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Abstract

This paper considers the asymptotic form for high frequencies of the equations of propagation on a nonuniform N-conductor transmission line. By considering the case of perfect conductions in uniform, isotropic media all N propagation velocities are the same, but the characteristic impedance matrix is allowed to vary with position along the line. Closed-form solutions are obtained for some cases of interest.

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HIGH-FREQUENCY PROPAGATION ON NONUNIFORM MULTICONDUCTOR TRANSMISSION LINES IN UNIFORM MEDIA

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SUMMARY

The asymptotic form, for high frequencies, of the equations of propagation on a nonuniform N-conductor transmission line is considered. Under the assumption of perfect conditions in uniform, isotropic media, all N propagation velocities are the same, but the characteristic impedance matrix is allowed to vary with position along the line. Closed-form solutions are obtained for some cases of interest.

1. INTRODUCTION

In the theory of multiconductor transmission lines one can readily solve for the propagation on such a system in terms of an exponential matrix function, provided the per-unit-length impedance and admittance are independent of position z along the particular transmission line (or tube). If these matrices are allowed to be functions of z the differential equations are considerably complicated.

If, however, one looks at the propagation at high frequencies the equations simplify somewhat. This is particularly so if the propagation matrix yields a single propagation speed, as is the case for perfect conductors in a uniform medium (such as free space). The resulting matrix differential equation is first order in z, but the coefficient matrix is a function of z. While this is solvable, at least numerically, it does admit of closed-form solutions for special, but interesting, cases of the coefficient matrix. This generalizes the well-known result for a single-conductor transmission line.

2. MULTICONDUCTOR TRANSMISSION LINE

Consider a multiconductor transmission line with per-unit-length model as shown in Figure 1. This represents one incremental section in such a transmission line. When considered as one transmission line in a network of such transmission lines (containing various numbers of conductors) we can think of each multiconductor line as one 'tube' in the network.[1] For the present, let us consider a single such tube.

The telegrapher equations are

$$\frac{\mathrm{d}}{\mathrm{d}z}(\tilde{V}_n(z,s)) = -(\tilde{Z}'_{n,m}(z,s))\cdot(\tilde{I}_n(z,s)) + (\tilde{V}_n^{(s)\prime}(z,s))$$
$$\frac{\mathrm{d}}{\mathrm{d}z}(\tilde{I}_n(z,s)) = -(\tilde{Y}'_{n,m}(z,s))\cdot(\tilde{V}_n(z,s)) + (\tilde{I}_n^{(s)\prime}(z,s)) \quad (1)$$

where

z	position along tube
$(\tilde{V}_n(z,s))$	voltage vector at z
$(\tilde{I}_n(z,s))$	current vector at z
$(\tilde{Z}'_{n,m}(z,s))$	per-unit-length series impedance matrix

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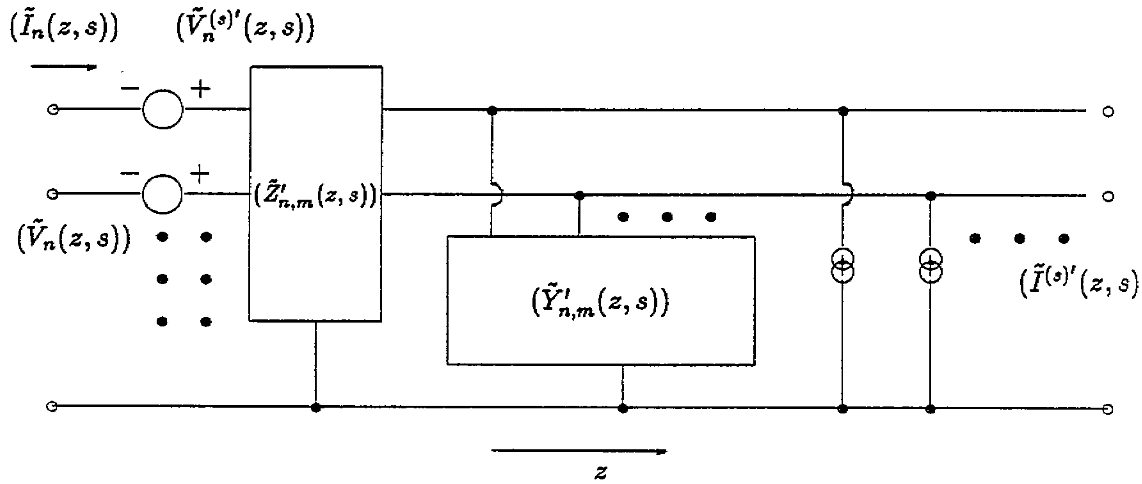

Figure 1. Per-Unit-Length Model of a Multiconductor Transmission line

$(\tilde{Y}'_{n,m}(z,s))$	per-unit-length shunt admittance matrix
$(\tilde{V}_n^{(s)\prime}(z,s))$	per-unit-length series voltage source vector
$(\tilde{I}_n^{(s)\prime}(z,s))$	per-unit-length shunt current source vector
s	Laplace transform variable (complex frequency) for transform over time (t)

For our N-wire transmission line all vectors are of dimension N, and all matrices are $N \times N$.

Defining the propagation matrix as

$$(\tilde{\gamma}_{c_{n,m}}(z,s)) \equiv \{(\tilde{Z}'_{n,m}(z,s))\cdot(\tilde{Y}'_{n,m}(z,s))\}^{1/2} \tag{2}$$

where the matrix square root is taken in the principal value or positive real (p.r.) sense[1] by diagonalizing $(\tilde{\gamma}_{c_{n,m}})^2$ and then taking p.r. square roots of the eigenvalues. From this we also derive the characteristic impedance matrix and characteristic admittance matrix

$$\begin{aligned}(\tilde{Z}_{c_{n,m}}(z,s)) &= (\tilde{\gamma}_{c_{n,m}}(z,s))\cdot(\tilde{Y}'_{n,m}(z,s))^{-1} = (\gamma_{c_{n,m}}(z,s))^{-1}\cdot(\tilde{Z}'_{n,m}(z,s))\\ (\tilde{Y}_{c_{n,m}}(z,s)) &= (\tilde{Y}'_{n,m}(z,s))\cdot(\tilde{\gamma}_{c_{n,m}}(z,s))^{-1} = (\tilde{Z}_{n,m}(z,s))^{-1}\cdot(\tilde{\gamma}_{c_{n,m}}(z,s)) = (\tilde{Z}_{c_{n,m}}(z,s))^{-1}\end{aligned} \tag{3}$$

Combined voltage and source vectors are then defined by

$$\begin{aligned}(\tilde{V}_n(z,s))_q &= (\tilde{V}_n(z,s)) + q(\tilde{Z}_{c_{n,m}}(z,s))\cdot(\tilde{I}_n(z,s)) \qquad q = \pm 1 \text{ (separation index)}\\ (\tilde{V}_n^{(s)\prime}(z,s))_q &= (\tilde{V}_n^{(s)\prime}(z,s)) + q(\tilde{Z}_{c_{n,m}}(z,s))\cdot(\tilde{I}_n^{(s)\prime}(z,s))\end{aligned} \tag{4}$$

where $\pm$ corresponds to waves propagation in the $\pm z$ directions.

