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Interaction Notes

Note 472

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On the Eigenmode Expansion Method for Electromagnetic Scattering and Antenna Problems
Part II: Asymptotic Expansion of Eigenmode-Expansion Parameters in the Complex-Frequency Plane

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Abstract

This paper extends the analysis of the eigenmode-expansion-method (EEM) parameters to their asymptotic forms in the left and right s-planes. The geometric properties of the object of interest are dominant in these asymptotic forms. The minimum circumscribing sphere gives an optimal choice of coordinate center for best convergence properties over all angles of incidence.

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I. Introduction

A previous paper [3] introduced the basic concept of the eigenmode expansion method (EEM) and related this to the singularity expansion method (SEM). This material has also been summarized in a review paper [4] and a book chapter [12].

In trying to understand the EEM it is useful to consider the asymptotic forms of the various terms as $\mathrm{Re}[s] \to \pm\infty$ in the complex frequency (i.e., s or Laplace transform) plane. There have been papers addressing parts of this problem [10,11]. Another paper considered some similar problems for SEM terms [9].

As before the Laplace transform (two-sided) is denoted by

$$\tilde{f}(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

$$f(t) = \frac{1}{2\pi j} \int_{Br} \tilde{f}(s)\, e^{st}\, dt \tag{1.1}$$

$Br \equiv$ Bromwich contour in strip of convergence in s plane

$s = \Omega + j\omega \equiv$ complex frequency

This paper extends the previous results and finds some relatively simple forms for the asymptotic forms of the various EEM parameters as $\mathrm{Re}[s] \to \pm\infty$. We first consider the eigenvalues and find the left-half-plane behavior determined by the maximum linear dimension L_0 of the object of interest. In the right half s-plane the eigenvalues go to simple constant values. The various terms in the EEM form of the response are examined and found to give rather simple exponential forms involving the dimensions of the object of interest. The general asymptotic results are found to agree in detail with the exact results for the perfectly conducting sphere.

II. Eigenmode Expansion

A general integral equation for the response of an antenna or scatterer in free space takes the form

$$\left\langle \tilde{\overleftrightarrow{T}}\left(\vec{r}_s , \vec{r}_s\,' ; s\right) ; \tilde{J}_s\left(\vec{r}_s\,'\right)\right\rangle = \tilde{\vec{I}}\left(\vec{r}_s , s\right) \tag{2.1}$$

where $\vec{J}_s$ represents the surface current density on a perfectly conducting (or at least thin-impedance-sheet) object, and $\vec{I}$ represents some incident-field parameter (such as tangential electric or magnetic field). The domain of integration is the surface S of the body. However, some of the present results relate to volume integral equations for the cases that currents are distributed in a volume as well.

We assume that the body has a finite size with maximum linear dimension L_0. This allows us to write a method-of-moments (MoM) equation in the form

$$\left(\tilde{T}_{n,m}(s)\right)\cdot\left(\tilde{J}_{s_n}(s)\right) = \left(\tilde{I}_n(s)\right) \tag{2.2}$$

The kernels of the various integral equations involve the free-space Green function

$$\tilde{G}_o\left(\vec{r}_s , \vec{r}_s\,' ; s\right) = \gamma\,\frac{e^{-\zeta}}{4\pi\zeta}$$

$$\nabla\tilde{G}_o\left(\vec{r}_s , \vec{r}_s\,' ; s\right) = \vec{1}_R\,\frac{\gamma^2}{4\pi}\left[-\zeta^{-2}-\zeta^{-1}\right]e^{-\zeta}$$

$$\tilde{\overleftrightarrow{G}}_o\left(\vec{r}_s , \vec{r}_s\,' ; s\right) = \left[\overleftrightarrow{1} - \gamma^{-2}\nabla\nabla\right]\tilde{G}_o\left(\vec{r}_s , \vec{r}_s\,' ; s\right)$$

$$= \frac{\gamma}{4\pi}\left\{\left[-2\zeta^{-3}-2\zeta^{-2}\right]e^{-\zeta}\vec{1}_R\vec{1}_R\right.$$

$$\left.+\left[\zeta^{-3}+\zeta^{-2}+\zeta^{-1}\right]e^{-\zeta}\left[\overleftrightarrow{1} - \vec{1}_R\vec{1}_R\right]\right\}$$

$$+\,\gamma^{-2}\vec{1}_S\vec{1}_S\,\delta\left(\vec{r} - \vec{r}_s\,'\right) \tag{2.3}$$

$$\zeta \equiv \gamma R \quad , \quad R \equiv \left|\vec{r}_s - \vec{r}_s\,'\right| \quad , \quad \vec{1}_R = \frac{\vec{r}_s - \vec{r}_s\,'}{R}$$

$$\gamma = \frac{s}{c}$$

Noting the appearance of $e^{-\gamma R}$ in all the kernels as the dominant term as $s \to \infty$ in the left-half plane the finite size of the object is important in that R is bounded (by L_0) for pairs of positions on the body as they appear in the matrix elements in (2.2). Here we have included the integration near $\vec{r} = \vec{r}_s$ as given by Yaghjian [7] for a surface type body with $\vec{1}_S$ as the outward pointing normal.

In the general form of the integral equation (2.1) we define eigenmodes and eigenvalues via [3,4,12]

$$\left\langle \overleftrightarrow{T}\left(\vec{r}_s, \vec{r}\,'_s ; s\right) ; \tilde{\vec{j}}_{s_\beta}\left(\vec{r}\,'_s , s\right)\right\rangle = \tilde{\lambda}_\beta(s)\ \tilde{\vec{j}}_s\ \left(\vec{r}_s , s\right)$$

$$\left\langle \tilde{\vec{\mu}}_\beta\left(\vec{r}_s , s\right)\ ;\ \tilde{\overleftrightarrow{T}}\left(\vec{r}_s , \vec{r}\,'_s ; s\right)\right\rangle = \tilde{\lambda}_\beta(s)\ \ \tilde{\vec{\mu}}_s\left(\vec{r}\,'_s , s\right) \tag{2.4}$$

$$\left\langle \vec{\mu}_{\beta_1}\left(\vec{r}_s , s\right)\ ;\ \tilde{\vec{j}}_{s_{\beta_2}}\left(\vec{r}_s , s\right)\right\rangle = 1_{\beta_1, \beta_2}\ \text{(biorthonormalization)}$$

This can be used to represent various powers of the kernel as

$$\tilde{\overleftrightarrow{T}}^n\left(\vec{r}_s , \vec{r}\,'_s ; s\right) = \sum_\beta\ \tilde{\lambda}^n_\beta(s)\ \ \tilde{\vec{j}}_s\left(\vec{r}_s , s\right)\ \ \tilde{\vec{\mu}}_\beta\left(\vec{r}\,'_s , s\right) \tag{2.5}$$

with special cases as $\tilde{\overleftrightarrow{T}}$, the identify, and $\tilde{\overleftrightarrow{T}}^{-1}$. This allows us to write

$$\tilde{\vec{J}}_s\left(\vec{r}_s , s\right) \sum_\beta\ \tilde{\lambda}_\beta^{-1}(s)\ \left\langle \tilde{\vec{\mu}}_\beta\left(\vec{r}\,'_s , s\right) ; \tilde{\vec{I}}\left(\vec{r}\,'_s , s\right)\right\rangle\ \tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s , s\right) \tag{2.6}$$

Eigenvalues are also expressible as [3]

$$\tilde{\lambda}_\beta(s) = \left\langle \tilde{\vec{\mu}}_\beta\left(\vec{r}_s , s\right)\ ;\ \tilde{\overleftrightarrow{T}}\ \left(\vec{r}_s , \vec{r}\,'_s ; s\right) ; \tilde{\vec{j}}_s\left(\vec{r}\,'_s , s\right)\right\rangle$$

$$\frac{d}{ds}\tilde{\lambda}_\beta = \left\langle \tilde{\vec{\mu}}_\beta\left(\vec{r}_s , s\right) ; \left[\frac{d}{ds}\tilde{\overleftrightarrow{T}}\ \left(\vec{r}_s , \vec{r}\,'_s ; s\right)\right] ; \tilde{\vec{j}}_{s_\beta}\left(\vec{r}\,'_s , s\right)\right\rangle \tag{2.7}$$

The identity is

$$\overleftrightarrow{T}^{o}\left(\vec{r}_s\,,\vec{r}_s\,'\,;s\right)=\sum_{\beta}\tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s\,,s\right)\;\tilde{\vec{\mu}}_{\beta}\left(\vec{r}_s\,',s\right)=\overleftrightarrow{1}_t\left(\vec{r}_s\right)\delta_s\left(\vec{r}_s-\vec{r}_s\,'\right)$$
$$=\overleftrightarrow{1}_t\left(\vec{r}_s\,'\right)\delta_s\left(\vec{r}_s-\vec{r}_s\,'\right) \tag{2.8}$$

Here δ_s is the surface (two-dimensional) delta function, and the transverse dyad is

$$\overleftrightarrow{1}_t\left(\vec{r}_s\right)=\overleftrightarrow{1}-\vec{1}_S\left(\vec{r}_s\right)\vec{1}_S\left(\vec{r}_s\right) \tag{2.9}$$

When the identity is dot multiplied by a vector function of $\vec{r}_s\,'$ on S and integrated over S the vector function is returned as a function of $\vec{r}_s$. Note at this point that the above expansion is possibly incomplete in that one may conceivably need what are termed root vectors [8] in some cases. However, it is yet to be shown that such are actually or practically necessary for finite-sized perfectly conducting scatterers in free space. Further investigation would be helpful here.

In MoM form the above takes the usual matrix-algebra form

$$\begin{aligned}
\left(\tilde{T}_{n,m}(s)\right)\cdot\left(\tilde{j}_{s_n}(s)\right)_{\beta} &= \tilde{\lambda}_{\beta}(s)\left(\tilde{j}_{s_n}(s)\right)_{\beta}\\
\left(\tilde{\mu}_n(s)\right)_{\beta}\cdot\left(\tilde{T}_{n,m}(s)\right) &= \tilde{\lambda}_{\beta}(s)\left(\tilde{\mu}_n(s)\right)_{\beta}\\
\left(\tilde{\mu}_n(s)\right)_{\beta_1}\cdot\left(\tilde{j}_{s_n}(s)\right)_{\beta_2} &= 1_{\beta_1,\beta_2}
\end{aligned} \tag{2.10}$$

Assuming NxN matrices we have

$$\left(\tilde{T}_{n,m}(s)\right)^{n}=\sum_{\beta=1}^{N}\lambda_{\beta}^{n}(s)\left(\tilde{j}_{s_n}(s)\right)_{\beta}\left(\tilde{\mu}_n(s)\right)_{\beta} \tag{2.11}$$

This points out, of course, that the MoM form is but an approximation of the integral-equation form in which there is an infinite number of eigenvalues. Note that the above representation is complete (as a representation of the approximate solution of the integral equation) as long as there are N linearly independent eigenvectors [3] and it follows that

$$\left(\tilde{\mu}_n(s)\right)_\beta \cdot \left(\tilde{j}_{s_n}(s)\right)_\beta \neq 0 \text{ for all } \beta \tag{2.12}$$

in such a case. For distinct eigenvalues the above is always so. If for some s two eigenvalues are equal (degeneracy) there can still be independent eigenmodes as some cases can attest (e.g., sheer, thin wire [4]). As of yet no counterexamples for simple electromagnetic scatterers have been found where eigenmode degeneracy implies the eigenvectors not spanning the surface.

