INTERACTION NOTES

Note 462

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Transfer of Norms Through Black Boxes

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Abstract

This note addresses the contribution of exciting waveforms and filter characteristics to the response of some ideal black box. Taking the excitation, filter, and response in norm sense, various characteristics of the excitation are examined for their effect on response. We find that the frequency spectral characteristics of the excitation waveform are very important for this purpose. This is true whether one is considering some ideal excitation specification, or one is considering a set of damped sinusoidal excitations.

I. Introduction

The relationship between time and frequency (or Laplace-transform) domains is elementary in modern electrical engineering. Its general formalism is expressed in terms of the Laplace (or Fourier) integral (two sided) as

$$\tilde{f}(s) \equiv \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

$$f(t) = \frac{1}{2\pi j} \int_{Br} \tilde{f}(s)\, e^{st}\, ds$$

$t \equiv$ time (1.1)

$s \equiv \Omega + j\omega \equiv$ complex frequency or Laplace-transform variable

$Br \equiv$ Bromwich contour in strip of convergence of two-sided Laplace transform (i.e. for $\Omega_- < \Omega < \Omega_+$)

$\sim \equiv$ Laplace transform

Now this transform is extremely powerful in linear systems or systems which can be approximated as linear. In addition if the system can be approximated (in time regions of interest) as time invariant, then the fundamental property is the convolution property

$$F(t) \equiv g(t) \circ f(t)$$

$$= \int_{-\infty}^{\infty} g(t-t')\, f(t')\, dt'$$

$$= \int_{-\infty}^{\infty} g(t')\, f(t-t')\, dt'$$

$\circ \equiv$ convolution

$g(t)\circ \equiv$ convolution operator (1.2)

$f(t) \equiv$ excitation

$F(t) \equiv$ response

In complex-frequency domain this assumes the elegant form

$$\tilde{F}(s) = \tilde{g}(s)\, \tilde{f}(s) \qquad (1.3)$$

While these relationships are fundamental for electrical engineering, they are sometimes obscure. There is an integral relationship between frequency and time domains, but this does not necessarily give a simple relationship. In order to simplify this relationship let us restrict the forms of f(t) and g(t) so that a small number of parameters characterize them. In addition let us consider the norms [6] of these to further reduce the number of parameters which characterize the response.

As a common example f(t) can be considered to be one or more damped sinusoids corresponding to a few pole pairs in complex-frequency domain. A similar approximation can be given to g(t). With such canonical forms norms can be computed and appropriate norms and hence bounds can be found. As is discussed later such general restrictions on the forms of f(t) and g(t) can reduce the integral relationships to algebraic ones.

II. Black Box Modelled as Linear Filter

As indicated in fig. 2.1 consider some general black box modelled as a transfer function $\tilde{T}(s)$ from some input port with excitation $V_+^{(1)}$ giving a signal (incident wave) $V_+^{(2)}$ at some other port with

$$\tilde{V}_+^{(2)}(s) = \tilde{T}(s)\ \tilde{V}_+^{(1)}(s)$$

$$V_+^{(2)}(t) = T(t) \circ V_+^{(1)}(t) \tag{2.1}$$

The variables here are combined voltages (waves) as linear combinations of voltages and currents of the form [1]

$$V_\pm(t) = V(t) \pm RI(t) \tag{2.2}$$

where there is a resistance R to normalize the combination [3]. As in fig. 2.1 these waves propagate in 2 directions with subscripts

$$\begin{aligned} + &\Longrightarrow \text{propagation to right} \\ - &\Longrightarrow \text{propagation to left} \end{aligned} \tag{2.3}$$

Now let us assume that $V_-^{(2)}$ can be neglected because either:

a. $\tilde{Z}_t = R$ making the reflection zero from $\tilde{Z}_t$, or

b. $V_-^{(2)}$ is terminated to the left in the black box, including any transfer (of significance) to the left through the black box

One candidate for such a "black box" is the constant resistance filter discussed in [5] which approximates condition b.

While this black box is modelled as linear, this need only be interpreted in the looser sense of "linearity to failure" discussed in [2]. In this case the termination on the right approximates the "failure port" discussed there [2]. Our concern is the characteristics of the signal arriving there (including in norm sense), and the relation of this signal to the signal (wave) exciting the black box.

Now let the exciting wave be given by a set of poles as

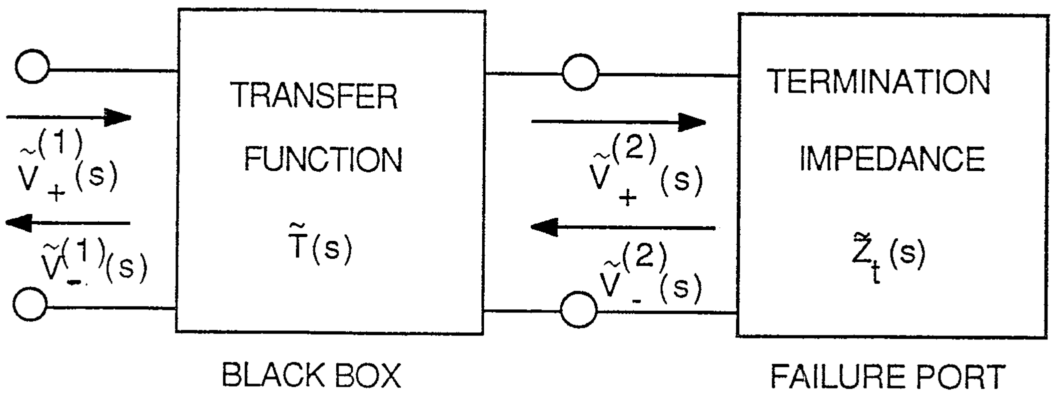

FIG. 2.1 BLACK BOX MODELLED AS LINEAR FILTER

$$V_+^{(1)}(t) = \sum_\alpha V_\alpha e^{s_\alpha t} u(t), \quad \tilde{V}_+^{(1)}(s) = \sum_\alpha V_\alpha [s - s_\alpha]^{-1} \tag{2.4}$$

$$\mathrm{Re}[s_\alpha] \leqslant 0 \text{ for all } \alpha$$

with all poles for $\mathrm{Im}[s_\alpha] \neq 0$ occurring in complex conjugate pairs so that $V_+^{(1)}(t)$ is real valued. While more general types of waveforms are also possible one can also represent some such types of waveforms as distributions of poles (such as branch integrals [7]). So let us take (2.4) as a canonical type of exciting waveform.

