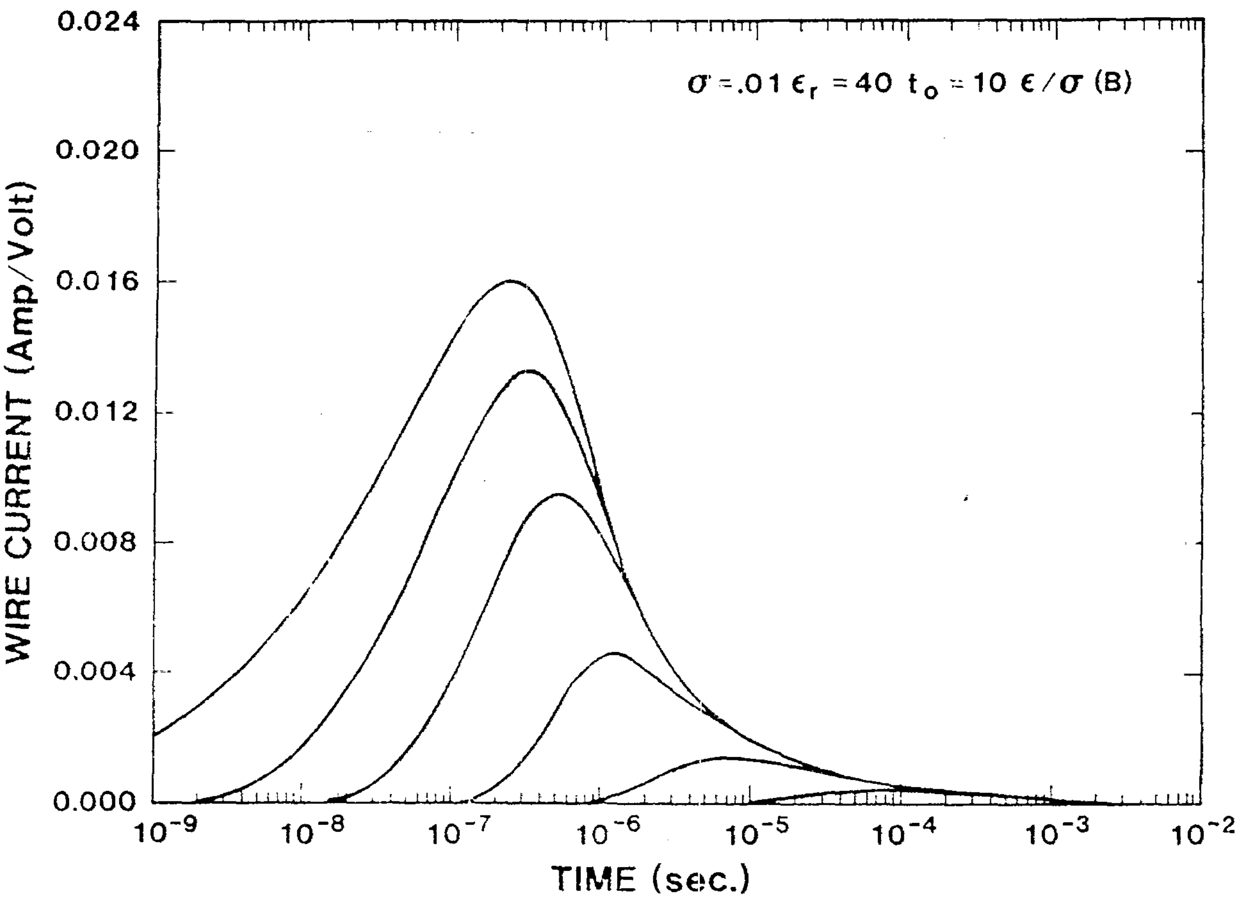


Figure 4b. Wire Currents for σ = 0.01 S/m, and ϵ_r = 40, t_o = $10t_e$

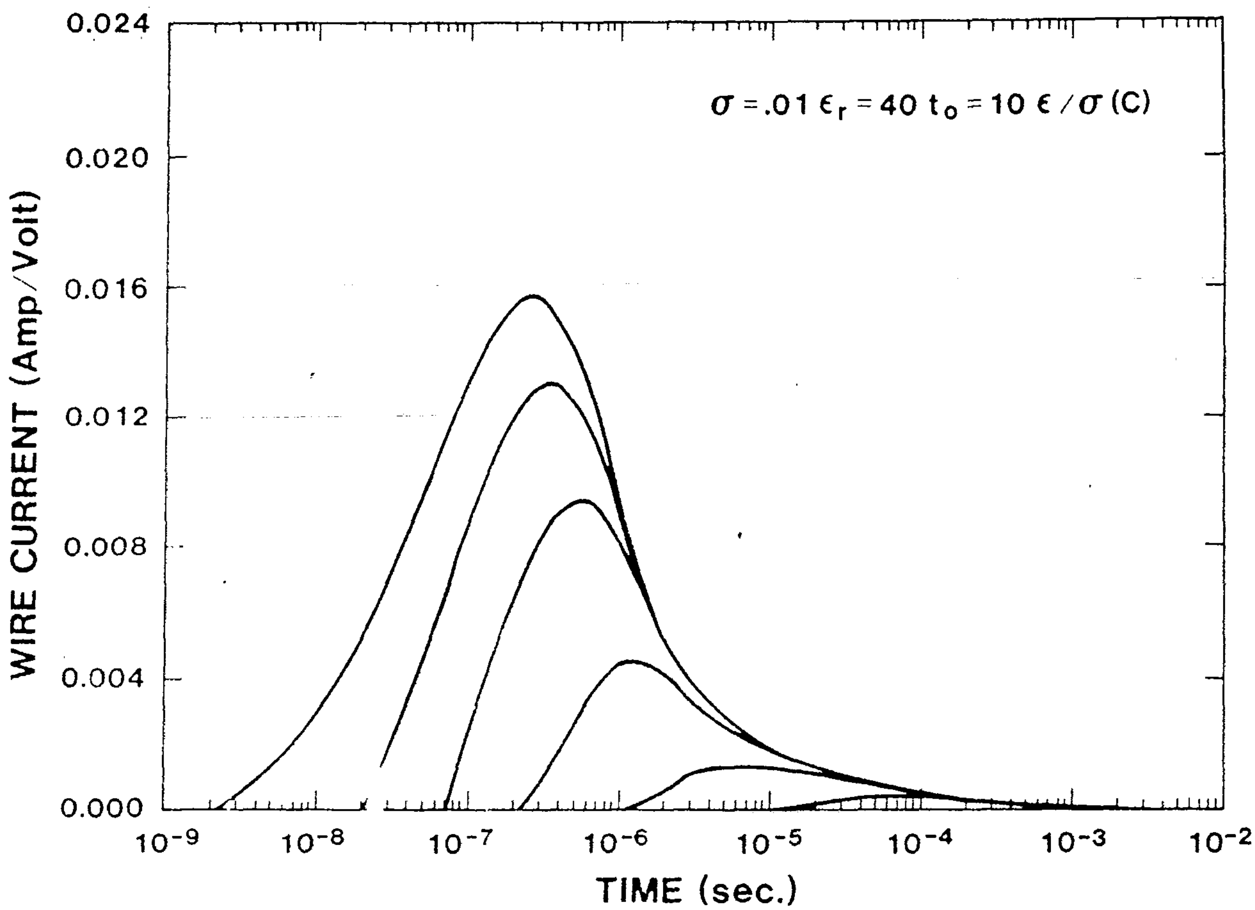


Figure 4c. Wire Currents for $\sigma = 0.01$ S/m, and $\varepsilon_r = 40$, $t_o = 10t_e$