One advantage of the MoM form is that we can write [3]

$$\begin{aligned}
D_N(s,\lambda) &\equiv \det\left(\left(\tilde{T}_{n,m}(s)\right) - \lambda\left(1_{n,m}\right)\right) \\
&= \sum_{p=0}^{N} \tilde{a}_p(s)\lambda^p \\
&= a_N \prod_{\beta=1}^{N}\left(\lambda - \tilde{\lambda}_\beta(s)\right) \\
a_N &= (-1)^N \\
\tilde{a}_o(s) &= \det\left(\left(\tilde{T}_{n,m}(s)\right)\right) = D_N(s,0) = \prod_{\beta=1}^{N}\tilde{\lambda}_\beta(s)
\end{aligned} \tag{2.13}$$

The N eigenvalues can be calculated given $\left(\tilde{T}_{n,m}(s)\right)$ by various methods.

The impedance, or E-field, integral equation takes the form

$$\begin{aligned}
&\left\langle \tilde{\overleftrightarrow{Z}}\left(\vec{r}_s, \vec{r}_s\,'; s\right); \tilde{\vec{J}}_s\left(\vec{r}_s\,', s\right)\right\rangle = \overleftrightarrow{1}_t\left(\vec{r}_s\right) \cdot \tilde{\vec{E}}^{(inc)}\left(\vec{r}_s, s\right) \equiv \tilde{\vec{E}}_t^{(inc)}\left(\vec{r}_s, s\right) \\
&\tilde{\overleftrightarrow{Z}}\left(\vec{r}_s, \vec{r}_s\,'; s\right) \equiv -s\,\mu_o \overleftrightarrow{1}_t\left(\vec{r}_s\right) \cdot \tilde{\overleftrightarrow{G}}_o\left(\vec{r}_s, \vec{r}_s\,'; s\right) \cdot \overleftrightarrow{1}_t\left(\vec{r}_s\,'\right) \equiv \text{impedance kernel}
\end{aligned} \tag{2.14}$$

Note that the transverse dyad removes the delta-function term from (2.3) since

$$\overleftrightarrow{1}_t\left(\vec{r}_s\right) \cdot \vec{1}_S\left(\vec{r}_s\right) = \vec{1}_S\left(\vec{r}_s\right) \cdot \overleftrightarrow{1}_t\left(\vec{r}_s\right) = 0 \tag{2.15}$$

For this integral equation we use a superscript E on the modes and note

$$\tilde{\vec{\mu}}_{\beta}^{(E)}\left(\vec{r}_s\,,s\right) = \tilde{\vec{j}}_{s_\beta}^{(E)}\left(\vec{r}_s\,,s\right) \tag{2.16}$$

since the kernel is symmetric. The eigenvalues are now dubbed eigenimpedances [3,4,12] so that

$$\begin{aligned} &\tilde{\lambda}_{\beta}^{(E)}(s) \equiv \tilde{Z}_{\beta}(s) \equiv Z_o\,\tilde{z}_{\beta}(s) \equiv \text{eigenimpedance} \\ &\tilde{z}_{\beta}(s) \equiv \text{normalized eigenimpedance} \end{aligned} \tag{2.17}$$

Note now that the eigenmodes are an orthonormal set

$$\left\langle \tilde{\vec{j}}_{s_{\beta_1}}\left(\vec{r}_s\,,s\right)\,;\,\tilde{\vec{j}}_{s_{\beta_2}}\left(\vec{r}_s\,,s\right)\right\rangle = 1_{\beta_1,\,\beta_2} \tag{2.18}$$

with the additional result

$$\left\langle \left[\frac{d}{ds}\,\tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s\,,s\right)\right];\,\tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s\,,s\right)\right\rangle = 0 \tag{2.19}$$

The eigenvalues are then

$$\tilde{Z}_{\beta}(s) = \left\langle \tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s\,,s\right)\,;\,\tilde{\overleftrightarrow{Z}}\left(\vec{r}_s\,,\vec{r}_s\,';s\right);\,\tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s\,',s\right)\right\rangle \tag{2.20}$$

since the eigenimpedances represent impedances and are p.r. functions as discussed in [3,4,12] due to the passive nature of the scatterer. This implies various things like conjugate symmetry and no singularities (or zeros) in the right half s plane.

The impedance integral equation can be modified by the addition of a uniform sheet-impedance loading $\tilde{Z}_{\ell}(s)$ to give an integral equation

$$\begin{aligned} \left\langle \tilde{\overleftrightarrow{Z}}\left(\vec{r}_s\,,\vec{r}_s\,';s\right) + Z_{\ell}(s)\overleftrightarrow{1}_t\left(\vec{r}_s\right)\delta_s\left(\vec{r}_s - \vec{r}_s\,'\right)\,;\,\tilde{\vec{J}}\left(\vec{r}_s\,',s\right)\right\rangle \\ = \overleftrightarrow{1}_t\left(\vec{r}_s\right)\cdot\tilde{\vec{E}}^{(inc)}\left(\vec{r}_s\,,s\right) \end{aligned} \tag{2.21}$$

This integral equation has the same eigenmodes $\tilde{\vec{j}}_{s_\beta}$, but the eigenvalues have been changed to $\tilde{Z}_{\beta}(s) + \tilde{Z}_{\ell}(s)$ [3,4,12].

The magnetic-field integral equation takes the form

$$\left\langle \tilde{\overleftrightarrow{L}}\left(\vec{r}_s, \vec{r}_s\,' ; s\right) ; \tilde{\vec{J}}\left(\vec{r}_s\,', s\right)\right\rangle = \tilde{\vec{J}}_s^{(inc)}\left(\vec{r}_s, s\right)$$

$$\tilde{\vec{J}}_s^{(inc)}\left(\vec{r}_s, s\right) \equiv \vec{1}_s\left(r_s\right) \times \tilde{\vec{H}}^{(inc)}\left(\vec{r}_s, s\right)$$

$$\tilde{\overleftrightarrow{L}}\left(\vec{r}_s, \vec{r}_s\,' ; s\right) = \frac{1}{2}\overleftrightarrow{1}_t\left(\vec{r}_s\right)\delta_s\left(\vec{r}_s - \vec{r}_s\,'\right) + \tilde{\overleftrightarrow{M}}\left(\vec{r}_s, \vec{r}_s\,' ; s\right) \tag{2.22}$$

$$\tilde{\overleftrightarrow{M}}\left(\vec{r}_s, \vec{r}_s\,' ; s\right) = -\vec{1}_S\left(\vec{r}_s\right) \times \left[\nabla \tilde{G}_o\left(\vec{r}_s, \vec{r}_s\,' ; s\right) \times \overleftrightarrow{1}_t\left(\vec{r}_s\,'\right)\right]$$

We have assumed an externally incident field; for internal incidence there are some sign changes. For this integral equation we use a superscript H on the eigenmodes and eigenvalues.

Let us now mention a key assumption concerning the present development. This concerns the behavior of the eigenmodes as functions of s, the complex frequency. We assume that for any fixed β the local variation of the mode (on S) is small over a wavelength as $s \to \infty$. In cases of simple bodies such as spheres and thin wires the mode index β is related to the number of half wave-like variations of the mode over the body. Of course for general finite-size perfectly conducting scatterers the $\tilde{\vec{j}}_{s_\beta}$ and $\tilde{\vec{\mu}}_{s_\beta}$ are known to be complex (varying phase over the body) [2], and the associated natural modes for say the prolate sphere are a little different from each other, implying variation as a function of s. Stated another way, let us assume that

$$\begin{gathered}\left|\left[\vec{1}_v \cdot \nabla_s\right]\vec{j}_{s_\beta}\right|\sqrt{A} \ll |\gamma| \text{ as } s \to \infty \text{ in both half planes}\\ \vec{1}_v \equiv \text{any unit vector tangential to } s\\ A \equiv \text{area of } S\end{gathered} \tag{2.23}$$

and that in the limit the ratio goes to zero. The introduction of $\sqrt{A}$ is to make $\vec{j}_{s_\beta}\sqrt{A}$ of order 1 due to the normalization of the modes. Of course, we want the body to be smooth so there are no singularities of the surface current density in the above expression (but this is not important since for an eigenmode the current and excitation vectors are related by the same eigenvalue all over S).