Next consider the filter which represents the black box. Let us similarly represent it as

$$\tilde{T}(s) = T_\infty + \sum_\beta T_\beta [s - s'_\beta]^{-1} \tag{2.5}$$

$$\mathrm{Re}\,[s'_\beta] \leqslant 0 \text{ for all } \beta$$

with all poles for $\mathrm{Im}[s'_\beta]$ occurring in complex conjugate pairs. In time domain the corresponding function (as distinct from convolution operator) is

$$T(t) = T_\infty \delta(t) + \sum_\beta T_\beta e^{s'_\beta t} u(t) \tag{2.6}$$

which we also require to be real valued. Another constraint which may be imposed on the filter is passivity, in which case we require [5]

$$\left|\tilde{T}(j\omega)\right| \leqslant 1 \text{ for all } \omega(\text{real}) \tag{2.7}$$

For poles near the $j\omega$ axis this requires

$$\left|T_\beta\right| [-\mathrm{Re}[s'_\beta]]^{-1} \leqslant 1 \text{ for all } \beta \text{ as } \mathrm{Re}[s'_\beta] \to 0 \tag{2.8}$$

Also note the presence of the constant term (for a finite number of poles) with

$$0 \leqslant T_\infty \leqslant 1 \tag{2.9}$$

This term allows for the possibility that the filter function tends to some positive constant (instead of only zero) as $\omega \to \infty$.

With distinct poles in $\tilde{V}_+^{(1)}$ and $\tilde{T}$ (i.e. $s_\alpha \neq s'_\beta$ for any α, β) we have using partial fraction expansions

$$
\begin{aligned}
V_+^{(2)}(t) &= \sum_{\alpha,\beta} T_\beta V_\alpha [s_\alpha - s'_\beta]^{-1} e^{s_\alpha t} u(t) + T_\infty V_+^{(1)}(t) \\
&\qquad + \sum_{\alpha,\beta} T_\beta V_\alpha [s'_\beta - s_\alpha]^{-1} e^{s'_\beta t} u(t) \\
&= \sum_\alpha \Big\{ \sum_\beta T_\beta [s_\alpha - s'_\beta]^{-1} \Big\} V_\alpha e^{s_\alpha t} u(t) + T_\infty V_+^{(1)}(t) \\
&\qquad + \sum_\beta \Big\{ \sum_\alpha V_\alpha [s'_\beta - s_\alpha]^{-1} \Big\} T_\beta e^{s'_\beta t} u(t) \\
&= \sum_\alpha \tilde{T}(s_\alpha) V_\alpha e^{s'_\beta t} u(t) \text{ (excitation term)} \\
&\qquad + \sum_\beta \tilde{V}_+^{(1)}(s'_\beta) T_\beta e^{s'_\beta t} u(t) \text{ (filter term)} \qquad (2.10)
\end{aligned}
$$

s_α

Here we have separated the response into two parts identified as the "excitation term" and the "filter term". In complex-frequency domain this is

$$
\begin{aligned}
\tilde{V}_+^{(2)}(s) &= \sum_\alpha \tilde{T}(s_\alpha) V_\alpha [s - s_\alpha]^{-1} \text{ (excitation term)} \\
&\qquad + \sum_\beta \tilde{V}_+^{(1)}(s'_\beta) T_\beta [s - s'_\beta]^{-1} \text{ (filter term)} \qquad (2.11)
\end{aligned}
$$

Look now at the forms of these results. Consider first the excitation term. Basically each pole s_α of the excitation has its amplitude (residue) scaled by $\tilde{T}(s_\alpha)$, i.e. by the transfer function of the filter evaluated at each natural frequency of the excitation. This is a generalization into time domain (and complex-frequency domain) of the usual concept of a filter. The filter attenuates the excitation, but now does it pole by pole (including damped sinusoid by damped sinusoid).

Consider second the filter term. The filter itself contributes poles s'_β to the response. Each such pole has its amplitude (residue) scaled by

$\tilde{V}_{+}^{(1)}(s'_{\beta})$, i.e. by the spectrum (in complex-frequency domain) of the excitation. Now the filter contributes to the response its own set of damped sinusoids in time domain. It is important to note that each such filter pole is very dependent on the excitation. If the excitation, for whatever reason, has a "notch" in the spectrum at or near some s'_{β}, then the corresponding damped sinusoid will be similarly reduced in the response $V_{+}^{(2)}(t)$. If $V_{+}^{(1)}(t)$ is interpreted as some environmental specification for testing purposes, then it is important to consider its properties in a frequency context so that there are no significant "notches" since such would result in a significant under test.

III. Residue Norm and ∞-Norm

As discussed in [6] one can define a residue norm (or r-norm) as the sum of the residue magnitudes for functions characterized by poles. Provided the poles are all distinct (and not too close to each other) this form of the residue norm approximates the ∞-norm. The r-norm can be generalized to include cases of higher-order poles, branch integrals, and closely spaced poles [6].

Considering the case of distinct (ant not closely spaced) poles we have the r-norm of the excitation as

$$\left\| V_+^{(1)}(t) \right\|_r = \sum_\alpha \left| V_\alpha \right| \tag{3.1}$$

Similarly for the response we have

$$\begin{aligned} \left\| \tilde{V}_+^{(2)}(t) \right\|_r = & \sum_\alpha \left| \tilde{T}(s_\alpha) \right| \left| V_\alpha \right| \text{ (excitation term)} \\ & + \sum_\beta \left| \tilde{V}_t^{(1)}(s'_\beta) \right| \left| T_\beta \right| \text{ (filter term)} \end{aligned} \tag{3.2}$$

