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Splitting of Degenerate Natural Frequencies in Coupled Two-Conductor Lines by Distance Variation

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ABSTRACT

Natural frequencies of strongly coupled cables do not only experience a shift from their unperturbed values but also experience splitting due to the resolution of degenerate natural modes. There are various physical mechanisms associated with frequency splitting. In this paper the influence of distance variation between cables over an infinitely conducting plane on the natural frequencies is investigated. Applying perturbation theory on the transmission line equations for lossless, identical, and non-parallel lines the differential and common modes of a simple two conductor system are determined. The highly symmetrical system under consideration enforces a perturbation expansion up to second order in the perturbation parameter θ. The trajectories of the (new) natural frequencies are straight lines on the imaginary axis of the complex frequency plane.

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I. Introduction

Innumerable measurements of currents on single cables as well as on cable bundles in simple and complex systems have resulted in an unexpectedly complicated frequency pattern, reflecting more the interaction between various subsystems rather than showing up in simple resonance peaks according to the geometry of the conductors. There have been many encouraging attempts [2,8,9,10] to explain this behavior, but there are still several questions without satisfactory answers. Especially, the different physical mechanisms which may induce the resolution of degenerate natural modes of complex systems with certain symmetries have not extensively been discussed in literature. In [9] it has been shown that coupling between the conductors together with their (natural) properties as (e.g.) internal inductance and resistance lead to frequency splitting.

It is the purpose of this paper to present another physical mechanism associated with natural frequency splitting: Coupling between conductors and space dependency of the mutual impedances and admittances of the interacting cables. This is done on the basis of transmission line theory and some kind of perturbation analysis often used in quantum mechanics. Thereby approximate perturbation-formulae for the splitting of natural frequencies are derived. Moreover, this paper gives an additional (non-trivial and analytical) example of the peculiar behavior of SEM-pole parameters [2] in the complex plane of natural frequencies.

The outline of our paper is as follows: In Section II we start with the description of two uniform and symmetrical conductor transmission lines over a perfectly conducting ground plane. In Section III we extend the former description to non-uniform conductor lines and provide the space dependent elements of the per-unit-length impedance- and admittance-matrices. In Section IV we insert these matrices into the non-uniform transmission line equation for the current vector and subsequently scalarize this equation. For the two-conductor system under consideration, there is mainly one function which is needed to express local dependent coupling and which enters the perturbation ansatz for the current modes. This function is subjected to a perturbation expansion in Section V. The most important section is Section VI. Here we systematically apply perturbation theory on the scalar transmission-line equations. The solutions for the perturbation of the natural frequencies as well as those for the corresponding natural modes are obtained in the successive (power-) order of the expansion parameter θ. A numerical example is presented to illustrate the ideas and relationships proffered in this and in the previous sections. Finally, in Section VII, we discuss the results of our paper and draw some conclusions.

II. Two Uniform and Symmetrical Conductor Transmission Lines above a Perfectly Conducting Ground Plane

One of the simpler models to apply transmission line theory is the two-conductor plus reference line in a homogeneous, lossless medium like free space. We consider two identical, lossless, parallel, and open ended conductors tightened over a perfectly conducting ground plane which in our case is taken as reference conductor. The lines are uniform and of length ℓ. For this situation, the homogeneous transmission line equation for the current vector reads [4,7]:

$$\frac{d^2}{dx^2}\left(I_n(x)\right)_0 - \left(\frac{s}{c}\right)_0^2 \left(I_n(x)\right)_0 = \left(0_n\right) \tag{2.1}$$

$(\gamma := (s/c)$ propagation constant$)$

Note that the simplicity of this equation essentially results from the mutual reciprocity of the impedance and admittance per-unit-length matrices.

$$\left(Z'_{n,m}\right) = s\left(L'_{n,m}\right) = s\,\mu_o\left(f^{(0)}_{g_{n,m}}\right) \tag{2.2}$$

$$\left(Y'_{n,m}\right) = s\left(C'_{n,m}\right) = s\,\varepsilon_o\left(f^{(0)}_{g_{n,m}}\right)^{-1} \tag{2.3}$$

with

$$\left(C'_{n,m}\right)\cdot\left(L'_{n,m}\right) = \varepsilon_o\,\mu_o\left(1_{n,m}\right) \tag{2.4}$$

and

$$\varepsilon_o\,\mu_o = c^{-2} \tag{2.5}$$

$(c = \text{speed of light} \cong 3 \cdot 10^8\ \frac{m}{s})$

The matrix elements of $(L'_{n,m})$ and $(C'_{n,m})$ are of purely geometrical nature and are given by [1]

$$f^{(0)}_{g_{1,1}} = f^{(0)}_{g_{2,2}} = \frac{1}{2\pi}\,\ell n\left(\frac{2h}{r}\right)\ ,\ \frac{r}{h} \ll 1\ ,\ \frac{r}{a_o} \ll 1 \tag{2.6}$$

and

$$f^{(0)}_{g_{1,2}} = f^{(0)}_{g_{2,1}} = \frac{1}{4\pi} \ln \left(1 + \left(\frac{h}{a_o}\right)^2\right) , \frac{r}{h} << 1 , \frac{r}{a_o} << 1 \tag{2.7}$$

Here h denotes the height of the wires above the ground plane, r their radius, and $2a_o$ is their constant spacing.

Assuming the transmission line is of length ℓ with coordinate x limited by

$$-\frac{\ell}{2} \leq x \leq \frac{\ell}{2} \tag{2.8}$$

with boundary conditions at $x = \pm (\ell/2)$ of

$$\left(I_n \left(\frac{\ell}{2}\right)\right)_0 = \left(I_n \left(-\frac{\ell}{2}\right)\right)_0 = \left(0_n\right) \tag{2.9}$$

then the homogeneous solution of (2.1) is easily found to be

$$\left(I_n\right)_{m,0} = \frac{1}{\sqrt{\ell}} \cdot \begin{cases} \cos\left(m \frac{\pi x}{\ell}\right) \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} , \text{ if } m=1,3,5,\ldots\text{(odd)} \\ \sin\left(m \frac{\pi x}{\ell}\right) \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} , \text{ if } m=2,4,6,\ldots\text{(even)} \end{cases}$$

$$s_{m,0} = j \frac{m\pi c}{\ell} \tag{2.10}$$

To normalize the current vector we take

$$\int_{-\frac{\ell}{2}}^{\ell/2} dx < \left(I_n(x)\right)_0 ; \left(I_n(x)\right)_0 > = 1 \tag{2.11}$$

For that what follows we restrict ourselves on the dominant ground mode (m=1) solution of (2.10) and use the notation

$$\left(I_n\right)_0 : = \left(I_n\right)_{1,0} = \frac{1}{\sqrt{\ell}} \cos\left(\frac{\pi x}{\ell}\right) \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

$$s_0 : = s_{1,0} = j \frac{\pi c}{\ell} \tag{2.12}$$

The vectors $\binom{1}{1}$ and $\binom{1}{-1}$ refer to the symmetric (common) and antisymmetric (differential) natural modes, respectively.

Observe that already the solutions of the homogeneous equation (2.1) yield the natural frequencies and the natural modes. These in turn form the basis functions necessary to express an induced (e.g., by an incident electromagnetic field) surface current on the wires.

III. Nonuniform Two-Conductor Lines

We now consider two straight wires (open ended) above a perfectly conducting plane which symmetrically diverge from each other with respect to a symmetry plane P (see Figure 1). The wire axes include an angle of θ with the plane P of symmetry (x-y plane), and their respective centers are at $z = \pm a_o$. More precisely, we have a second symmetry plane which is the conducting plane itself. These kinds of symmetry are also referred to as reflection (mirror) symmetry which have the related reflection dyads [2,3,5]

$$\overleftrightarrow{R}_v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } \overleftrightarrow{R}_h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.1}$$

($v \equiv$ vertical , $h \equiv$ horizontal)

However, in all what follows we only refer to the P-symmetry, and the decomposition of the currents and frequencies into symmetric (subscript sy) and antisymmetric (subscript as) parts is performed with respect to P.