III. Asymptotic Form of Eigenvalues in Left Half s-Plane

Beginning with the form of the eigenvalues in (2.7) let us consider the case that $\mathrm{Re}[s] \to -\infty$, i.e., left half-plane asymptotics. Following a concept in [9], let us consider the case that the dominant contribution in the kernel in the left half plane is given by maximizing the exponential appearing in the kernel. As indicated in fig. 3.1 let us define

$$L_0 \equiv \sup_{\vec{r}_s ,\, \vec{r}\,'_s \in S} \left| \vec{r}_s - \vec{r}\,'_s \right| \equiv \sup_{\vec{r}_s ,\, \vec{r}\,'_s \in S} R$$

$$\equiv \text{maximum linear dimension of scatterer} \tag{3.1}$$

Writing an eigenvalue as

$$\tilde{\lambda}_\beta(s) = \left\langle \tilde{\vec{\mu}}_\beta \left(\vec{r}_s , s\right) ; \tilde{\vec{T}} \left(\vec{r}_s , \vec{r}\,'_s ; s\right) ; \tilde{\vec{j}}_{s_\beta}\left(\vec{r}\,'_s , s\right) \right\rangle \tag{3.2}$$

note that the kernel takes the form of $e^{-\gamma R}$ times powers of ζ as in (2.3). As $\mathrm{Re}[s] \to -\infty$ the exponential term dominates the kernel. This exponential is largest when $R = L_0$. So as in [9] the significant integration occurs when $\vec{r}_s$ and $\vec{r}\,'_s$ are near the opposite extremities of the scatterer. The normalization of the eigenmodes as in (2.4) makes the modes of "average" amplitude $A^{-\frac{1}{2}}$ (as used in (2.23)). Integrating near the extremal points gives contributions proportional to $A^{-\frac{1}{2}}$ for each integration over $\vec{r}_s$ and $\vec{r}\,'_s$. Of course, the variation of $e^{-\gamma R}$ near each of these points makes the integration over distances like γ^{-1} of significance. Details of the geometry near these points also contribute as do the details of the modal variation. However, the fundamental smoothness assumption of the eigenmodes in (2.23) makes the modal variation small compared to the exponential variation in the kernel. So our estimate of (3.2) is the same as the corresponding term (D_α) in [9] except for general s giving

$$\tilde{\lambda}_\beta(s) = \tilde{f}_0(s) e^{-\gamma L_0} \text{ as } \mathrm{Re}[s] \to -\infty \tag{3.3}$$

with $\tilde{f}_0(s)$ bounded above and below by algebraic quantities and where the exponential term is the most significant. Now this is an asymptotic bound since $\tilde{\lambda}_\beta(s)$ can have zeros in the left half plane. Of course, as the restraint of (2.23) takes effect so that the eigenmode oscillations are over larger distances than

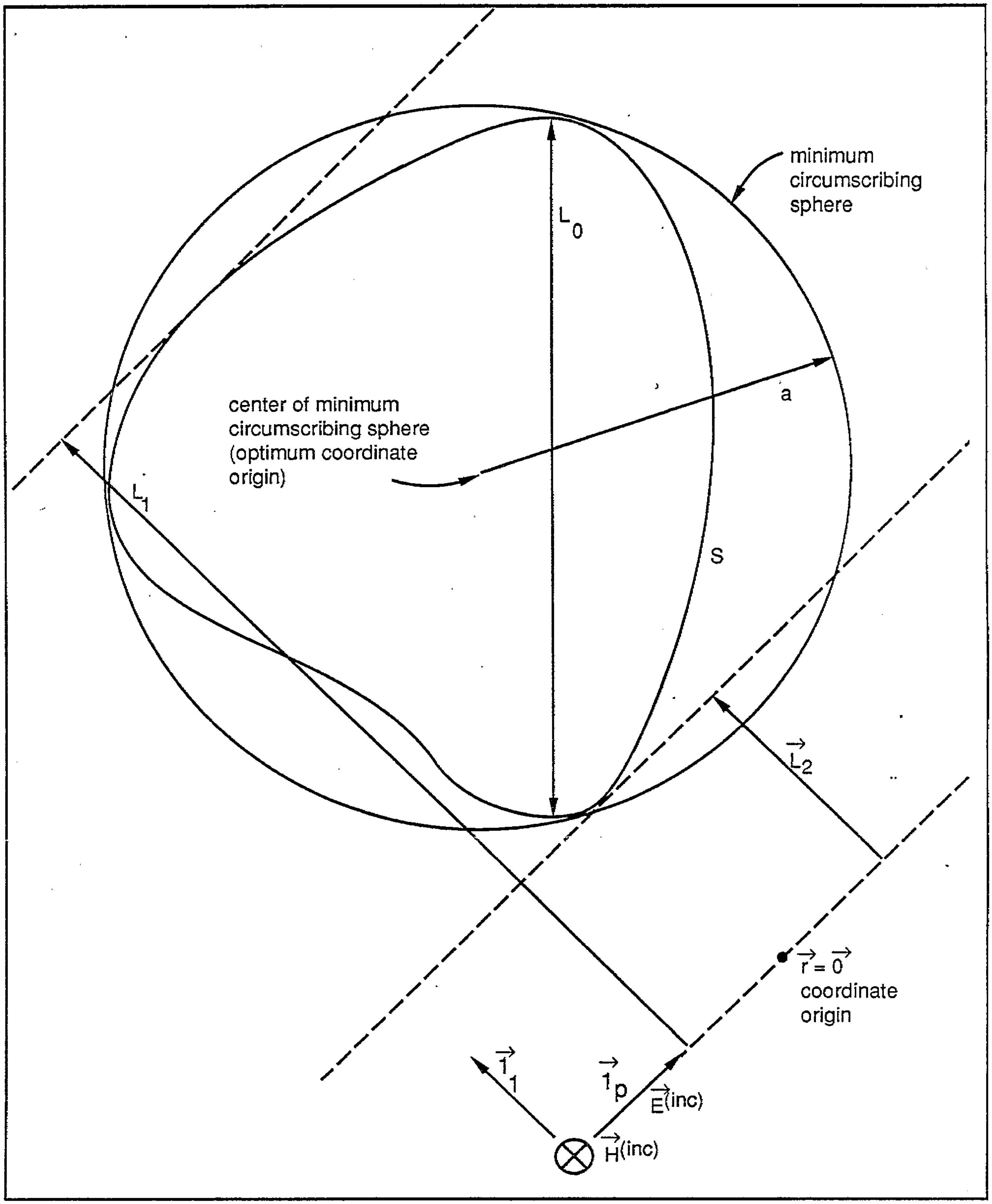

Fig. 3.1. General Surface Scatterer with Maximum Dimension L_0, Clearance Distance L_1, and Contact Distance L_2

$\left|\gamma^{-1}\right|$ such zeros should not appear and the magnitude of $\tilde{\lambda}_{\beta}(s)$ should behave like $e^{-\mathrm{Re}[\gamma L_o]}$ times a slower varying function of s as in (3.3).

As in (2.6) the reciprocal eigenvalues appear in the expansion for the surface currents or for the resolvent kernel (inverse or $n = -1$ in (2.5)). The present results then agree with the results obtained in [10,11].

A note of caution is that there are special cases in which the extremal points are on portions of S which are degenerate in the sense that S (near one or more of these) collapses to a single surface (no interior). If the modal currents are equal and opposite on the two sides for some modes then they do not contribute to the integration in (3.2). Cases when such surfaces are on planes of symmetry can produce such conditions [6]. In such cases the parameter L_o must be reduced for these modes to extremal positions for which the modal currents do contribute to the integration. This sheds some insight onto the assumption of a convex body in [11]. While the body need not be strictly convex there are degenerate cases (as above which are definitely not convex) which require some modification of the results.

As a check, consulting appendices A and B, these results are confirmed for the perfectly conducting sphere for both E and H field integral equations. In this case, $L_o = 2a$ and the term is multiplied by a constant (1/2 times a sign depending on mode index).

IV. Asymptotic Form of Eigenvalues in Right Half s-Plane

In [10,11] the asymptotic behavior of the inverse kernel is also discussed for large s in the right half plane. The authors note that instead of an exponential behavior the inverse kernel (and for our purposes the inverse eigenvalues) have behavior bounded by a rational function in s. Here we find asymptotic estimates for the eigenvalues of the E and H equations.

Referring to fig. 4.1 consider some small region of S and consider that there is some particular βth eigenmode of concern. Now for our βth eigenmode we consider very small wavelengths in the right half plane under the assumption of a bounded variation of the eigenmode as in (2.23). Consider some position on S away from any zero of the βth eigenmode. Then we can think of an outward propagating wavefront as a plane wave propagation in the $+\vec{1}_S$ direction.

Of course these are not strictly plane waves but can be thought of as such as $\mathrm{Re}[s] \to +\infty$ for which we confine our attention to a small region of S and the surrounding space. The size of this region needs to be small compared to both wavelength and the local radii of curvature of S. Of course, the eigenmode is assumed to vary negligibly over this region. Viewed another way, one may consider the situation in fig. 4.1 in time domain for say an eigenmode surface-current-density which behaves as a step function at early times. Then for sufficiently small times the fields are not influenced by portions of S outside of our region of interest.

Now incident and scattered fields at high frequencies do not in general propagate parallel to the local $\vec{1}_S$. Such fields are composed of an infinite number of eigenmodes. As more and more eigenmodes are included (larger and larger β) an arbitrary distribution of some incident or scattered field can be arbitrarily approximated, including waves not normal to local S. The result that eigenmodes propagate parallel to $\vec{1}_S$ near S as $\mathrm{Re}[s] \to +\infty$ is for fixed β. If one first fixes s and then lets $\beta \to \infty$ the results can be quite different.

Considering first the E-field integral equation, note that the eigenimpedances relate the surface current density to the tangential incident electric field, which is the negative of the tangential scattered electric field. Here we use labels "ex" and "in" for exterior and interior scattered fields respectively. For the βth eigenmode we have on the exterior