Expanding this we have from (2.10)

$$\begin{aligned} \left\| V_+^{(2)}(t) \right\|_r = & \left| T_\infty \right| \sum_\alpha \left| V_\alpha \right| + 2 \sum_{\alpha,\beta} \left| T_\beta \right| \left| V_\alpha \right| \left| s_\alpha - s'_\beta \right|^{-1} \\ & \leq \left\{ \left| T_\infty \right| + \max_\beta 2 \left| T_\beta \right| \left| s_\alpha - s'_\beta \right|^{-1} \right\} \sum_\alpha \left| V_\alpha \right| \\ & = \left\{ T_\infty + \max_\beta 2 \left| T_\beta \right| \left| s_\alpha - s'_\beta \right|^{-1} \right\} \left\| V_+^{(1)}(t) \right\|_r \end{aligned} \tag{3.3}$$

$$T_\infty \geq 0$$

Considering the filter as a convolution operator, note first that the delta-function part has the norm as the identity convolution operator

$$\left\| \delta(t) \circ \right\|_r = 1 \tag{3.4}$$

Then using [6] for the residue norm of a convolution operator with (3.4) we have

$$\left\| T(t)o \right\|_r < \{ T_\infty + \max_\beta 2 \left| T_\beta \right| \{-\mathrm{Re}[s'_\beta]\}^{-1} \} \tag{3.5}$$

by adjusting s_α with $\mathrm{Re}[s_\alpha] < 0$. Considering the possibility of equality consider a specific excitation as a single pole pair matched to the maximum $\left| T_\beta \right| \{-\mathrm{Re}[s'_\beta]\}^{-1}$ in the sense of

$$\begin{aligned} \mathrm{Re}[s_\alpha] &= 0 \\ I_m[s_\alpha] &= I_m[s'_\beta] \end{aligned} \tag{3.6}$$

In this case equality holds and

$$\left\| T(t)o \right\|_r = T_\infty + \max_\beta 2 \left| T_\beta \right| \{-\mathrm{Re}[s'_\beta]\}^{-1} \tag{3.7}$$

If we also impose the passivity requirement of (2.8) for poles near the $j\omega$ axis this gives

$$\left\| T(t)o \right\|_r < T_\infty + 2 \tag{3.8}$$

There are other points one can make concerning the r-norm as in [6]. One can consider higher order poles and closely spaced poles as well as more general types of terms such as branch integrals. Most interesting is the case of the close approach of two poles appearing in a convolution. As discussed in [6] there is a term, which in current notation, is of the form

$$\begin{aligned} &\left\| [T_\beta e^{s'_\beta t} u(t)] \circ [V_\alpha e^{s_\alpha t} u(t)] \right\|_r \\ &\equiv \frac{1}{e} \frac{2}{-\mathrm{Re}[s'_\beta]-\mathrm{Re}[s_\alpha]} \left| T_\beta \right| \left| V_\alpha \right| \end{aligned} \tag{3.9}$$

which characterizes this case. Including the conjugate poles and noting that the cross terms between s'^*_β and s_α, and between s'_β and s^*_α, are negligible, we see that the term $2\left| T_\beta \right| \{-\mathrm{Re}[s'_\beta]\}^{-1}$ in (3.7) is replaced by $(2/e) \left| T_\beta \right| \{-\mathrm{Re}[s'_\beta]\}^{-1}$ as $\mathrm{Re}[s_\alpha] \to 0$. So as poles come closely together (for poles near the $j\omega$ axis) there is a factor of e^{-1} which is included in $\|T(t)o\|_r$.

For comparison to the r-norm results consider the closely related ∞-norm. As shown in [6]

$$\|T(t)o\|_\infty = \|T(t)\|_1$$

$$= \int_{0-}^{\infty} \left| \{T_\infty \delta(t) + \sum_\beta T_\beta e^{s'_\beta t} u(t)\} \right| dt$$

$$= T_\infty + \int_0^\infty \left| \sum_\beta T_\beta e^{s'_\beta t} u(t) \right| dt \tag{3.10}$$

Now considering the case of only one significant pole pair, and that near the $j\omega$ axis, we have

$$\|T(t)o\|_\infty = T_\infty + \int_0^\infty \left| T_\beta e^{s'_\beta t} + T^*_\beta e^{s'^*_\beta t} \right| dt$$

$$= T_\infty + \int_0^\infty e^{Re[s'_\beta]t} \left| T_\beta e^{I_m[s'_\beta]t} + T^*_\beta e^{-I_m[s'_\beta]t} \right| dt \tag{3.11}$$

Assuming that $\left| I_m[s'_\beta] \right| >> \left| Re[s'_\beta] \right|$ then the term $e^{Re[s'_\beta]t}$ changes negligibly during one cycle of the oscillatory term in the integrand. Noting that $\left| \cos\ [I_m[s'_\beta]t] \right|$ has an average value of $2/\pi$ we have

$$\|T(t)o\|_\infty \simeq T_\infty + \frac{4}{\pi} \left| T_\beta \right| \{-Re[s'_\beta]\}^{-1} \tag{3.12}$$

for poles near the $j\omega$ axis. Note the factor of $2/\pi$ in comparing this result to that of (3.7) for the r-norm so that the infinity norm is less than the r-norm in this case.

The reader can note that since [6]

$$\|T(t)o\|_1 = \|T(t)\|_1 \tag{3.13}$$

The above results apply to the 1-norm of the filter as well.

IV. 2-Norm

Taking a different tack consider the 2-norm related to energy. Remember first the Parseval theorem as discussed in [4(Appendix B)]. Briefly stated this is

$$\left\| f(t) \right\|_{2,t} = \frac{1}{\sqrt{2\pi}} \left\| \tilde{f}(j\omega) \right\|_{2,\omega} \tag{4.1}$$

where integration over time t is distinguished from integration over frequency ω. More generally we have

$$\begin{aligned} \left\| f(t) \right\|^2_{2,t} &= \int_{-\infty}^{\infty} f^2(t)\, dt \\ &= \frac{1}{2\pi j} \int_{Br} \tilde{f}(s)\, \tilde{f}(-s)\, ds \end{aligned} \tag{4.2}$$

provided $\tilde{f}(s)$ and $\tilde{f}(-s)$ have a common strip of convergence containing the Bromwich (Br) contour parallel to the $j\omega$ axis in the complex frequency plane. Note that in this form the integrand $\tilde{f}(s)\, \tilde{f}(-s)$ is an analytic function of s (except at singularities) allowing for contour deformation in the s plane.

