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Interaction Notes

Note 479

8 December 1989

The Self-Complementary Rotation Group

Carl E. Baum
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Abstract

Self-complementary antennas have long been of interest because of their resistive, frequency-independent impedance. Such structures can be generalized to sheet admittances involving other than specially shaped perfectly conducting sheets. This paper explores these possibilities in the context of symmetry: the self-complementary rotation group C_{Nc}. Special results are found for C_{2c} symmetry involving the symmetry of the fields and special resistive structures.

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Self-complementary antennas have long been of interest because of their resistive, frequency-independent impedance. Such structures can be generalized to sheet admittances involving other than specially shaped perfectly conducting sheets. This paper explores these possibilities in the context of symmetry: the self-complementary rotation group C_{Nc}. Special results are found for C_{2c} symmetry involving the symmetry of the fields and special resistive structures.

I. Introduction

The Babinet principle is a well-known concept in electromagnetics in which an aperture in a perfectly conducting sheet can be replaced by the complementary disk with an interchange of electric and magnetic fields to give an equivalent boundary value problem [7]. This relies on duality for the electric and magnetic fields (appendix A) and complementary structures (or more general sheet admittances) on a plane (appendix B) [3, 7, 9]. One goes from this to find types of self-complementary antennas comprised of conductors on a plane, these having a frequency-independant input impedance [1, 7, 8, 12].

In the generalized form of the Babinet principle rather general forms of sheet admittances can be used and their complements found (appendix B) [3, 9]. As shown in the present paper there are various interesting forms self-complementary sheet admittances can take, including self inverse and self rotated inverse (section 3). This is fit into the context of C_N and C_{Nc} symmetry (section 2). It is noted that, while one can define C_{Nc} symmetry for the sheet admittance the electromagnetic-field symmetry of interest is C_{2c} (section 4). Furthermore C_{2c} symmetry give various special results (section 5). Including an incident plane wave and the resulting equivalent sources (appendix C) one can also define self-complementary scattering (section 6).

II. Self-Complementary Rotation Group

Now, of course, we consider the case that the screen is self complementary, i.e., the complement is the same as the original except for a rotation. This has the well-known result that the input impedance of a self-complementary antenna with two separate conductors (C_2 symmetry), and hence a single terminal pair, is just $Z_o / 2$ independent of frequency. This can be generalized to N-conductor self-complementary antennas with N terminals (C_N symmetry) in which case the various input impedances (in a matrix) are all frequency independent but of various special values [8]. Note that in [3, 9] and this paper we have a generalized form of complementarity and hence self complementarity. All previous results concerning the input impedance of self complementary antennas still hold with the use of the generalized self-complementary structures (e.g. using resistive and/or anisotropic sheets).

Write (B.13) for the complementary admittance (normalized) using cylindrical coordinates on S (the x, y plane) as

$$x = \Psi \cos(\phi)$$

$$y = \Psi \sin(\phi)$$

$$\tilde{\tilde{y}}_s^{(c)}(\Psi, \phi; s) = \tilde{\tilde{\tau}}_d \cdot \tilde{\tilde{y}}_s^{-1}(\Psi, \phi; s) \cdot \tilde{\tilde{\tau}}_d^T \tag{2.1}$$

Then, imposing the self-complementary condition with self complement at a rotation by ϕ_c, we have

$$\tilde{\tilde{y}}_s(\Psi, \phi + \phi_c; s) = \tilde{\tilde{\tau}}_c \cdot \tilde{\tilde{y}}_s^{(c)}(\Psi, \phi; s) \cdot \tilde{\tilde{\tau}}_c^T$$

$$\tilde{\tilde{\tau}}_c = \begin{pmatrix} \cos(\phi_c) & -\sin(\phi_c) \\ \sin(\phi_c) & \cos(\phi_c) \end{pmatrix}$$

$$\tilde{\tilde{\tau}}_c^{-1} = \tilde{\tilde{\tau}}_c^T \tag{2.2}$$

Combining these we have

$$\tilde{\tilde{y}}_s(\Psi, \phi + \phi_c; s) = \tilde{\tilde{\tau}}_c \cdot \tilde{\tilde{\tau}}_d \cdot \tilde{\tilde{y}}_s^{-1}(\Psi, \phi; s) \cdot \tilde{\tilde{\tau}}_d^T \cdot \tilde{\tilde{\tau}}_c^T \tag{2.3}$$

as the basic self-complementary equation for $\tilde{\tilde{y}}_s$.

Note that applying this twice we have

$$\tilde{\tilde{y}}_s(\Psi,\ \phi + 2\phi_c;\ s) = \tilde{\tau}_c \cdot \tilde{\tau}_d \cdot \tilde{\tilde{y}}_s^{-1}(\Psi,\ \phi+\phi_c;\ s) \cdot \tilde{\tau}_d^T \cdot \tilde{\tau}_c^T$$

$$= \tilde{\tau}_c \cdot \tilde{\tau}_d \cdot \tilde{\tau}_c \cdot \tilde{\tau}_d \cdot \tilde{\tilde{y}}_s(\Psi,\ \phi;\ s) \cdot \tilde{\tau}_d^T \cdot \tilde{\tau}_c^T \cdot \tilde{\tau}_d^T \cdot \tilde{\tau}_c^T$$

$$= \tilde{\tau}_c^2 \cdot \tilde{\tilde{y}}_s(\Psi,\ \phi;\ s) \cdot \tilde{\tau}_c^{-2} \tag{2.4}$$

which uses the commutativity of the two-dimensional rotation matrices as well as

$$\tilde{\tau}_d^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\left(1_{n,m}\right) \tag{2.5}$$

We also have [5, 6]

$$\tilde{\tau}_c \cdot \tilde{\tau}_d = \begin{pmatrix} \cos\left(\phi_c + \frac{\pi}{2}\right) & -\sin\left(\phi_c + \frac{\pi}{2}\right) \\ \sin\left(\phi_c + \frac{\pi}{2}\right) & \cos\left(\phi_c + \frac{\pi}{2}\right) \end{pmatrix} = \left(C_{n,m}\left(\phi_c + \frac{\pi}{2}\right)\right)$$

$$\tilde{\tau}_c^2 = \begin{pmatrix} \cos(2\phi_c) & -\sin(2\phi_c) \\ \sin(2\phi_c) & \cos(2\phi_c) \end{pmatrix} = \left(C_{n,m}(2\phi_c)\right) \tag{2.6}$$