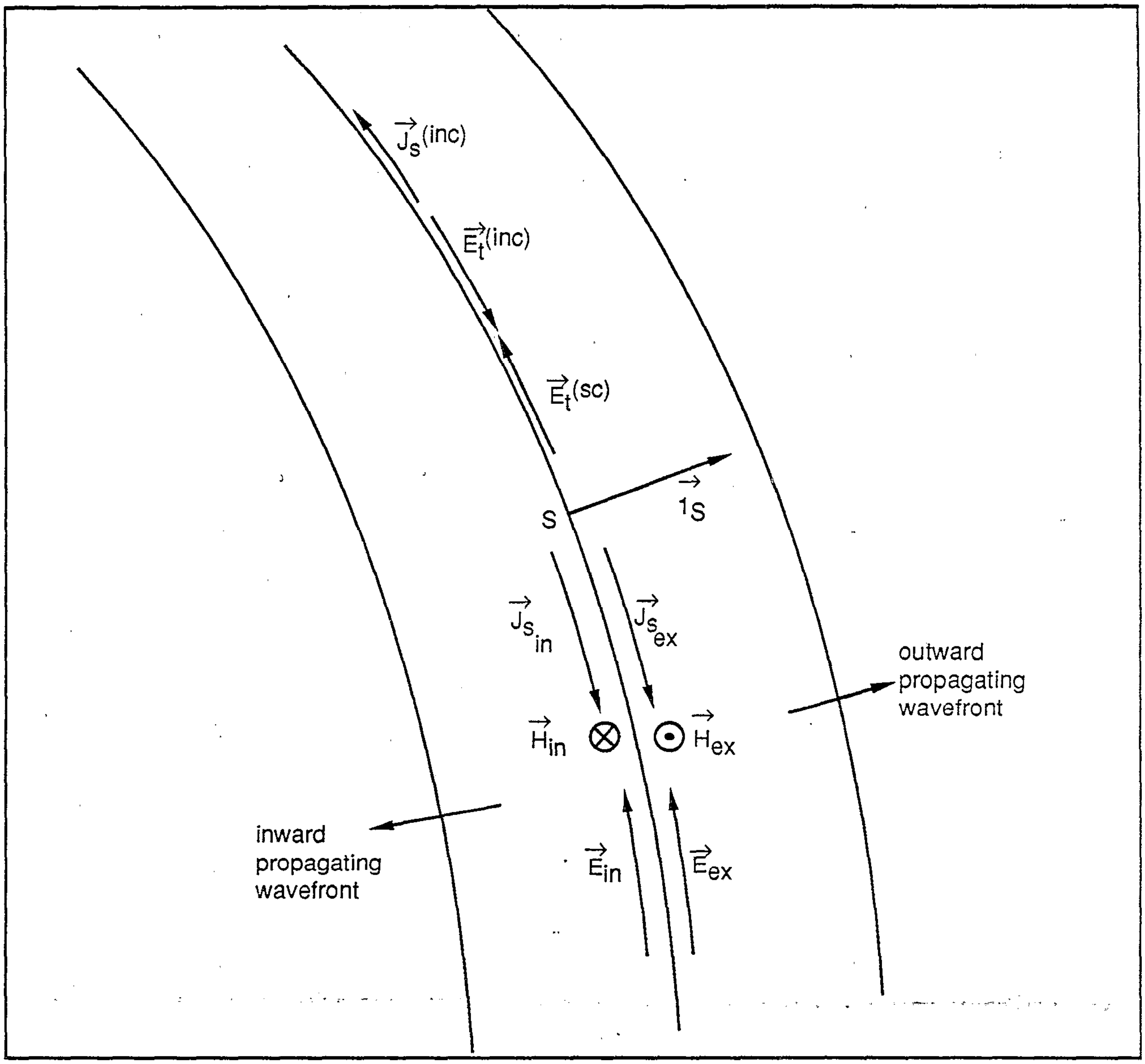

Fig. 4.1. Small Region of S with Early-Time/High-Frequency Behavior of Local Fields for Some Particular Eigenmode

$$\tilde{\vec{J}}_{s_{ex}} = \vec{1}_S \times \tilde{\vec{H}}_{ex} \text{ as } \mathrm{Re}[s] \to +\infty$$

$$\tilde{\vec{E}}_{ex} = -\tilde{Z}_{\beta}^{(ex)}(s) \; \tilde{\vec{J}}_{s_{ex}} = -Z_o \vec{1}_S \times \tilde{\vec{H}}_{ex} \text{ as } \mathrm{Re}[s] \to +\infty \tag{4.1}$$

$$\tilde{Z}_{\beta}^{(ex)}(s) = Z_o \text{ as } \mathrm{Re}[s] \to +\infty$$

Similarly for the inward propagating $\left(-\vec{1}_S\right)$ plane wave we have

$$\tilde{\vec{J}}_{s_{in}} = -\vec{1}_S \times \tilde{\vec{H}}_{in} \text{ as } \mathrm{Re}[s] \to +\infty$$

$$\tilde{\vec{E}}_{in} = -\tilde{Z}_{\beta}^{(in)}(s) \; \tilde{\vec{J}}_{s_{in}} = Z_o \vec{1}_S \times \tilde{\vec{H}}_{in} \text{ as } \mathrm{Re}[s] \to +\infty \tag{4.2}$$

$$\tilde{Z}_{\beta}^{(in)}(s) = Z_o \text{ as } \mathrm{Re}[s] \to +\infty$$

So we have effectively split the eigenimpedance into exterior and interior parts. Recombining we have

$$\tilde{\vec{E}}_{ex} = \tilde{\vec{E}}_{in} = -\tilde{Z}_{\beta}(s) \; \tilde{\vec{J}}_s$$

$$\tilde{\vec{J}}_s = \tilde{\vec{J}}_{s_{in}} + \tilde{\vec{J}}_{s_{ex}} \tag{4.3}$$

Combining with the previous equations we have

$$\tilde{Z}_{\beta}(s) = \left[\tilde{Z}_{\beta}^{(ex)^{-1}}(s) + \tilde{Z}_{\beta}^{(in)^{-1}}(s) \right]^{-1}$$

$$= \frac{Z_o}{2} \text{ as } \mathrm{Re}[s] \to +\infty \tag{4.4}$$

In normalized form we have

$$\tilde{z}_\beta(s) \equiv \frac{1}{Z_o}\tilde{Z}_\beta(s) = \frac{1}{2} \text{ as } \operatorname{Re}[s] \to +\infty$$

$$\tilde{z}_\beta^{(ex)}(s) \equiv \frac{1}{Z_o}\tilde{Z}_\beta^{(ex)} = 1 \text{ as } \operatorname{Re}[s] \to +\infty \tag{4.5}$$

$$\tilde{z}_\beta^{(in)}(s) \equiv \frac{1}{Z_o}\tilde{Z}_\beta^{(in)} = 1 \quad \text{as } \operatorname{Re}[s] \to +\infty$$

As a check, consulting appendix A, the results for $\tilde{z}_\beta(s)$ as $\operatorname{Re}[s] \to +\infty$ are confirmed for the perfectly conducting sphere. This is true for both E modes and H modes of the E-field integral equation.

Considering second the H-field integral equation, now the eigenvalues relate the surface current density to $\vec{J}^{(inc)}$ as defined in (2.22) as $\vec{1}_S \times \tilde{\vec{H}}^{(inc)}$. For this integral equation the boundary condition has the total magnetic field inside S as 0. For the βth eigenmode we then have on the exterior

$$\tilde{\vec{J}}_{s_{ex}} = \vec{1}_S \times \tilde{\vec{H}}_{ex}$$

$$\tilde{\vec{J}}_s^{(inc)} = \vec{1}_S \times \tilde{\vec{H}}^{(inc)} = \tilde{\lambda}_\beta^{(H,\,ex)}(s)\tilde{\vec{J}}_{s_{ex}} \quad \text{as } \operatorname{Re}[s] \to +\infty \tag{4.6}$$

On the interior we have

$$\tilde{\vec{J}}_{s_{in}} = -\vec{1}_S \times \tilde{\vec{H}}_{in} = \vec{1}_S \times \tilde{\vec{H}}^{(inc)} = \tilde{\vec{J}}_s^{(inc)}$$

$$\tilde{\vec{J}}_s^{(inc)} = \tilde{\lambda}_\beta^{(H,\,in)}(s)\,\tilde{\vec{J}}_{s_{in}} \tag{4.7}$$

$$\tilde{\lambda}_\beta^{(H,\,in)}(s) = 1$$

Combining the surface current densities

$$\tilde{\vec{J}}_s = \tilde{\vec{J}}_{s_{in}} + \tilde{\vec{J}}_{s_{ex}} \tag{4.8}$$

and noting

$$\tilde{\vec{J}}_s^{(inc)} = \tilde{\lambda}_\beta^{(H)}(s)\;\tilde{\vec{J}}_s \tag{4.9}$$

we have

$$\tilde{\lambda}_{\beta}^{(H)}(s) = \left[\tilde{\lambda}_{\beta}^{(H,ex)^{-1}}(s) + \tilde{\lambda}_{\beta}^{(H,in)^{-1}}(s)\right]^{-1}$$

$$= \left[\tilde{\lambda}_{\beta}^{(H,ex)^{-1}} + 1\right]^{-1} \tag{4.10}$$

Furthermore, noting that at high frequencies we have the well-known doubling of the tangential magnetic field on the incident side (exterior for the βth eigenmode), then the exterior scattered field equals the incident field giving

$$\vec{\tilde{J}}_{s}^{(inc)}(s) = \vec{\tilde{J}}_{s_{ex}}(s) \text{ as } \mathrm{Re}[s] \to +\infty$$

$$\tilde{\lambda}_{\beta}^{(H,ex)}(s) = 1 \text{ as } \mathrm{Re}[s] \to +\infty \tag{4.11}$$

$$\tilde{\lambda}_{\beta}^{(H)}(s) = \frac{1}{2} \text{ as } \mathrm{Re}[s] \to +\infty$$

As a check consulting appendix B, the results for $\tilde{\lambda}_{\beta}^{(H)}(s)$ as $\mathrm{Re}[s] \to +\infty$ are confirmed for the perfectly conducting sphere. This is true for both E modes and H modes of the H-field integral equation.

While [10,11] obtained results for the eigenvalues of the E-field integral equation as $\mathrm{Re}[s] \to +\infty$, these were of a general form which could be used in a bounding sense. Here (4.5) gives an exact asymptotic results (i.e., 1/2 for the normalized eigenimpedances). Furthermore, we have shown a similar result for the eigenvalues of the H-field integral equation.

V. Coupling to Incident Plane Wave

As indicated in fig. 3.1 we have an incident plane wave given by

$$\vec{E}^{(inc)}(\vec{r},t) = E_o f\left(t - \frac{\vec{1}_1 \cdot \vec{r}}{c}\right)\vec{1}_p$$

$$\vec{H}^{(inc)}(\vec{r},t) = \frac{E_o}{Z_o} f\left(t - \frac{\vec{1}_1 \cdot \vec{r}}{c}\right)\vec{1}_1 \times \vec{1}_p$$

$$\tilde{\vec{E}}^{(inc)}(\vec{r},s) = E_o \tilde{f}(s)\vec{1}_p e^{-\gamma \vec{1}_1 \cdot \vec{r}}$$

$$\tilde{\vec{H}}^{(inc)}(\vec{r},s) = \frac{E_o}{Z_o} \tilde{f}(s)\vec{1}_1 \times \vec{1}_p e^{-\gamma \vec{1}_1 \cdot \vec{r}} \tag{5.1}$$

$\vec{1}_p \equiv$ polarization vector

$\vec{1}_1 \equiv$ direction of incidence

$\vec{1}_1 \cdot \vec{1}_p = 0$

In normalized form we can define

$$\tilde{\vec{e}}^{(inc)}(\vec{r},s) \equiv \vec{1}_p e^{-\gamma \vec{1}_1 \cdot \vec{r}}$$

$$\vec{e}^{(inc)}(\vec{r},t) = \vec{1}_p \delta\left(t - \frac{\vec{1}_1 \cdot \vec{r}}{c}\right)$$

$$\tilde{\vec{h}}^{(inc)}(\vec{r},s) \equiv \vec{1}_1 \times \vec{1}_p e^{-\gamma \vec{1}_1 \cdot \vec{r}} \tag{5.2}$$

$$\vec{h}^{(inc)}(\vec{r},t) = \vec{1}_1 \times \vec{1}_p \delta\left(t - \frac{\vec{1}_1 \cdot \vec{r}}{c}\right)$$