Now, in Reference 1, under the assumption that $(\tilde{Z}'_{n,m}(s))$ and $(\tilde{Y}'_{n,m}(s))$ are *not* functions of z (allowing $(\tilde{Z}_{c_{n,m}}(s))$ to pass through the z derivative), we obtain the combined-voltage differential equation

$$\left\{(1_{n,m})\frac{\mathrm{d}}{\mathrm{d}z} + q(\tilde{\gamma}_{c_{n,m}}(s))\right\}\cdot(\tilde{V}_n(z,s))_q = (\tilde{V}_n^{(s)\prime}(z,s))_q \tag{5}$$

where $(\tilde{\gamma}_{c_{n,m}}(s))$ is also *not* a function of z. This is readily solved[2] to give

$$(\tilde{V}_n(z,s))_q = \exp\{-q(\tilde{\gamma}_{c_{n,m}}(s))z\}\cdot(\tilde{V}_n(0,s))_q + \int_0^z \exp\{-q(\tilde{\gamma}_{c_{n,m}}(z))[z-z']\}\cdot(\tilde{V}_n^{(s)\prime}(z',s))\mathrm{d}z' \tag{6}$$

where without loss of generality we have taken the initial condition as $(V_n(0,s))$, but any other z_0 could be used.

In this paper we assume there to be no sources $(\tilde{V}_n^{(s)\prime})$ and $(\tilde{I}_n^{(s)\prime})$ along the tube, the only 'excitation' coming from conditions at some coordinate we take as z=0. In this case (6) reduces to

$$(\tilde{V}_n(z,s))_q = \exp\{-q(\tilde{\gamma}_{c_{nm}}(s))z\}\cdot(\tilde{V}_n(0,s))_q \tag{7}$$

as the solution of the homogeneous form of (5). This is a fairly simple result for propagation along a uniform N-wire transmission line.

3. NONUNIFORM TUBE AT HIGH FREQUENCIES

Letting the per-unit-length impedance and admittance matrices in (1) be functions of z, we can write a second-order differential equation, say in terms of the voltage vector, as

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{(\tilde{Z}'_{n,m}(z,s))^{-1}\cdot\frac{\mathrm{d}}{\mathrm{d}z}(\tilde{V}_n(z,s))\right\} = (\tilde{Y}'_{n,m}(z,s))\cdot(\tilde{V}_n(z,s)) \tag{8}$$

which can be manipulated into the form

$$\begin{aligned}(\tilde{Z}'_{n,m}(z,s))\cdot\left\{\frac{\mathrm{d}}{\mathrm{d}z}[(\tilde{Z}'_{n,m}(z,s))^{-1}]\right\}\cdot\frac{\mathrm{d}}{\mathrm{d}z}(\tilde{V}_n(z,s))\\ +\left\{(1_{n,m})\frac{\mathrm{d}^2}{\mathrm{d}z^2}-(\tilde{Z}'_{n,m}(z,s))\cdot(\tilde{Y}'_{n,m}(z,s))\right\}\cdot(\tilde{V}_n(z,s)) = (0_n)\end{aligned} \tag{9}$$

Identifying the propagation matrix, let us assume that all modes have the same speed of propagation v, i.e. that

$$(\tilde{\gamma}_{c_{n,m}}(z,s))^2 = (\tilde{Z}'_{n,m}(z,s))\cdot(\tilde{Y}'_{n,m}(z,s)) = \frac{s^2}{v^2(z)}(1_{n,m}) \tag{10}$$

This is the case if the tube consists of N perfect conductors immersed in a uniform isotropic medium (at each cross-section), the $(Z'_{n,m})$ and $(Y'_{n,m})$ being frequency-independent real matrices times functions of the constitutive parameters of the medium. Furthermore, if we let the constitutive parameters be taken as independent of z, we have

$$(\tilde{\gamma}_{c_{n,m}}(s))^2 = \frac{s^2}{v^2}(1_{n,m}) \tag{11}$$

independent of z. Note that the speed may be a complex function of s if the medium is lossy. However, for many practical problems the medium can be approximated by real constants ϵ and μ with $\sigma = 0$ for which case v is a constant. With this form (9) becomes

$$(\tilde{Z}_{c_{n,m}}(z,s))\cdot\left\{\frac{\mathrm{d}}{\mathrm{d}z}(\tilde{Y}_{c_{n,m}}(z,s))\right\}\cdot\frac{\mathrm{d}}{\mathrm{d}z}(\tilde{V}_n(z,s)) + \left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}-\frac{s^2}{v^2}\right\}(\tilde{V}_n(z,s)) = (0_n) \tag{12}$$

This equation is still fairly complicated. Let us try a substitution of the form (appropriate for waves propagating in the positive z direction)

$$\begin{aligned}(\tilde{V}_n(z,s)) &= (\tilde{\phi}_{n,m}(z,s))\cdot\exp\{-(\gamma_{c_{n,m}})z\}\cdot(\tilde{V}_n(0,s))\\ &= \exp(-sz/v)\,(\phi_{n,m}(z,s))\cdot(\tilde{V}_n(0,s))\end{aligned} \tag{13}$$

giving

$$\begin{aligned}\Bigg\{(\tilde{Z}_{c_{n,m}}(z,s))\cdot\left\{\frac{\mathrm{d}}{\mathrm{d}z}(\tilde{Y}_{c_{n,m}}(z,s))\right\}\cdot\left[-\frac{s}{v}+\frac{\mathrm{d}}{\mathrm{d}z}\right](\tilde{\phi}_{n,m}(z,s))\\ -\frac{2s}{v}\frac{\mathrm{d}}{\mathrm{d}z}(\tilde{\phi}_{n,m}(z,s))+\frac{\mathrm{d}^2}{\mathrm{d}z^2}(\tilde{\phi}_{n,m}(z,s))\Bigg\}\cdot(\tilde{V}_n(0,s)) = (0_n)\end{aligned} \tag{14}$$

If we take N linearly independent choices of $(\tilde{V}_n(0,s))$ as the columns of a matrix, we can replace $(\tilde{V}_n(0,s))$ by such a nonsingular matrix which we can remove from (14) to give

$$\begin{aligned}(\tilde{Z}_{c_{n,m}}(z,s))\cdot\left\{\frac{\mathrm{d}}{\mathrm{d}z}(\tilde{Y}_{c_{n,m}}(z,s))\right\}\cdot\left[-\frac{s}{v}+\frac{d}{dz}\right](\phi_{n,m}(z,s))\\ -\frac{2s}{v}\frac{\mathrm{d}}{\mathrm{d}z}(\tilde{\phi}_{n,m}(z,s))+\frac{\mathrm{d}^2}{\mathrm{d}z^2}(\tilde{\phi}_{n,m}(z,s)) = (0_{n,m})\end{aligned} \tag{15}$$