So the self-complementary rotation is like rotation by $\phi_c + \frac{\pi}{2}$. However, the twice application of this leads to rotation by $2\phi_c$, or equivalently by $2\phi_c + \pi$. (Note that any 2 x 2 matrix is invariant to rotation by π which is just the negative identity as in (2.5)). Since rotation by $2\phi_c$ reproduces the original object then the object must have C_N symmetry where

$$2\phi_c N = 2\pi \tag{2.7}$$

C_N symmetry has [5, 6, 11]

$$C_N = \left\{(C_N)_\ell \mid \ell = 1, 2, \cdots, N\right\}$$

$$(C_N)_\ell = (C_N)_1^\ell \tag{2.8}$$

$$(C_N)_N = (C_N)_1^N = (1) \equiv \text{identity}$$

It has a convenient matrix representation

$$(C_N)_\ell \to \left(C_{n,m}\left(\frac{2\pi\ell}{N}\right)\right) = \begin{pmatrix} \cos\left(\frac{2\pi\ell}{N}\right) & -\sin\left(\frac{2\pi\ell}{N}\right) \\ \sin\left(\frac{2\pi\ell}{N}\right) & \cos\left(\frac{2\pi\ell}{N}\right) \end{pmatrix}$$

$$= \left(C_{n,m}\left(\frac{2\pi}{N}\right)\right)^\ell \tag{2.9}$$

$$\ell = 1, 2, \cdots, N$$

Here we have

$$2\phi_c = \frac{2\pi}{N} \tag{2.10}$$

Note that it is rotation by $2\phi_c$, not by ϕ_c, which defines the C_N symmetry.

Turning to rotation by ϕ_c let us define another group element (C_c) corresponding to the transformation in (2.3), i.e. inversion followed by rotation by $\overleftrightarrow{\tau}_c \cdot \overleftrightarrow{\tau}_d$ or $\phi_c + \pi/2$. So (C_c) does not have a simple matrix representation as $(C_N)_\ell$ in (2.9). In a more general operator representation we have

$$(C_N)_\ell \to \left(C_{n,m}\left(\frac{2\pi\ell}{N}\right)\right) \cdot (\quad) \cdot \left(C_{n,m}\left(\frac{-2\pi\ell}{N}\right)\right)$$

$$(C_c) \to \overleftrightarrow{\tau}_c \cdot \overleftrightarrow{\tau}_d \cdot (\quad)^{-1} \cdot \overleftrightarrow{\tau}_d^T \cdot \overleftrightarrow{\tau}_c^T$$

$$= \left(C_{n,m}\left(\phi_c + \frac{\pi}{2}\right)\right) \cdot (\quad)^{-1} \cdot \left(C_{n,m}\left(-\phi_c - \frac{\pi}{2}\right)\right) \tag{2.11}$$

$$(C_c)^2 = (C_N)_1$$

Here $\overleftrightarrow{\tilde{y}}_s$ goes in the empty parentheses to produce the rotation as in $(C_N)_\ell$ or the self complement as in (C_c) as well as combinations of these. Note that (C_c) corresponds to rotation by ϕ_c (self-complementary rotation angle) plus $\pi/2$ (corresponding to interchange of E and H) as well as inverse (of the sheet admittance dyad).

So now define the self-complementary rotation group (order 2N) by

$$C_{Nc} \equiv \left\{(C_N)_\ell, (C_N)_\ell\,(C_c) \mid \ell = 1, 2, \cdots, N\right\} \tag{2.12}$$

This has C_N (order N) as a subgroup of index 2. It is isomorphic to (has the same group structure as) rotation-reflection symmetry S_{2N} [11] where the operation (C_c) is replaced by (R_t) which is reflection through a plane transverse (perpendicular) to the symmetry axis. Note that all the elements of C_{Nc} commute, i.e.,

$$\begin{gathered}
(C_N)_{\ell_1}\,(C_N)_{\ell_2} = (C_N)_{\ell_2}\,(C_N)_{\ell_1} \\
(C_N)_\ell\,(C_c) = (C_c)\,(C_N)_\ell \\
(C_N)_{\ell_1}\,(C_N)_{\ell_2}\,(C_c) = (C_N)_{\ell_2}\,(C_c)\,(C_N)_{\ell_1} \\
(C_N)_{\ell_1}\,(C_c)\,(C_N)_{\ell_2}\,(C_c) = (C_N)_{\ell_2}\,(C_c)\,(C_N)_{\ell_1}\,(C_c)
\end{gathered} \tag{2.13}$$

so that the group is commutative (or abelian).

III. Acceptable Kinds of Sheet Admittances

A. Scalar sheet admittance

Let us first choose a scalar form for the sheet admittance as

$$\begin{aligned} &\overleftrightarrow{\tilde{y}}_s(\Psi, \phi; s) \equiv \tilde{y}_s(\Psi, \phi; s) \overleftrightarrow{1}_t \\ &\overleftrightarrow{1}_t \equiv \vec{1}_x \vec{1}_x + \vec{1}_x \vec{1}_x \equiv \text{transverse identity} \end{aligned} \tag{3.1}$$

Then we have the required properties of $\tilde{y}_s$ as

$$\begin{aligned} &\tilde{y}_s^{(c)}(\Psi, \phi; s) \equiv \tilde{y}_s^{-1}(\Psi, \phi; s) \\ &\tilde{y}_s^{(c)}(\Psi, \phi + \phi_c; s) = \tilde{y}_s^{-1}(\Psi, \phi; s) = \tilde{y}_s^{(c)}(\Psi, \phi; s) \\ &\tilde{y}_s(\Psi, \phi + 2\phi_c; s) = \tilde{y}_s(\Psi, \phi; s) \end{aligned} \tag{3.2}$$

Here the rotation matrices are not required (except on the coordinates to rotate ϕ). Clearly C_{Nc} symmetry can be realized with $\tilde{y}_s$ meeting (3.2). Note that a realizable admittance can be any p.r. (positive real) function of s and the reciprocal of a p.r. function is itself p.r. [10].