As the plane wave interacts with the scatterer, the leading edge of the incident wave just passes the entire object at the clearance distance defined by

$$L_1 \equiv \sup_{\vec{r}_s \in S} \vec{1}_1 \cdot \vec{r} \tag{5.3}$$

and it first reaches the object at the contact distance defined by

$$L_2 \equiv \inf_{\vec{r}_s \in S} \vec{1}_1 \cdot \vec{r} \tag{5.4}$$

In temporal units we have the clearance time

$$t_1 \equiv \frac{L_1}{c} \tag{5.5}$$

and the contact time

$$t_2 \equiv \frac{L_s}{c} \tag{5.6}$$

The coupling term as in (2.6) is the symmetric product of a left eigenmode with the incident field. Using the normalized incident wave as in (5.2) we notice that in the left half plane we maximize $e^{-\gamma \vec{1}_1 \cdot \vec{r}}$ and observe

$$\left\langle \tilde{\vec{\mu}}_\beta\left(\vec{r}_s\ , s\right) ; \vec{I}\left(\vec{r}_s\ , s\right)\right\rangle = \tilde{f}_1(s)e^{-\gamma L_1} = \tilde{f}_1(s)e^{-st_1} \quad \text{as } \mathrm{Re}[s] \to +\infty \tag{5.7}$$

similar to the result for $\tilde{\lambda}_\beta(s)$ in (3.3) except that only one extremal position is used, one with respect to the incident wave. Similarly we minimize $e^{-\gamma \vec{1}_1 \cdot \vec{r}}$ in the right half plane and observe

$$\left\langle \tilde{\vec{\mu}}_\beta\left(\vec{r}_s\ , s\right) ; \vec{I}\left(\vec{r}_s\ , s\right)\right\rangle = \tilde{f}_s(s)e^{-\gamma L_2} = \tilde{f}_2(s)e^{-st_2} \tag{5.8}$$

In obtaining these asymptotic estimates one avoids special cases in which the dot product of the incident field with the mode is zero over some non-zero region near the clearance and contact positions respectively. For example, a protruding fin with ideally zero thickness could have its surface perpendicular to the incident electric field; this would be such a case.

While the coordinate origin ($\vec{r} = \vec{0}$) can be chosen arbitrarily, there is some benefit in trying to choose this optimally. Note that as the direction of incidence $\vec{1}_1$ is changed over 4π steradians both L_1 and L_2 change. If $\vec{r} = \vec{0}$ is chosen inside the scatterer this variation is reduced. Suppose we try to minimize the maximum value that L_1 achieves over all $\vec{1}_1$. This minimizes the left-half-plane exponentially growing coupling term in (5.7). Just as in [9] we have

$$
a = \inf_{\text{for all } \vec{r} = \vec{0}} L_{1_{\sup}}
$$

$$
L_{1_{\sup}} = \sup_{\vec{1}_1} L_1 \tag{5.9}
$$

This length defines the radius of the smallest circumscribing sphere for the scatterer and the coordinate origin is the center of this sphere. With this choice we have

$$
0 \le L_1 \le a
$$

$$
0 \le t_1 \le \frac{a}{c} \tag{5.10}
$$

It may not be immediately apparent that there is this lower limit of 0. However, this can be obtained by assuming the converse for some $\vec{1}_1$ and noting that this places the entire scatterer to one side of a diameter of the bounding sphere and noting that there is now a new bounding sphere of radius less than a (a contradiction).

Similarly one can look at L_2. In the right half plane the exponential dominates the asymptotic form of the coupling term as in (5.8). The minimum value of this for all $\vec{1}_1$ is found by minimizing the maximum value of L_2 over all $\vec{1}_1$ giving

$$
-a = \sup_{\text{for all } \vec{r} = \vec{0}} L_{2_{\inf}}
$$

$$
L_{2_{\inf}} = \inf_{\vec{1}_1} L_2 \tag{5.11}
$$

Note that under this condition L_2 is negative as

$$
0 \ge L_2 \ge -a
$$

$$
0 \ge t_2 \ge -\frac{a}{c} \tag{5.12}
$$

This also defines the same coordinate center as the center of the minimum circumscribing sphere with radius a.

Consulting appendices A and B for the perfectly conducting sphere note first that in this case the coordinate center is already optimally chosen. In this case

$$
L_1 = -L_2 = a \tag{5.13}
$$

and appropriate foregoing inequalities are now equalities.

VI. Asymptotic Forms of Eigenterm

Consider first the kernel as in (2.5). The dyadic product of eigenmodes is reasonably well behaved in both half planes as $s \to \infty$ due to the restriction in (2.23). Each term in the inverse kernel then behaves as

$$\tilde{\lambda}_\beta^{-1}(s)\ \tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s\,,s\right)\ \tilde{\vec{\mu}}_\beta\left(\vec{r}_s\,,s\right) = \tilde{f}_o^{-1}(s)e^{\gamma L_o}\ \tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s\,,s\right)\ \tilde{\vec{\mu}}_\beta\left(\vec{r}_s\,,s\right)$$

$$\text{as } \mathrm{Re}[s] \to -\infty \tag{6.1}$$

From (5.7) we can include the incident field to get the βth term in the expansion of the surface current density in (2.6) as

$$\tilde{\lambda}_\beta^{-1}(s)\ \left\langle \tilde{\vec{\mu}}_\beta\left(\vec{r}_s\,,s\right);\ \tilde{\vec{I}}\left(\vec{r}_s\,,s\right)\right\rangle\ \tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s\,,s\right) = \tilde{f}_o^{-1}(s)\tilde{f}_1(s)e^{\gamma[L_0 - L_1]}\ \tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s\,,s\right)$$

$$\text{as } \mathrm{Re}[s] \to -\infty \tag{6.2}$$

So in the left half plane the dyadic terms in (6.1) go asymptotically (exponentially) to zero since L_0 is definitely positive.

The term $L_0 - L_1$ in (6.2) is also positive, so the βth term in the incident field expansion also goes to zero exponentially. To see this note first an upper bound for L_0 *as*

$$L_0 \le 2a \tag{6.3}$$

which merely requires that the scatterer touch diametrically opposed positions on the minimum circumscribing sphere. Examples of this case are given in [9]. However, examples can also be produced for which L_0 is smaller than the diameter. To obtain a lower bound choose some point $\vec{r}_1$ on the object corresponding to one end of some line of length L_0 which also touches the object somewhere else (say $\vec{r}_2$). Consider $\vec{r}_1$ as the center of a sphere of radius L_0. Note that $\vec{r}_2$ lies on the sphere. Now other points on the body may touch the sphere but no points of the scatterer lie outside the sphere since the distance from such points to $\vec{r}_1$ would be greater than L_0, the maximum linear dimension of the body (a contradiction). So one circumscribing sphere for the scatterer has radius L_0, and the minimum circumscribing sphere of radius a has

$$a \le L_0 \tag{6.4}$$

This is not a tight bound since if one takes a line of length a, touching a sphere of radius a at both ends one can construct another line from the center of this one, through the center of the sphere to touch the sphere. This line has length greater than a, a length which is less than L_0. So we have

$$a < L_0 \leq 2a \tag{6.5}$$

A more detailed analysis could establish a tight lower bound on L_0. An equilateral triangular plate has $\sqrt{3}\,a \simeq 1.73\,a$ for L_0. A regular tetrahedron has $2\sqrt{2/3}\,a \simeq 1.63a$ for L_0. This latter case appears to be the best lower bound for L_0.

Now from (5.10) we have bounds for L_1. Combining with the above gives

$$0 < L_0 - L_1 \leq 2a \tag{6.6}$$

Using the result for the tetrahedron the lower bound for this is $(2\sqrt{2/3} - 1)a \simeq .63a$. So (6.2) converges to zero exponentially in the left half plane.

In the right half s plane each term in the inverse kernel behaves as

$$\tilde{\lambda}_\beta^{-1}(s)\ \vec{\vec{j}}_{s_\beta}\left(\vec{r}_s\,,s\right)\ \vec{\vec{\tilde{\mu}}}_\beta\left(\vec{r}_s\,,s\right) = 2\,\vec{\vec{j}}_{s_\beta}\left(\vec{r}_s\,,s\right)\ \vec{\vec{\tilde{\mu}}}_\beta\left(\vec{r}_s\,,s\right) \text{ as } \mathrm{Re}[s] \to +\infty \tag{6.7}$$

provided the eigenvalues are taken as the normalized eigenimpedances (4.5) of the E-field integral equation or the eigenvalues (4.11) of the H-field integral equation. Including the incident field the βth term in the expansion of the surface current density is from (5.8)

$$\tilde{\lambda}_\beta^{-1}(s)\ \left\langle \vec{\vec{\tilde{\mu}}}_\beta\left(\vec{r}_s\,,s\right);\ \vec{\vec{I}}\left(\vec{r}_s\,,s\right)\right\rangle = 2\tilde{f}_2(s)e^{-\gamma L_2} \text{ as } \mathrm{Re}[s] \to +\infty \tag{6.8}$$

again using the eigenvalues (normalized) as above. With L_2 bounded between 0 and -a as in (5.12) then (6.8) diverges exponentially unless L_2 is zero.

By a time shift, equivalent to a turn-on time

$$t_o \equiv -\frac{a}{c} \tag{6.9}$$

we can then multiply the incident wave by $e^{-\gamma a}$. This additional factor makes (6.8) converge in the right half plane since

$$0 \leq a + L_2 \leq a \tag{6.10}$$

Applying the same factor to the left-half-plane, then the length in the (6.2) exponent has, from (6.6)

$$-a < L_0 - L_1 - a \leq a \tag{6.11}$$

where this lower bound is actually somewhat tighter. If $L_0 - L_1 - a$ is negative then there is divergence in the left half plane. If the scatterer has $L_0 = 2a$ then the lower bound is zero giving no exponential divergence in the left half plane.

Another approach would use the result that

$$0 \le L_1 - L_2 \le L_0 \text{ for all } \vec{1}_1 \tag{6.12}$$

which is apparent from fig. 3.1 as $L_1 - L_2$ is the distance the incident wave travels across the object. The lower bound corresponds to allowing the scatterer to be a disk and having the wave normally incident. Looking at (6.2) and (6.8) and multiplying by $e^{\gamma L_2}$ gives no exponential in (6.8) but

$$0 \le L_0 - L_1 + L_2 \le 0 \tag{6.13}$$

as the coefficient of γ in (6.2). This corresponds to choosing the turn-on time as the time the incident wave first touches the scatterer. In such a form the βth term in the eigenmode expansion of the surface current density is well behaved as $\mathrm{Re}[s] \to \pm \infty$, i.e., in both half planes.