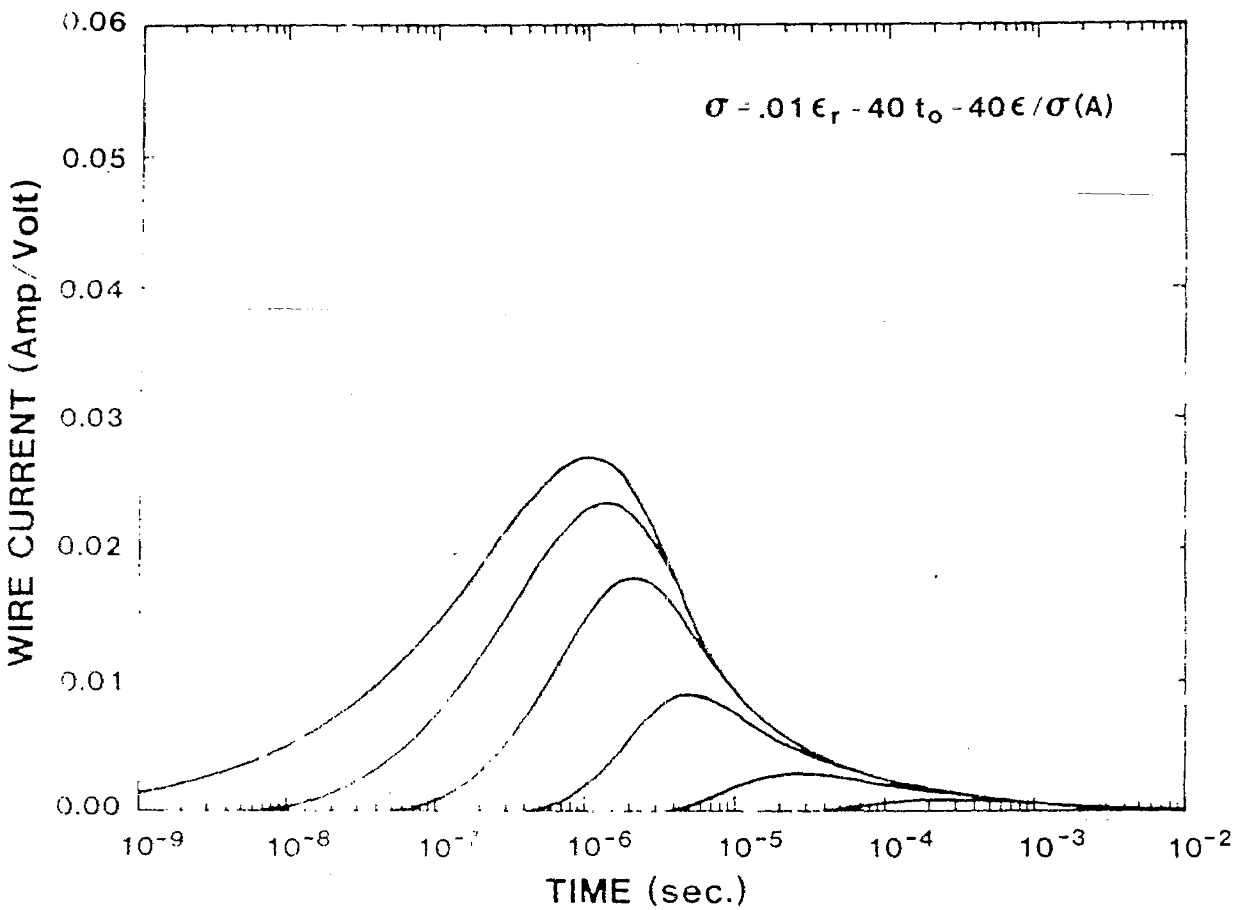


Figure 5a. Wire Currents for σ = 0.01 S/m, and ϵ_r = 40, t_o = $40t_e$

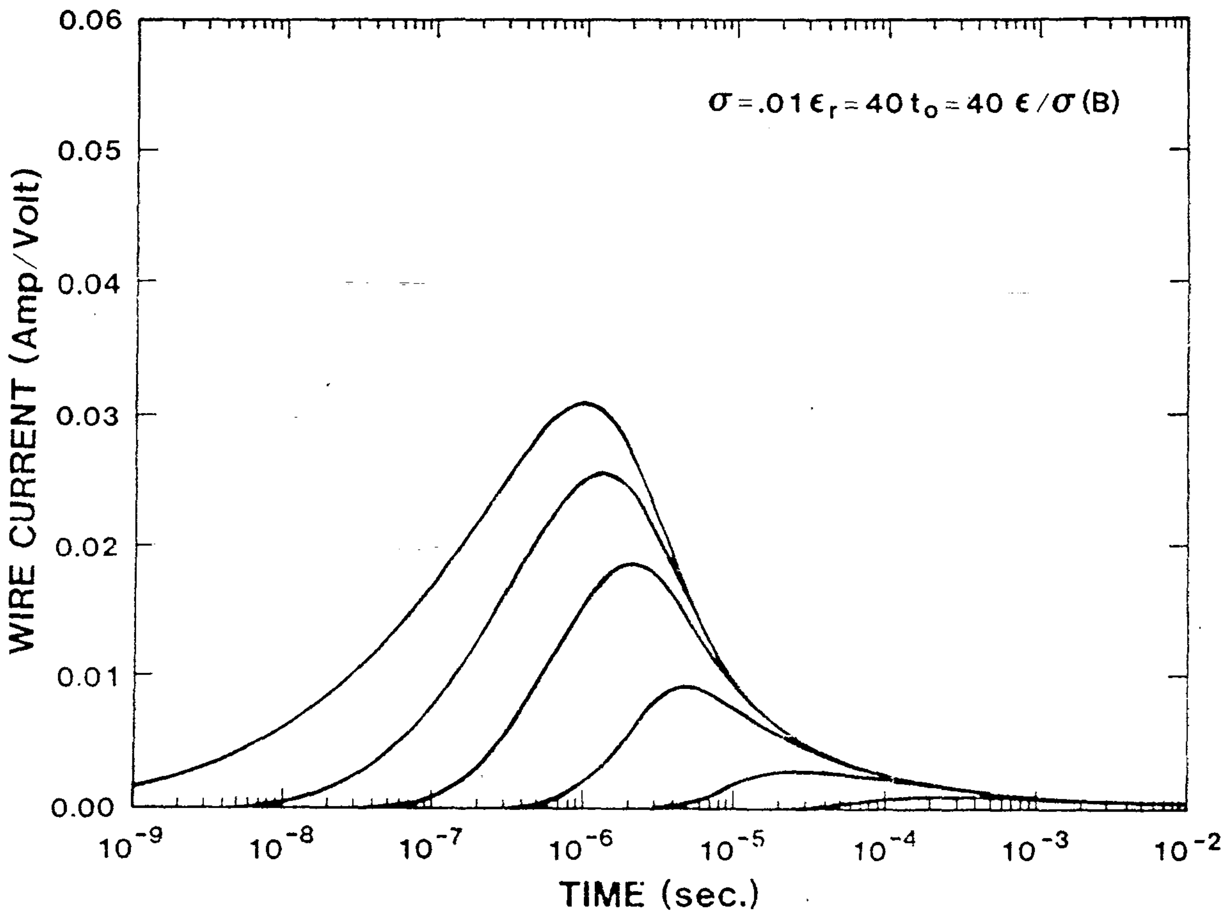


Figure 5b. Wire Currents for $\sigma = 0.01$ S/m, and $\epsilon_r = 40$, $t_o = 40t_e$

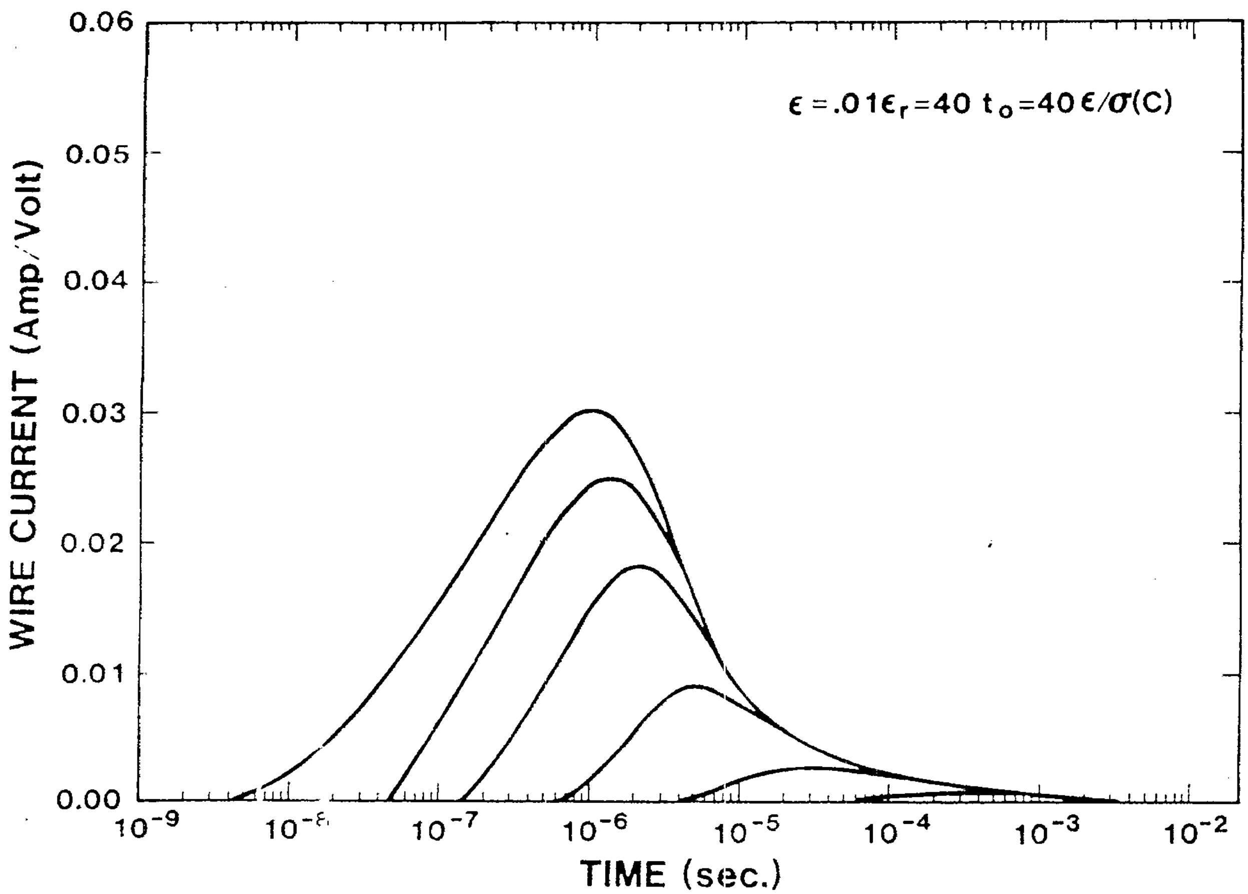


Figure 5c. Wire Currents for σ = 0.01 S/m, and ε_r = 40, t_o = 40t_e

approximation. In Parts a and b the wavefront arrives at $t = o$, because of the intrinsic nature of a diffusion approximation, the early-time waveform is not correctly given. It is possible to remedy this, if one replaces (1) the diffusion singularity, $(t-t')^{1/2}$ in Equation (11) by $\left[(t-t')^2 - z^2/c^2\right]^{-1/4}$ and (2) the lower limit of integration o by z/c. The resulting numerical integration is indistinguishable from that obtained by using Equation (6), and is shown in Part c. Also, in Part c, $z = \sqrt{2t_{on}\alpha^{-1}A}$. Six curves in Part c are generated by six values of A (or z). The basic conclusion is that when $t_o \geq 5t_e$ (1) the early-time waveform cannot be obtained by the diffusion approximations, (2) the peak value can be obtained with very good accuracy by using a diffusion approximation with the impedance inside the integrand, and (3) the early-time waveform is correctly obtained by Part c.