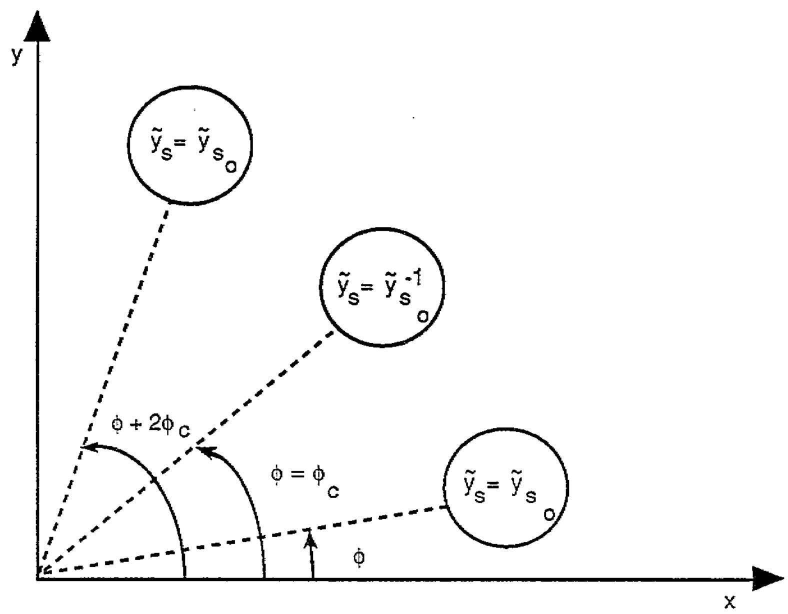
As indicated in Fig. 3.1A, if the sheet admittance is $\tilde{y}_{s_o}$ at some angle ϕ, then the self-complementary pattern has values $\tilde{y}_{s_o}$ and $\tilde{y}_{s_o}^{-1}$ alternating for each increase of ϕ by $\phi_c = \pi / N$. As indicated in fig. 3.1B, there is a special case defined by

$$\tilde{y}_s(\Psi, \phi + \phi_c; s) \equiv \tilde{y}_s(\Psi, \phi; s) = 1 \tag{3.3}$$

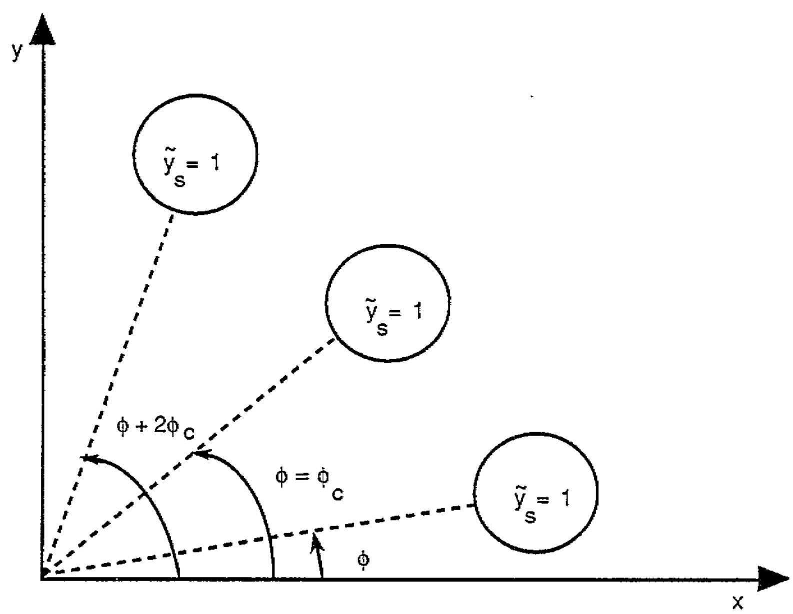
This is more than self complementary since now

$$\tilde{y}_s(\Psi, \phi; s) = \tilde{y}_s^{-1}(\Psi, \phi; s) = 1 \tag{3.4}$$

If there is some region of the z = 0 plane with this normalized admittance, it repeats on rotation by ϕ_c for a total of 2N such regions. This is a special kind of self complementarity which we can call self inverse.



A. Self complementary



B. Self inverse

Fig. 3.1. Scalar Sheet Admittance

B. Dyadic sheet admittance

A more general form for the sheet admittance is found by a diagonal form as

$$\tilde{\vec{\vec{y}}}_s(\Psi, \phi; s) = \tilde{y}_{s_a}(\Psi, \phi; s)\ \vec{1}_a(\phi)\ \vec{1}_a(\phi) = \tilde{y}_{s_b}(\Psi, \phi; s)\ \vec{1}_b(\phi)\ \vec{1}_b(\phi) \tag{3.5}$$

where $\vec{1}_a$ and $\vec{1}_b$ are assumed real. As eigenvectors we have

$$\vec{1}_a(\phi) \cdot \vec{1}_b(\phi) = 0 \tag{3.6}$$

Being at right angles in the (x, y) plane let us take

$$\begin{aligned} \vec{1}_b(\phi) &= \left(C_{n,m}\left(\frac{\pi}{2}\right)\right) \cdot \vec{1}_a(\phi) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \vec{1}_a(\phi) = \vec{\vec{\tau}}_d \cdot \vec{1}_a(\phi) \\ \vec{1}_a(\phi) &= \vec{\vec{\tau}}_d^{\,T} \cdot \vec{1}_b(\phi) = -\vec{\vec{\tau}}_d \cdot \vec{1}_b(\phi) \end{aligned} \tag{3.7}$$

This assumed form has a convenient approximate realization as a grid of admittance elements, $\tilde{y}_{s_a}$ conducting currents in the $\vec{1}_a$ direction and $\tilde{y}_{s_b}$ conducting currents in the $\vec{1}_b$ direction.

Noting that the inverse takes the form

$$\tilde{\vec{\vec{y}}}_s^{-1}(\Psi, \phi; s) = \tilde{y}_{s_a}^{-1}(\Psi, \phi; s)\ \vec{1}_a(\phi)\ \vec{1}_a(\phi) + \tilde{y}_{s_b}^{-1}(\Psi, \phi; s)\ \vec{1}_b(\phi)\ \vec{1}_b(\phi) \tag{3.8}$$

then the complementary sheet admittance in (2.1) takes the form

$$\begin{aligned} \tilde{\vec{\vec{y}}}_s^{(c)}(\Psi, \phi; s) &= \vec{\vec{\tau}}_d \cdot \tilde{\vec{\vec{y}}}_s^{-1}(\Psi, \phi; s) \cdot \vec{\vec{\tau}}_d^{\,T} \\ &= \tilde{y}_{s_a}^{-1}(\Psi, \phi; s)\vec{\vec{\tau}}_d \cdot \vec{1}_a(\phi)\ \vec{1}_a(\phi) \cdot \vec{\vec{\tau}}_d^{\,T} + \tilde{y}_{s_b}^{-1}(\Psi, \phi; s)\vec{\vec{\tau}}_d \cdot \vec{1}_b(\phi)\ \vec{1}_b(\phi) \cdot \tau_d^T \\ &= \tilde{y}_{s_a}^{-1}(\Psi, \phi; s)\vec{1}_b(\phi)\ \vec{1}_b(\phi) + \tilde{y}_{s_b}^{-1}\ \vec{1}_a(\phi)\ \vec{1}_a(\phi) \end{aligned} \tag{3.9}$$