Consider first the excitation as in (2.4). The 2-norm is then given by

$$\left\| V_+^{(1)}(t) \right\|_2^2 = \frac{1}{2\pi j} \int_{Br} \tilde{V}_+^{(1)}(s)\, \tilde{V}_+^{(1)}(-s)\, ds \tag{4.3}$$

Require for the moment that

$$\mathrm{Re}[s_\alpha] \leqslant b < 0 \text{ for all } \alpha \tag{4.4}$$

then there is a strip of convergence of the integrand in (4.3) of width 2b centered on the $j\omega$ axis. Deforming the Bromwich contour to the left note that as $|s| \to \infty$ the integrand is $O(s^{-2})$ and the integral is $O(s^{-1})$ which is zero for the contour at infinity. As the contour sweeps by the poles s_α in the left half plane there is a residue $V_\alpha \tilde{V}_\alpha^{(1)}(-s_\alpha)$ giving

$$\left\| V_+^{(1)}(t) \right\|_2^2 = \sum_\alpha V_\alpha\, \tilde{V}_\alpha^{(1)}(-s_\alpha) \tag{4.5}$$

If now we restrict the excitation to only one pole pair (s_α with its conjugate), then we have

$$\|V_t^{(1)}(t)\|_2^2 = V_\alpha \tilde{V}_\alpha^{(1)}(-s_\alpha) + V_\alpha^* \tilde{V}_\alpha^{(1)}(-s_\alpha^*)$$

$$= 2 \text{ Re } [V_\alpha \tilde{V}_\alpha^{(1)}(-s_\alpha)]$$

$$= \text{Re } [V_\alpha^2[-s_\alpha]^{-1} + |V_\alpha|^2 \; [-\text{Re}[s_\alpha]]^{-1}]$$

$$= -\text{Re}[\frac{V_\alpha^2}{s_\alpha}] - \frac{|V_\alpha|2}{\text{Re}[s_\alpha]} \tag{4.6}$$

Note that the second term is dominant for $|\text{Re}[s_\alpha]| << |\text{Im}[s_\alpha]|$, in which case we have

$$\|V_+^{(1)}(t)\|_2 \simeq |V_\alpha| \{-\text{Re}[s_\alpha]\}^{-1/2} \tag{4.7}$$

Similarly consider the response as in (2.10) and (2.11). The 2-norm is given by

$$\|V_+^{(2)}(t)\|_2^2 = \frac{1}{2\pi j} \int_{Br} \tilde{V}_+^{(2)}(s) \; \tilde{V}_+^{(2)}(-s) \; ds \tag{4.8}$$

Requiring for the moment that

$$\begin{aligned} &\text{Re}[s_\alpha] \leqslant b < 0 \text{ for all } \alpha \\ &\text{Re}[s'_\beta] \leqslant b < 0 \text{ for all } \beta \\ &s_\alpha \neq s_\beta \text{ for all } \alpha,\beta \end{aligned} \tag{4.9}$$

there is a strip of convergence of width 2b centered on the $j\omega$ axis. Again deforming the contour to the left there is no contribution from ∞. As the contour sweeps by the poles s_α and s_β in the left half plane we get

$$\|V_+^{(2)}(t)\|_2^2 = \sum_\alpha \tilde{T}(s_\alpha)\ V_\alpha\ \tilde{V}_+^{(2)}(-s_\alpha) \text{ (excitation term)}$$

$$+ \sum_\beta \tilde{V}_+^{(1)}(s'_\beta)\ T_\beta \tilde{V}_+^{(2)}(-s'_\beta) \text{ (filter term)} \qquad (4.10)$$

Note as in (2.10) and (2.11) the response is separated into a direct and a filter term corresponding respectively to s_α and s'_β. However, in contrast to (3.2) for the r-norm in the case of the 2-norm we have the square root of the sum (or sum of squares if you will).

Now restricting the excitation to a single pole pair (s_α and its conjugate) and the filter to a constant term plus a single pole pair (s'_β and its conjugate) we have

$$\|V_+^{(2)}(t)\|_2^2 = \tilde{T}(s_\alpha)\ V_\alpha\ \tilde{T}(-s_\alpha)\ \tilde{V}_+^{(1)}(-s_\alpha) + \tilde{T}(s_\alpha^*)\ V_\alpha^*\ \tilde{T}(-s_\alpha^*)\ \tilde{V}_+^{(1)}(-s_\alpha^*)$$

$$+ \tilde{V}_+^{(1)}(s'_\beta)\ T_\beta\ \tilde{T}(-s'_\beta)\ \tilde{V}_+^{(1)}(-s'_\beta) + \tilde{V}_+^{(1)}(s'^*_\beta)\ T_\beta^*\ \tilde{T}(-s'^*_\beta)\ \tilde{V}_+^{(1)}(-s'^*_\beta)$$

$$= 2\ \mathrm{Re}[\tilde{T}(s_\alpha)\ V_\alpha\ \tilde{T}(-s_\alpha)\ \tilde{V}_+^{(1)}(-s_\alpha)] \text{ (excitation term)}$$

$$+ 2\ \mathrm{Re}[\tilde{V}_+^{(1)}(s'_\beta)\ T_\beta\ \tilde{T}(-s'_\beta)\ \tilde{V}_+^{(1)}(-s'_\beta)] \text{ (filter term)} \qquad (4.11)$$

In order to simplify the results let us consider the special case that s_α and s_β are near each other and near the $j\omega$ axis, giving

$$\begin{aligned} &\left|s_\alpha - s'_\beta\right| \ll \left|s_\alpha\right|, \left|s'_\beta\right| \\ &\left|-s_\alpha - s'^*_\beta\right| \ll \left|s_\alpha\right|, \left|s'_\beta\right| \\ &0 < -s_\alpha - s_\alpha^* = -2\mathrm{Re}[s_\alpha] \ll \left|s_\alpha\right|, \left|s'_\beta\right| \\ &0 < -s'_\beta - s'^*_\beta = -2\ \mathrm{Re}[s'^*_\beta] \ll \left|s_\alpha\right|, \left|s'_\beta\right| \end{aligned} \qquad (4.12)$$