When $t_o < 5t_e$, the numerical result is not as consistent. Figure 2 shows the case with $t_o = 2t_e$. In this case, the correct peak value given in Part c is somewhere between that given by Part a and that by Part b.

Section 3. The Effect of an Air-Ground Interface

Since the diffusion limit of the wire current is shown to give very good approximate solutions, we shall pursue the diffusion approximation further. The effect of the air-ground interface will be taken into account under that limit (Figures 6a and 6b) by using an image consideration.

It was shown in Reference 2 that a rigorous transmission line theory can be obtained by considering a radial electric field of

$$E_\rho(\rho, z, t) = \frac{\partial I/\partial z}{2\pi\varepsilon\rho} e^{-\rho^2/2\delta^2} \tag{14}$$

where I is given by Equation (9). The electric field in the dissipative half space can now be obtained by adding that due to the wire [Equation (14)] and that due to the image located in the air as shown in Figure 6b.

As in the case without interface, the transmission line voltage can be defined by integrating the electric field from the wire surface to infinity. The integration can be carried out most easily along the x-axis or the y-axis; therefore, denote $\mathscr{V}_0(z,t)$ as the voltage for the case when the interface is not present and the image term disappears.

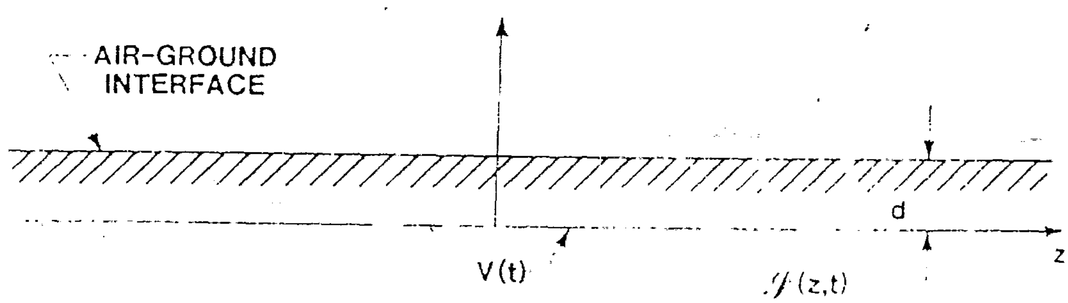
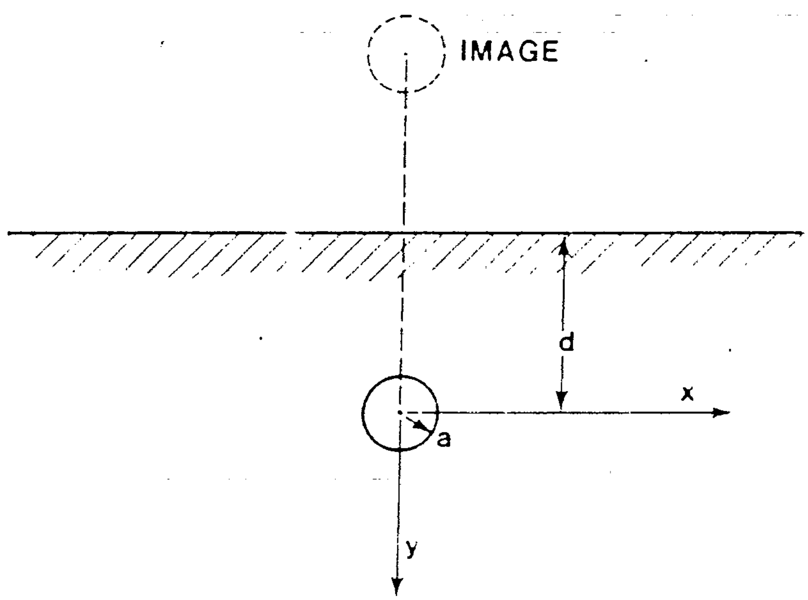


Figure 3. The Geometry of a Buried Wire With an Air-Ground Interface



F gure 6b. The Side View of a Buried Wire With Its Image

$$\mathcal{V}_o(z,t) = -\int_a^\infty E_\rho(\rho,z,t)\, d\rho = \frac{-1}{2\pi\varepsilon}\frac{\partial I}{\partial z} E_1\left(\frac{a^2}{2\delta^2}\right) \tag{15}$$

$$\sim \frac{1}{2\pi\varepsilon}\frac{\partial I}{\partial z}\left[2\ln\frac{\delta}{a} + \ln 2 - \gamma\right]$$

where E_1 is the exponential integral. When the interface is present and the image is included, the transmission line voltage denoted by $\mathcal{V}(z,t)$ is

$$\mathcal{V}(z,t) = -\int_a^\infty E_y(y,z,t)\, dy - \int_{2d+a}^\infty E_y(y,z,t)\, dy \tag{16}$$

$$- \frac{1}{2\pi\varepsilon}\frac{\partial I}{\partial z}\left\{E_1\left(\frac{a^2}{2\delta^2}\right) + E_1\left[\frac{(2d+a)^2}{2\delta^2}\right]\right\}$$

$$\sim -\frac{1}{2\pi\varepsilon}\frac{\partial I}{\partial z}\left[2\ln\frac{\delta}{a} + \ln 2 - \gamma + 2\ln\frac{\delta}{2d} + \ln 2 - \gamma\right] .$$

The last step follows, because in considering a signal with a characteristic early time of 10^{-6} sec and a characteristic late time of 10^{-3} sec, the diffusion depth δ is found to be about 30 m and 1000 m, respectively for $\sigma = 10^{-3}$ S/m. Thus, a small argument approximation of E_1 is allowed.

The ratio of $\mathcal{V}$ to $\mathcal{V}_o$ as given in Equations (15) and (16) is the ratio of characteristic impedance for the two cases. The wire current is seen to be inversely proportional to this ratio, which is

$$R = \frac{2\left[\ln\frac{\delta}{a} + \ln\frac{\delta}{2d} + \ln 2 - \gamma\right]}{2\ln\frac{\delta}{a} + \ln 2 - \gamma} .$$

When $\ell n\ \delta/a \gg \ell n\ 2d/a$, $R \sim 2$. This indicates that the equivalent transmission line for the case with the interface has half of the shunt conductance as in the case without the interface. When $\ell n\ \delta/a \gg \ell r\ \delta/2d$, $R \sim 1$. This indicates that the interface has no effect on the wire current. This is the case described in most of the literature.[1,6]