Observe the interchange of the eigenvectors along with the inverse of the eigenvalues. From (2.2) the self-complementary condition gives

$$\tilde{\tilde{y}}_s(\Psi,\ \phi+\phi_c;\ s) = \tilde{\tau}_c \cdot \tilde{\tilde{y}}_s^{(c)}(\Psi,\ \phi;\ s) \cdot \tilde{\tau}_c^T$$

$$= \tilde{y}_{s_a}^{-1}(\Psi,\ \phi;\ s)\tilde{\tau}_c \cdot \vec{1}_b(\phi)\ \vec{1}_b(\phi) \cdot \tilde{\tau}_c^T + \tilde{y}_{s_b}^{-1}(\Psi,\ \phi;\ s)\tilde{\tau}_c \cdot \vec{1}_a(\phi)\ \vec{1}_a(\phi) \cdot \tilde{\tau}_c^T$$

$$= \tilde{y}_{s_a}^{-1}(\Psi,\ \phi;\ s)\ \vec{1}_b(\phi+\phi_c)\ \vec{1}_b(\phi+\phi_c) + \tilde{y}_{s_b}^{-1}(\Psi,\ \phi;\ s)\ \vec{1}_a(\phi+\phi_c)\ \vec{1}_a(\phi+\phi_c) \quad (3.10)$$

where we have also required the eigenvectors to rotate by ϕ_C in going from ϕ to ϕ_C as

$$\vec{1}_a(\phi + \phi_c) = \tilde{\tau}_c \cdot \vec{1}_a(\phi) \quad , \quad \vec{1}_b(\phi + \phi_c) = \tilde{\tau}_c \cdot \vec{1}_b(\phi) \quad (3.11)$$

Writing out the sheet admittance at $\phi + \phi_C$ gives

$$\tilde{\tilde{y}}_s(\Psi,\ \phi + \phi_c;\ s) = \tilde{\tilde{y}}_{s_a}(\Psi,\ \phi + \phi_c;\ s)\ \vec{1}_a(\phi+\phi_c)\ \vec{1}_a(\phi + \phi_c)$$

$$+ \tilde{y}_{s_b}(\Psi,\ \phi + \phi_c\ ;s)\ \vec{1}_b(\phi + \phi_c)\ \vec{1}_b(\phi + \phi_c) \quad (3.12)$$

Comparing this to 3.10) gives

$$\tilde{y}_{s_a}(\Psi,\ \phi + \phi_c;\ s) \quad = \quad \tilde{y}_{s_b}^{-1}(\Psi,\ \phi;\ s)$$

$$\tilde{y}_{s_b}(\Psi,\ \phi + \phi_c;\ s) \quad = \quad \tilde{y}_{s_a}^{-1}(\Psi,\ \phi;\ s) \quad (3.13)$$

Since $\tilde{y}_{s_a}$ and $\tilde{y}_{s_b}$ are assumed to be realizable p.r. admittances, then so are their reciprocals. Then the self-complementary dyadic admittance in (3.10) is also realizable, giving C_{Nc} symmetry.

Applying (3.10) twice (or (2.4) and (3.5)) gives

$$\tilde{\tilde{y}}_s(\Psi,\ \phi+2\phi_c;\ s) = \tilde{\tau}_c \cdot \tilde{\tilde{y}}_s^{(c)}(\Psi,\ \phi+\phi_c;\ s) \cdot \tilde{\tau}_c^T = \tilde{\tau}_c^2 \cdot \tilde{\tilde{y}}_s(\Psi,\ \phi;\ s) \cdot \left(\tilde{\tau}_c^2\right)^T$$

$$= \tilde{y}_{s_a}(\Psi,\ \phi;\ s)\vec{1}_a(\phi+2\phi_c)\ \vec{1}_a(\phi+2\phi_c) + \tilde{y}_{s_b}(\Psi,\ \phi;\ s)\ \vec{1}_b(\phi+2\phi_c)\ \vec{1}_b(\phi+2\phi_c)$$

$$\vec{1}_a(\phi+2\phi_c) = \tilde{\tau}_c^2 \cdot \vec{1}_a(\phi) \quad , \quad \vec{1}_b(\phi+2\phi_c) = \tilde{\tau}_c^2 \cdot \vec{1}_b(\phi) \quad (3.14)$$

This is just the required C_N symmetry.

As indicated in fig. 3.2A, if the sheet admittance has $\tilde{y}_{s_1}$ and $\tilde{y}_{s_2}$ for $\tilde{y}_{s_a}$ and $\tilde{y}_{s_b}$ respectively at some angle ϕ, then the self complementary pattern has $\tilde{y}_{s_1}$ and $\tilde{y}_{s_2}$ alternating with $\tilde{y}_{s_2}^{-1}$ and $\tilde{y}_{s_1}^{-1}$ respectively parallel to the directions $\vec{1}_a$ and $\vec{1}_b$ respectively. The directions, of course, rotate by ϕ_C at each step in the sequence.

As indicated in fig. 3.2B there is a special case defined by

$$\tilde{\tilde{y}}_s(\Psi,\ \phi + \phi_c) = \tilde{\tau}_c \cdot \tilde{\tilde{y}}_s(\Psi,\ \phi;\ s) \cdot \tilde{\tau}_c^T \tag{3.15}$$

Accounting for rotation in going from ϕ to $\phi + \phi_C$ this is a requirement of periodicity on rotation by ϕ_C instead of only rotation by $2\phi_C$. With the self-complementary condition (2.2) we now have

$$\tilde{\tilde{y}}_s^{(c)}(\Psi,\ \phi; s) = \tilde{\tilde{y}}_s(\Psi,\ \phi;\ s) \tag{3.16}$$

With (2.1) this becomes

$$\tilde{\tilde{y}}_s(\Psi,\ \phi; s) = \tilde{\tau}_d \cdot \tilde{\tilde{y}}_s^{-1}(\Psi,\ \phi;\ s) \cdot \tilde{\tau}_d^T \tag{3.17}$$