For the experimental arrangement of Figure 1, the inductance- and capacitance-matrices per unit length are still symmetric and real but no longer inverse to each other. They become space dependent along the lines and explicitly read:

$$\left(L'_{n,m}\right) = \mu_o \left\{ f_{g_{1,1}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + f_{g_{1,2}} \left(x_{\frac{1}{2}}, \theta\right) \cos^2(\theta) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \tag{3.2}$$

and

$$\left(C'_{n,m}\right) = \varepsilon_o \, \Delta^{-1}\left(x_{\frac{1}{2}}, \theta\right) \; f_{g_{1,1}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - f_{g_{1,2}}\left(x_{\frac{1}{2}}, \theta\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.3}$$

with

$$\Delta\left(x_{\frac{1}{2}}, \theta\right) \equiv \det\left(f_{g_{n,m}}\right) = f^2_{g_{1,1}} - f^2_{g_{1,2}}\left(x_{\frac{1}{2}}, \theta\right) \tag{3.4}$$

Observe that the $(C'_{m,m})$ - matrix conserved its form-structure (compared to that one for uniform lines) whereas in the $(L'_{n,m})$ - matrix an additional $\cos^2(\theta)$ - function appears. This function arises because of the modification

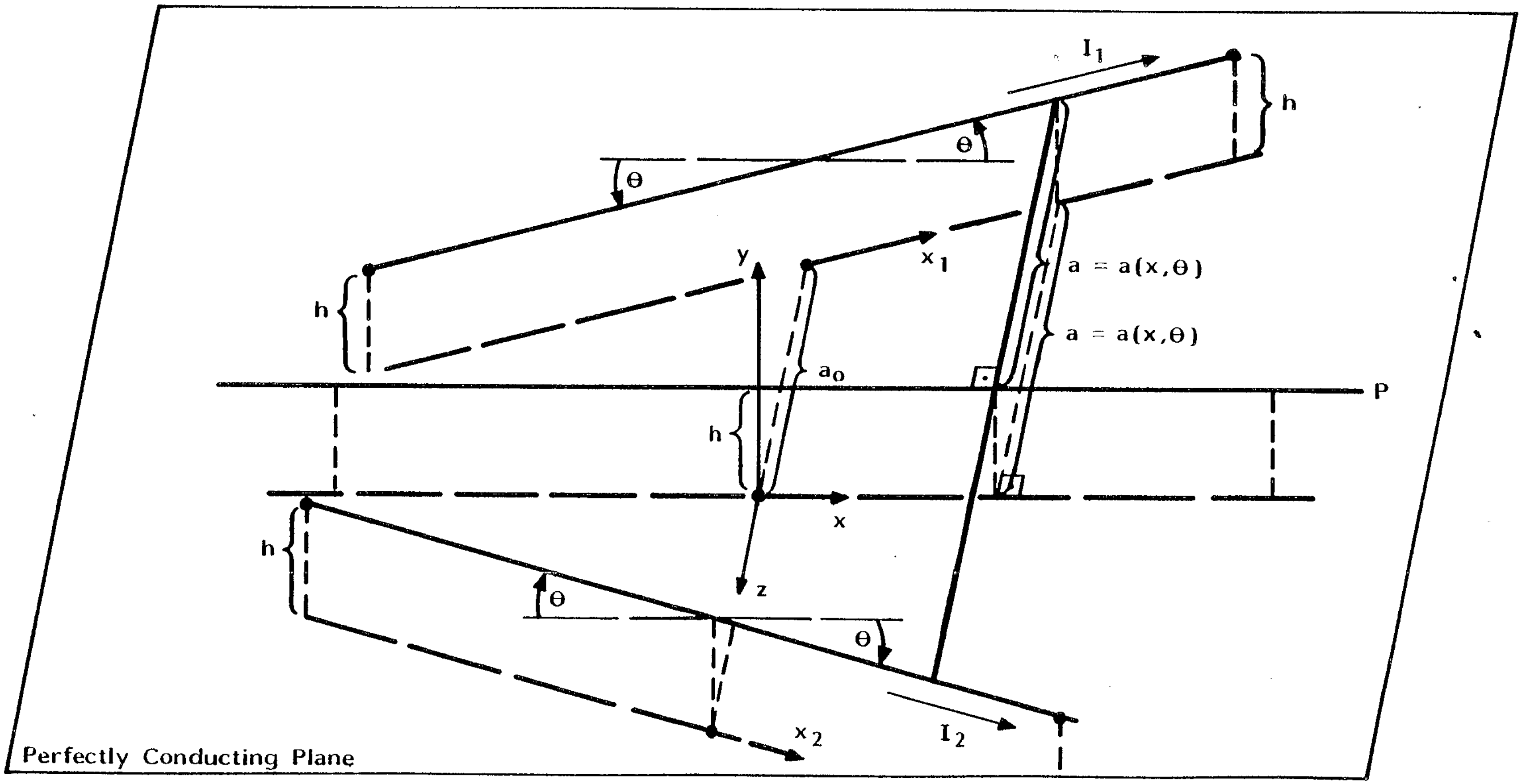

Figure 1. Two thin non-parallel conductors above an infinitely conducting ground plane.

of the interaction between the local infinitesimal magnetic dipole elements of the wires. Consider an elementary length of wire

$$\Delta x_{\frac{1}{2}} = \Delta x/\cos(\theta) \tag{3.5}$$

An elementary current, say at x_1 on wire 1 has the magnetic field reduced for $x_2 = x_1$ on wire 2 by a factor $\cos(\theta)$. The orientation of wire 2 further reduces the magnetic flux coupled to wire 2 by the same factor $\cos(\theta)$, for an overall coupling reduction of $\cos^2(\theta)$.

The wire orientation angle θ induces another change. If the overall length of the wire is fixed as ℓ $\left(\underline{\text{not}}\ \ell/\cos(\theta)\right)$, then x extends as

$$-\frac{\ell}{2}\cos(\theta) \leq x \leq \frac{\ell}{2}\cos(\theta) \tag{3.6}$$

while

$$-\frac{\ell}{2} < x_{\frac{1}{2}} < \frac{\ell}{2} \tag{3.7}$$

Now assuming most of the energy is confined to near the wires, or equivalently

$$f_{g_{1,1}} >> f_{g_{1,2}} > 0 \tag{3.8}$$

then propagation is better considered to be along the coordinate x_1 (or x_2). Note that if the second wire is removed, then on the first wire it does not matter what value of θ is used (by symmetry); the unperturbed natural frequency and mode of Section II still applies. However, with the inclusion of mutual coupling this is only an approximation.

In what follows x_1 and x_2 are replaced by the parameter x for simplicity. This is not the old x (slightly different), but does have the simple boundaries as in (3.7).

Since the orientation of a single conductor above a conducting plane has no influence on its external inductance per unit length, we have

$$f_{g_{1,1}} = f^{(0)}_{g_{1,1}} = (2\pi)^{-1}\ \ell n\left(\frac{2h}{r}\right) \tag{3.9}$$

However, the function $f_{g_{1,2}}$ depends on the spacing between the lines:

$$f_{g_{1,2}} \equiv f_{g_{1,2}}(x,\theta) = (4\pi)^{-1}\ \ell n\left(1 + \left(\frac{h}{a}\right)^2\right) \tag{3.10}$$

with

$$a \equiv a(x,\theta) = a_o + x \cdot \sin(\theta) \tag{3.11}$$

For that what follows it turns out to be convenient and useful to introduce the coupling function $\kappa = \kappa\ (x,\theta)$ defined by

$$\kappa \equiv \kappa(x,\theta) = \frac{f_{g_{1,2}}(x,\theta)}{f_{g_{1,1}}} = \frac{\ell n\left(1 + \left(\frac{h}{a}\right)^2\right)}{2\ \ell n\left(\frac{2h}{r}\right)} \tag{3.12}$$

It will be shown that all results derived in the subsequent sections can essentially be expressed in terms of κ and some of its derivatives with respect to x and θ.

IV. Scalarization of the Transmission Line Equations

For the two lines described in Section III, the transmission line equations get a more complicated structure than in (2.1). Due to the space dependency of the impedance - and admittance-matrix elements we now obtain an additional term which contains the first derivatives with respect to x of the current vector (I_n) and of the admittance matrix $(Y'_{n,m})$. We start with the transmission line equations [4,7] without sources

$$\frac{d}{dx}(V_n) = -(Z'_{n,m}) \cdot (I_n) \tag{4.1}$$

$$\frac{d}{dx}(I_n) = -(Y'_{n,m}) \cdot (V_n) \tag{4.2}$$

Resolve (4.2) with respect to (V_n) and replace this quantity in (4.1) giving

$$\frac{d}{dx}\left((Y'_{n,m})^{-1} \cdot \frac{d}{dx}(I_n)\right) = (Z'_{n,m}) \cdot (I_n) \tag{4.3}$$

Applying the chain rule of differentiation on (4.3) we obtain

$$\frac{d}{dx}\left((Y'_{n,m})^{-1}\right) \cdot \left(\frac{d}{dx}(I_n)\right) + (Y'_{n,m})^{-1} \cdot \frac{d^2}{dx^2}(I_n) = (Z'_{n,m}) \cdot (I_n) \tag{4.4}$$