Note that, to satisfy (13) for all $(\tilde{V}_n(0,s))$,

$$(\tilde{\phi}_{n,m}(0,s)) = (1_{n,m}) \tag{16}$$

Now let us take $s \to \infty$ and neglect $\mathrm{d}(\tilde{\phi}_{n,m})/\mathrm{d}z$ with respect to $(s/v)(\tilde{\phi}_{n,m})$, giving

$$\frac{\mathrm{d}}{\mathrm{d}z}(\phi_{n,m}(z)) = \left\{-\frac{1}{2}(Z_{c_{n,m}}(z))\cdot\frac{\mathrm{d}}{\mathrm{d}z}(Y_{c_{n,m}}(z))\right\}\cdot(\phi_{n,m}(z)) \tag{17}$$

as our high-frequency equation, which is now only first-order. Note that we have assumed that the coefficient matrix has gone to some asymptotic form as $s \to \infty$, and the s dependence is dropped from $(\phi_{n,m}(z))$ as it applies only at high frequencies. Noting that

$$(\tilde{Z}_{c_{n,m}}(z,s))\cdot(\tilde{Y}_{c_{n,m}}(z,s)) = (1_{n,m}) \tag{18}$$

we have

$$\left[\frac{\mathrm{d}}{\mathrm{d}z}(\tilde{Z}_{c_{n,m}}(z,s))\right]\cdot(\tilde{Y}_{c_{n,m}}(z,s))+(\tilde{Z}_{c_{n,m}}(z,s))\cdot\frac{\mathrm{d}}{\mathrm{d}z}(\tilde{Y}_{c_{n,m}}(z,s)) = (0_{n,m}) \tag{19}$$

This gives an alternative form of (17) as

$$\frac{\mathrm{d}}{\mathrm{d}z}(\phi_{n,m}(z)) = \left\{\frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}z}(Z_{c_{n,m}}(z))\right]\cdot(Y_{c_{n,m}}(z))\right\}\cdot(\phi_{n,m}(z)) \tag{20}$$

Another form of this equation is found by first defining

$$\begin{aligned}(\tilde{z}_{c_{n,m}}(z,s)) &= (\tilde{Z}_{c_{n,m}}(z,s))^{1/2}\\ (\tilde{y}_{c_{n,m}}(z,s)) &= (\tilde{Y}_{c_{n,m}}(z,s))^{1/2} = (\tilde{z}_{c_{n,m}}(z,s))^{-1}\end{aligned} \tag{21}$$

with the p.r. square root as before. Note that we have

$$\begin{aligned}(\tilde{z}_{c_{n,m}}(z,s))\cdot(\tilde{y}_{c_{n,m}}(z,s)) = (\tilde{y}_{c_{n,m}}(z,s))\cdot(\tilde{z}_{c_{n,m}}(z,s)) &= (1_{n,m})\\ \left[\frac{\mathrm{d}}{\mathrm{d}z}(\tilde{z}_{c_{n,m}}(z,s))\right]\cdot(\tilde{y}_{c_{n,m}}(z,s)) + (\tilde{z}_{c_{n,m}}(z,s))\cdot\frac{\mathrm{d}}{\mathrm{d}z}(\tilde{y}_{c_{n,m}}(z,s)) &= (0_{n,m})\\ \left[\frac{\mathrm{d}}{\mathrm{d}z}(\tilde{y}_{c_{n,m}}(z,s))\right]\cdot(\tilde{z}_{c_{n,m}}(z,s)) + (\tilde{y}_{c_{n,m}}(z,s))\cdot\frac{\mathrm{d}}{\mathrm{d}z}(\tilde{z}_{c_{n,m}}(z,s)) &= (0_{n,m})\end{aligned} \tag{22}$$

Again, for high frequencies let us assume that these matrices go to an asymptotic form for which the s dependence can be neglected.

Now, defining

$$\begin{aligned}(\Phi_{n,m}(z)) &= (y_{c_{n,m}}(z))\cdot(\phi_{n,m}(z))\cdot(z_{c_{n,m}}(0))\\ (\Phi_{n,m}(0)) &= (1_{n,m})\end{aligned} \tag{23}$$

gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}(\Phi_{n,m}(z)) &= (C_{n,m}(z))\cdot(\Phi_{n,m}(z))\\ (C_{n,m}(z)) &= -\frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}z}(y_{c_{n,m}}(z))\right]\cdot(z_{c_{n,m}}(z)) + \frac{1}{2}(z_{c_{n,m}}(z))\cdot\frac{\mathrm{d}}{\mathrm{d}z}(y_{c_{n,m}}(z))\\ &= \frac{1}{2}(y_{c_{n,m}}(z))\cdot\frac{\mathrm{d}}{\mathrm{d}z}(z_{c_{n,m}}(z)) - \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}z}(z_{c_{n,m}}(z))\right]\cdot(y_{c_{n,m}}(z))\end{aligned} \tag{24}$$

If we assume that only reciprocal media are used, then $(\tilde{Z}_{c_{n,m}}(z,s))$ and $(\tilde{Y}_{c_{n,m}}(z,s))$ are symmetric and consequently $(z_{c_{n,m}}(z))$ and $(y_{c_{n,m}}(z))$ are symmetric. This implies that

$$(C_{n,m}(z)) = -(C_{n,m}(z))^{\mathrm{T}} \tag{25}$$

that is, it is skew-symmetric, as can be seen from (24). For many problems of interest the characteristic impedance is real (for example, for lossless media) so that $(C_{n,m}(z))$ is also real.

Utilizing (46) (see Appendix A), we have,

$$\det[(\Phi_{n,m}(z))] = \exp\left\{\int_0^z \mathrm{tr}[(C_{n,m}(z'))]\mathrm{d}z'\right\} \tag{26}$$

$$\begin{aligned}\mathrm{tr}[(C_{n,m}(z))] &= \sum_{n=1}^{N} C_{n,m}(z) = \sum_{n=1}^{N} 0 = 0\\ \det[(\Phi_{n,m}(z))] &= 1\\ &= \prod_{\beta=1}^{N} \lambda_\beta[(C_{n,m}(z))]\end{aligned} \tag{27}$$

where λ_β is an eigenvalue of $(C_{n,m}(z))$. So our normalization in (23) has led to the property that the determinant of $(\Phi_{n,m})$ is conserved (with value 1) for all z. This applies for all N.

The normalization in (23) is interesting. If we regard $(\Phi_{n,m})$ as a matrix of voltage vectors, then $(\Phi_{n,m})$ takes the form of the square root of admittance times voltages. Going back to (4), where combined voltages (waves) are defined, note that if we multiply this by the square root of $(\tilde{Y}_{c_{n,m}})$ we multiply the voltage vector by this matrix, but multiply the current vector by the square root of $(\tilde{Z}_{c_{n,m}})$, thereby intrducing a certain symmetry into the formulation of a combined voltage/current wave.