Finally, since the reflected wave from the interface does not affect the wire current until the arrival of the reflected wave, the wire current is

$$\mathscr{I}(z,t) \sim \left(\sigma/\varepsilon - t_o^{-1}\right) 2\varepsilon \left(\frac{\pi}{\mu\sigma}\right)^{1/2} \int_o^t \frac{f(t-t')}{(t-t')^{1/2}}\, e^{-\frac{z^2\sigma\mu}{4(t-t')} - t'/t_o}\, dt'$$

where
$$f(t) = \begin{cases} \dfrac{1}{\ell n\left[\dfrac{2t}{\sigma\mu a^2}\right] + \ell n\ 2 - \gamma}, & \text{for } t < \dfrac{2d}{c} \\[2ex] \dfrac{1}{\ell n\left[\dfrac{2t}{\sigma\mu a^2}\right] + \ell n\left[\dfrac{2t}{\sigma\mu d^2}\right] + 2\ell n\ 2 - 2\gamma}, & \text{for } t > \dfrac{2d}{c} \end{cases}.$$

When $f(t-t')$ is approximated by $f(t)$, the resulting formula is

$$\mathscr{I}(z,t) \sim \frac{4\sqrt{\pi}(t_o - t_{rel})\ f(t)\ g(z,t)}{\zeta_o (t_o t_{rel})^{1/2}} \qquad (17)$$

where $g(z,t)$ is given by Equation (13).

Section 4. The Current for a Wire Buried in a Soil With a Realistic Dielectric Constant

Consider the problem of determining the wire current for a wire with a radius $a = 10^{-2}$ meter buried 1 meter below the air-ground interface when an input voltage is impressed at the gap (Figures 6a and b). Let us assume the input voltage to be

$$V(t) = \left(e^{-t/t_f} - e^{-t/t_r}\right) u(t) \qquad (18)$$

where $t_f = 10^{-3}$ sec

$t_r = 10^{-6}$ sec.

The relative dielectric constant as shown in Figure 7 is more realistic than the dielectric constant used before.[7] The theoretical reasoning of its variation is given in Reference 8. The ground conductivity is assumed to be

$$\sigma = 2 \times 10^{-3} \text{ S/m} \quad .$$

Due to the wide variation of the relative dielectric constant over the whole frequency range, it is necessary to decompose the input voltage into a superposition of many double exponential waveforms such that each double exponential waveform is applied to the wire buried in an assumed medium with a uniform relative dielectric constant. Note that the larger the time constant, the larger the response current. Therefore, begin with the decay portion of the double exponential. The relative dielectric constant

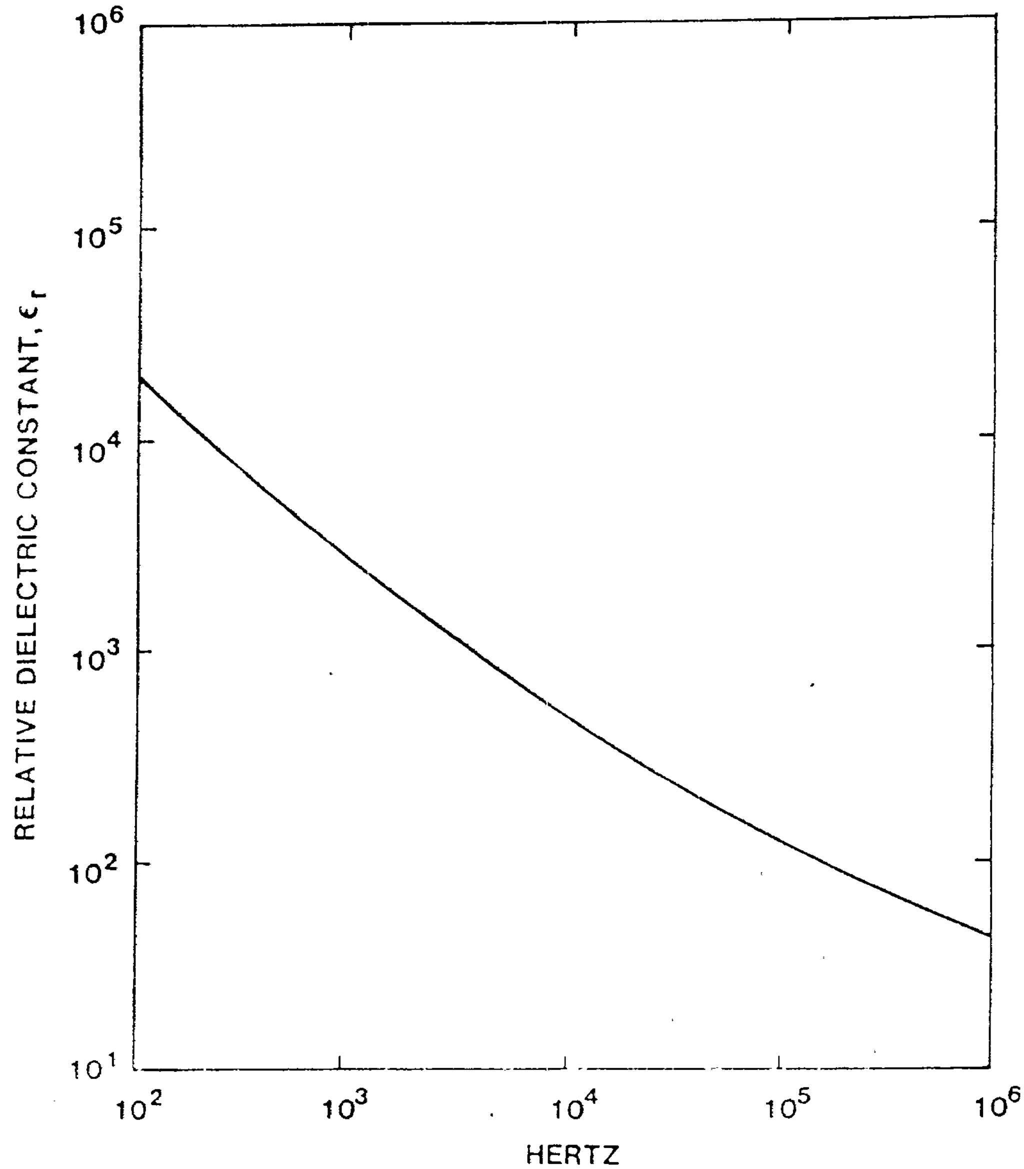


Figure 7. The Relative Dielectric Constant ϵ_r Versus Frequency for 10 Percent by Volume of the Water Content

corresponding to $t_f = 10^{-3}$ sec (or $f_f = 1/2\pi t_f \sim 160$ Hz) is $\varepsilon_{rf} = 10^4$. In order to have an input voltage without discontinuity, let the dominating voltage be

$$\mathcal{V}_1(t) = \left(e^{-t/t_f} - e^{-t/t_1}\right) u(t) \tag{19}$$

where $t_1 = \varepsilon_f/\sigma = 4.428 \times 10^{-5}$ sec.