Then considering the excitation term in (4.11) we have

$$\tilde{T}(s_\alpha)\ V_\alpha\ \tilde{T}(-s_\alpha)\ \tilde{V}_+^{(1)}(-s_\alpha)$$

$$= \{T_\infty + T_\beta[s_\alpha - s'_\beta]^{-1} + T_\beta^*[s_\alpha - s'^*_\beta]^{-1}\}$$

$$\{T_\infty + T_\beta[-s_\alpha - s'_\beta]^{-1} + T_\beta^*[-s_\alpha - s'^*_\beta]^{-1}\}$$

$$V_\alpha\{V_\alpha[-s_\alpha - s_\alpha]^{-1} + V_\alpha^*[-s_\alpha - s_\alpha^*]^{-1}\}$$

$$\simeq \left|T_\beta\right|^2 [s_\alpha - s'_\beta]^{-1} [-s_\alpha - s'^*_\beta]^{-1} \left|V_\alpha\right|^2 [-2\mathrm{Re}[s_\alpha]]^{-1} \tag{4.13}$$

$$2\mathrm{Re}[\tilde{T}(s_\alpha)\ V_\alpha\ \tilde{T}(-s_\alpha)\ \tilde{V}_+^{(1)}(-s_\alpha)]$$

$$\simeq \frac{\left|T_\beta\right|^2 \left|V_\alpha\right|^2}{-\mathrm{Re}[s_\alpha]}\ \mathrm{Re}[[s_\alpha - s'_\beta]^{-1} [-s_\alpha - s'^*_\beta]^{-1}]$$

Similarly the filter term in (4.11) gives

$$\tilde{V}_+^{(1)}(s'_\beta)\ T_\beta\ \tilde{T}(-s'_\beta)\ \tilde{V}_+^{(1)}(-s'_\beta)$$

$$= \{V_\alpha[s'_\beta - s_\alpha]^{-1} + V_\alpha^*\ [s'_\beta - s_\alpha^*]^{-1}\}$$

$$\{V_\alpha[-s'_\beta - s_\alpha]^{-1} + V_\alpha^*[-s'_\beta - s_\alpha^*]^{-1}\}$$

$$T_\beta\{T_\infty + T_\beta[-s'_\beta - s_\beta]^{-1} + T_\beta^*[-s'_\beta - s_\beta^*]^{-1}\}$$

$$\simeq \left|V_\alpha\right|^2 [s'_\beta - s_\alpha]^{-1} [-s'_\beta - s_\alpha^*]^{-1} \left|T_\beta\right|^2 [-2\ \mathrm{Re}[s'_\beta]]^{-1} \tag{4.14}$$

$$2\ \mathrm{Re}[\tilde{V}_+^{(1)}(s'_\beta)\ T_\beta\ \tilde{T}(-s'_\beta)\ \tilde{V}_+^{(1)}(-s'_\beta)]$$

$$\simeq \frac{\left|T_\beta\right|^2 \left|V_\alpha\right|^2}{-\mathrm{Re}[s'_\beta]}\ \mathrm{Re}[[s'_\beta - s_\alpha]^{-1} [-s'_\beta - s_\alpha^*]^{-1}]$$

Note that the excitation and filter terms are very similar with the roles of

s_α and s'_β interchanged. Based on the form of these results we can say that which of these two terms is dominant depends on how small $|\mathrm{Re}[s_\alpha]|$ and $|\mathrm{Re}[s'_\beta]|$ are relatively speaking.

Combining the excitation and filter terms we have

$$\|v_+^{(2)}(t)\|_2^2 \simeq |T_\beta|^2 \; |V_\alpha|^2 \; \mathrm{Re}[[s_\alpha - s'_\beta]^{-1} \; \{[s_\alpha + s'^{*}_\beta]^{-1} \; [\mathrm{Re}[s_\alpha]]^{-1}$$

$$-[s^*_\alpha + s'_\beta]^{-1}[\mathrm{Re}[s_\beta]]^{-1}\}]$$

$$= |T_\beta|^2 \; |V_\alpha|^2 \mathrm{Re}[\frac{[s^*_\alpha - s^*_\beta][\mathrm{Re}[s'_\beta][s^*_\alpha + s'_\beta] - \mathrm{Re}[s_\alpha][s_\alpha + s'^{*}_\beta]]}{\mathrm{Re}[s_\alpha]\mathrm{Re}[s_\beta]\,|s_\alpha + s'^{*}_\beta|^2 \; |s_\alpha - s'_\beta|^2}]$$

$$= |T_\beta|^2 \; |V_\alpha|^2 \; \mathrm{Re}[\frac{[s^*_\alpha - s'^{*}_\beta] \; \mathrm{Re}[s_\alpha + s'_\beta][s'_\beta - s_\alpha]}{\mathrm{Re}[s_\alpha]\mathrm{Re}[s'_\beta] \; |s_\alpha + s'^{*}_\beta|^2 \; |s_\alpha - s'_\beta|^2}]$$

$$= - \; |T_\beta|^2 \; |V_\alpha|^2 \; \frac{\mathrm{Re}[s_\alpha + s'_\beta]}{\mathrm{Re}[s_\alpha] \; \mathrm{Re}[s'_\beta]} \; |s_\alpha + s'^{*}_\beta|^{-2} \tag{4.15}$$

Note that the outermost Re operator has been unnecessary since its argument has been shown to be real (and positive). Then we have

$$\|v_+^{(2)}(t)\|_2 \simeq \frac{|T_\beta| \; |V_\alpha|}{|s_\alpha + s'^{*}_\beta|} \; \{\frac{-\mathrm{Re}[s_\alpha + s'_\beta]}{\mathrm{Re}[s_\alpha] \; \mathrm{Re}[s'_\beta]}\}^{1/2} \tag{4.16}$$

as a rather compact result under the assumption of single pole pairs for excitation and filter with poles closely approaching each other and the $j\omega$ axis as in (4.12).