Dot multiply this equation from the left with the matrix $(Y'_{n,m})$ and finally get

$$\frac{d^2}{dx^2}(I_n) + (Y'_{n,m}) \cdot \left(\frac{d}{dx}(Y'_{n,m})^{-1}\right) \cdot \frac{d}{dx}(I_n) - (Y'_{n,m}) \cdot (Z'_{n,m}) \cdot (I_n) = (0_n) \tag{4.5}$$

Assuming again the wires to be identical, lossless and embedded in a lossless, homogeneous medium (4.5) simplifies to

$$\frac{d^2}{dx^2}(I_n) + (C'_{n,m}) \cdot \left(\frac{d}{dx}(C'_{n,m})^{-1}\right) \cdot \frac{d}{dx}(I_n) - s^2(C'_{n,m}) \cdot (L'_{n,m})(I_n) = (0_n) \tag{4.6}$$

with the matrices $(L'_{n,m})$ and $(C'_{n,m})$ given by (3.2) and (3.3), respectively. Performing explicitly the matrix multiplication $(C'_{n,m}) \cdot (L'_{n,m})$ with formulae (3.2) and (3.3), we arrive after some simple algebraic manipulations at the result

$$\left(C'_{n,m}\right) \cdot \left(L'_{n,m}\right) = \frac{\varepsilon_o \mu_o}{\left[1-\kappa^2(x,\theta)\right]} \begin{pmatrix} 1-\kappa^2(x,\theta)\cos^2(\theta) & -\kappa(x,\theta)\sin^2(\theta) \\ -\kappa(x,\theta)\sin^2(\theta) & 1-\kappa^2(x,\theta)\cos^2(\theta) \end{pmatrix} \tag{4.7}$$

In order to obtain the product $\left(C'_{n,m}\right) \cdot \frac{d}{dx}\left(C'_{n,m}\right)^{-1}$ in terms of κ and its derivatives we recall the geometric-impedance-factor matrix from Section III as

$$\mu_o \left(f_{g_{n,m}}\right) = \mu_o \, f_{g_{1,1}} \begin{pmatrix} 1 & \kappa(x,\theta) \\ \kappa(x,\theta) & 1 \end{pmatrix} \tag{4.8}$$

and observe that

$$\left(C'_{n,m}\right) = \varepsilon_o \left(f_{g_{n,m}}\right)^{-1} \tag{4.9}$$

With the aid of (3.3) we then easily can calculate the product

$$\left(C'_{n,m}\right) \cdot \left(\frac{d}{dx}\left(C'_{n,m}\right)^{-1}\right) = \frac{-\left(\frac{\partial \kappa(x,\theta)}{\partial x}\right)}{\left[1-\kappa^2(x,\theta)\right]} \begin{pmatrix} \kappa(x,\theta) & -1 \\ -1 & \kappa(x,\theta) \end{pmatrix} \tag{4.10}$$

With the expressions (4.7) and (4.10) for the matrix products we rewrite the field equation (4.6):

$$\frac{d^2}{dx^2}\left(I_n\right) - \frac{\frac{\partial \kappa(x,\theta)}{\partial x}}{\left[1-\kappa^2(x,\theta)\right]} \begin{pmatrix} \kappa(x,\theta) & -1 \\ -1 & \kappa(x,\theta) \end{pmatrix} \cdot \left(\frac{d}{dx}\left(I_n\right)\right)$$

$$- \frac{s^2}{c^2} \frac{1}{\left[1-\kappa^2(x,\theta)\right]} \begin{pmatrix} 1-\kappa^2(x,\theta)\cos^2(\theta) & -\kappa(x,\theta)\sin^2(\theta) \\ -\kappa(x,\theta)\sin^2(\theta) & 1-\kappa^2(x,\theta)\cos^2(\theta) \end{pmatrix} \cdot \left(I_n\right) = \left(0_n\right) \tag{4.11}$$

This is a second order common differential equation with known space-dependent matrices which has to be solved with appropriate boundary conditions for the current vector (I_n). Since we are interested in analytical rather than numerical solutions we shall try to find solutions of (4.11) by use of perturbation-theoretical approximations. But before we apply perturbation theory on equation (4.11) we reformulate this equation taking into account the symmetry of our two-wire problem mentioned in Section III. Due to this symmetry, it is convenient and physically reasonable to decompose the current vector (I_n) into two parts, designated symmetric (sy ≡ common mode) and antisymmetric (as ≡ differential mode). We define

$$\begin{pmatrix} I_n \end{pmatrix}_\chi \equiv I_\chi \begin{pmatrix} 1 \\ \chi \end{pmatrix} \tag{4.12}$$

where the parameter χ is introduced to indicate symmetric or antisymmetric current modes by the following choice:

$$\chi = \begin{cases} +1 & \text{symmetric (or common) mode} \\ -1 & \text{antisymmetric (or differential) mode} \end{cases} \tag{4.13}$$

Insert equation (4.12) into (4.11), apply the matrices on the vector $\begin{pmatrix} 1 \\ \chi \end{pmatrix}$ and find

$$\left[\frac{d^2}{dx^2} I_\chi + \chi \cdot \frac{\left(\frac{\partial\kappa(x,\theta)}{\partial x}\right)}{[1+\chi\kappa(x,\theta)]} \left(\frac{d}{dx} I_\chi\right) - \frac{s^2}{c^2} \frac{[1+\chi\kappa(x,\theta)\cos^2(\theta)]}{[1+\chi\kappa(x,\theta)]} I_\chi\right] \cdot \begin{pmatrix} 1 \\ \chi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.14}$$

Thus from (4.14) we immediately read off the scalar equation for I_χ:

$$\frac{d^2}{dx^2} I_\chi + \chi \frac{\left(\frac{\partial\kappa(x,\theta)}{\partial x}\right)}{[1+\chi\kappa(x,\theta)]} \left(\frac{d}{dx} I_\chi\right) - \frac{s^2}{c^2} \frac{[1+\chi\kappa(x,\theta)\cos^2(\theta)]}{[1+\chi\kappa(x,\theta)]} I_\chi = 0 \tag{4.15}$$

Equation (4.15) can analytically be solved at the same time for both scalar modes

$$I_{\substack{1 \\ -1}} \equiv I_{\substack{sy \\ as}} \equiv I_{\substack{c.m. \\ d.m.}} \tag{4.16}$$

We are looking for solutions of (4.15) by using successively different orders of a perturbation-theoretical approach. For this purpose we assume that we deal with analytic functions which allow a Taylor-series-like expansion with respect to the perturbation parameter θ. Indeed, the angle θ occurring in the equations has to be understood as a given (i.e., fixed but arbitrary) parameter rather than as an independent variable. Nevertheless, in order to indicate derivatives uniquely, we frequently choose the partial derivative sign instead of the common one. In addition, in the following perturbation ansaetze for the quantities I_χ, κ, $(\partial\kappa/\partial x)$, and s_χ the angle θ is considered to be a continuous parameter, and we assume the derivatives of these quantities with respect to θ to exist up to a certain order. We write for $\theta \to 0$

$$I_\chi = \sum_{p=0}^{p_0} \frac{1}{p!} I_\chi^{(p)} \theta^p + O\left(\theta^{p_0+1}\right) \tag{4.17}$$

$$\kappa(x,\theta) = \sum_{k=0}^{k_0} \frac{1}{k!} \kappa^{(k)} \theta^{(k)} + O\left(\theta^{k_0+1}\right) \tag{4.18}$$

$$\left(\frac{\partial\kappa(x,\theta)}{\partial x}\right) = \sum_{\ell=0}^{\ell_0} \frac{1}{\ell!} \left(\frac{\partial\kappa(x,\theta)}{\partial x}\right)^{(\ell)} \theta^\ell + O\left(\theta^{\ell_0+1}\right) \tag{4.19}$$

The natural frequencies which are associated with the natural modes (I_χ) are designated by s_χ. They are approximated by the following expansion

$$s_\chi = \sum_{m=0}^{m_0,2} \frac{1}{m!} s_\chi^{(m)} \theta^{(m)} + O\left(\theta^{m_0+2}\right) \tag{4.20}$$

The number 2 on top of the sum sign indicates that only even numbers for m needed to be chosen. The lowest order quantity $s_\chi^{(0)}$ for both modes $\chi = \pm 1$ is just given by

$$s_\chi^{(0)} \equiv s_0 = j\,\frac{\pi c}{\ell} \tag{4.21}$$

All the perturbation contributions to s_χ which carry an odd upper index number, like e.g. $s_\chi^{(1)}$ and $s_\chi^{(3)}$, do not appear in (4.20) due to the symmetry of the two-wire arrangement. The solutions for

$$\Delta s_\chi^{(m)} \equiv \frac{1}{m!} s_\chi^{(m)} \theta^m \tag{4.22}$$

have to be even (i.e. m=2n; n=1,2,3,...) in θ because a change of the angle from $\theta \to (-\theta)$ does not alter the physics of our system. In addition, the result $\Delta s_\chi^{(1)} = 0$ very well fits into that obtained in [2] for the numerical example.