In this section the development has been in terms of the voltage vector as a wave propagating in the $+z$ direction. Note that this applies also to a combined voltage wave (with $q = +1$). Of course $(Z_{c_{n,m}})$ varies with z, but so does $(y_{c_{n,m}})$ used in the normalization in (23). So one may think of a normalized combined voltage wave

$$(\tilde{y}_{c_{n,m}}(z,s))\cdot(\tilde{V}_n(z,s))_q = (\tilde{y}_{c_{n,m}}(z,s))\cdot(\tilde{V}_n(z,s)) + q(\tilde{z}_{c_{n,m}}(z,s))\cdot(\tilde{I}_n(z,s)) \tag{28}$$

as an appropriate form of wave for high-frequency purposes.

Having determined $(\Phi_{n,m}(z))$, by means to be discussed below, the voltage vector is computed (for high frequencies) as

$$\begin{aligned}(\tilde{V}_n(z,s)) &= \exp(-sv/z)(\phi_{n,m}(z))\cdot(\tilde{V}_n(0,s))\\ &= \exp(-sz/v)(z_{c_{n,m}}(z))\cdot(\Phi_{n,m}(z))\cdot(y_{c_{n,m}}(0))\cdot(\tilde{V}_n(0,s)).\end{aligned} \tag{29}$$

In the time domain for lossless, dispersionless transmission lines the factor $\exp(-sz/v)$ can be removed by placing the result in retarded time:

$$t_r \equiv t - z/v \tag{30}$$

Note that (28) applies to a combined voltage wave, as in (4) for $q = +1$ (propagation in the positive z direction) if $(\tilde{V}_n(z,s))$ and $(\tilde{V}_n(0,s))$ are reinterpreted as $(\tilde{V}_n(z,s))_+$ and $(\tilde{V}_n(0,s))_+$ respectively. For the normalized combined voltage wave as in (27), we have the simpler result

$$[(y_{c_{n,m}}(z))\cdot(\tilde{V}_n(z,s))_+] = \exp(-sz/v)(\Phi_{n,m}(z))\cdot[(y_{c_{n,m}}(0))\cdot(\tilde{V}_n(0,s))_+] \tag{31}$$

and similarly for $(y_{c_{n,m}}(z))\cdot(\tilde{V}_n(z,s))$.

An alternative approach to deriving (24) uses the BLT equation,[1] in which the nonuniform tube is approximated by a series of short uniform tubes (say of length Δz) connected by junctions. By scattering the combined waves off a typical nth junction, the transmitted wave can be used to obtain a differential equation for the leading edge of the wave (conveniently thought of as being in the time domain). This leads to the same result as above.

4. SOLUTION FOR $N = 1$

For the simplest case of a single-conductor, nonuniform transmission line, (24) reduces to

$$C(z) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\Phi(z) = 0, \qquad \Phi(z) = \Phi(0) \tag{32}$$

This is interpreted in terms of voltage as

$$\phi(z) = \frac{z_c(z)}{z_c(0)}\phi(0) = \left[\frac{Z_c(z)}{Z_c(0)}\right]^{1/2} \phi(0) \tag{33}$$

or, as $s \to \infty$ (in the right half-plane),

$$\tilde{V}(z,s) = \exp(-sz/v)\left[\frac{\tilde{Z}_c(z,s)}{\tilde{Z}_c(0,s)}\right]^{1/2} \tilde{V}(0,s). \tag{34}$$

In the lossless case this result corresponds to the simple concept of conservation of energy at the wavefront.
This type of high-frequency solution is used in quantum mechanics, where it is referred as the WKB approximation.[2] In transmission-line theory this is also a well-known result.

5. SOLUTION FOR $N = 2$

Now, for a two-conductor, nonuniform transmission line let us use the property that the coefficient matrix is skew-symmetric:

$$(C_{n,m}(z)) = \begin{pmatrix} 0 & h(z) \\ -h(z) & 0 \end{pmatrix} = h(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{35}$$

This makes the coefficient matrix a scalar function of z times a constant matrix. From Appendix B, we have the exact solution

$$(\Phi_{n,m}(z)) = \exp\left\{\int_0^z (C_{n,m}(z'))dz'\right\} = \exp\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g(z)\right\}$$
$$g(z) \equiv \int_0^z h(z')\mathrm{d}z' \tag{36}$$

Noting that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -(1_{n,m}) \tag{37}$$

an alternate representation is found from

$$(\Phi_{n,m}(z)) = \sum_{n=0}^{\infty} \frac{1}{n!}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^n g^n(z)$$

$$= \left\{\sum_{n=0}^{\infty,2} \frac{1}{\mathrm{n}!}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^n \mathrm{g}^n(\mathrm{z})\right\} + \left\{\sum_{n=1}^{\infty,2} \frac{1}{\mathrm{n}!}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathrm{g}^n(\mathrm{z})\right\}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\left\{\sum_{n=0}^{\infty,2} \frac{(-1)^{n/2}}{\mathrm{n}!}\mathrm{g}^n(\mathrm{z})\right\} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left\{\sum_{n=1}^{\infty,2} \frac{(-1)^{(n-1)/2}}{\mathrm{n}!}\mathrm{g}^n(\mathrm{z})\right\}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(g(z)) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\sin(g(z))$$

$$= \begin{pmatrix} \cos(g(z)) & \sin(g(z)) \\ -\sin(g(z)) & \cos(g(z)) \end{pmatrix} \tag{38}$$

Now isn't that compact! One can easily verify that the determinant is 1.
For $N = 2$, $h(z)$ can be computed from

$$(z_{c_{n,m}}(z)) = \begin{pmatrix} z_{c_{1,1}}(z) & z_{c_{1,2}}(z) \\ z_{c_{2,1}}(z) & z_{c_{2,2}}(z) \end{pmatrix}$$
$$(y_{c_{n,m}}(z)) = (z_{c_{n,m}}(z))^{-1}$$
$$= [z_{c_{1,1}}(z)z_{c_{2,2}}(z) - z_{c_{1,2}}(z)z_{c_{2,1}}(z)]^{-1}\begin{pmatrix} z_{c_{2,2}}(z) & -z_{c_{2,1}}(z) \\ -z_{c_{1,2}}(z) & z_{c_{1,1}}(z) \end{pmatrix} \tag{39}$$

Then, using (24), we have

$$
\begin{aligned}
(C_{n,m}(z)) &= \frac{1}{2}\left\{(y_{c_{n,m}}(z))\cdot\frac{\mathrm{d}}{\mathrm{d}z}(z_{c_{n,m}}(z)) - \left[\frac{\mathrm{d}}{\mathrm{d}z}(z_{c_{n,m}}(z))\right]\cdot(y_{c_{n,m}}(z))\right\} \\
&= \frac{1}{2}[z_{c_{1,1}}(z)z_{c_{2,2}}(z) - z_{c_{1,2}}(z)z_{c_{2,1}}(z)]^{-1}
\end{aligned}
\tag{40}
$$

$$
\left\{
\begin{bmatrix}
z_{c_{2,2}}(z)\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{1,1}}(z)-z_{c_{2,1}}(z)\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{2,1}}(z) & z_{c_{2,2}}(z)\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{1,2}}(z)-z_{c_{2,1}}(z)\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{2,2}}(z) \\
z_{c_{1,2}}(z)\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{1,1}}(z)+z_{c_{1,1}}(z)\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{2,1}}(z) & -z_{c_{1,2}}(z)\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{1,2}}(z)+z_{c_{1,1}}(z)\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{2,2}}(z)
\end{bmatrix}
-
\begin{bmatrix}
z_{c_{2,2}}(z)\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{1,1}}(z)-z_{c_{1,2}}(z)\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{1,2}}(z) & -z_{c_{2,1}}(z)\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{1,1}}(z)+z_{c_{1,1}}(z)\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{1,2}}(z) \\
z_{c_{2,2}}(z)\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{2,1}}(z)-z_{c_{1,2}}(z)\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{2,2}}(z) & -z_{c_{2,1}}(z)\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{2,1}}(z)+z_{c_{1,1}}(z)\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{2,2}}(z)
\end{bmatrix}
\right\}
$$