In another form

$$\|v_+^{(2)}(t)\|_2 \simeq \frac{|T_\beta| \; |V_\alpha|}{|s_\alpha + s'^{*}_\beta|} \; \{- \frac{1}{\mathrm{Re}(s_\alpha]} - \frac{1}{\mathrm{Re}[s'_\beta]}\}^{1/2} \tag{4.17}$$

so that we can see the separate contributions of $\mathrm{Re}[s_\alpha]$ and $\mathrm{Re}[s'_\beta]$ to the size of the 2-norm of the response. Substituting from (4.7) for the 2-norm of the excitation as

$$\left\| V_+^{(2)}(t) \right\|_2 \simeq \frac{|T_\beta|}{\left| s_\alpha + s_\beta'^{*} \right|} \{1 + \frac{Re[s_\alpha]}{Re[s_\beta']}\}^{1/2} \left\| V_+^{(1)}(t) \right\|_2 \qquad (4.18)$$

we can note that the 2-norm of the filter (a convolution operator) is found by maximizing the coefficient of $\left\| V_+^{(1)}(t) \right\|_2$ giving

$$Im[s_\alpha] = Im[s_\beta'] \qquad (4.19)$$

$$\left\| T(t)o \right\|_2 \simeq \sup_{Re[s_\alpha] \leqslant 0} \frac{|T_\beta|}{-Re[s_\alpha + s_\beta']} \{\frac{Re[s_\alpha + s_\beta']}{Re[s_\beta']}\}^{1/2}$$

$$= \left| T_\beta \right| \{-Re[s_\beta']\}^{-1}$$

Note that this is consistent with the more general results [6] for the 2-norm of a convolution operator as

$$\left\| T(t)o \right\|_2 = \left| \tilde{T}(j\omega) \right|_{max} \qquad (4.20)$$

Considering the filter as in (2.5) for the case of a single pole pair we have

$$\left\| T(t)o \right\|_2 = \max_\omega \left| T_\infty + T_\beta [j\omega - s_\beta']^{-1} + T_\beta^* [j\omega - s_\beta'^{*}]^{-1} \right| \qquad (4.21)$$

Letting s_β' be restricted as in (4.12) we have for the maximum

$$\omega_{max} \simeq Im[s_\beta']$$

$$\left\| T(t)o \right\|_2 \simeq \left| T_\infty + T_\beta [-Re[s_\beta']]^{-1} + T_\beta^* [j2\omega_{max} - Re[s_\beta']]^{-1} \right|$$

$$\simeq \{\{T_\infty + \frac{Re[T_\beta]}{-Re[s_\beta']}\}^2 + \{\frac{Im[T_\beta]}{-Re[s_\beta']}\}^2\}^{1/2}$$

$$= \{T_\infty^2 + 2 \frac{T_\infty Re[T_\beta]}{-Re[s_\beta']} + \frac{|T_\beta|^2}{[-Re[s_\beta']]^2}\}^{1/2} \qquad (4.22)$$

Neglecting T_∞ in comparison to $|T_\beta|/[-\mathrm{Re}[\bar{s}'_\beta]]$ reproduces the result in (4.19).

V. Contribution of Excitation and Filter to Response Norms

Looking over the foregoing results, let us make some observations concerning the relative contributions of the excitation and filter to the response as reflected in norms. If we now look at the norms discussed in sections 3 and 4 we can see how the selecion of the excitation $V_+^{(1)}(t)$ for a given filter operator T(t)o can influence the response $V_+^{(2)}(t)$, now interpreted in norm sense.

Beginning with (2.10) we first observe that the damped sinusoids (in time domain) are of two kinds: those associated with s_α (the excitation poles) proportional to $\tilde{T}(s_\alpha)$, and those associated with s'_β (the filter poles) proportional to $\tilde{V}_+^{(1)}(s'_\beta)$. So far this indicates how the filter's natural frequencies enhance the excitation poles, and how the excitation natural frequencies enhance the filter poles.

In section 3 the relative contributions of the excitation and filter terms are treated in terms of the r-norm, an approximation to the ∞-norm. As indicated in (3.2) it shows that the excitation poles are again scaled by the filter evaluated at the s_α. Furthermore the filter poles are also scaled by the excitation evaluated at the s'_β.

In section 4 the use of the Parseval theorem in the complex s-plane has given exact representations for the 2-norm of the response in (4.10) and (4.11). In this case excitation and filter terms combine as the square root of the sum of the squares. However, these separate terms still have the property that the excitation poles are scaled by the filter evaluated at the s_α and $-s_\alpha$, and the filter poles are scaled by the excitation evaluated at the s'_β and $-s'_\beta$. Note that if the s_α and s'_β are near the $j\omega$ axis then

$$-s_\alpha \simeq s_\alpha^* \simeq -j\omega \quad , \quad -s'_\beta \simeq s'^*_\beta \simeq -j\omega'_\beta \tag{5.1}$$

which with the conjugate pairing of the poles allows us to regard the $-s_\alpha$ and $-s'_\beta$ as approximately equivalent to other s_α and s'_β respectively. However, this observation should be restricted to the case that the s_α and s'_β are not too close to each other.

Based on these observations for the norms considered there is an important concept for the case that the s_α and s_β are not too close to each other, but are both close to the $j\omega$ axis. The response scales according to:

a. $\tilde{T}(j\omega_\alpha)$, the filter evaluated at the excitation poles, and

b. $\tilde{V}_+^{(1)}(j\omega'_\beta)$, the excitation evaluated at the filter poles.

Carrying the argument further note that even if the s_α are not near the $j\omega$ axis, the filter term still scales by $\tilde{V}_+^{(1)}(j\omega'_\beta)$ for s'_β near the $j\omega$ axis. In a more general case $V_+^{(1)}(t)$ may be some specified environment waveform. The present results show:

It is important that the excitation not be deficient in its frequency spectrum, particularly if the location of the $j\omega'_\beta$ (the frequencies for which the transfer through the filter is greatest) are not known beforehand.

Note that an experimental time-domain waveform that one uses for test purposes is only an approximation to a specified waveform. It is quite possible for the experimental waveform to be a good approximation of a specified waveform in time domain, but a poor approximation in some portion of the frequency domain. Letting $V_+^{(a)}(t)$ be some approximation of $V_+^{(1)}(t)$ so that

$$\Delta V_+(t) \equiv V_+^{(1)}(t) - V_+^{(a)}(t)$$
$$0 \leqslant |\Delta V_+^{(1)}(t)| \leqslant \nu \text{ for } -\infty < t < \infty \tag{5.2}$$
$$\nu << \sup_t |V_+^{(1)}(t)|$$

This does not imply that $\tilde{V}_+^{(a)}(j\omega)$ is non zero for some ω for which $\tilde{V}_+^{(1)}(j\omega)$ is nonzero.

As a simple example suppose

$$V_+^{(a)}(t) = V_+^{(1)}(t) + [A\, e^{s_a t} + A^*\, e^{s_a^* t}]\, u(t) \tag{5.3}$$

Then (5.1) requires for s_a near the $j\omega$ axis

$$|A| \leqslant \frac{\nu}{2} \tag{5.4}$$

as a constraint making the size of the damped sinusoid sufficiently small.