There it turned out that the so-called image coefficient $\nu_{1,1}$ varies approximately with the inverse of $2\ln(\ell/r)$ and approaches zero (and thereby also $\Delta s_\chi^{(1)}$) for very thin scatterers.

We finally conclude this section with a remark concerning the normalization of the current vectors in their successive approximations. We start with (compare (2.12)) the lowest order approximation

$$I_\chi^{(0)} \equiv \frac{1}{\sqrt{\ell}} \cos\left(\frac{\pi x}{\ell}\right) \tag{4.23}$$

which fulfills the condition (2.11)

$$\int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} dx < I_\chi^{(0)} \begin{pmatrix} 1 \\ \chi \end{pmatrix}; I_\chi^{(0)} \begin{pmatrix} 1 \\ \chi \end{pmatrix} > = 1 \; ; \quad \int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} dx \, I_\chi^{(0)} \, I_\chi^{(0)} = \frac{1}{2} \tag{4.24}$$

Now, for the higher order approximations we require the normalization condition to be fulfilled up to the considered order. Deviations from unity may occur in the next higher order, i.e.

$$\int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} dx < \left(I_\chi^{(0)} + I_\chi^{(1)}\theta + \ldots + \frac{1}{n!} I_\chi^{(n)}\theta^n\right)\begin{pmatrix} 1 \\ \chi \end{pmatrix} ; \left(I_\chi^{(0)} + I_\chi^{(1)}\theta + \ldots + \frac{1}{n!} I_\chi^{(n)}\theta^n\right)\begin{pmatrix} 1 \\ \chi \end{pmatrix} >$$

$$= 1 + 0\left(\theta^{n+1}\right) \tag{4.25}$$

as $\theta \to 0$

This requirement imposes some conditions on our successive solutions. For θ^1 equation (4.25) implies that

$$\int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} I_\chi^{(0)} \, I_\chi^{(1)} \, dx = 0 \tag{4.26}$$

whereas in the second order (θ^2) approach the equation

$$\int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} I_\chi^{(0)} I_\chi^{(2)} dx = - \int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} I_\chi^{(1)} I_\chi^{(1)} dx \qquad (4.27)$$

must hold.

In cases where the condition (4.25) is not automatically fulfilled, we use the integration constants to achieve this condition.

V. Perturbation of the Coupling Function

This section is devoted to the Taylor-series expansion of the coupling function κ and its derivative $(\frac{\partial\kappa}{\partial x})$. These expansions are needed to solve the field equation (4.15) for $I_\chi^{(n)}$. Since we aim at the calculation of the lowest order frequency shift $\Delta s_\chi^{(2)}$, we only have to expand the above mentioned functions at most up to second order in θ. Later it will turn out (see next section), however, that it is sufficient to know the expansion-coefficients of $\kappa(x,\theta)$ only up to first order. The expansions read:

$$\kappa(x,\theta) = \kappa(x,\theta=0) + \left(\frac{\partial}{\partial\theta}\kappa(x,\theta)\right)\bigg|_{\theta=0} \theta + O(\theta^2) \quad \text{as } \theta\to 0 \tag{5.1}$$

and

$$\left(\frac{\partial}{\partial x}\kappa(x,\theta)\right) = \left(\frac{\partial}{\partial x}\kappa(x,\theta)\right)\bigg|_{\theta=0} + \left(\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial x}\kappa(x,\theta)\right)\right)\bigg|_{\theta=0}\theta + \frac{1}{2}\left(\frac{\partial^2}{\partial\theta^2}\left(\frac{\partial\kappa}{\partial x}\right)\right)\bigg|_{\theta=0}\theta^2 + O(\theta^3) \quad \text{as } \theta\to 0 \tag{5.2}$$

We identify (compare (4.18), (4.19))

$$\kappa^{(0)} \equiv \kappa(x,0) \;;\; \kappa^{(1)} \equiv \left(\frac{\partial}{\partial\theta}\kappa(x,\theta)\right)\bigg|_{\theta=0} \tag{5.3a,b}$$

$$\kappa^{(2)} \equiv \left(\frac{\partial^2}{\partial\theta^2}\kappa(x,\theta)\right)\bigg|_{\theta=0} \tag{5.4}$$

$$\left(\frac{\partial\kappa}{\partial x}\right)^{(0)} \equiv \left(\frac{\partial}{\partial x}\kappa(x,\theta)\right)\bigg|_{\theta=0}, \quad \left(\frac{\partial\kappa}{\partial x}\right)^{(1)} \equiv \left(\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial x}\kappa(x,\theta)\right)\right)\bigg|_{\theta=0} \tag{5.5a,b}$$

$$\frac{\partial\kappa}{\partial x}^{(2)} \equiv \left(\frac{\partial^2}{\partial\theta^2}\left(\frac{\partial\kappa}{\partial x}\right)\right)\bigg|_{\theta=0} \tag{5.6}$$

In the following we explicitly represent the necessary derivatives. We begin with the first derivatives and have

$$\frac{\partial}{\partial x} \left(\kappa(x,\theta)\right) = \left(\frac{\partial a}{\partial x}\right) \left(\frac{d\kappa}{da}\right) \tag{5.7}$$

$$\frac{\partial}{\partial \theta} \left(\kappa(x,\theta)\right) = \left(\frac{\partial a}{\partial \theta}\right) \left(\frac{d\kappa}{da}\right) \tag{5.8}$$

Observe the common factor ($d\kappa/da$) in both expressions. Explicit calculations yield

$$\left(\frac{d\kappa}{da}\right) = - \frac{1}{\ell n\left(\frac{2h}{r}\right)} \frac{1}{h} \left\{\frac{a}{h} + \left(\frac{a}{h}\right)^3\right\}^{-1} \tag{5.9}$$

and

$$\left(\frac{\partial a}{\partial x}\right) = \sin(\theta) \ ; \ \left(\frac{\partial a}{\partial \theta}\right) = x \cos(\theta) \tag{5.10a,b}$$

The expressions for the second derivatives become somewhat longer:

$$\frac{\partial}{\partial \theta} \left(\frac{\partial \kappa}{\partial x}\right) = \left(\frac{\partial a}{\partial \theta}\right) \left(\frac{\partial a}{\partial x}\right) \left(\frac{d^2\kappa}{da^2}\right) + \left(\frac{d\kappa}{da}\right) \left(\frac{\partial}{\partial \theta} \left(\frac{\partial a}{\partial x}\right)\right) \tag{5.11}$$

with

$$\left(\frac{d^2\kappa}{da^2}\right) = \frac{1}{\ell n\left(\frac{2h}{r}\right)} \frac{1}{h^2} \left(1 + 3 \left(\frac{a}{h}\right)^2\right) \left\{\frac{a}{h} + \left(\frac{a}{h}\right)^3\right\}^{-2} \tag{5.12}$$

Inserting (5.12) into (5.11) we find

$$\frac{\partial}{\partial \theta} \left(\frac{\partial \kappa}{\partial x}\right) = x \sin(\theta) \cos(\theta) \frac{1}{\ell n\left(\frac{2h}{r}\right)} \frac{1}{h^2} \left(1 + 3 \left(\frac{a}{h}\right)^2\right) \left\{\frac{a}{h} + \left(\frac{a}{h}\right)^3\right\}^{-2}$$

$$- \cos(\theta) \frac{1}{\ell n\left(\frac{2h}{r}\right)} \frac{1}{h} \left\{\frac{a}{h} + \left(\frac{a}{h}\right)^3\right\}^{-1} \tag{5.13}$$

and therefore

$$\left(\frac{\partial \kappa}{\partial x}\right)^{(1)} = \frac{-1}{\ell n\left(\frac{2h}{r}\right)} \frac{1}{h} \left\{\frac{a_o}{h} + \left(\frac{a_o}{h}\right)^3\right\}^{-1}$$

Eventually we need the derivative $(\frac{\partial^2}{\partial\theta^2}(\frac{\partial\kappa}{\partial x}))$ of κ. The general expression reads:

$$\frac{\partial^2}{\partial\theta^2}\left(\frac{\partial\kappa}{\partial x}\right) = \left(\frac{\partial^2 a}{\partial\theta^2}\right)\left(\frac{\partial a}{\partial x}\right)\left(\frac{d^2\kappa}{da^2}\right) + 2\left(\frac{\partial a}{\partial\theta}\right)\left(\frac{\partial^2 a}{\partial\theta\partial x}\right)\left(\frac{d^2\kappa}{da^2}\right)$$

$$+ \left(\frac{\partial a}{\partial\theta}\right)^2\left(\frac{\partial a}{\partial x}\right)\left(\frac{d}{da}\left(\frac{d^2\kappa}{da^2}\right)\right) + \left(\frac{d\kappa}{da}\right)\left(\frac{\partial^2}{\partial\theta^2}\left(\frac{\partial a}{\partial x}\right)\right) \tag{5.14}$$