Applying this to the diagonal forms (3.5) and (3.8) gives

$$\tilde{y}_{s_a}(\Psi,\ \phi;\ s)\ \vec{1}_a(\phi)\ \vec{1}_a(\phi) + \tilde{y}_{s_b}(\Psi,\ \phi;\ s)\ \vec{1}_b(\phi)\ \vec{1}_b(\phi)$$

$$= \tilde{y}_{s_a}^{-1}(\Psi,\ \phi;\ s)\ \vec{1}_b(\phi)\ \vec{1}_b(\phi) + \tilde{y}_{s_b}^{-1}(\Psi,\ \phi;\ s)\ \vec{1}_a(\phi)\ \vec{1}_a(\phi) \tag{3.18}$$

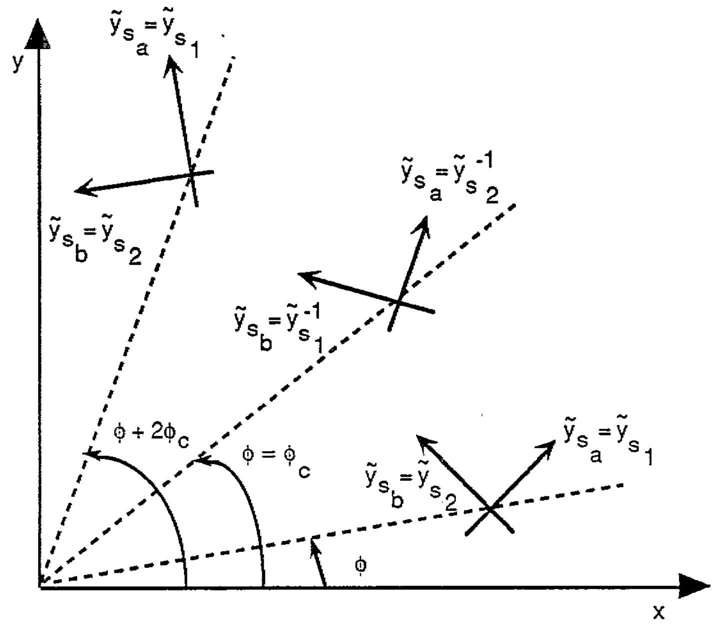
where the interchange of the unit vectors is given by the $\tilde{\tau}_d$ rotation in (3.7). The eigenvalues are then related as

$$\tilde{y}_{s_a}(\Psi,\ \phi;\ s) = \tilde{y}_{s_b}^{-1}(\Psi,\ \phi;\ s) \tag{3.19}$$

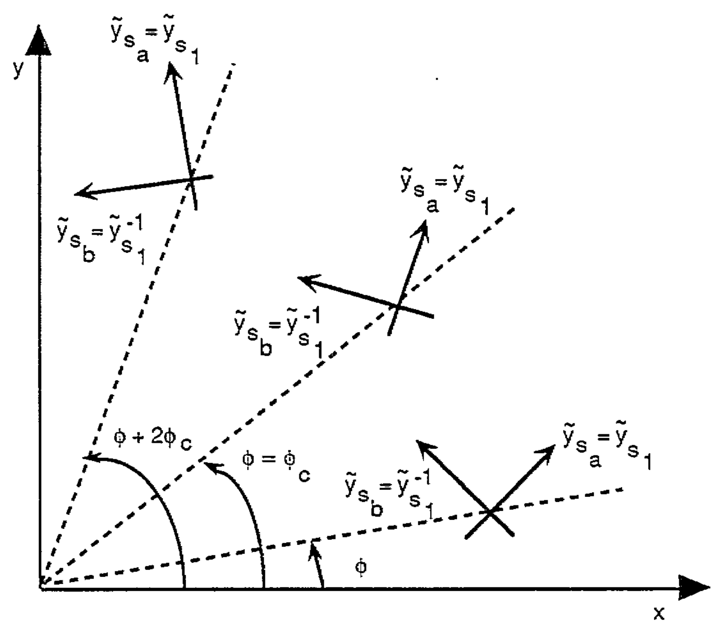
giving

$$\det\left(\tilde{\tilde{y}}_s(\Psi,\ \phi;\ s)\right) = \tilde{y}_{s_a}(\Psi,\ \phi;\ s)\ \tilde{y}_{s_b}(\Psi,\ \phi;\ s) = 1 \tag{3.20}$$

If there is some region of the z=0 plane with this special form of dyadic sheet admittance, it repeats on rotation by ϕ_C for a total of 2N such regions. This is a special kind of self complementarity which we can call a self rotated (by $\pi/2$) inverse.



A. Self complementary



B. Self rotated inverse

Fig. 3.2. Dyadic Sheet Admittance

As indicated in fig. 3.2B, if the sheet admittance has $\tilde{y}_{s_1}$ and $\tilde{y}_{s_1}^{-1}$ for $\tilde{y}_{s_a}$ and $\tilde{y}_{s_b}$ respectively at some angle ϕ, then this pattern repeats at each rotation by ϕ_C. An example of this kind of admittance sheet is a uniconducting sheet specified by

$$\tilde{y}_{s_a}(\Psi, \phi; s) = \infty \quad , \quad \tilde{y}_{s_b}(\Psi, \phi; s) = 0 \tag{3.21}$$

So a uniconducting sheet is its own complement. This is approximately realized by a set of parallel highly conducting wires or strips oriented in the $\vec{1}_a$ direction on the z=0 plane. The spacing between the conductors prevents currents (except for a small capacitive term related to wire or strip width) from flowing in the $\vec{1}_b$ direction.

IV. Complementary Sources and Fields

From (B.14) the surface-current-density sources on S are

$$\vec{J}_s^{(c,s)}(\Psi,\ \phi;\ t) = 2\ \tilde{\tau}_d\ \cdot \vec{H}_s^{(d)}(\Psi,\ \phi;\ t) = -\frac{2}{Z_o}\ \tilde{\tau}_d\ \cdot \vec{E}_s(\Psi,\ \phi;\ t)$$

$$\vec{J}_s^{(s)}(\Psi,\ \phi;\ t) = 2\ \tilde{\tau}_d\ \cdot \vec{H}_s(\Psi,\ \phi;\ t) = \frac{2}{Z_o}\ \tilde{\tau}_d\ \cdot \vec{E}_s^{(d)}(\Psi,\ \phi;\ t) \tag{4.1}$$

Now for a self-complementary structure as in section 3, the sources need not have the same detailed symmetry, but may be more arbitrarily specified.