With the symmetry condition (reciprocity related)

$$
(z_{c_{2,1}}(z)) = (z_{c_{1,2}}(z)) \tag{41}
$$

we have

$$
(C_{n,m}(z)) = \frac{1}{2}[z_{c_{1,1}}(z)z_{c_{2,2}}(z) - z^2_{c_{1,2}}(z)]^{-1} \tag{42}
$$

$$
\begin{bmatrix}
0 & [z_{c_{2,2}}(z)-z_{c_{1,1}}(z)]\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{1,2}}(z) - [\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{2,2}}(z)-\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{1,1}}(z)]z_{c_{1,2}} \\
-[z_{c_{2,2}}(z)-z_{c_{1,1}}(z)]\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{1,2}}(z) + [\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{2,2}}(z)-\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{1,1}}(z)]z_{c_{1,2}} & 0
\end{bmatrix}
$$

From this we immediately identify

$$
h(z) = \frac{1}{2}\,\frac{[z_{c_{2,2}}(z)-z_{c_{1,1}}(z)]\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{1,2}}(z)-[\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{2,2}}(z)-\frac{\mathrm{d}}{\mathrm{d}z}z_{c_{1,1}}(z)]z_{c_{1,2}}(z)}{z_{c_{1,1}}(z)z_{c_{2,2}}(z)-z^2_{c_{1,2}}(z)} \tag{43}
$$

6. CONCLUDING REMARKS

For $N \geq 3$ the discussion in Appendix C applies. There is the result that at least N linearly independent, commuting constant matrices (specifically, simultaneously diagonalizable matrices) can be constructed. In this form the matrizant takes the simple exponential form (68). So, provided the coefficient matrix can be expressed as a set of scalar functions of z times such constant matrices, such an explicit solution is possible.

Fortunately the skew-symmetric coefficient matrix $(C_{n,m}(z))$ developed in Section 3 has at most $\frac{1}{2}N(N-1)$ independent elements, namely just those on one side of the diagonal (not including the diagonal elements, which are all zero). For $N = 2$ there was only one such element. For $N = 3$ there are three such elements which may still be handled. For $N = 4$ there are six such elements, too many for N matrices unless there are special relations between the six elements. Furthermore, if $(C_{n,m}(z))$ is real then, by (25), it is also normal (it commutes with its adjoint), which makes the use of normal matrices (as in Appendix C) appropriate.

For particular examples of multiconductor transmission lines one can investigate the particular forms of coefficient matrices that result. The foregoing results can then be used to solve the particular case at high frequencies, or to constrain the case so that certain desirable behaviour is synthesized.

APPENDIX A: GENERAL PROPERTIES OF THE DIFFERENTIAL EQUATION

Consider the $N \times N$ matrix differential equation

$$\frac{\mathrm{d}}{\mathrm{d}z}(X_{n,m}(z)) = (A_{n,m}(z)\cdot(X_{n,m}(z)), \qquad (X_{n,m}(0)) = (1_{n,m}) \tag{44}$$

In this canonical form the solution $(X_{n,m}(z))$ is referred to as the *matrizant*, with various special properties.

If the coefficient matrix is a constant matrix we have the well-known result

$$(X_{n,m}(z)) = \exp\{(A_{n,m})z\}. \tag{45}$$

However, for $(A_{n,m}(z))$ not constant things are much more complicated. This and other results in this section are discussed in various texts.[3–6]

One of the general results gives the determinant as

$$\det[(X_{n,m}(z))] = \exp\left\{\int_0^z \mathrm{tr}[(A_{n,m}(z'))]\mathrm{d}z'\right\} \tag{46}$$

where $\mathrm{tr}[(A_{n,m}(z))]$ is the trace of $(A_{n,m}(z))$, equal to

$$\sum_{n=1}^{N} A_{n,n}(z) = \sum_{\beta=1}^{N} \lambda_\beta[(A_{n,m}(z))]$$

and λ_β is the eigenvalue of $(A_{n,m}(z))$.

In more general form this is

$$\det[(X_{n,m}(z))] = \det[(X_{n,m}(z_0))]\exp\left\{\int_{z_0}^z \mathrm{tr}(A_{n,m}(z'))]dz'\right\} \tag{47}$$

Note that the above establishes that

$$\det[(X_{n,m}(z))] \neq 0,\infty \tag{48}$$

for reasonably well behaved $(A_{n,m}(z))$.

The general solution of (44) is given by

$$\begin{aligned}
(X_{n,m}(z)) &= \sum_{n=0}^{\infty} (X_{n,m})_n \\
(X_{n,m})_0 &= (1_{n,m}) \\
(X_{n,m})_{n+1} &= \int_0^z (A_{n,m}(z')\cdot(X_{n,m}(z'))_n \mathrm{d}z' \\
(X_{n,m})_n &= \int_0^z (A_{n,m}(z_1))\cdot\int_0^{z_1} (A_{n,m}(z_2))\cdot\int_0^{z_2}\cdots\int_0^{z_{n-1}} (A_{n,m}(z_n))\mathrm{d}z_n\mathrm{d}z_{n-1}\cdots\mathrm{d}z_1
\end{aligned} \tag{49}$$

While this is an explicit series involving repeated integrals, it can require much computation.

Another important property of the matrizant is derived from shifting the independent variable z via

$$(X_{n,m}(z)) = (X_{n,m}(z))\cdot(X_{n,m}(z_0))^{-1}\cdot(X_{n,m}(z_0)) \tag{50}$$

Here $(X_{n,m}(z_0))$ is the solution for $z = z_0$. By shifting z_0 to 0, we can solve (44), obtaining $(X_{n,m}(z))\cdot(X_{n,m}(z_0))^{-1}$. In (49) this means that we can integrate from 0 to z_0 in order to obtain $(X_{n,m}(z_0))$; then we can integrate from z_0 to z in order to obtain $(X_{n,m}(z))\cdot(X_{n,m}(z_0))^{-1}$. In (44) this leads to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}(X_{n,m}(z,z_0)) &= (A_{n,m}(z))\cdot(X_{n,m}(z,z_0)) \\ (X_{n,m}(z_0,z_0)) &= (1_{n,m}) \\ (X_{n,m}(z,z_0)) &= (X_{n,m}(z))\cdot(X_{n,m}(z_0))^{-1} \end{aligned} \tag{51}$$

where now

$$(X_{n,m}(z)) = (X_{n,m}(z,z_0))\cdot(X_{n,m}(z_0)) \tag{52}$$

This is generalized to

$$(X_{n,m}(z)) = (X_{n,m}(z,z-\Delta z))\cdot(X_{n,m}(z-\Delta z,z-2\Delta z))\cdot\cdot\cdot(X_{n,m}(\Delta z,0)) \tag{53}$$

If z is divided into N' intervals so that

$$z = N'\Delta z \tag{54}$$

this leads to an approximate solution for the matrizant. If $(A_{n,m}(z))$ is approximately constant over the n'th interval, then

$$(X_{n,m}(z,n'\Delta z)) \simeq \exp\{(A_{n,m}([n'-\tfrac{1}{2}]\Delta z))\Delta z\} \tag{55}$$

evaluating the coefficient matrix in the centre of the interval. In (53) this gives a product of exponential matrices. Note that the exponents cannot in general be directly added since the matrices do not in general commute.