Taken at $\theta=0$, we obtain

$$\left(\frac{\partial^2}{\partial\theta^2}\left(\frac{\partial\kappa}{\partial x}\right)\right)_{|\theta=0} = 2\left[\left(\frac{\partial a}{\partial\theta}\right)\left(\frac{\partial^2 a}{\partial\theta\partial x}\right)\left(\frac{d^2\kappa}{da^2}\right)\right]_{|\theta=0} = \left(\frac{\partial\kappa}{\partial x}\right)^{(2)}$$

$$= \frac{2x}{\ell n(\frac{2h}{r})}\frac{1}{h^2}\left(1 + 3\left(\frac{a_o}{h}\right)^2\right)\left\{\frac{a_o}{h} + \left(\frac{a_o}{h}\right)^3\right\}^{-2} \tag{5.15}$$

The remaining expansion coefficients are

$$\left(\frac{\partial\kappa}{\partial x}\right)^{(0)} = 0 \;;\; \kappa^{(0)} = \frac{\ell n\left(1 + \left(\frac{h}{a_o}\right)^2\right)}{2\,\ell n(\frac{2h}{r})} \tag{5.16a,b}$$

$$\kappa^{(1)} = -\frac{x}{\ell n(\frac{2h}{r})}\frac{1}{h}\left\{\frac{a_o}{h} + \left(\frac{a_o}{h}\right)^3\right\}^{-1} \tag{5.17}$$

For purposes in the next section we define the functions

$$x\,\kappa_x^{(1)} \equiv \kappa^{(1)} \quad \text{and} \quad x\left(\frac{\partial\kappa}{\partial x}\right)_x^{(2)} \equiv \left(\frac{\partial\kappa}{\partial x}\right)^{(2)} \tag{5.18a,b}$$

and observe the relation

$$\kappa_x^{(1)} = \left(\frac{\partial\kappa}{\partial x}\right)^{(1)} \tag{5.19}$$

Note that the coefficients $\kappa^{(1)}$ and $(\frac{\partial\kappa}{\partial x})^{(2)}$ are proportional to the coördinate x. Thus, also in our perturbation-theoretical approach of low order, we have x-dependent coefficient-functions in the differential equation for I_χ.

VI. Perturbation Solutions

In order to find perturbation solutions we proceed as follows: In a first step we are looking for first order perturbation solutions for the natural modes. Secondly, with the solutions of these modes we derive second order solutions for the natural frequencies. Finally, we offer an alternate solution for the shifts of the natural frequencies and show how these alternate solutions converge into the former ones.

A. First Order Perturbation

We insert the ansaetze

$$I_{\chi,1} \equiv I_\chi^{(0)} + I_\chi^{(1)}\theta \tag{6.1}$$

$$\kappa_1(x,\theta) \equiv \kappa^{(0)} + \kappa^{(1)}\theta \tag{6.2}$$

$$s_{\chi,1} \equiv s_\chi^{(0)} = s_0 \tag{6.3}$$

into the differential equation (4.15) and reorder the resulting equation collecting only terms up to first order in θ. We then obtain the second order common differential equation for $I_\chi^{(1)}$

$$\left(\frac{d^2}{dx^2} I_\chi^{(1)}\right)\theta - \left(\frac{s_0}{c}\right)^2 I_\chi^{(1)}\theta = -\chi \frac{\left(\frac{\partial\kappa}{\partial x}\right)^{(1)}\theta}{\left(1+\chi\kappa^{(0)}\right)}\left(\frac{d}{dx} I_\chi^{(0)}\right) \tag{6.4}$$

Here we used already the fact that $I_\chi^{(0)}$ fulfills the differential equation (2.1).

Taking into account (4.23) for $I_\chi^{(0)}$ we have to solve the equation

$$\frac{d^2}{dx^2}\left(I_\chi^{(1)}\right) - \left(\frac{s_0}{c}\right)^2 I_\chi^{(1)} = A_\chi \sin\left(\frac{\pi x}{\ell}\right) . \tag{6.5}$$

with

$$A_\chi \equiv \frac{\chi\left(\frac{\partial\kappa}{\partial x}\right)^{(1)}}{\left(1+\chi\kappa^{(0)}\right)} \frac{1}{\sqrt{\ell}} \left(\frac{\pi}{\ell}\right) \tag{6.6}$$

and the boundary conditions

$$I_\chi^{(1)}\left(-\frac{\ell}{2}\right) = I_\chi^{(1)}\left(\frac{\ell}{2}\right) = 0 \qquad (6.7)$$

The subsidiary condition (4.26) as well has to be met.

The general solution of (6.5) is composed by the general solution of the homogeneous equation

$$\frac{d^2}{dx^2}\left(I^{(1)}_{\chi,\text{homog.}}\right) - \left(\frac{s_0}{c}\right)^2 I^{(1)}_{\chi,\text{homog.}} = 0$$

and a special solution of (6.5). With usual methods we obtain

$$I^{(1)}_{\chi,\text{homog.}} = C_1 \cos\left(\frac{\omega_0 x}{c}\right) + C_2 \sin\left(\frac{\omega_0 x}{c}\right) \qquad (6.8)$$

$$\left(j\omega_0 \equiv s_0 \; ; \; C_i = \text{integration constants} \; ; \; i = 1,2\right)$$

The special solution is found with the method of the variation of the constants and by calculation of the Wronskian determinant. It reads

$$I^{(1)}_{\chi,\text{special}} = C_1(x) \cos\left(\frac{\omega_0 x}{c}\right) + C_2(x) \sin\left(\frac{\omega_0 x}{c}\right)$$

with

$$C_1(x) = \int_{-\frac{\ell}{2}}^{x} \frac{\left(-\sin\left(\frac{\omega_0}{c}\,\xi\right)\right)\left(A_\chi \sin\left(\frac{\pi\xi}{\ell}\right)\right)}{\left(\frac{\omega_0}{c}\right)}\, d\xi \qquad (6.9)$$

and

$$C_2(x) = \int_{-\frac{\ell}{2}}^{x} \frac{\cos\left(\frac{\omega_0}{c}\,\xi\right)\left(A_\chi \sin\left(\frac{\pi\xi}{\ell}\right)\right)}{\left(\frac{\omega_0}{c}\right)}\, d\xi \qquad (6.10)$$

Together we have

$$I_\chi^{(1)}(x) = I^{(1)}_{\chi,\text{homog.}}(x) + I^{(1)}_{\chi,\text{special}}(x) \qquad (6.11)$$

Depending on whether (ω_0/c) is equal or not equal to (π/ℓ) we get two different solutions for (6.11). First we assume equality, i.e. $(\omega_0/c) = (\pi/\ell)$. This is in accordance with the fact $\Delta s_\chi^{(1)} = 0$.

We also solve (6.9/10) for $(\omega_o/c) \neq (\pi/\ell)$ in a separate subsection. This, however, is not consistent with the lowest order solution $I_\chi^{(0)}(x)$. Nevertheless, we will present these solutions in order to indicate how to proceed on the basis of a more exact ground mode solution.

In the first case $[\text{i.e. } (\omega_o/c) = (\pi/\ell)]$ we obtain as special solution

$$I^{(1)}_{\chi,\text{special}} = - \frac{A_\chi}{(\frac{\pi}{\ell})} \left\{ \frac{1}{2}\, x - \frac{1}{4(\frac{\pi}{\ell})} \sin(\frac{2\pi x}{\ell}) + \frac{1}{4}\, \ell \right\} \cos(\frac{\pi x}{\ell})$$

$$+ \frac{A_\chi}{2(\frac{\pi}{\ell})^2} \left\{ \sin^2(\frac{\pi x}{\ell}) - 1 \right\} \sin(\frac{\pi x}{\ell}) \qquad (6.12)$$

It remains to determine the integration constants C_1 and C_2. The condition (6.7) only fixes $C_2 = 0$, and we are left with the constant C_1. This in turn is fixed by the condition (4.26) and becomes

$$C_1 = (\frac{\ell}{4}) \frac{A_\chi}{(\frac{\pi}{\ell})} \qquad (6.13)$$

Thus, the natural modes $\Delta I_\chi^{(1)}$ of first order in θ are given by

$$\Delta I_\chi^{(1)}(x) \equiv I_\chi^{(1)}(x)\ \theta = - \frac{A_\chi \theta}{(\frac{\pi}{\ell})} \left\{ \frac{1}{2}\, x - \frac{1}{4(\frac{\pi}{\ell})} \sin(\frac{2\pi x}{\ell}) \right\} \cos(\frac{\pi x}{\ell})$$

$$+ \frac{A_\chi\, \theta}{2(\frac{\pi}{\ell})^2} \left\{ \sin^2(\frac{\pi x}{\ell}) - 1 \right\} \sin(\frac{\pi x}{\ell}) \qquad (6.14)$$

In the following subsection we use this solution for $I_\chi^{(1)}(x)$ to estimate the natural frequency shifts.