For general C_{Nc} symmetry we may design an antenna with N currents (sum zero) going into N conductors near Ψ=0 [8]. There are various choices of the currents one may wish to choose including various subgroups of C_N. However, one may ask if sources and associated fields in (4.1) can be self complementary.

Imposing the self-complementary condition rotate the fields and surface current density at ϕ, by ϕ_c' and equate to the dual fields at $\phi + \phi_c'$ giving

$$\vec{J}_s^{(c,s)}(\Psi,\ \phi + \phi_c';\ t) = \tilde{\tau}_c' \cdot \vec{J}_s^{(s)}(\Psi,\ \phi;\ t)$$

$$\vec{E}_s^{(d)}(\Psi,\ \phi + \phi_c';\ t) = \tilde{\tau}_c' \cdot \vec{E}_s(\Psi,\ \phi;\ t)$$

$$\vec{H}_s^{(d)}(\Psi,\ \phi + \phi_c';\ t) = \tilde{\tau}_c' \cdot \vec{H}_s^{(d)}(\Psi,\ \phi;\ t)$$

$$\tilde{\tau}_c' = \begin{pmatrix} \cos(\phi_c') & -\sin(\phi_c') \\ \sin(\phi_c') & \cos(\phi_c') \end{pmatrix} \tag{4.2}$$

Here ϕ_c' can be chosen as ϕ_C or any other angle (such as $3\phi_C$, $5\phi_C$, etc.) for which the self-complementary condition for the structure (sheet admittance) holds as in sections 2 and 3.

From (4.1) we also have the additional relation for the electric field (with (4.2) applied for the surface current density)

$$\vec{E}_s^{(d)}(\Psi,\ \phi + \phi_c';\ t) = -\tilde{\tau}_c' \cdot \vec{E}_s^{(d)}(\Psi,\ \phi;\ t) \tag{4.3}$$

This also takes the form

$$\vec{E}_s^{(d)}(\Psi,\ \phi;\ t) = -\overleftrightarrow{\tau}_c'^{-1} \cdot \vec{E}_s(\Psi,\ \phi + \phi_c';\ t)$$

$$\vec{E}_s^{(d)}(\Psi,\ \phi - \phi_c;\ t) = -\overleftrightarrow{\tau}_c'^{-1} \cdot \vec{E}_s(\Psi,\ \phi;\ t) \tag{4.4}$$

which can be interpreted as negative rotation $\left(-\phi_c' \text{ and } \overleftrightarrow{\tau}_c'^{-1}\right)$, but with a sign inversion.

Combining the results for the electric field from (4.2) and (4.3) gives

$$\vec{E}_s(\Psi,\ \phi;\ t) = -\overleftrightarrow{\tau}_c'^{2} \cdot \vec{E}_s(\Psi,\ \phi;\ t) \tag{4.5}$$

For arbitrary orientation of $\vec{E}_s$ we then have

$$\overleftrightarrow{\tau}_c'^{2} = -\overleftrightarrow{1}_t \equiv \text{rotation by } \pm \pi$$

$$\overleftrightarrow{\tau}_c' = \pm\overleftrightarrow{\tau}_d \equiv \text{rotation by } \pm \pi/2$$

$$\phi_c' = \pm \pi/2 \tag{4.6}$$

This restricts the case of self-complementary sources and fields to the case of C_{2c} symmetry.

Now a self-complementary structure has in general C_{Nc} symmetry. For certain cases C_{Nc} symmetry in the structure allows C_{2c} fields. In particular for

$$\phi_c' = \phi_c,\ 3\phi_c,\ 5\phi_c, \cdots = (2M+1)\ \phi_c$$

$$= \pi/2 \tag{4.7}$$

then

$$\phi_c = \frac{\pi}{N} = \frac{\pi}{2(2M+1)}$$

$$M = 0,\ 1,\ 2, \cdots$$

$$N = 4M+2 = 2,\ 6,\ 10, \cdots \tag{4.8}$$

gives the only possibilities for self-complementary fields. While one can have self-complementary sources and fields as above, other distributions of sources and fields are also possible.

To see this point consider the examples in figs. 4.1 and 4.2. A simple case of C_{2c} symmetry is illustrated in fig. 4.1. In this case the C_{2c} applies to the fields as shown by transforming the original sources and fields in fig. 4.1A to their duals in fig. 4.1B. This transformation is equivalent to rotating both the structure and fields by $+\pi/2$. A similar case of C_{3c} symmetry is illustrated in fig. 4.2. In this case the C_{3c} does not apply to the fields as seen in the transformation of the original sources and fields in fig. 4.2A to their duals in fig. 4.2B. Noting that more than one field pattern is possible on such a structure, the example has one that is symmetric [2, 4] with respect to the y, z plane while the complementary fields are antisymmetric with respect to this plane (noting the interchange of $\vec{E}$ and $\vec{H}$). In the original problem (fig. 4.2A) equal currents are collected from two of the three conducting arms and delivered to the third arm. In the complementary problem (fig. 4.2B) current is collected from one conducting arm and passed to a second, while the third conducting arm has no net current. Clearly these two sets of currents are not self-complementary.

Another interesting feature of C_{2c} symmetry concerns the fields on the z axis. The complementary coordinate is

$$\vec{r}_c = \vec{\vec{\tau}}_d \cdot \vec{r}$$

$$\vec{\vec{\tau}}_d = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{(three dimensional sense)} \tag{4.9}$$

The complementary fields at $\vec{r}_c$ are set equal to the original fields at $\vec{r}$ rotated by $\pi/2$ giving the self-complementary relation