An alternative approximation would just use the leading terms in the expansion of the exponential, giving

$$(X_{n,m}(z,n'\Delta z)) \simeq (1_{n,m}) + (A_{n,m}([n'-\tfrac{1}{2}]\Delta z))\Delta z \tag{56}$$

Provided Δz is sufficiently small, the approximations are adequate if $(A_{n,m})$ is well behaved.

APPENDIX B: COEFFICIENT MATRIX AS SCALAR FUNCTION TIMES CONSTANT MATRIX

Consider the case that in (44)

$$(A_{n,m}(z)) \equiv h(z)\,(d_{n,m}) \tag{57}$$

where $h(z)$ is a scalar function of z and $(d_{n,m})$ a constant $N \times N$ matrix. Noting that scalars commute with any matrices, we define

$$g(z) \equiv \int_0^z h(z')\mathrm{d}z' \tag{58}$$

Now we compute

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}z}\exp\{(d_{n,m})g(z)\} \\ &= \lim_{\Delta z\to 0}\frac{1}{\Delta z}\left(\exp\{(d_{n,m})\left[g(z)+\frac{\mathrm{d}}{\mathrm{d}z}g(z)\Delta z+O((\Delta z)^2)\right]\} - \exp\{(d_{n,m})g(z)\}\right) \\ &= \lim_{\Delta z\to 0}\frac{\exp\{(d_{n,m})g(z)\}}{\Delta z}\cdot\left(\exp\{(d_{n,m})\left[\frac{\mathrm{d}}{\mathrm{d}z}g(z)\Delta z+O((\Delta z)^2)\right]\} - (1_{n,m})\right) \\ &= \exp\{(d_{n,m})g(z)\}\cdot(d_{n,m})\frac{\mathrm{d}}{\mathrm{d}z}g(z) \\ &= h(z)(d_{n,m})\cdot\exp\{(d_{n,m})g(z)\} \end{aligned} \tag{59}$$

where the series expansion of the exponential is used, as is the fact that $(d_{n,m})$ commutes with itself.

Looking at (44), we then have the solution for the matrizant as

$$\begin{aligned} (X_{n,m}(z)) &= \exp\{(d_{n,m})g(z)\} = \exp\{(d_{n,m})\int_0^z h(z')\mathrm{d}z'\} \\ &= \exp\{\int_0^z (A_{n,m}(z'))\mathrm{d}z'\} \end{aligned} \tag{60}$$

This is then a case of a coefficient matrix leading to an exact (and simple) expression for the matrizant.

APPENDIX C: COEFFICIENT MATRIX AS LINEAR COMBINATION OF SCALAR FUNCTIONS TIMES CONSTANT COMMUTING MATRICES

Consider the case that in (44)

$$(A_{n,m}(z)) = \sum_{p=1}^{P} h_p(z)(d_{n,m})_p \tag{61}$$

where $h_p(z)$ are scalar functions of z and $(d_{n,m})_p$ are constant $N \times N$ matrices, with the commutation condition

$$(d_{n,m})_{p_1}\cdot(d_{n,m})_{p_2} = (d_{n,m})_{p_2}\cdot(d_{n,m})_{p_1}, \qquad p_1,p_2 = 1,2,\ldots,P \tag{62}$$

Furthermore, we define

$$g_p(z) \equiv \int_0^z h_p(z')\mathrm{d}z' \tag{63}$$

so that

$$\int_0^z (A_{n,m}(z'))\mathrm{d}z' = \sum_{p=1}^{P} g_p(z)(d_{n,m})_p \tag{64}$$

Since the $(d_{n,m})_p$ commute, we have

$$\exp\{\sum_{p=1}^{P} g_p(z)(d_{n,m})_p\} = \exp\{g_1(z)(d_{n,m})_1\}\cdot\exp\{g_2(z)(d_{n,m})_2\}\cdot\ \cdot\exp\{g_p(z)(d_{n,m})_P \tag{65}$$

Note that this product can be taken in any order, as is seen by the arbitrariness of the ordering of the index p. In shorthand form,

$$\exp\{\sum_{p=1}^{P} g_p(z)(d_{n,m})_p\} = \bigodot_{p=1}^{P} \exp\{g_p(z)(d_{n,m})_p\} \tag{66}$$

where $\odot$ with indices indicates a generalized dot product over the given range of indices.

Now, we compute

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}z}\exp\{\sum_{p=1}^{P}(d_{n,m})_p g_p(z)\}\\
&= \lim_{\Delta z\to 0}\frac{1}{\Delta z}\cdot\left(\exp\{\sum_{p=1}^{P}(d_{n,m})_p\left[g_p(z)+\frac{\mathrm{d}}{\mathrm{d}z}g_p(z)\Delta z+O((\Delta z)^2)\right]\} - \exp\{\sum_{p=1}^{P}(d_{n,m})_p g_p(z)\}\right)\\
&= \lim_{\Delta z\to 0}\frac{\exp\{\sum_{p=1}^{P}(d_{n,m})_p g(z)\}}{\Delta z}\cdot\left(exp\{\sum_{p=1}^{P}(d_{n,m})_p\left[\frac{\mathrm{d}}{\mathrm{d}z}g_p(z)\Delta z+O((\Delta z)^2)\right]\} - (1_{n,m})\right)\\
&= \exp\{\sum_{p=1}^{P}(d_{n,m})_p g_p(z)\}\cdot\left\{\sum_{p=1}^{P}(d_{n,m})_p\frac{\mathrm{d}}{\mathrm{d}z}g_p(z)\right\}\\
&= \left\{\sum_{p=1}^{P}h_p(z)(d_{n,m})_p\right\}\cdot\exp\{\sum_{p=1}^{P}(d_{n,m})_p g_p(z)\}
\end{aligned} \tag{67}$$

Comparing this with (44), we have

$$\begin{aligned}
(X_{n,m}(z)) &= \exp\{\sum_{p=1}^{P}(d_{n,m})g(z)\} = \exp\{\sum_{p=1}^{P}(d_{n,m})\int_0^z h(z')\mathrm{d}z'\}\\
&= \exp\{\int_0^z (A_{n,m}(z'))\mathrm{d}z'\}
\end{aligned} \tag{68}$$

So in the simple form of the solution for the matrizant, as in (68), carries through at least to the case in which $(A_{n,m}(z))$ can be represented as in (61), i.e. as a sum of scalar functions times constant commuting matrices. Well, how general is this? Can we represent any matrix function of z in this way? Clearly we cannot (except for $N = 1$) since any combination as in (61) commutes with any other linear combination of the $(d_{n,m})_p$. However, we can construct various $(A_{n,m}(z))$ which do not commute with their intergrals, so not all conceivable $(A_{n,m}(z))$ can be so treated.