Next, begin with the second term in Equation (18). The relative dielectric constant corresponding to $t_1 = 4.428 \times 10^{-5}$ ($f_1 \sim 3.6 \times 10^3$) is $\varepsilon_{r1} = 10^3$. Thus, the next voltage to consider is

$$\mathcal{V}_2(t) = \left(e^{-t/t_1} - e^{-t/t_2}\right) u(t) \tag{20}$$

where $t_2 = \varepsilon_1/\sigma = 4.428 \times 10^{-6}$ sec.

Proceeding as before, the relative dielectric constant corresponding to t_2 is $\varepsilon_{r2} = 200$, and the voltage to use is

$$\mathcal{V}_3(t) = \left(e^{-t/t_2} - e^{-t/t_3}\right) u(t) \tag{21}$$

where $t_3 = \varepsilon_2/\sigma = 8.855 \times 10^{-7}$ sec.

Notice the input voltage $\mathcal{V}(t)$ can be approximated by

$$\mathcal{V}(t) \sim \mathcal{V}_1(t) + \mathcal{V}_2(t) + \mathcal{V}_3(t) \quad .$$

The current is the sum of the individual current due to $\mathcal{V}_i(t)$ in its corresponding medium ε_i and σ. The total wire current in

Amps, which includes the interface effect for an input voltage given by Equation (18) at the gap, is shown in Figure 8. The figure shows the wire currents at locations z = 100 m, 316 m, 1000 m, and 3160 m from the source.

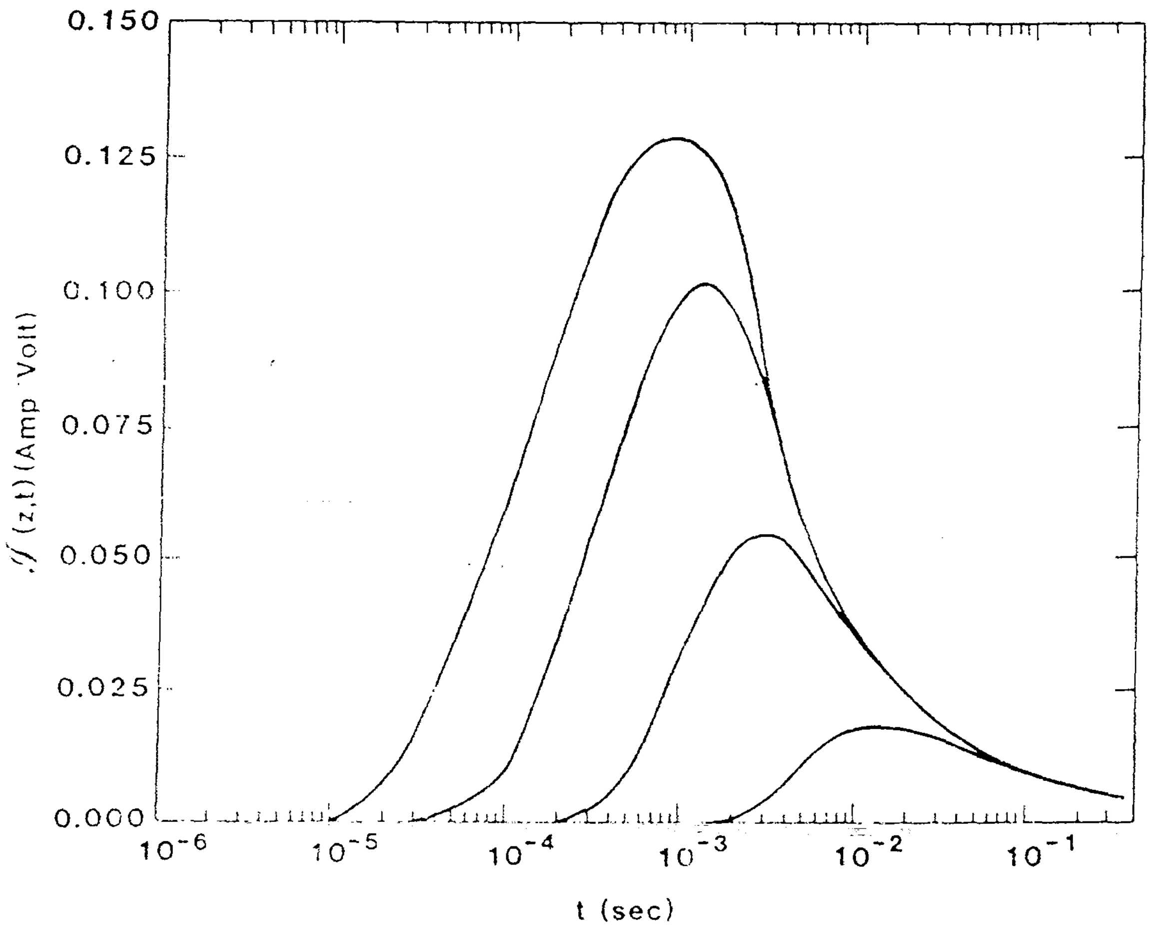


Figure 8. Wire Currents for an Input Voltage at the Gap Given by Equation 16. The locations for the given currents are shown in the figure: from the left z = 100, 316, 1000, and 3160 m. The wire configuration is shown in Figures 6a and 6b.

Section 5. The Current for a Wire Buried Below an Air-Ground Interface and Under a Normal Incident Electric Field

Consider the problem of calculating the current on a wire with a wire radius $a = 10^{-2}$ meter buried 1 meter below the air-ground interface, when an electric field of

$$E(t) = \left(e^{-t/t_f} - e^{-t/t_r}\right) u(t) \tag{22}$$

where $t_f = 2.5 \times 10^{-7}$ sec

$t_r = 10^{-8}$ sec

is normally incident on the interface.