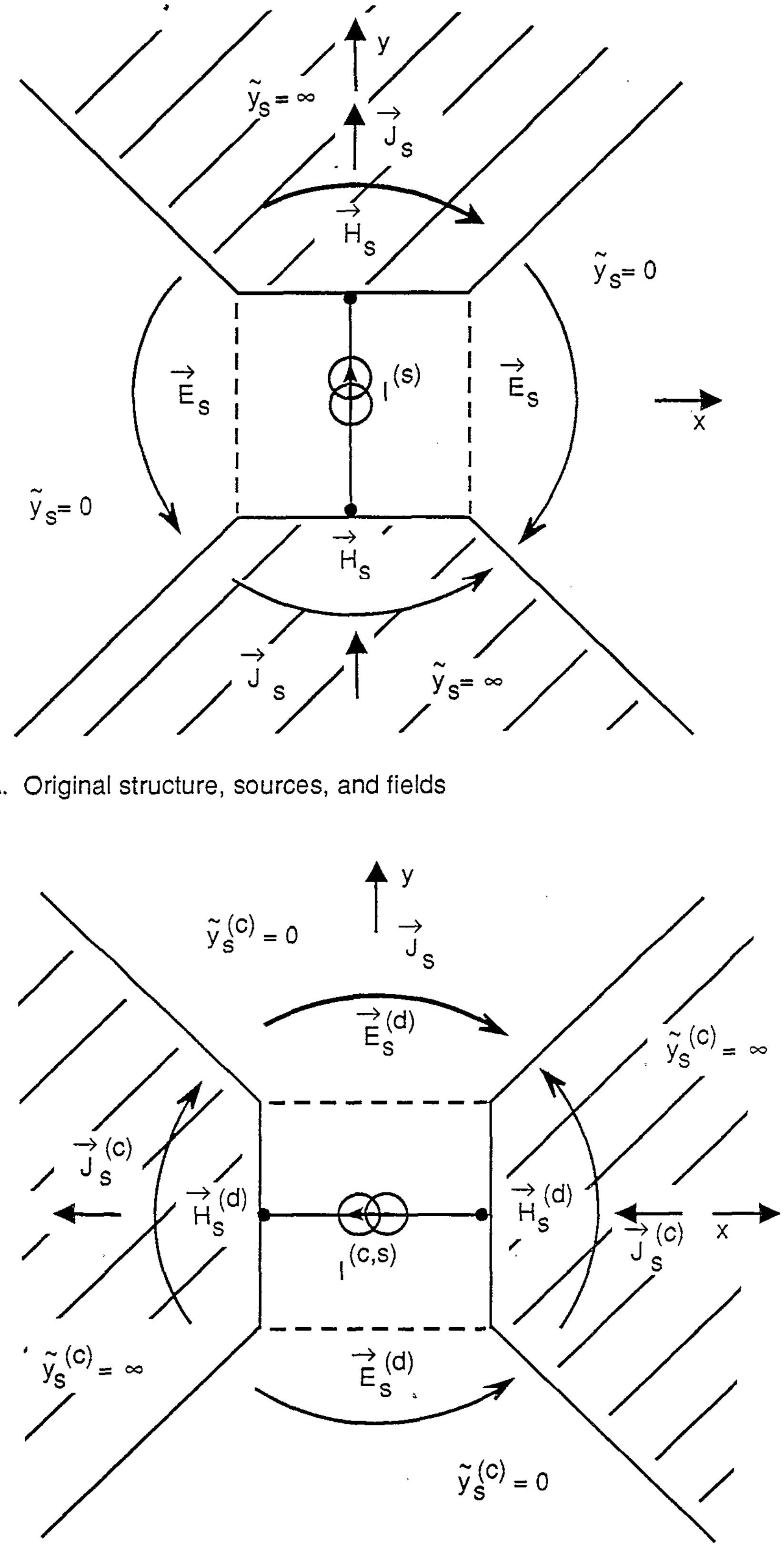
$$\vec{\vec{\tau}}_d \cdot \vec{E}(\vec{r}, t) = \vec{E}^{(d)}(\vec{r}_c, t) = Z_o \; \vec{H}(\vec{r}_c, t)$$

$$\vec{\vec{\tau}}_d \cdot \vec{H}(\vec{r}, t) = \vec{H}^{(d)}(\vec{r}_c, t) = -\frac{1}{Z_o} \; \vec{E}(\vec{r}_c, t) \tag{4.10}$$

In cylindrical coordinates this is

$$\vec{\vec{\tau}}_d \cdot \vec{E}(\Psi, \phi, z; t) = \vec{E}^{(d)}(\Psi, \phi + \frac{\pi}{2}, z; t) = Z_o \; \vec{H}(\Psi, \phi + \frac{\pi}{2}, z; t)$$

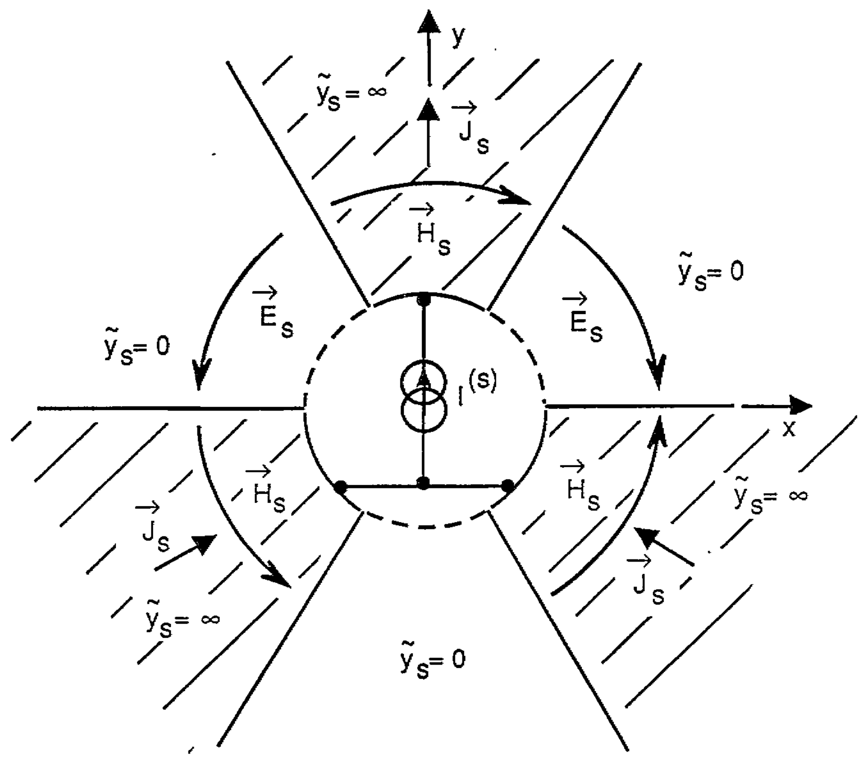
$$\vec{\vec{\tau}}_d \cdot \vec{H}(\Psi, \phi, z; t) = \vec{H}^{(d)}(\Psi, \phi + \frac{\pi}{2}, z; t) = -\frac{1}{Z_o} \; \vec{E}(\Psi, \phi + \frac{\pi}{2}, z; t) \tag{4.11}$$