Now, an $N \times N$ matrix function of z can be represented by N^2 linearly independent constant matrices times functions of z. Merely choose each matrix as all zeros, except for one element which is 1 (the n,mth element for the 'n,mth' matrix). So we need no more than N^2 commuting matrices, but how many can we construct?

One clue to our problem is the result[4] that normal matrices which commute are all simultaneously diagonalizable. If the $(d_{n,m})_p$ are normal, then

$$(d_{n,m})_p \cdot (d_{n,m})_p^\dagger = (d_{n,m})_p^\dagger \cdot (d_{n,m})_p \tag{69}$$

where $(d_{n,m})_p^\dagger$ is the adjoint of $(d_{n,m})_p$, the conjugate of the transpose; in other words, by definition the $(d_{n,m})_p$ commute with their adjoints. All the $(d_{n,m})_p$ take the common form

$$(d_{n,m})_p = (U_{n,m}) \cdot \begin{bmatrix} \lambda_1^{(p)} & & & 0 \\ & \lambda_2^{(p)} & & \\ & & \ddots & \\ 0 & & & \lambda_N^{(p)} \end{bmatrix} \cdot (U_{n,m})^{-1} \tag{70}$$

where the unitary matrix $(U_{n,m})$ has the property

$$(U_{n,m})^{-1} = (U_{n,m})^\dagger \tag{71}$$

and has as its columns the eigenvectors (right side) of all the $(d_{n,m})_p$, and rows of $(U_{n,m})^\dagger$ as left eigenvectors (equal to conjugate of right eigenvectors).

Now, (70) indicates that there are at most N linearly independent commuting normal matrices corresponding to the only N eigenvalues available. The eigenvalues on the diagonal in (70) are analogous to an N-component vector which can be represented by any N linearly independent vectors spanning the N-dimensional space.

There are various ways to choose the N eigenvalues in (70). As the identity commutes with any matrix, one choice, say labelled $p = 0$, is

$$(d_{n,m})_0 = (1_{n,m}) \tag{72}$$

Then choose the pth diagonal eigenvalue matrix to have all its elements as 1, except for $\lambda_p^{(p)}$ as -1. With this choice,

$$(d_{n,m})_p^2 = (U_{n,m}) \cdot \begin{bmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{bmatrix}^2 \cdot (U_{n,m})^{-1} = (1_{n,m}) \tag{73}$$

The N matrices $p = 1,2,\ldots,N$ are linearly independent, but the identity can be formed for the above definition as

$$(d_{n,m})_0 = \frac{1}{N-2} \sum_{p=1}^{N} (d_{n,m})_p, \quad N \neq 2 \tag{74}$$

Another way to construct a set of commuting matrices is to note that any set of simultaneously diagonalizable matrices commutes pairwise. Analogous to (70), one can construct a set of matrices via

$$(d_{n,m})_p \equiv (a_{n,m}) \cdot \begin{bmatrix} \lambda_1^{(p)} & & & \bigcirc \\ & \lambda_2^{(p)} & & \\ & & \ddots & \\ \bigcirc & & & \lambda_N^{(p)} \end{bmatrix} \cdot (a_{n,m})^{-1} \tag{75}$$

where $(a_{n,m})$ is chosen as any nonsingular matrix. That the $(d_{n,m})_p$ commute is seen from

$$\begin{aligned}
(d_{n,m})_{p_1} \cdot (d_{n,m})_{p_2} &= (a_{n,m}) \cdot \begin{bmatrix} \lambda_1^{(p_1)} & & & \bigcirc \\ & \lambda_2^{(p_1)} & & \\ & & \ddots & \\ \bigcirc & & & \lambda_N^{(p_1)} \end{bmatrix} \cdot (a_{n,m})^{-1} \cdot (a_{n,m}) \\
&\quad \cdot \begin{bmatrix} \lambda_1^{(p_2)} & & & \bigcirc \\ & \lambda_2^{(p_2)} & & \\ & & \ddots & \\ \bigcirc & & & \lambda_N^{(p_2)} \end{bmatrix} \cdot (a_{n,m})^{-1} \\
&= (a_{n,m}) \cdot \begin{bmatrix} \lambda_1^{(p_1)} & & & \bigcirc \\ & \lambda_2^{(p_1)} & & \\ & & \ddots & \\ \bigcirc & & & \lambda_N^{(p_1)} \end{bmatrix} \cdot \begin{bmatrix} \lambda_1^{(p_2)} & & & \bigcirc \\ & \lambda_2^{(p_2)} & & \\ & & \ddots & \\ \bigcirc & & & \lambda_N^{(p_2)} \end{bmatrix} \cdot (a_{n,m})^{-1} \\
&= (a_{n,m}) \cdot \begin{bmatrix} \lambda_1^{(p_2)} & & & \bigcirc \\ & \lambda_2^{(p_2)} & & \\ & & \ddots & \\ \bigcirc & & & \lambda_N^{(p_2)} \end{bmatrix} \cdot \begin{bmatrix} \lambda_1^{(p_1)} & & & \bigcirc \\ & \lambda_2^{(p_1)} & & \\ & & \ddots & \\ \bigcirc & & & \lambda_N^{(p_1)} \end{bmatrix} \cdot (a_{n,m})^{-1} \\
&= (a_{n,m}) \cdot \begin{bmatrix} \lambda_1^{(p_2)} & & & \bigcirc \\ & \lambda_2^{(p_2)} & & \\ & & \ddots & \\ \bigcirc & & & \lambda_N^{(p_2)} \end{bmatrix} \cdot (a_{n,m})^{-1} \cdot (a_{n,m}) \\
&\quad \cdot \begin{bmatrix} \lambda_1^{(p_1)} & & & \bigcirc \\ & \lambda_2^{(p_1)} & & \\ & & \ddots & \\ \bigcirc & & & \lambda_N^{(p_1)} \end{bmatrix} \cdot (a_{n,m})^{-1} \\
&= (d_{n,m})_{p_2} \cdot (d_{n,m})_{p_1}
\end{aligned} \tag{76}$$

where we have used the well-known property that diagonal matrices commute.