B. Second Order Perturbation

With the above result for $I_\chi^{(1)}$ (see (6.14)) we are now prepared to proceed with the second step: Seeking the solution for $\Delta s_\chi^{(2)}$. This time we expand the differential equation (4.15) up to second order in θ. This is a somewhat tedious and lengthy calculation. Nevertheless, for a better understanding of the reader, we elaborate on the important steps in our derivation. We start expanding all quantities up to second order and insert them into (4.15)

$$\frac{d^2}{dx^2}\left(\sum_{n=0}^{2} \frac{1}{n!} I_{\chi}^{(n)} \theta^n\right) + \chi \frac{\sum_{\ell=1}^{2} \frac{1}{\ell!}\left(\frac{\partial\kappa}{\partial x}\right)^{(\ell)} \theta^{\ell}}{\left(1 + \chi \sum_{m=0}^{2} \frac{1}{m!} \kappa^{(m)} \theta^m\right)} \frac{d}{dx}\left(\sum_{n=0}^{2} \frac{1}{n!} I_{\chi}^{(n)} \theta^n\right)$$

$$- \frac{1}{c^2}\left(\sum_{p=0}^{2,2} \frac{1}{p!} s_{\chi}^{(p)} \theta^p\right)^2 \frac{1 + \chi\left(\sum_{m=0}^{2} \frac{1}{m!} \kappa^{(m)} \theta^m\right)\left(1-\theta^2\right)}{\left(1 + \chi \sum_{m=0}^{2} \frac{1}{m!} \kappa^{(m)} \theta^m\right)} \left(\sum_{n=0}^{2} \frac{1}{n!} I_{\chi}^{(n)} \theta^n\right) = O(\theta^3) \tag{6.15}$$

The notation in (6.15) is self explanatory. Recall, however, that $s_{\chi}^{(1)} = 0$ and observe the expansion of $\cos^2(\theta)$ up to second order

$$\cos^2(\theta) = 1 - \theta^2 + O(\theta^4) \tag{6.16}$$

Equation (6.15) has to be reordered only taking into account the products up to second order in θ. In order to achieve this, we first calculate the ratio

$$\frac{1 + \chi\left(\sum_{m=0}^{2} \frac{1}{m!} \kappa^{(m)} \theta^m\right)\left(1-\theta^2\right)}{\left(1 + \chi \sum_{m=0}^{2} \frac{1}{m!} \kappa^{(m)} \theta^m\right)} = 1 - \frac{\chi\, \kappa^{(0)}\, \theta^2}{\left(1+\chi\kappa^{(0)}\right)} + O(\theta^3) \tag{6.17}$$

and the sum

$$\left(\sum_{p=0}^{2} \frac{1}{p!} s_{\chi}^{(p)} \theta^p\right)^2 = s_0^2 + s_0\, s_{\chi}^{(2)}\, \theta^2 + O(\theta^4) \tag{6.18}$$

Then the product of (6.17) and (6.18) becomes

$$\left(1 - \chi \frac{\kappa^{(0)}\, \theta^2}{\left(1+\chi\kappa^{(0)}\right)}\right)\left(s_0^2 + s_0\, s_{\chi}^{(2)}\, \theta^2\right) = s_0^2\left(1 - \chi \frac{\kappa^{(0)}\, \theta^2}{\left(1+\chi\kappa^{(0)}\right)}\right) + s_0\, s_{\chi}^{(2)}\, \theta^2 + O(\theta^4) \tag{6.19}$$

Note that the lowest order of the numerator in the second term of (6.15) is $O(\theta)$ and therefore the other factors in this term have to be taken up to order θ. We find for the ratio

$$\frac{\sum_{\ell=1}^{2} \frac{1}{\ell!} \left(\frac{\partial\kappa}{\partial x}\right)^{(\ell)} \theta^{\ell}}{1 + \chi \sum_{m=0}^{2} \frac{1}{m!} \kappa^{(m)} \theta^{m}} = \frac{1}{\left(1+\chi\kappa^{(0)}\right)} \left\{\left(\frac{\partial\kappa}{\partial x}\right)^{(1)} \theta + \frac{1}{2} \left(\frac{\partial\kappa}{\partial x}\right)^{(2)} \theta^2\right\}$$

$$- \frac{\chi\kappa^{(1)} \left(\frac{\partial\kappa}{\partial x}\right)^{(1)} \theta^2}{\left(1+\chi\kappa^{(0)}\right)^2} + O(\theta^3) \qquad (6.20)$$

Using the approximations (6.17)-(6.20) and inserting them into (6.15) we obtain up to second order in θ:

$$\left(\frac{d^2}{dx^2} I_\chi^{(0)} - \frac{s_0^2}{c^2} I_\chi^{(0)}\right) + \left[\frac{d^2}{dx^2} I_\chi^{(1)} + \frac{\chi}{\left(1+\chi\kappa^{(0)}\right)} \left(\frac{\partial\kappa}{\partial x}\right)^{(1)} \left(\frac{d}{dx} I_\chi^{(0)}\right) - \frac{s_0^2}{c^2} I_\chi^{(1)}\right]\theta$$

$$+ \frac{1}{2}\left[\frac{d^2}{dx^2} I_\chi^{(2)} - \frac{s_0^2}{c^2} I_\chi^{(2)} + \frac{\chi}{\left(1+\chi\kappa^{(0)}\right)} \left(\left(\frac{\partial\kappa}{\partial x}\right)^{(2)} - 2\chi \frac{\kappa^{(1)}\left(\frac{\partial\kappa}{\partial x}\right)^{(1)}}{\left(1+\chi\kappa^{(0)}\right)}\right) \left(\frac{d}{dx} I_\chi^{(0)}\right)\right.$$

$$\left. + 2 \frac{\chi}{\left(1+\chi\kappa^{(0)}\right)} \left(\frac{\partial\kappa}{\partial x}\right)^{(1)} \left(\frac{d}{dx} I_\chi^{(1)}\right)\right] \theta^2 + \left(\frac{s_0}{c}\right)^2 \frac{\chi\kappa^{(0)}}{\left(1+\chi\kappa^{(0)}\right)} I_\chi^{(0)} \theta^2$$

$$- \frac{s_0}{c^2} s_\chi^{(2)} I_\chi^{(0)} \theta^2 = 0 \qquad (6.21)$$

In the first two brackets we recover the differential equations for $I_\chi^{(0)}$ and $I_\chi^{(1)}$, respectively. Since $I_\chi^{(0)}$ and $I_\chi^{(1)}$ from (4.23) and (6.14) fulfill their respective differential equations, we finally get for $I_\chi^{(2)}$

$$\frac{d^2}{dx^2} I_\chi^{(2)} - \left(\frac{s_0}{c}\right)^2 I_\chi^{(2)} = - \frac{\chi}{\left(1+\chi\kappa^{(0)}\right)} \left[\left(\frac{\partial\kappa}{\partial x}\right)^{(2)} - 2 \frac{\kappa^{(1)} \left(\frac{\partial\kappa}{\partial x}\right)^{(1)}}{\left(1+\chi\kappa^{(0)}\right)} \right] \left(\frac{d}{dx} I_\chi^{(0)}\right)$$

$$- 2 \frac{\chi}{\left(1+\chi\kappa^{(0)}\right)} \left(\frac{\partial\kappa}{\partial x}\right)^{(1)} \left(\frac{d}{dx} I_\chi^{(1)}\right) - 2 \left(\frac{s_0}{c}\right)^2 \frac{\chi\kappa^{(0)}}{\left(1+\chi\kappa^{(0)}\right)} I_\chi^{(0)}$$

$$+ 2 \frac{s_0}{c^2} s_\chi^{(2)} I_\chi^{(0)} \qquad (6.22)$$

Equation (6.22) contains the two unknown quantities $I_\chi^{(2)}$ and $s_\chi^{(2)}$. As soon as we know one of them, the other can easily be determined. There is a way to calculate $s_\chi^{(2)}$ first. Multiply (6.22) with $I_\chi^{(0)}$ and perform the integration from $(-\frac{\ell}{2})$ to $(+\frac{\ell}{2})$ then it turns out that

$$\int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} dx \left\{ \frac{d^2}{dx^2} I_\chi^{(2)} - \left(\frac{s_0}{c}\right)^2 I_\chi^{(2)} \right\} I_\chi^{(0)} = 0 \qquad (6.23)$$