We shall solve this problem via the frequency domain by using the time-harmonic formula derived in Appendix B.

$$\tilde{E}(\omega) = \frac{1}{-i\omega + t_f^{-1}} - \frac{1}{-i\omega + t_r^{-1}} \ .$$

The transmission coefficient of a normal incident wave from the air to the earth with ε and σ as its dielectric permittivity and conductivity is

$$\tilde{T}(\omega) = \frac{2(\varepsilon_r - \sigma/i\omega\varepsilon_o)^{1/2}}{\varepsilon_r - \sigma/i\omega\varepsilon_o + (\varepsilon_r - \sigma/i\omega\varepsilon_o)^{1/2}} \tag{23}$$

$$\sim 2\varepsilon_r^{-1/2}(-i\omega)^{1/2}(-i\omega + \sigma/\varepsilon)^{-1/2} \ . \tag{24}$$

The frequency domain transfer function from just below the interface to a depth of d from the interface is

$$\tilde{T}_d(\omega) = e^{-ikd} \quad , \tag{25}$$

$$-ikc = \left[(-i\omega)(-i\omega + \sigma/\varepsilon)\right]^{1/2} \ .$$

The wire current transfer function due to a localized voltage E dz at z distance away is (Appendix B)

$$\tilde{\mathscr{I}}(z,\omega) = (-i\omega + \sigma/\varepsilon) \text{ (Equation B6)} \quad . \tag{26}$$

The total wire current in the frequency domain due to a normal incident electric field is

$$\tilde{\mathscr{I}}_T(\omega) = \frac{4c}{\zeta_o\sqrt{\varepsilon_r}}\left(\int_{-\infty}^{o} dz - \int_{o}^{\infty} dz\right) e^{-ik(d+z)} \ .$$

$$\arctan\left[\frac{-\pi}{\ell n \dfrac{\alpha}{<\tau>} + \dfrac{K_o(\alpha<\tau>)}{I_o(\alpha<\tau>)} - \ell n\ 2+\gamma}\right] \tilde{E}(\omega)dz$$

$$= \frac{4c}{\sqrt{\varepsilon_r}} \tilde{E}(\omega)\left\{\frac{2\varepsilon^{-ikd}}{\zeta_o ik} \arctan\left[\frac{-\pi}{\ell n \dfrac{\alpha}{<\tau>} + \dfrac{K_o(\alpha<\tau>)}{I_o(\alpha<\tau>)} - \ell n\ 2+\gamma}\right]\right\} \ . \tag{27}$$

The time-domain function for the frequency-domain function inside the parentheses is I(d,t) as defined in Reference 2, Equation 6 with z replaced by d.

The time-domain total current, which is the inverse transform of (27) can be easily written as

$$\mathscr{I}_T(z,t) = \frac{4c}{\sqrt{\varepsilon_r}} \int_0^t dt' \ I(d,t') \left[e^{-(t-t')/t_f} - e^{-(t-t')/t_r} \right] \quad . \tag{28}$$

Finally, in a diffusion limit, $\mathscr{I}_T(z,t)$ can be written as

$$\mathscr{I}_T(z,t) = Z_T(t_f,t)\ g_{t_f}(d,t) - Z_T(t_r,t)\ g_{t_r}(d,t) \tag{29}$$

where

$$Z_T(t_o,t) = \frac{16(\pi t_o t_e)^{1/2}}{\mu\left[2\ln\left(\frac{\delta}{a}\right) + \ln 2 - \gamma\right]} \tag{30}$$

and $g_{t_o}(d,t)$ is given by Equation (11) and it is a function of t_o because of $A = \frac{z^2\sigma\mu}{4t_o}$.

Notice t_e is defined as $t_e = \varepsilon_o/\sigma$. Equation (30) differs by a factor of 2 from that given in Reference 6. It appears that a factor of 2 has been dropped on page 385 of that reference. The effect of the air-ground interface can be easily accounted for by using Equation (15) given in Section 3. An important comment is in order: the wire current solved in Section 2 and that solved in this section have identical normalized admittance functions. The difference in the two cases is the admittance factor. The interpretation of z and d should be the separation distance in the dissipative medium from the source point to the observation point.

A numerical example is helpful in appreciating the value of an induced wire current due to an HAB EMP. Notice Z_T in Equation (30) is proportional to $\sqrt{t_o}$. Therefore, dropping the second term in Equation (29) results only in about 10 percent accuracy. Consider an earth conductivity of $\sigma = 10^{-3}$ S/m.

Thus, $t_e = \epsilon_o/\sigma \sim 10^{-8}$ sec,

$$Z_T(t_f, t) = \frac{16\sqrt{\pi \times 10^{-8} \times 2.5 \times 10^{-7}}}{4\pi \times 10^{-7} \times 16} = 0.07 \quad .$$

In obtaining the above number, the interface effect has been included. Also, $A \sim 10^{-2}$, and from Figure 1, the peak value of $g_{t_f}(d,t) \sim 0.5$.

When accounting for a multiplicative constant of $E_o = 5 \times 10^4$ V/m in Equation (21) for an HAB EMP and the error introduced by using Figure 1, the peak value of the current is approximately given by 2 kA.

Section 6. Conclusions

This report illustrated the use of a new theory developed in Reference 2. First, the diffusion approximation was used for both a localized excitation and a double exponential plane wave excitation. Under that approximation, universal curves were given for obtaining numerical results in applications. Second, the more accurate new theory was used to ascertain the limitations of the diffusion approximation. Third, the clarification of an inconsistency in the early-time solution in the literature was noteworthy. Fourth, the time-harmonic formula was obtained previously only in the limiting case. The most important conclusion is that for most situations with buried wires under lightning or nuclear EMP excitations, the diffusion approximation with impedance included inside the integrand (Figures 2b through 5b) gives the correct peak value. When an accurate early-time waveform is required, the present theory is no more complicated than the diffusion theory. It only requires a simple integration. Typical results are shown in Figures 2c through 5c.

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APPENDIX A. An Early-Time Solution

Previous investigations of the early-time behavior for free space are described in References 3, 4, and 5. In order to study the early-time behavior of $I(z,t)$, use Reference 1 Equation (3) to write $I(z,t)$ as

$$I(z,t) = I_1 + I_2$$

$$\sim \frac{2\pi}{\zeta_o} e^{-\sigma t/2\varepsilon} \int_{\tau^{-\beta}}^{\infty} \frac{d\eta}{\eta} \frac{I_o(\eta)}{K_o(\eta)} \frac{K_o\left(\tau\sqrt{a^2 + \eta^2}\right)}{\left\{[K_o(\eta)]^2 + \pi^2[I_o(\eta)]^2\right\}} \quad . \qquad \text{(A1)}$$

When $\tau << 1$, $I(z,t)$ is contributed mostly from the last term of Equation (A1). Furthermore, β is chosen to satisfy $0 < \beta < 1$, such that, as $\tau \to 0$, $(\tau^{-\beta} \to \infty)$ $I_o(\eta)$ and $K_o(\eta)$ can be replaced by the asymptotic expansions.