Thus objects with $L_0 = 2a$ have special properties in the asymptotic behavior of the eigenterms. Such scatterers also have special properties in the convergence of the SEM series [9]. This is associated with the choice of a turn-on time t_0 which is independent of $\vec{1}_1$.

As in appendices A and B for the perfectly conducting sphere all the foregoing results can be seen to apply.

VII. Asymptotic Forms of Kernel, Inverse Kernel, and Response

Now consider the properties of various summations over the various types of eigenterms. From (2.5) we have an expression for various powers of the kernel. In (2.8) we have the 0 power which is the identity on S; this expression is not asymptotic, but rather valid for all s.

Consider, however, the kernel

$$\tilde{\overleftrightarrow{T}}\left(\vec{r}_s , \vec{r}_s\,' ; s\right) = \sum_{\beta} \tilde{\lambda}_\beta(s)\, \tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s , s\right)\; \tilde{\vec{\mu}}_\beta\left(\vec{r}_s\,' , s\right) \tag{7.1}$$

In the left half plane this has the formal result from (3.3)

$$\begin{aligned}\tilde{\overleftrightarrow{T}}\left(\vec{r}_s , \vec{r}_s\,' ; s\right) &= \sum_{\beta} \tilde{f}_{0,\beta}(s) e^{-\gamma L_0}\; \tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s , s\right)\; \tilde{\vec{\mu}}_\beta\left(\vec{r}_s\,' , s\right) \\ &= e^{-\gamma L_0} \sum_{\beta} \tilde{f}_{0,\beta}(s)\; \tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s , s\right)\; \tilde{\vec{\mu}}_\beta\left(\vec{r}_s\,' , s\right)\end{aligned} \tag{7.2}$$

Note that $\tilde{f}_0$ has another subscript β since the result (3.3) might conceivably give a different function for each β. This points out part of the problem in that summing over β the asymptotic results may not be uniform over all β. If β is proportional to the number of oscillations on the body for the βth eigenmode, then for larger and larger β one must go to more and more negative Re[s] before the restriction in (2.23) is met.

So we have the problem of the order of $\beta \to \infty$ and $\mathrm{Re}[s] \to -\infty$, i.e., which limit is taken first. Now from (2.3) we can write

$$\tilde{\overleftrightarrow{T}}\left(\vec{r}_s , \vec{r}_s\,' ; s\right) = \tilde{\overleftrightarrow{f}}_3\left(\vec{r}_s , \vec{r}_s\,' ; s\right) e^{-\gamma\left|\vec{r}_s - \vec{r}_s\,'\right|} \quad \text{as } \mathrm{Re}[s] \to -\infty \tag{7.3}$$

where $\tilde{\overleftrightarrow{f}}_3$ is rational in s. Note that L_0 in (7.2) bounds $\left|\vec{r}_s - \vec{r}_s\,'\right|$ in (7.3) so that these two are consistent in the sense that for some $\vec{r}_s$, $\vec{r}_s\,'$ (7.3) blows up as fast as (7.2) in the left half plane.

Similarly the inverse kernel has

$$\tilde{\overleftrightarrow{T}}^{-1}\left(\vec{r}_s , \vec{r}_s\,' ; s\right) = \sum_{\beta} \tilde{\lambda}_\beta^{-1}(s)\, \tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s , s\right)\; \tilde{\vec{\mu}}_\beta\left(\vec{r}_s\,' , s\right) \tag{7.4}$$

In the left half plane this has the formal result

$$\overset{\leftrightarrow}{\tilde{T}}\left(\vec{r}_s, \vec{r}_s\,'; s\right) = \sum_\beta \tilde{f}_{0,\beta}^{-1}(s) e^{\gamma L_0} \; \tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s, s\right) \; \tilde{\vec{\mu}}_\beta\left(\vec{r}_s\,', s\right)$$

$$= e^{\gamma L_0} \sum_\beta \tilde{f}_{0,\beta}^{-1}(s) \; \tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s, s\right) \; \tilde{\vec{\mu}}_\beta\left(\vec{r}_s\,', s\right) \text{as } \mathrm{Re}[s] \to -\infty \tag{7.5}$$

This one is somewhat problematical since $e^{\gamma L_0}$ goes to zero. However, $\tilde{f}_{0,\beta}^{-1}$ should be increasing as $\beta \to \infty$ so this relates to the order-of-the-limits problem. In [11] there is the conjecture that

$$\overset{\leftrightarrow}{\tilde{T}}\left(\vec{r}_s, \vec{r}_s\,'; s\right) = \overset{\leftrightarrow}{\tilde{f}}_4 \, e^{\gamma\left|\vec{r}_s - \vec{r}_s\,'\right|} \text{as } \mathrm{Re}[s] \to -\infty \tag{7.6}$$

where $\overset{\leftrightarrow}{\tilde{f}}_4$ is rational in s. Note that this is large when $\vec{r}_s$ is near $\vec{r}_s\,'$. It can be reconciled with (7.5) by the contributions to the sum for large β.

In the right half plane we have the formal result for the kernel

$$\overset{\leftrightarrow}{\tilde{T}}\left(\vec{r}_s, \vec{r}_s\,'; s\right) = \sum_\beta \frac{1}{2} \; \tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s, s\right) \; \tilde{\vec{\mu}}_\beta\left(\vec{r}_s\,', s\right)$$

$$= \frac{1}{2} \overset{\leftrightarrow}{1}_t\left(\vec{r}_s\right) \delta_s\left(\vec{r}_s - \vec{r}_s\,'\right) \text{as } \mathrm{Re}[s] \to +\infty \tag{7.7}$$

Similarly we have the formal result for the inverse kernel

$$\overset{\leftrightarrow}{\tilde{T}}{}^{-1}\left(\vec{r}_s, \vec{r}_s\,'; s\right) = \sum_\beta 2 \; \tilde{\vec{j}}_{s_\beta}\left(\vec{r}_s, s\right) \; \tilde{\vec{\mu}}_\beta\left(\vec{r}_s\,', s\right)$$

$$= 2 \overset{\leftrightarrow}{1}_t\left(\vec{r}_s\right) \delta_s\left(\vec{r}_s - \vec{r}_s\,'\right) \text{as } \mathrm{Re}[s] \to +\infty \tag{7.8}$$

where in both cases the eigenvalues as in (6.7) are taken as the normalized eigenimpedances of the E-field integral equation or the eigenvalues of the H-field integral equation.

Now (7.7) and (7.8) are clearly inverse to each other. However, each result is only partly correct, and that concerns the illuminated side of the scatterer. Assuming a simple shape for the scatterer (such as convex and smooth) divide S as

$$\begin{aligned} S &= S_{il} \cup S_{sh} \\ S_{il} &\equiv \text{illuminated portion of } S \\ &\equiv \text{portion of S first reached by rays parallel to (and in same direction as) } \vec{1}_1 \\ S_{sh} &\equiv \text{shadowed portion of } S \end{aligned} \tag{7.9}$$

Now we have on S the usual physical optics result

$$\tilde{\vec{J}}_s(\vec{r}_s) = \begin{cases} \vec{1}_S(\vec{r}_s) \times \tilde{\vec{H}}^{(inc)}(\vec{r}_s, s) = 2\,\tilde{\vec{J}}_s^{(inc)}(\vec{r}_s, s) & for\ \vec{r}_s \in S_{il} \\ \vec{0} \quad for\ \vec{r}_s \in S_{sh} \end{cases}$$

$$\text{as } \mathrm{Re}[s] \to +\infty \tag{7.10}$$

If we apply (7.5) in the context of the H-field integral equation (2.22) we have the formal result

$$\begin{aligned}\tilde{\vec{J}}_s(\vec{r}_s, s) &= \left\langle \tilde{\overleftrightarrow{T}}^{-1}\left(\vec{r}_s, \vec{r}_s\,'; s\right);\ \tilde{\vec{J}}_s^{(inc)}\left(\vec{r}_s\,', s\right)\right\rangle \\ &= \left\langle 2\ \overleftrightarrow{1}_t^{(inc)}(\vec{r}_s)\,\delta_s\left(\vec{r}_s - \vec{r}_s\,'\right);\ \tilde{\vec{J}}_s^{(inc)}\left(\vec{r}_s\,', s\right)\right\rangle \text{ as } \mathrm{Re}[s] \to +\infty \\ &= 2\ \tilde{\vec{J}}_s^{(inc)}(\vec{r}_s, s) \text{ as } \mathrm{Re}[s] \to +\infty \end{aligned} \tag{7.11}$$

This of course is only correct in the illuminated region. One could also operate termwise on the incident field to give

$$\begin{aligned}\tilde{\vec{J}}_s(\vec{r}_s, s) &= \sum_\beta \tilde{\lambda}_\beta^{-1}(s) \left\langle \vec{\mu}_\beta\left(\vec{r}_s\,', s\right);\ \tilde{\vec{J}}_s^{(inc)}\left(\vec{r}_s\,', s\right)\right\rangle\ \tilde{\vec{j}}_{s_\beta}(\vec{r}_s, s) \\ &= \sum_\beta 2\left\langle \vec{\mu}_\beta\left(\vec{r}_s\,', s\right);\ \vec{J}_s^{(inc)}\left(\vec{r}_s\,', s\right)\right\rangle\ \tilde{\vec{j}}_{s_\beta}(\vec{r}_s, s) \text{ as } \mathrm{Re}[s] \to +\infty \end{aligned} \tag{7.12}$$

One could also use the asymptotic expansion of the symmetric product as in (5.8) giving a leading term of $e^{-\gamma L_2}$. In any event the asymptotic form of the kernel, inverse kernel and response have the fundamental order-of-the-limits problem because β has to be summed to ∞ before $\mathrm{Re}[s] \to \pm\infty$.