In order to obtain equation (6.23) we integrate twice by parts

$$\int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} \left(\frac{d^2}{dx^2} I_\chi^{(2)}\right) I_\chi^{(0)} \, dx = \frac{d}{dx} I_\chi^{(2)} I_\chi^{(0)} \Bigg|_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} - \int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} \left(\frac{d}{dx} I_\chi^{(2)}\right) \left(\frac{d}{dx} I_\chi^{(0)}\right) dx$$

$$= - \int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} \left(\frac{d}{dx} I_\chi^{(2)}\right) \left(\frac{d}{dx} I_\chi^{(0)}\right) dx = - I_\chi^{(2)} \left(\frac{d}{dx} I_\chi^{(0)}\right) \Bigg|_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} + \int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} I_\chi^{(2)} \frac{d^2}{dx^2} I_\chi^{(0)} \, dx$$

$$= \int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} I_\chi^{(2)} \frac{d^2}{dx^2} I_\chi^{(0)} \, dx \qquad (6.24)$$

and observe the differential equation (2.1) for $I_\chi^{(0)}$. Moreover the boundary conditions for $I_\chi^{(0)}$ and $I_\chi^{(2)}$

$$\left(\text{i.e. } I_\chi^{(0)}\left(-\frac{\ell}{2}\right) = I_\chi^{(0)}\left(\frac{\ell}{2}\right) = 0 \quad \text{and} \quad I_\chi^{(2)}\left(-\frac{\ell}{2}\right) = I_\chi^{(2)}\left(\frac{\ell}{2}\right) = 0\right)$$

have to be taken into consideration.

Now the integration of the r.h.s. of (6.22) can be performed and the resulting equation can be resolved with respect to $s_\chi^{(2)}$. The integration yields

$$\left(\frac{s_0}{c^2}\right)s_\chi^{(2)} - \left(\frac{s_0}{c}\right)^2 \frac{\chi\kappa^{(0)}}{\left(1+\chi\kappa^{(0)}\right)} - \frac{\chi}{\left(1+\chi\kappa^{(0)}\right)} \int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} dx \left(\frac{\partial\kappa}{\partial x}\right)^{(2)} - 2\chi \frac{\kappa^{(1)}\left(\frac{\partial\kappa}{\partial x}\right)^{(1)}}{\left(1+\chi\kappa^{(0)}\right)} \frac{d}{dx} I_\chi^{(0)} I_\chi^{(0)}$$

$$- 2 \frac{\chi}{\left(1+\chi\kappa^{(0)}\right)} \int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} \left(\frac{\partial\kappa}{\partial x}\right)^{(1)} \left(\frac{d}{dx} I_\chi^{(1)}\right) I_\chi^{(0)} \, dx = 0 \tag{6.25}$$

For the above integration we need to know $I_\chi^{(0)}$, $\left(\frac{d}{dx} I_\chi^{(0)}\right)$, and $\left(\frac{d}{dx} I_\chi^{(1)}\right)$ explicitly. Differentiation of (4.23) and (6.14) yields in turn

$$\frac{d}{dx} I_\chi^{(0)} = \left(-\frac{\pi}{\ell}\right) \frac{1}{\sqrt{\ell}} \sin\left(\frac{\pi x}{\ell}\right) \tag{6.26}$$

$$\frac{d}{dx} I_\chi^{(1)} = \frac{1}{2} \frac{A_\chi}{\left(\frac{\pi}{\ell}\right)} \sin^2\left(\frac{\pi x}{\ell}\right) \cos\left(\frac{\pi x}{\ell}\right) + A_\chi \left\{\frac{1}{2} x - \frac{1}{4\left(\frac{\pi}{\ell}\right)} \sin\left(\frac{2\pi x}{\ell}\right)\right\} \sin\left(\frac{\pi x}{\ell}\right)$$

$$- \frac{A_\chi}{2\left(\frac{\pi}{\ell}\right)} \cos\left(\frac{\pi x}{\ell}\right), \quad \left(\frac{\omega_0}{c} = \frac{\pi}{\ell}\right) \tag{6.27}$$

The following integrals will therefore be used for the integration of (6.25)

$$\int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} dx\ x \sin\left(\frac{\pi x}{\ell}\right) \cos\left(\frac{\pi x}{\ell}\right) = \left(\frac{\ell}{2}\right)^2 \left(\frac{1}{\pi}\right) \tag{6.28}$$

$$\int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} dx\ \sin^2\left(\frac{\pi x}{\ell}\right) \cos^2\left(\frac{\pi x}{\ell}\right) = \frac{1}{8}\ell \tag{6.29}$$

$$\int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} dx\ \sin\left(\frac{2\pi x}{\ell}\right) \sin\left(\frac{\pi x}{\ell}\right) \cos\left(\frac{\pi x}{\ell}\right) = \frac{1}{4}\ell \tag{6.30}$$

$$\int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} \cos^2\left(\frac{\pi x}{\ell}\right) dx = \frac{\ell}{2} \tag{6.31}$$

$$\int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} dx\ \cos\left(\frac{\omega_0}{c} x\right) \cos\left(\frac{\pi x}{\ell}\right) = \frac{-2\left(\frac{\pi}{\ell}\right) \cos\left(\frac{\omega_0}{c}\frac{\ell}{2}\right)}{\left[\left(\frac{\omega_0}{c}\right)^2 - \left(\frac{\pi}{\ell}\right)^2\right]} \tag{6.32}$$

After integration of (6.25) and resolution for $s_\chi^{(2)}$ we arrive at

$$s_\chi^{(2)} = \chi \frac{s_0 \kappa^{(0)}}{\left(1+\chi\kappa^{(0)}\right)} + \frac{1}{2}\left(\frac{c^2}{s_0}\right) \chi^2 \frac{\left(\kappa_x^{(1)}\right)\left(\frac{\partial\kappa}{\partial x}\right)^{(1)}}{\left(1+\chi\kappa^{(0)}\right)^2} - \frac{1}{4}\left(\frac{c^2}{s_0}\right)\chi^2 \frac{\left(\left(\frac{\partial\kappa}{\partial x}\right)^{(1)}\right)^2}{\left(1+\chi\kappa^{(0)}\right)^2}$$

$$- \frac{1}{4}\left(\frac{c^2}{s_0}\right) \frac{\chi \left(\frac{\partial\kappa}{\partial x}\right)_x^{(2)}}{\left(1+\chi\kappa^{(0)}\right)} \tag{6.33}$$

Observing the relation (5.19) we can combine the second and third term in (6.33) and finally end up with the compact result

$$\Delta s_\chi^{(2)} = \frac{1}{2}\, s_\chi^{(2)}\, \theta^2 = \frac{1}{2}\, \frac{\chi\, s_0\, \theta^2}{\left(1+\chi\kappa^{(0)}\right)} \left\{ \kappa^{(0)} + \frac{1}{4}\left(\frac{c}{s_0}\right)^2 \left[\frac{\chi\left(\kappa_x^{(1)}\right)^2}{\left(1+\chi\kappa^{(0)}\right)} - \left(\frac{\partial\kappa}{\partial x}\right)_x^{(2)} \right] \right\} \tag{6.34}$$

The frequency shift $\Delta s_\chi^{(2)}$ is caused by two different effects: The first effect arises from the derivatives of the capacitance-matrix elements (expressed by the derivatives of κ) whereas the second effect results because of the non-reciprocal relation of the matrices $(C'_{n,m})$ and $(L'_{n,m})$. The corresponding term which takes into consideration this effect is the first term (no derivative of κ!) in (6.34). It turns out, however, that the summands containing the derivatives of κ considerably predominate the summand containing only κ itself (see below the numerical example). As expected (we did not include any losses) the frequency shifts are purely imaginary and therefore only take place along the imaginary axis in the complex s-plane. This is different from the spiral behavior of the natural frequencies in [2].