As a result, for $\tau << 1$,

$$I(z,t) \sim \frac{4}{\zeta_o \pi} e^{-\sigma t/2\varepsilon} \int_{\tau^{-\beta}}^{\infty} K_o(\tau\eta)\, d\eta = \frac{4}{\zeta_o \pi\tau} e^{-\sigma t/2\varepsilon} \int_{\tau^{1-\beta}}^{\infty} K_o(x)\, dx$$

$$\sim \frac{2}{\zeta_o \tau} e^{-\sigma t/2\varepsilon} \quad . \qquad \text{(A2)}$$

Notice in Equation (A2) the lower limit $\tau^{1-\beta}$ in the last integration can be replaced by zero. When $\sigma \to 0$, Equation (7) reduces to the lossless case studied by Wu[3] but differs from Lee and Latham,[4,5] whose early-time current is a factor of $\sqrt{2\pi}$ too large.

Alternatively, Equation (A2) could be obtained by using Reference 1 Equation (2) as follows:

$$I(z,t) \sim \dots + \frac{4}{\pi\zeta_o} e^{-\sigma t/2\varepsilon} \int_{\tau-\beta}^{\infty} \frac{J_o\left(\tau\sqrt{\eta^2 - \alpha^2}\right)}{\left\{[J_o(\eta)]^2 + [Y_o(\eta)]^2\right\}} \frac{d\eta}{\eta}$$

$$\sim \frac{4}{\zeta_o \pi\tau} e^{-\sigma t/2\varepsilon} \int_{\tau\sqrt{1-\beta}}^{\infty} J_o(x)\, dx = \frac{2}{\zeta_o \tau} e^{-\sigma t/2\varepsilon} .$$

APPENDIX B. A Time-Harmonic Solution

To obtain a time-harmonic solution for the wire current in a dissipative medium, it is necessary to consider the following integral:[9]

$$\int_1^\infty (x^2 - 1)^{\nu/2} e^{-px} I_\nu\left(q\sqrt{x^2 - 1}\right) dx$$

$$= \sqrt{\frac{2}{\pi}} q^\nu (p^2 - q^2)^{-1/2\nu-1/4} K_{\nu+1/2}\left(\sqrt{p^2 - q^2}\right) \quad . \qquad (B1)$$

Differentiate Equation (B1) with respect to ν and then set $\nu = o$. The following equation emerges:

$$\frac{1}{2} \int_1^\infty \ln(x^2 - 1) e^{-px} I_o\left(q\sqrt{x^2 - 1}\right) dx = \int_1^\infty e^{-px} K_o\left(q\sqrt{x^2 - 1}\right) dx$$

$$+ \sqrt{\frac{2}{\pi}} \ln \frac{q}{\sqrt{p^2 - q^2}} \frac{K_{1/2}\left(\sqrt{p^2 - q^2}\right)}{\left(p^2 - q^2\right)^{1/4}} + \sqrt{\frac{2}{\pi}} \frac{\frac{\partial}{\partial \nu} K_{\nu+1/2}\left(\sqrt{p^2 - q^2}\right)\Big|_{\nu = 0}}{\left(p^2 - q^2\right)^{1/4}} \quad . \qquad (B2)$$

The right-hand side of Equation (B2) can be evaluated and an averaging for $1/2 \ln (x^2 - 1)$ can be defined as

$$\left\langle \frac{1}{2} \ln(x^2 - 1) \right\rangle = \frac{\text{Equation B2}}{\int_1^\infty e^{-px} I_o\left(q\sqrt{x^2 - 1}\right) dx}$$

$$= \ln\left(\frac{p + \sqrt{p^2 - q^2}}{q}\right) - E_i\left(-2\sqrt{p^2 - q^2}\right) e^{2\sqrt{p^2-q^2}}$$

$$+ \sqrt{\frac{2}{\pi}} \ln\left(\frac{q}{\sqrt{p^2 - q^2}}\right) K_{1/2}\left(\sqrt{p^2 - q^2}\right)\left(p^2 - q^2\right)^{1/4} e^{\sqrt{p^2-q^2}}$$

$$\sim \ln\left(\frac{p + \sqrt{p^2 - q^2}}{\sqrt{p^2 - q^2}}\right) . \tag{B3}$$

The time-harmonic wire current is related to the forward Fourier transform of Reference 2 Equation (6) as

$$\tilde{I}(z,\omega) = \frac{2}{\zeta_o} \int_0^\infty dt\, e^{i\omega t - \sigma t/2\varepsilon} I_o\left(\frac{\sigma}{2\varepsilon}\sqrt{t^2 - z^2/c^2}\right) .$$

$$\arctan\left[\frac{-\pi}{\ln \frac{\alpha}{\tau} + \frac{K_o(\alpha\tau)}{I_o(\alpha\tau)} - \ln 2 + \gamma}\right] . \tag{B4}$$

To evaluate Equation (B4), let

$$p = \left(-i\omega + \frac{\sigma}{2\varepsilon} \frac{z}{c}\right)$$

$$q = \frac{\sigma z}{2\varepsilon c} .$$

Equation (B4) can be approximated by replacing τ by its averaging. Notice

$$< \frac{1}{2} \ln(x^2 - 1)> \sim \ln \frac{\frac{\sigma}{2\varepsilon}}{\sqrt{(-i\omega)\left(-i\omega + \frac{\sigma}{\varepsilon}\right)}} = \ln \frac{\sigma}{-i2\varepsilon kc}$$

and

$$<\tau> \sim \frac{z\sigma}{-i2a\varepsilon kc} \, , \quad -ikc = \sqrt{(-i\omega)\left(-i\omega + \frac{\sigma}{\varepsilon}\right)} \; . \tag{B5}$$

Use of Equation (B5) n Equation (B4) gives

$$\tilde{I}(z,\omega) \sim \frac{2}{\zeta_o} \frac{e^{-ikz}}{(-ik)} \arctan\left[\frac{-\pi}{\ln \frac{\alpha}{<\tau>} + \frac{K_o(\alpha<\tau>)}{I_o(\alpha<\tau>)} - \ln 2 + \gamma}\right] \; . \tag{B6}$$

When $\alpha<\tau> \gg 1$, Equation (B6) reduces to

$$\tilde{I}(z,\omega) \sim \frac{-1}{\zeta_o} \frac{e^{-ikz}}{kc} \ln\left[1 - \frac{i2\pi}{\ln(2kz) - 2\ln(ka) - \gamma + i\frac{3\pi}{2}}\right] \; . \tag{B7}$$

On using Reference 1 Equation (1) the time-harmonic solution is given by

$$\tilde{\mathscr{I}}(z,\omega) \sim \left(-i\omega + \frac{\sigma}{\varepsilon}\right) \tilde{I}(z,\omega) \quad . \tag{B8}$$

Equations (B7) and (B8) agree with Wu's result.[10] Notice the time-harmonic solution for an infinite wire in free space is given by Equation (B8) with $k = \omega/c$ and $\sigma = 0$.

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