Rewriting (75) as

$$(d_{n,m})_p \cdot (a_{n,m}) = (a_{n,m}) \cdot \begin{bmatrix} \lambda_1^{(p)} & & & \bigcirc \\ & \lambda_2^{(p)} & & \\ & & \ddots & \\ \bigcirc & & & \lambda_N^{(p)} \end{bmatrix} \tag{77}$$

we can see that the columns of $(a_{n,m})$ are eigenvectors (right side) of $(d_{n,m})_p$, i.e.

$$(d_{n,m})_p \cdot (x_n)_\beta = \lambda_\beta^{(p)} (x_n)_\beta, \qquad x_{n;\beta} = a_{n,\beta} \tag{78}$$

where these eigenvectors are not functions of p, the index on our set of matrices, that is, the eigenvectors are common to all these matrices. Similarly, by writing

$$(b_{n,m}) \equiv (a_{n,m})^{-1} \tag{79}$$

$$(b_{n,m})\cdot(d_{n,m})_p = \begin{bmatrix} \lambda_1^{(p)} & & & \bigcirc \\ & \lambda_2^{(p)} & & \\ & & \ddots & \\ \bigcirc & & & \lambda_N^{(p)} \end{bmatrix} \cdot (b_{n,m})$$

we can see that the rows of $(b_{n,m})$ are left eigenvectors of $(d_{n,m})_p$, i.e.

$$(y_n)_\beta\cdot(d_{n,m})_p = \lambda_\beta^{(p)}(y_n)_\beta, \qquad y_{m;\beta} = b_{\beta,m} \tag{80}$$

So we construct from the above the right and left eigenvector matrices as

$$(a_{n,m}) = ((x_n)_1,(x_n)_2,\ldots,(x_n)_N) \tag{81}$$

$$(b_{n,m}) = \begin{bmatrix} (y_n)_1 \\ (y_n)_2 \\ \vdots \\ (y_n)_N \end{bmatrix} = (a_{n,m})^{-1}$$

respectively. These left and right eigenvectors are a biorthonormal set since

$$(b_{n,m})\cdot(a_{n,m}) = (1_{n,m}) \tag{82}$$

which is written as

$$(y_n)_{\beta_1}\cdot(x_n)_{\beta_2} = 1_{\beta_1,\beta_2} \tag{83}$$

Note also that

$$(a_{n,m})\cdot(b_{n,m}) = (1_{n,m}) \tag{84}$$

i.e. $(a_{n,m})$ commutes with its inverse, so that the rows of $(a_{n,m})$ and columns of $(b_{n,m})$ also form a biorthonormal set of vectors.

Carrying the argument a step further, we can see that, provided one of the set of commuting matrices has distinct eigenvalues, then (75) is the most general form for this set. To see this, let $(d_{n,m})_p$ be the matrix with distinct eigenvalues (guaranteeing diagonalizability). Then we have

$$\begin{aligned} (d_{n,m})_p\cdot(x_n)_\beta &= \lambda_\beta^{(p)}(x_n)_\beta \\ (y_n)_\beta\cdot(d_{n,m})_p &= \lambda_\beta^{(p)}(x_n)_\beta \\ (y_n)_{\beta_1}\cdot(x_n)_{\beta_2} &= 1_{\beta_1,\beta_2} \\ (d_{n,m})_p &= \sum_{\beta=1}^{N} \lambda_\beta^{(p)}(x_n)_\beta(y_n)_\beta. \end{aligned} \tag{85}$$

where $\lambda_\beta^{(p)}$ is distinct for $\beta = 1,2,\ldots,N$.

Considering another matrix, say $(d_{n,m})_{p'}$, in this set, commutation gives

$$\begin{aligned} (d_{n,m})_p\cdot(d_{n,m})_{p'} &= (d_{n,m})_{p'}\cdot(d_{n,m})_p \\ (d_{n,m})_p\cdot(d_{n,m})_{p'}\cdot(x_n)_\beta &= (d_{n,m})_{p'}\cdot(d_{n,m})_p\cdot(x_n)_\beta \\ &= \lambda_\beta^{(p)}(d_{n,m})_{p'}\cdot(x_n)_\beta \end{aligned} \tag{86}$$

This says that the vector $(d_{n,m})_{p'}\cdot(x_n)_\beta$ is an eigenvector of $(d_{n,m})_p$ with eigenvalue $\lambda_\beta^{(p)}$. However, since the eigenvalues of $(d_{n,m})_p$ are distinct, the corresponding eigenvectors are unique except for a scaling constant. Hence

$$(d_{n,m})_{p'}\cdot(x_n)_\beta = \lambda_\beta^{(p')}(x_n)_\beta \tag{87}$$

where evidently this constant is the corresponding eigenvalue of $(d_{n,m})_{p'}$. Similarly, by taking dot products in (86) on the left we obtain

$$\begin{aligned} (y_n)_\beta\cdot(d_{n,m})_{p'}\cdot(d_{n,m})_p &= (y_n)_\beta\cdot(d_{n,m})_p\cdot(d_{n,m})_{p'} \\ &= \lambda_\beta^{(p)}\,(y_n)_\beta\cdot(d_{n,m})_{p'} \end{aligned} \tag{88}$$

Thus the vector $(y_n)_\beta \cdot (d_{n,m})_{p'}$ is a left eigenvector of $(d_{n,m})_p$, and so must be equal to a constant times $(y_n)_\beta$, giving

$$(y_n)_\beta \cdot (d_{n,m})_{p'} = \lambda_\beta^{(p')} (y_n)_\beta \tag{89}$$

where the constant is the corresponding eigenvalue of $(d_{n,m})_{p'}$. Thus we have

$$(d_{n,m})_{p'} = \sum_{\beta=1}^{N} \lambda_\beta^{(p')} (x_n)_\beta (y_n)_\beta \tag{90}$$

as the general (dyadic) representation of all the commuting matrices in the set. The commuting matrices are all simultaneously diagonalized since they all have the same eigenvectors, but with (in general) different eigenvalues. The explicit form of the simultaneously diagonalizing matrices is given in (81).

Note that commuting matrices have the property that

$$\begin{aligned}(d_{n,m})_{p_1} \cdot (d_{n,m})_{p_2} &= (d_{n,m})_{p_2} \cdot (d_{n,m})_{p_1} \\ (d_{n,m})_{p_1} &= (d_{n,m})_{p_2} \cdot (d_{n,m})_{p_1} \cdot (d_{n,m})_{p_2}^{-1}\end{aligned} \tag{91}$$

provided $(d_{n,m})_{p_2}$ is nonsingular. Thus we have that any member of the set of commuting matrices is similarity-transformed into itself by any nonsingular matrix in the set.

Noting that

$$(1_{n,m} = \sum_{\beta=1}^{N} (y_n)_\beta (x_n)_\beta = \sum_{\beta=1}^{N} (x_n)_\beta (y_n)_\beta \tag{92}$$

we find the known result that the identity can be one of the $(d_{n,m})_{p'}$. In addition, the identity commutes with all $N \times N$ matrices. Furthermore, for any members of the commuting set

$$\alpha_1 (d_{n,m})_{p_1} + \alpha_2 (d_{n,m})_{p_2}, \qquad (d_{n,m})_{p_1} \cdot (d_{n,m})_{p_2} \tag{93}$$

where α_1 and α_2 are scalars, are also members of the set, as seen by substitution in (75). However, as shown by the form of (75), the N choosable eigenvalues allow at most N linearly independent $(d_{n,m})_p$.

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