Looking at the E-field integral equation (2.14) we have the formal result

$$\begin{aligned}\tilde{\vec{J}}_s(\vec{r}_s, s) &= \left\langle \tilde{\overleftrightarrow{Z}}^{-1}\left(\vec{r}_s, \vec{r}_s\,'; s\right);\ \tilde{\vec{E}}_t^{(inc)}\left(\vec{r}_s\,', s\right)\right\rangle \\ &= \left\langle \frac{2}{Z_o}\ \overleftrightarrow{1}_t(\vec{r}_s)\,\delta_s\left(\vec{r}_s - \vec{r}_s\,'\right);\ \tilde{\vec{E}}_t^{(inc)}\left(\vec{r}_s\,', s\right)\right\rangle \text{ as } \mathrm{Re}[s] \to +\infty \\ &= \frac{2}{Z_o}\ \tilde{\vec{E}}_t^{(inc)}(\vec{r}_s, s) \text{ as } \mathrm{Re}[s] \to +\infty \\ &= \frac{2}{Z_o}\ \overleftrightarrow{1}_t \cdot \tilde{\vec{E}}_t^{(inc)}(\vec{r}_s, s) \end{aligned}$$

$$
\begin{aligned}
&= -2\, \overleftrightarrow{1}_t \left(\vec{r}_s\right) \cdot \left[\vec{1}_1 \times \overset{\rightharpoonup}{\tilde{H}}^{(inc)}\left(\vec{r}_s\,,s\right)\right] \\
&= 2\, \vec{1}_S\left(\vec{r}_s\right) \times \left[\vec{1}_S\left(\vec{r}_s\right) \times \left[\vec{1}_1 \times \overset{\rightharpoonup}{\tilde{H}}^{(inc)}\left(\vec{r}_s\,,s\right)\right]\right] \\
&\quad - 2\left[\vec{1}_S\left(\vec{r}_s\right) \cdot \vec{1}_1\right]\left[\vec{1}_S\left(\vec{r}_s\right) \times \overset{\rightharpoonup}{\tilde{H}}^{(inc)}\left(\vec{r}_s\,,s\right)\right] \\
&= 2\left[\vec{1}_S\left(\vec{r}_s\right) \times \vec{1}_1\right]\left[\vec{1}_S\left(\vec{r}_s\right) \cdot \overset{\rightharpoonup}{\tilde{H}}^{(inc)}\left(\vec{r}_s\,,s\right)\right] \\
&\quad - 2\left[\vec{1}_S\left(\vec{r}_s\right) \cdot \vec{1}_1\right]\vec{J}_s^{(inc)}\left(\vec{r}_s\,,s\right)
\end{aligned}
\tag{7.13}
$$

This result differs from (7.11) unless there is normal incidence (i.e., $\vec{1}_1 = -\vec{1}_S$). So this result has problems not only on S_{sh}, but on S_{il} as well. Due to this order-of-the-limits problem (7.6) looks better for the kernel and inverse kernel in the right half plane.

In the case of the perfectly conducting sphere (appendices A and B), we know by the uniqueness of the solution to the Maxwell equations and the completeness of the eigenmodes that the result is exact for all frequencies. For any given $\vec{r}_s$ on the sphere, however, one needs to sum an infinite series. As frequency is increased more and more terms are needed to obtain an accurate answer. In principle an infinite number of terms is needed for each finite frequency for an exact answer. The limit of this exact answer as $\text{Re}[s] \to +\infty$ is what is wanted. This again illustrates the order-of-the-limits problem.

VIII. Concluding Remarks

Here we have found some results of interest and rather simple forms concerning the asymptotic behavior of the EEM terms in both left and right s-planes. What is interesting is not only the result but the questions that are raised.

As we have seen in evaluating the eigenvalues as $\mathrm{Re}[s] \to +\infty$ there are separable contributions from the exterior and interior fields. Perhaps this extends throughout the s plane. In the right half s-plane the symmetric product (integral) of the eigenmode with the kernel is dominated by the local contribution near $\vec{r}\,'_s = \vec{r}_s$. An asymptotic evaluation of this integral over S might lead to a simpler (perhaps differential) equation over S for the eigenmodes in the limit of $\mathrm{Re}[s] \to +\infty$.

Appendix A. E-Field-Integral-Equation EEM Parameters for the Perfectly Conducting Sphere

As discussed in [1,4,5] the various EEM parameters for a perfectly conducting sphere of radius a are summarized for the E-field integral equation (2.14). The normalized eigenimpedances are

$$\tilde{z}_{e,n}(s) = Z_0^{-1}\tilde{Z}_{e,n}(s) = -\left[\gamma a i_n(\gamma a)\right]'\left[\gamma a k_n(\gamma a)\right]'$$

$$\tilde{z}_{h,n}(s) = Z_0^{-1}\tilde{Z}_{h,n}(s) = -\left[\gamma a i_n(\gamma a)\right]\left[\gamma a k_n(\gamma a)\right] \tag{A.1}$$

$$n = 1,2,\ldots$$

with a prime indicating differention with respect to the argument of the bessel function (i.e., γa). The eigenmodes are

$$\vec{\tilde{j}}_{s_{e,n,m,\sigma}}(\vec{r}_s, s) = d_{e,n,m,\sigma}\vec{Q}_{n,m,\sigma}(\theta,\phi)$$

$$\vec{\tilde{j}}_{s_{h,n,m,\sigma}}(\vec{r}_s, s) = -d_{h,n,m,\sigma}\vec{R}_{n,m,\sigma}(\theta,\phi)$$

$$\vec{\tilde{j}}_{s_{h,n,m,\sigma}}(\vec{r}_s, s) = \vec{1}_r(\vec{r}_s) \times \vec{\tilde{j}}_{s_{e,n,m,\sigma}}(\vec{r}_s, s) \tag{A.2}$$

$$\vec{\tilde{j}}_{s_{e,n,m,\sigma}}(\vec{r}_s, s) = -\vec{1}_r(\vec{r}_s) \times \vec{\tilde{j}}_{s_{h,n,m,\sigma}}(\vec{r}_s, s)$$

Here we have organized the eigenmode index as a set

$$\beta \equiv \left\{ \begin{matrix} e \\ h \end{matrix}, n, m, \sigma \right\} \tag{A.3}$$

with the first index indicating whether the mode is an E or H mode [4]. The unit normal is

$$\vec{1}_S(\vec{r}_s) = \vec{1}_r(\vec{r}_s) \tag{A.4}$$

The modes are developed from

$$Y_{n,m,{1 \atop 2}}(\theta,\phi) \equiv P_n^{(m)}(\cos(\theta))\left\{ \begin{matrix} \cos(m\phi) \\ \sin(m\phi) \end{matrix} \right\}$$

$$\vec{P}_{n,m,\sigma}(\theta,\phi) \equiv Y_{n,m,\sigma}(\theta,\phi)\vec{1}_r(\theta,\phi)$$

$$\vec{Q}_{n,m,\sigma}(\theta,\phi) \equiv a\nabla_s Y_{n,m,\sigma}(\theta,\phi)$$
$$= \vec{1}_\theta \frac{\partial}{\partial\theta} Y_{n,m,\sigma}(\theta,\phi) + \vec{1}_\theta \frac{1}{\sin(\theta)} \frac{\partial}{\partial\theta} Y_{n,m,\sigma}(\theta,\phi)$$
$$\vec{R}_{n,m,\sigma}(\theta,\phi) \equiv a\nabla_s \times \left[\vec{1}_r(\theta,\phi) Y_{n,m,\sigma}(\theta,\phi)\right] \tag{A.5}$$
$$= \vec{1}_\theta \frac{1}{\sin(\theta)} \frac{\partial}{\partial\theta} Y_{n,m,\sigma}(\theta,\phi) - \vec{1}_\theta \frac{\partial}{\partial\theta} Y_{n,m,\sigma}(\theta,\phi)$$
$$\vec{Q}_{n,m,\sigma}(\theta,\phi) = \vec{1}_r(\theta,\phi) \times \vec{R}_{n,m,\sigma}(\theta,\phi)$$
$$\vec{R}_{n,m,\sigma}(\theta,\phi) = -\vec{1}_r(\theta,\phi) \times \vec{Q}_{n,m,\sigma}(\theta,\phi)$$

Note that $\sigma = 1, 2$ corresponds to the choice of $\cos(m\phi)$ or $\sin(m\phi)$ in the first equation in (A.5). From [1] we now have the normalization coefficients

$$d_{e,n,m,\sigma} = d_{h,n,m,\sigma}$$
$$= \left\{\left[1 + \left[1_{1,\sigma} - 1_{2,\sigma}\right] 1_{0,m}\right] 2\pi a^2 \frac{n(n+1)}{2n+1} \frac{(n+m)!}{(n-m)!}\right\}^{-\frac{1}{2}} \tag{A.6}$$

The bessel functions are

$$k_n(\gamma a) = \frac{e^{-\gamma a}}{\gamma a} \sum_{n'=0}^{n} \frac{(n+n')!}{n'!(n-n')!} (2\gamma a)^{-n'}$$
$$i_n(\gamma a) = \frac{1}{2}\left\{-k_n(-\gamma a) + (-1)^{n+1} k_n(\gamma a)\right\} \tag{A.7}$$

From these we construct the normalized eigenimpedances in (A.1).

Considering the E modes we have for low frequencies

$$\tilde{z}_{e,n}(s) = -\left[\gamma a\, i_n(\gamma a)\right]' \left[\gamma a\, k_n(\gamma a)\right]'$$
$$= -\left[\frac{n+1}{(2n+1)!!}(\gamma a)^n \left[1 + O\left((\gamma a)^2\right)\right]\right]\left[-n(2n-1)!!(\gamma a)^{-n-1}\left[1 + O\left((\gamma a)^2\right)\right]\right]$$
$$= \frac{n(n+1)}{2n+1}(\gamma a)^{-1}\left[1 + O\left((\gamma a)^2\right)\right] \text{ as } \gamma a \to 0 \tag{A.8}$$

This is capacitive with

$$Z_o \tilde{z}_{e,\,n}(s) = \frac{1}{sC_n}\left[1 + O\left((\gamma a)^2\right)\right]$$

$$C_n = \frac{2n+1}{n(n+1)}\varepsilon_o a \tag{A.9}$$

For high frequencies we have in the left half s plane

$$\tilde{z}_{e,\,n}(s) = \frac{(-1)^n}{2}e^{-2\gamma a}\left[1 + O\left((\gamma a)^{-1}\right)\right] \text{ as } \mathrm{Re}[\gamma a] \to -\infty \tag{A.10}$$

In the right half plane we have

$$\tilde{z}_{e,\,n}(s) = \frac{1}{2}\left[1 + O\left((\gamma a)^{-1}\right)\right] \text{ as } \mathrm{Re}[\gamma a] \to +\infty \tag{A.11}$$

Unnormalized this last result is resistive with value $Z_o/2$.