C. Alternate Solution

In order to complete our first order results we put the current solutions $\Delta I_\chi^{(1)}$ for the case that $(\omega_o/c) \neq (\pi/\ell)$ to our disposal. The calculations needed to be performed are completely equivalent to those of (6.8) through (6.14). For the special solution we now find with the help of (6.9) and (6.10)

$$I_{\chi,\text{special}}^{(1)} = A_\chi \frac{\sin\left(\frac{\pi x}{\ell}\right)}{\left(\frac{\omega_o}{c}\right)^2 - \left(\frac{\pi}{\ell}\right)^2} \tag{6.35}$$

and thus the general solution is

$$I_\chi^{(1)} = C_1 \cos\left(\frac{\omega_o}{c}\, x\right) + C_2 \sin\left(\frac{\omega_o}{c}\, x\right) + A_\chi \frac{\sin\left(\frac{\pi x}{\ell}\right)}{\left[\left(\frac{\omega_o}{c}\right)^2 - \left(\frac{\pi}{\ell}\right)^2\right]} \tag{6.36}$$

This time the boundary conditions (6.7) fix the constants C_1 and C_2:

$$C_1 = 0 \quad \text{and} \quad C_2 = - \frac{A_\chi}{\sin\left(\frac{\omega_o}{c}\frac{\ell}{2}\right)\left[\left(\frac{\omega_o}{c}\right)^2 - \left(\frac{\pi}{\ell}\right)^2\right]} \tag{6.37}$$

This leads to the result

$$I_\chi^{(1)}(x) = \frac{A_\chi}{\left[\left(\frac{\omega_o}{c}\right)^2 - \left(\frac{\pi}{\ell}\right)^2\right]} \left\{\sin\left(\frac{\pi x}{\ell}\right) - \frac{\sin\left(\frac{\omega_o}{c}x\right)}{\sin\left(\frac{\omega_o}{c}\frac{\ell}{2}\right)}\right\} \tag{6.38}$$

$$\left(\left(\frac{\omega_o}{c}\right) \neq \left(\frac{\pi}{\ell}\right)\right)$$

In this case the condition (4.26) is automatically fulfilled by $I_\chi^{(1)}$ of (6.38) and by $I_\chi^{(0)}$ of (4.23).

Analogously to our former calculations we now differentiate (6.38)

$$\frac{d}{dx} I_\chi^{(1)} = \frac{A_\chi}{\left[\left(\frac{\omega_o}{c}\right)^2 - \left(\frac{\pi}{\ell}\right)^2\right]} \left\{\frac{\pi}{\ell}\cos\left(\frac{\pi x}{\ell}\right) - \frac{\omega_o}{c}\frac{\cos\left(\frac{\omega_o}{c}x\right)}{\sin\left(\frac{\omega_o}{c}\frac{\ell}{2}\right)}\right\} \tag{6.39}$$

$$\left(\left(\frac{\omega_o}{c}\right) \neq \left(\frac{\pi}{\ell}\right)\right),$$

insert (6.39) instead of (6.27) into equation (6.25) and obtain after integration and resolution with respect to $\Delta s_\chi^{(2)}$

$$\frac{1}{2} s_\chi^{(2)} \theta^2 = \Delta s_\chi^{(2)} = \frac{1}{2}\chi \frac{s_0 \kappa^{(0)} \theta^2}{\left(1+\chi\kappa^{(0)}\right)} + \frac{1}{4}\left(\frac{c^2}{s_0}\right)\chi^2 \frac{\left(\kappa_x^{(1)}\right)\left(\frac{\partial\kappa}{\partial x}\right)^{(1)}}{\left(1+\chi\kappa^{(0)}\right)^2}\theta^2$$

$$- \frac{1}{8}\left(\frac{c^2}{s_0}\right)\frac{\chi\left(\frac{\partial\kappa}{\partial x}\right)_x^{(2)}}{\left(1+\chi\kappa^{(0)}\right)}\theta^2 + \frac{1}{2}\left(\frac{c^2}{s_0}\right)\chi^2 \frac{\left(\left(\frac{\partial\kappa}{\partial x}\right)^{(1)}\right)^2}{\left(1+\chi\kappa^{(0)}\right)^2} \frac{\left(\frac{\pi}{\ell}\right)^2\theta^2}{\left[\left(\frac{\omega_o}{c}\right)^2 - \left(\frac{\pi}{\ell}\right)^2\right]}$$

$$\left\{1 + 4\left(\frac{\omega_o}{c}\right)\frac{\cot\left(\frac{\omega_o}{c}\frac{\ell}{2}\right)}{\ell\left[\left(\frac{\omega_o}{c}\right)^2 - \left(\frac{\pi}{\ell}\right)^2\right]}\right\} \tag{6.40}$$

Comparing the results (6.33) and (6.40) with each other we observe that they agree in all summands except one: The factors in front of $((\partial\kappa/\partial x)^{(1)})^2$ are considerably different. One might think that there is a singularity for $(\omega_o/c) = (\pi/\ell)$ in formula (6.40). This, however, is not the case. Formula (6.40) has a well defined limit for $(\omega_o/c)\to(\pi/\ell)$. Applying de l'Hospital's rule two times we get

$$\lim_{\left(\frac{\omega_o}{c}\right)\to\left(\frac{\pi}{\ell}\right)} \left\{ \frac{\frac{\pi}{\ell}^{2}}{\left[\left(\frac{\omega_o}{c}\right)^2 - \left(\frac{\pi}{\ell}\right)^2\right]} \left(1 + \frac{4\,\frac{\omega_o}{c}}{\ell\left[\left(\frac{\omega_o}{c}\right)^2 - \left(\frac{\pi}{\ell}\right)^2\right]} \cot\left(\frac{\omega_o}{c}\;\frac{\ell}{2}\right)\right)\right\} = -\frac{1}{4} \tag{6.41}$$

and thus recover with (6.40) (as it should be) our old result (modulo multiplication with $2\theta^{-2}$) (6.33). This establishes the consistency of our calculation.

Finally, we can go back with our solution $s_\chi^{(2)}$(compare (6.34)) to (6.22) and solve this equation with the usual methods for $I_\chi^{(2)}$. Since we do not intend, however, to extend our calculation for $s_\chi^{(n)}$ to higher than second order in θ, we abstain from calculating $I_\chi^{(2)}$. Note from (4.20) that s_χ is already accurate to third order, the error being $O(\theta^4)$.

VII. Discussion and Concluding Remarks

We begin our discussion with a numerical example. For this purpose we consider two copper wires of radius 0.4 mm above the conducting plane. The lines are 3m long (and open ended), the height above ground is 3 cm and their spacing $2a_o = 1$ cm. We also chose these numbers in ref. [9]. With these numbers we get:

$$\kappa^{(0)} = 0.36$$

$$\kappa_x^{(1)} = -38.84 \; [m^{-1}] \tag{7.1}$$

$$\left(\frac{\partial\kappa}{\partial x}\right)_x^{(2)} = 16.38 \cdot 10^3 \; [m^{-2}]$$

For the angle θ we choose the value

$$\theta = 1.67 \cdot 10^{-3} \tag{7.2}$$

This angle is obtained assuming that the closer ends of the wires are 0.5 cm apart from each other. Inserting the above numbers into (6.34) we find the following results $[(\omega_o/c) = (\pi/\ell)]$:

$$\Delta f_{sy}^{(2)} = 0.18 \text{ MHz} \qquad \chi = \begin{cases} 1 & sy \\ -1 & as \end{cases} \tag{7.3}$$

$$\Delta f_{as}^{(2)} = -0.47 \text{ MHz}$$

with

$$2\pi\Delta f_\chi^{(2)} \, j \equiv \Delta s_\chi^{(2)} \tag{7.4}$$

Thus the (single) frequency $f_o = 50$ MHz is split into two distinct natural frequencies $(f_o + \Delta f_{sy}^{(2)}) = 50.18$ MHz and $(f_o + \Delta f_{as}^{(2)}) = 49.53$ MHz which are 0.65 MHz separated from each other. The amount of separation as well as the relative position of the two natural frequencies $\Delta f_{sy}^{(2)}$ and $\Delta f_{as}^{(2)}$ with respect to each other depend on various system-parameters like a_o, h, and r. (Compare the "competition" between the second and third term in (6.34).)

The above splitting of frequencies is analogous to the splitting of energy levels in quantum mechanics. There degeneration occurs for forces of high(er) symmetry. Breaking those symmetries leads to resolution of degeneration. In our case the space dependency of the mutual impedance - and admittance - matrix elements of the two wires is responsible for the frequency splitting.

We dealt with lossless transmission lines and thereby disregarded radiation losses. Moreover, neither internal self inductances nor internal resistances of the conductors were included to make s_χ complex. Therefore, the ground frequency as well as the frequency shifts $\Delta s_\chi^{(2)}$ turned out to be purely imaginary.

We found that small changes in the spacing between the conductors can produce small shifts in resonant frequencies (but with negligible damping). This can greatly modify the coupling into a multiconductor cable by moving one resonance away from another. In propagating signals through systems one maximizes the various norms (related to time-domain peak, square root of energy, etc.) by lining up the various resonances in the transmission path, as well as the excitation frequency (of say a damped sinusoid) [6]. What one thinks are small changes in dimensions can produce small shifts in resonant frequencies. However, with negligible damping this can make the resonances not significantly overlap, thereby reducing the propagation through the system.

All our calculations refer to open ended cables, short-circuit terminations, being also lossless will give similar results. In more typical situations intermediate terminating impedances, including losses (such as 50Ω), are likely.

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