Then set

$$
\begin{aligned}
0 &= \tilde{V}_+^{(1)}(j\omega_a) + \frac{A}{j\omega_a - s_a} + \frac{A^*}{j\omega_a - s_a^*} \\
&= \tilde{V}_+^{(1)}(j\omega_a) + \frac{A}{-\mathrm{Re}[s_a]} + \frac{A^*}{j2\omega_a - \mathrm{Re}[s_a]} \\
&\simeq \tilde{V}_+^{(1)}(j\omega_a) - \frac{A}{\mathrm{Re}[s_a]}
\end{aligned}
\qquad (5.5)
$$

So setting $|\mathrm{Re}[s_a]|$ sufficiently small with

$$
\mathrm{Re}[s_a] \simeq \frac{A}{\tilde{V}_+^{(1)}(j\omega_a)} \qquad (5.6)
$$

subject to (5.4) gives us a case for which $V_+^{(a)}(t)$ is deficient in that it has a poor frequency content (a notch) at $s \simeq j\omega_a$ while departing minimally from $V_+^{(1)}(t)$ in time domain. Other examples can also be constructed; the above one merely has a simple form.

Of course we have now introduced excitation poles at s_a and s_a^* which, if they are near the filter poles s'_β and s'^*_β complicate the matter considerably. The case of such close approach of poles is treated in [6] and in the previous sections. So the response is significantly changed from the simple case of separate excitation and filter terms discussed above. In any event the response is significantly changed in time domain by what one might think is a small perturbation of a time-domain excitation which, however, has a comparatively big effect in frequency domain.

VI. CHOOSING EXCITATION TO TEST THE VARIETIES OF RESPONSE

Now suppose one wishes to test a black box by some excitation waveform. The question arises as to what is an appropriate waveform. Now if $V_+^{(1)}(t)$ is to represent some waveform appearing somewhere within a linear system then some filter (which represents such a system) tends to make $V_+^{(1)}(t)$ consist of a set of damped sinusoids, although other shapes are possible.

Let us now take a single damped sinusoid of the form

$$V_+^{(1)}(t) = \{V_\alpha e^{s_\alpha t} + V_\alpha^* e^{s_\alpha^* t}\} u(t) \tag{6.1}$$

as a canonical waveform. Now putting this into a black box characterized by a filter $\tilde{T}(s)$ what should V_α and s_α be? Well, as the previous development indicates, that depends on $\tilde{T}(s)$ if the behavior of $V_+^{(2)}(t)$ is important (say in norm sense). If any s'_β are near the $j\omega$ axis then choosing s_α near such an s'_β can make a dramatic difference in $\|V_+^{(2)}(t)\|$.

Now if small changes in s_α can produce large changes in $\|V_+^{(2)}(t)\|$, and if failure is dependent basically on $\|V_+^{(2)}(t)\|$ for some norm, then it is important to choose s_α in a way that maximizes the norm of the response. Let us assume that $\|V_+^{(1)}(t)\|$, the excitation norm, is not to change very much for such small changes in s_α. Then we can consider the ratio of the response norm to the excitation norm. However the maximum this ratio achieves is just the filter norm as

$$\sup_{V_+^{(1)}(t)\not\equiv 0} \frac{\|V_+^{(2)}(t)\|}{\|V_+^{(1)}(t)\|} = \sup_{V_+^{(1)}(t)\not\equiv 0} \frac{\|T(t)\circ V_+^{(1)}(t)\|}{\|V_+^{(1)}(t)\|} \equiv \|T(t)\circ\| \tag{6.2}$$

Thus one way to consider this problem is to see how close the ratio approaches $\|T(t)\circ\|$ as a measure of how well $V_+^{(1)}(t)$ (and thus s_α) has been chosen.

Reviewing our previous results let T(t) take the canonical form

$$T(t) = T_\infty \delta(t) + \{T_\infty e^{s'_\beta t} + T_\beta^* e^{s'^*_\beta t}\} u(t) \tag{6.3}$$

including only one damped sinusoid. Considering the case that s_α is near s'_β let us define

$$\eta \equiv \frac{\left\| V_+^{(2)}(t) \right\|}{\left\| V_+^{(1)}(t) \right\|} \left\{ \left\| T(t)o \right\| \right\}^{-1} \tag{6.4}$$

with appropriate subscripts for the norm under consideration.

From section 3 we have the ∞-norm (and related r-norm) with

$$\mathrm{Re}[s_\alpha] \leqslant 0 \ , \ \mathrm{Re}[s'_\beta] \leqslant 0 \tag{6.5}$$

$$|\mathrm{Re}[s_\alpha]| << |\mathrm{Im}[s_\alpha]|, \ |\mathrm{Re}[s'_\beta]| << |\mathrm{Im}[s'_\beta]|$$

giving

$$\left\| V_+^{(1)}(t) \right\|_\infty \simeq 2 \ |V_\alpha| \tag{6.6}$$

and for s_α not near s'_β

$$\left\| V_+^{(2)}(t) \right\|_\infty \simeq 2|\tilde{T}(s_\alpha)| \ |V_\alpha| + 2|V_+^{(1)}(s'_\beta)| \ |T_\beta| \tag{6.7}$$

and for s_α near s'_β (with imaginary parts equal)

$$\left\| V_+^{(2)}(t) \right\|_\infty \simeq \frac{1}{e} \frac{4}{-\mathrm{Re}[s'_\beta] - \mathrm{Re}[s_\alpha]} \ |T_\beta| \ |V_\alpha| \tag{6.8}$$

The filter norm is

$$\left\| T(t)o \right\|_\infty \simeq T_\infty + \frac{4}{\pi} \ |T_\beta| \ \{-\mathrm{Re}[s'_\beta]\}^{-1}$$

$$\simeq \frac{4}{\pi} \ |T_\beta| \ \{-\mathrm{Re}[s'_\beta]\}^{-1} \tag{6.9}$$

provided T_∞ can be neglected compared to the second term.