Considering the H modes we have for low frequencies

$$\tilde{z}_{h,\,n}(s) = [\gamma a\, i_n(\gamma a)][\gamma a k_n(\gamma a)]$$

$$= \left[\frac{(\gamma a)^{n+1}}{(2n+1)!!}\left[1 + O\left((\gamma a)^2\right)\right]\right]\left[(2n-1)!!(\gamma a)^{-n}\left[1 + O\left((\gamma a)^2\right)\right]\right]$$

$$= \frac{1}{2n+1}(\gamma a)\left[1 + O\left((\gamma a)^2\right)\right] \text{ as } \gamma a \to 0 \tag{A.12}$$

This is inductive with

$$Z_o \tilde{z}_{h,\,n}(s) = sL_n\left[1 + O\left((\gamma a)^2\right)\right] \text{ as } \gamma a \to 0$$

$$L_n = \frac{\mu_o a}{2n+1} \tag{A.13}$$

For high frequencies we have in the left half s plane

$$\tilde{z}_{h,\,n}(s) = \frac{(-1)^{n+1}}{2}e^{-2\gamma a}\left[1 + O\left((\gamma a)^{-1}\right)\right] \text{ as } \mathrm{Re}[\gamma a] \to -\infty \tag{A.14}$$

In the right half plane we have

$$\tilde{z}_{h,\,n}(s) = \frac{1}{2}\left[1 + O\left((\gamma a)^{-1}\right)\right] \text{ as } \mathrm{Re}[\gamma a] \to +\infty \tag{A.15}$$

This is also resistive with unnormalized value $Z_o/2$.

The coupling term for E modes (in normalized form as in (5.2)) is

$$\left\langle \vec{\vec{j}}_{s_{e,n,m,\sigma}}\left(\vec{r}_s\, , s\right) ;\ \vec{\vec{e}}^{(inc)}\left(\vec{r}_s\, , s\right)\right\rangle = d^{-1}_{e,n,m,\sigma}\frac{\left[\gamma a\, i_n(\gamma a)\right]'}{\gamma a}A_{2,n,m,\sigma,p} \tag{A.16}$$

where $A_{2,n,m,\sigma,p}$ is a function of the indices and the direction of incidence and polarization of the incident wave, but is independent of frequency. For low frequencies

$$\frac{\left[\gamma a\, i_n(\gamma a)\right]'}{\gamma a} = \frac{n+1}{(2n+1)!!}(\gamma a)^{n-1}\left[1 + O\left((\gamma a)^2\right)\right] \text{ as } \gamma a \to 0 \tag{A.17}$$

which goes to a constant for $n = 1$, but to zero for $n > 1$. For high frequencies we have in the left half s plane

$$\frac{\left[\gamma a\, i_n(\gamma a)\right]'}{\gamma a} = (-1)^n \frac{e^{-\gamma a}}{2\gamma a}\left[1 + O\left((\gamma a)^{-1}\right)\right] \text{ as } \mathrm{Re}[\gamma a] \to -\infty \tag{A.18}$$

In the right half s plane we have

$$\frac{\left[\gamma a\, i_n(\gamma a)\right]'}{\gamma a} = \frac{e^{\gamma a}}{2\gamma a}\left[1 + O\left((\gamma a)^{-1}\right)\right] \text{as } \mathrm{Re}[\gamma a] \to +\infty \tag{A.19}$$

One can see here the symmetric form in the last two equations.

The coupling term for H modes (in normalized form as in (5.2)) is

$$\left\langle \vec{\vec{j}}_{s_{h,n,m,\sigma}}\left(\vec{r}_s\, , s\right) ;\ \vec{\vec{e}}^{(inc)}\left(\vec{r}_s\, , s\right)\right\rangle = d^{-1}_{h,n,m,\sigma}\, i_n(\gamma a) A_{1,n,m,\sigma} \tag{A.20}$$

where $A_{1,n,m,\sigma}$ is again independent of frequency. For low frequencies

$$\begin{aligned} i_n(\gamma a) &= \frac{(\gamma a)^n}{(2n+1)!!}\left[1 + O\left((\gamma a)^2\right)\right] \text{ as } \gamma a \to 0 \\ &\to 0 \text{ as } s \to 0 \text{ for } n = 1,2,\ldots \end{aligned} \tag{A.21}$$

For high frequencies we have in the left half s plane

$$i_n(\gamma a) = (-1)^{n+1}\frac{e^{-\gamma a}}{2\gamma a}\left[1 + O\left((\gamma a)^{-1}\right)\right] \text{ as } \mathrm{Re}[\gamma a] \to -\infty \tag{A.22}$$

In the right half s plane we have

$$i_n(\gamma a) = \frac{e^{\gamma a}}{2\gamma a}\left[1 + O\left((\gamma a)^{-1}\right)\right] \text{ as } \mathrm{Re}[\gamma a] \to +\infty \tag{A.23}$$

The eigenterm for E modes is

$$\tilde{Z}_{e,n}^{-1}(s) \left\langle \vec{\vec{j}}_{s_\beta}\left(\vec{r}_s, s\right) ; \vec{\vec{E}}_t^{(inc)}\left(\vec{r}_s, s\right)\right\rangle$$

$$= - d_{e,n,m,\sigma}^{-1} \; \vec{\vec{j}}_{s_{e,n,m,\sigma}}\left(\vec{r}_s, s\right)\left[\gamma a\left[\gamma a \, k_n(\gamma a)\right]'\right]^{-1} A_{2,n,m,\sigma,p} \tag{A.24}$$

At high frequencies the combination of coupling term and eigenimpedance gives interesting results. In the left and right half s planes we have

$$-\left[\gamma a\left[\gamma a \, k_n(\gamma a)\right]'\right]^{-1} = \frac{e^{\gamma a}}{\gamma a}\left[1 + O\left((\gamma a)^{-1}\right)\right] \text{ as } \mathrm{Re}[\gamma a] \to \pm\infty \tag{A.25}$$

If one multiplies the response by $e^{\gamma a}$ to give a time shift the eigenterms are well behaved in both half planes and only have pole singularities.

The eigenterm for H modes is

$$\tilde{Z}_{h,n}^{-1}(s) \left\langle \vec{\vec{j}}_{s_h}\left(\vec{r}_s, s\right) ; \vec{\vec{E}}^{(inc)}\left(\vec{r}_s, s\right)\right\rangle$$

$$= d_{h,n,m,\sigma}^{-1} \; \vec{\vec{j}}_{s_{h,n,m,\sigma}}\left(\vec{r}_s, s\right)\left[(\gamma a)^2 k_n(\gamma a)\right]^{-1} A_{1,n,m,\sigma,p} \tag{A.26}$$

In the left and right half s planes we have

$$\left[(\gamma a)^2 k_n(\gamma a)\right]^{-1} = \frac{e^{\gamma a}}{\gamma a}\left[1 + O\left((\gamma a)^{-1}\right)\right] \text{ as } \mathrm{Re}[\gamma a] \to \pm\infty \tag{A.27}$$

This is the same form as for the E modes and the same comments apply.

Appendix B. H-Field-Integral-Equation EEM Parameters for the Perfectly Conducting Sphere

As discussed in [2,5] the various EEM parameters for a perfectly conducting sphere of radius a are summarized for the H-field integral equation (pseudosymmetric) (2.22). The eigenimpedances are

$$\tilde{\lambda}_{e,n}^{(H)}(s) = -\left[\gamma a i_n(\gamma a)\right]\left[\gamma a k_n(\gamma a)\right]'$$

$$\tilde{\lambda}_{h,n}^{(H)(s)}(s) = -\left[\gamma a i_n(\gamma a)\right]'\left[\gamma a k_n(\gamma a)\right] \qquad \text{(B.1)}$$

$$n = 1,2,...$$

Note that these modes also separate into E and H modes indicated by the first subscripts. These modes are the same as those for the E equation in (A.2).

Considering the E modes we have for low frequencies

$$\tilde{\lambda}_{e,n}^{(H)}(s) = -\left[\gamma a i_n(\gamma a)\right]\left[\gamma a k_n(\gamma a)\right]'$$

$$= -\left[\frac{(\gamma a)^{n+1}}{(2n+1)!!}\left[1 + O\left((\gamma a)^2\right)\right]\right]\left[-n(2n-1)!!(\gamma a)^{-n-1}\left[1 + O\left((\gamma a)^2\right)\right]\right]$$

$$= \frac{n}{(2n+1)}\left[1 + O\left((\gamma a)^2\right)\right] \text{ as } \gamma a \to 0 \qquad \text{(B.2)}$$

This is not an impedance as it represents a relation between the response surface current density and the incident magnetic field as in (2.22). For high frequencies we have in the left half s plane

$$\tilde{\lambda}_{e,n}^{(H)}(s) = \frac{(-1)^{n+1}}{2} e^{-2\gamma a}\left[1 + O\left((\gamma a)^{-1}\right)\right] \text{ as } \mathrm{Re}[\gamma a] \to -\infty \qquad \text{(B.3)}$$

In the right half s plane we have

$$\tilde{\lambda}_{e,n}^{(H)}(s) = \frac{1}{2}\left[1 + O\left((\gamma a)^{-1}\right)\right] \text{ as } \mathrm{Re}[\gamma a] \to +\infty \qquad \text{(B.4)}$$

Considering the H modes we have for low frequencies

$$\tilde{\lambda}_{h,n}^{(H)}(s) = \left[\gamma a i_n(\gamma a)\right]'\left[\gamma a k_n(\gamma a)\right]$$

$$= \left[\frac{n+1}{(2n+1)!!}(\gamma a)^n\left[1 + O\left((\gamma a)^2\right)\right]\right]\left[(2n-1)!!(\gamma a)^{-n}\left[1 + O\left((\gamma a)^2\right)\right]\right]$$

$$= \frac{n+1}{(2n+1)}\left[1 + O\left((\gamma a)^2\right)\right] \text{ as } \gamma a \to 0 \qquad \text{(B.5)}$$

For high frequencies we have in the left half s plane

$$\tilde{\lambda}_{h,n}^{(H)}(s) = \frac{(-1)^n}{2} e^{-2\gamma a}\left[1 + O\left((\gamma a)^{-1}\right)\right] \text{as } \mathrm{Re}[\gamma a] \to -\infty \tag{B.6}$$

In the right half s plane we have

$$\tilde{\lambda}_{h,n}^{(H)}(s) = \frac{1}{2}\left[1 + O\left((\gamma a)^{-1}\right)\right] \text{as } \mathrm{Re}[\gamma a] \to +\infty \tag{B.7}$$

Since the eigenmodes of the H equation are the same as for the E equation and since the solution must be unique, the coefficients of the eigenmodes must be the same as in (A.24) and (A.26). The asymptotic expansions are the same as are the associated conclusions. while the coupling terms are different for the H modes they can be related to those for the E modes (as in (A.16) and (A.20)) by a ratio of H equation eigenvalues to the E equation eigenvalues.

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