Using the results for s_α near s'_β (with equal imaginary parts) we have

$$\eta_\infty \simeq \frac{\pi}{2e} \frac{-\mathrm{Re}[s'_\beta]}{-\mathrm{Re}[s'_\beta] - \mathrm{Re}[s_\alpha]} \tag{6.10}$$

This is maximized for $\text{Re}[s_\alpha] = 0$ and as it varies away to say the same value as $\text{Re}[s'_\beta]$ then η_∞ is cut in half. So this shows that as one approaches a filter pole variations of the order of $\text{Re}[s'_\beta]$ are important. Along the $j\omega$ axis this represents the width of the resonance peak. Note the coefficient $\pi/(2e) \simeq .578$ which is somewhat less than unity. This indicates that some other waveform than our simple damped sinusoid could achieve a slightly greater increase in the ∞-norm.

From section 4 we have the 2-norm with

$$\|V_+^{(1)}(t)\|_2 \simeq |V_\alpha| \{-\text{Re}[s_\alpha]\}^{1/2}$$

$$\|V_+^{(2)}(t)\|_2 \simeq \frac{|T_\beta|\ |V_\alpha|}{|s_\alpha + s'^*_\beta|} \left\{\frac{-\text{Re}[s_\alpha + s'_\beta]}{\text{Re}[s_\alpha]\ \text{Re}[s'_\beta]}\right\}^{1/2} \tag{6.11}$$

$$\|T(t)o\|_2 \simeq |T_\beta| \{-\text{Re}[s'_\beta]\}^{-1}$$

This gives

$$\eta_2 = \frac{\|V_+^{(2)}(t)\|_2}{\|V_+^{(1)}(t)\|_2} \{\|T(t)o\|_2\}^{-1}$$

$$\simeq |s_\alpha + s'^*_\beta|^{-1} \{\text{Re}[s'_\beta]\ [\text{Re}[s_\alpha + s'_\infty]\}^{1/2} \tag{6.12}$$

Letting

$$\text{Im}[s_\alpha] = \text{Im}[s_\beta]$$
$$\text{Re}[s_\alpha] = 0 \tag{6.13}$$

gives

$$\eta_2 \simeq 1 \tag{6.14}$$

as the maximum value of η_2.

Rearranging we have

$$\eta_2^2 \simeq \frac{Re[s_\beta'] \; Re[s_\alpha + s_\beta']}{\left| s_\alpha + s_\beta'^* \right|^2}$$

$$= \frac{Re[s_\beta'] \; Re[s_\alpha + s_\beta']}{Re^2[s_\alpha + s_\beta'] + Im^2[s_\alpha - s_\beta']} \qquad (6.15)$$

This allows us to see how η_2 approaches unity as s_α approaches the conditions in (6.13). First, for matching imaginary parts we have

$$Im[s_\alpha] = Im[s_\beta']$$

$$\eta_2^2 \simeq \frac{Re[s_\beta']}{Re[s_\alpha + s_\beta']} = \{1 + \frac{Re[s_\alpha]}{Re[s_\beta']}\}^{-1} \qquad (6.16)$$

$$\rightarrow 1 \text{ as } Re[s_\alpha] \rightarrow 0$$

Note that for

$$Re[s_\alpha] = Re[s_\beta'] \qquad (6.17)$$

$$\eta_2^2 \simeq \frac{1}{2}$$

Second, for pure imaginary s_α we have

$$Re[s_\alpha] = 0 \qquad (6.18)$$

$$\eta_2^2 \simeq \frac{Re^2[s_\beta']}{Re^2[s_\beta'] + Im^2[s_\alpha - s_\beta']} = \{1 + \frac{Im^2[s_\alpha - s_\beta']}{Re^2[s_\beta']}\}^{-1}$$

$$\rightarrow 1 \text{ as } Im[s_\alpha - s_\beta'] \rightarrow 0$$

Note that for

$$|Im[s_\alpha - s_\beta']| = |Re[s_\beta']| \qquad (6.19)$$

$$\eta_2^2 \simeq \frac{1}{2}$$

So for s_α deviating from $j\mathrm{Im}[s'_\beta]$ in magnitude $|\mathrm{Re}[s'_\beta]|$ then η_2 is reduced to $1/\sqrt{2}$. This shows that for maximizing $\|V_{+}^{(2)}(t)\|$ we need s_α this close to $j\mathrm{Im}[s'_\beta]$.

These results then show:

> It is important that to find the maximum response $\|V_{+}^{(2)}(t)\|$ it is necessary that the excitation natural frequencies s_α approach the filter $j\mathrm{Im}[s'_\beta]$ to within an amount related to $|\mathrm{Re}[s'_\beta]|$. Otherwise the response norm is significantly reduced.

Assuming that the s'_β for some black box are in general unknown a priori, it is then necessary that the s_α should be varied such that

$$|\mathrm{Re}[s_\alpha]| < |\mathrm{Re}[s'_\beta]| \quad \text{for all } s'_\beta \tag{6.20}$$

This in turn implies that some lower bound be placed on the $|\mathrm{Re}[s'_\beta]|$ that one may encounter in black boxes of interest. Furthermore, we need to select samples of s_α such that

$$|\Delta[\mathrm{Im}[s_\alpha]]| < |\mathrm{Re}[s'_\beta]| \tag{6.21}$$

where $\Delta[\mathrm{Im}[s_\alpha]]$ represents the change in $\mathrm{Im}[s_\alpha]$ from one test excitation waveform to the next. This change might itself be a function of $\mathrm{Im}[s_\alpha]$.

VII. Summary

Based on this simple model of a linear filter two important points can be made.

a. For a particular specified environmental excitation $V_+^{(1)}(t)$ it is important that its approximate realization $V_+^{(a)}(t)$ be such that in frequency domain $\tilde{V}_+^{(a)}(j\omega)$ closely approximate $\tilde{V}_+^{(1)}(j\omega)$ for all ω of interest. In particular $\tilde{V}_+^{(a)}(j\omega)$ should not have notches (near zeros) compared to $\tilde{V}_+^{(1)}(j\omega)$ for all ω of interest.

b. For damped-sinusoid testing one needs to place some lower bound on $|Re[s'_\beta]|$ (perhaps as a function of ω). Then fixing $|Re[s_\alpha]| < |Re[s'_\beta]|$ one needs to select a set of ω_α such that $|\Delta\omega_\alpha| < |Re[s'_\beta]|$ in order to be able to excite the filter sufficiently close to the $Im[s'_\beta]$ (unknown a priori) so as to approximate the maximum $\|V_+^{(2)}(t)\|$ across the frequency band of interest.

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