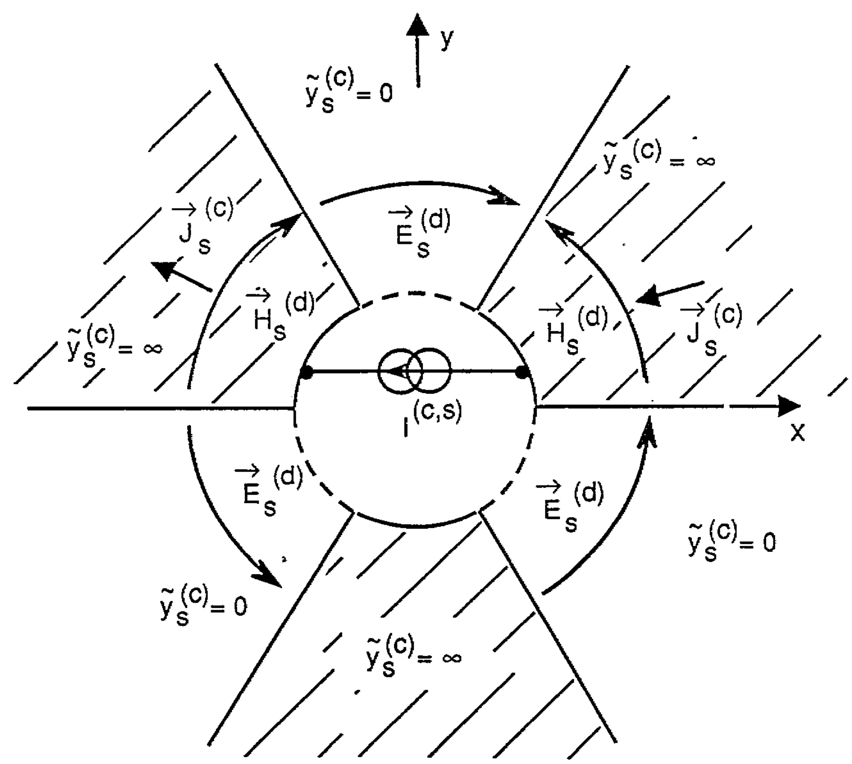
A. Original structure, sources, and fields

B. Self-complementary structure, sources, and fields

Fig. 4.1. Example of C_{2c} Structure, Sources, and Fields



A. Original structure, sources, and fields



B. Self-complementary structure, and complementary sources and fields

Fig. 4.2. Example of C_{3c} Structure with Complementary Sources and Fields

This exhibits the C_{2c} symmetry between the electric and magnetic fields. This applies for z>0, there being a sign reversal on the magnetic field for z<0.

Now (4.11) can combined in a way to give separate equations for electric and magnetic fields as

$$\vec{E}(\Psi,\ \phi+\pi,\ z;\ t) = -\overleftrightarrow{\tau}_d^2 \cdot \vec{E}(\Psi,\ \phi,\ z;\ t)$$

$$\vec{H}(\Psi,\ \phi+\pi,\ z;\ t) = -\overleftrightarrow{\tau}_d^2 \cdot \vec{H}(\Psi,\ \phi,\ z;\ t)$$

$$-\tau_d^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{4.12}$$

This exhibits the C_2 symmetry of the fields. Note that rotation by π is equivalent to two-dimensional inversion through the z axis. The transverse fields (x and y components) are invariant to this transformation while the z component is multiplied by -1. Note that if one considers the components with respect to Ψ and ϕ (instead of x and y) these components do not change sign because $\vec{1}_\Psi$ and $\vec{1}_\phi$ reverse direction under rotation by π.

As a special case consider fields on the z axis so that

$$\vec{r} = z\ \vec{1}_z = \vec{r}_c$$

$$(\Psi,\ \phi,\ z) = (0,\ \phi,\ z) \tag{4.13}$$

with the value of ϕ irrelevant. Then (4.11) reduces to

$$\overleftrightarrow{\tau}_d \cdot \vec{E}(0,\ \phi,\ z;\ t) = Z_o\ \vec{H}(0,\ \phi,\ z;\ t) \tag{4.14}$$

while (4.12) gives

$$E_z(0,\ \phi,\ z;\ t) = 0 \quad , \quad H_z(0,\ \phi,\ z;\ t) = 0 \tag{4.15}$$

Thus on the z axis the fields are purely transverse, and the electric and magnetic field are mutually orthogonal, related by the free-space wave impedance. This is the usual relation for an outward propagating TEM wave.

It is well known that two or more conical perfect conductors with a common apex propagate one or more TEM modes [13]. The example geometry in fig. 4.1 consists of two flat-plate cones in which case the fields are everywhere TEM, propagating radially outward. However symmetry is not restricted to conical shapes or perfect conductors. Various strange shapes (scimitars, saw-toothed shapes, etc.) are also allowed [8, 12]. In this case we still have the TEM result on the z axis, even in the near field.

V. Implications of C_{2c} Symmetry

A. Self-Complementary Antenna

There is already the well-known result of a self-complementary antenna [7, 8]. As indicated in fig. 5.1 this is typically taken as two perfectly conducting sheets with nothing on the complementary portions of S. Derivations can be found in various references [1, 3].

For the present the use of complementary sources as in section 4 and appendix B makes the derivation straightforward. In fig. 5.1 the region near $\Psi=0$ is taken as a source region which is traversed by contours C_e for voltage and C_h for current giving sources driving the antenna as

$$V^{(s)} = -\int_{C_e} \vec{E}_s \cdot d\vec{\ell} = Z_o \int_{C_e} \vec{H}_s^{(d)} \cdot d\vec{\ell}$$

$$I^{(s)} = -2\int_{C_h} \vec{H}_s \cdot d\vec{\ell} = \frac{2}{Z_o}\int_{C_h} \vec{E}_s^{(d)} \cdot d\vec{\ell} \qquad (5.1)$$

Note the factor of 2 in the current expression accounting for the closed contour integral surrounding the source region (z=0+ and z=0-). Note from (4.1) the current can also be expressed directly in terms of a source surface current density as

$$I^{(s)} = \int_{C_h}\left[\vec{\tau}_d \cdot \vec{J}_s^{(s)}\right] \cdot d\vec{\ell} = -\int_{C_h} \vec{J}_s^{(s)} \cdot \vec{\tau}_d \cdot d\vec{\ell} \qquad (5.2)$$

Next, take the complementary problem interchanging the position of conducting sheets with the empty portions of S. Now note that C_e and C_h are defined such that rotating C_e by $\pi/2$ (or by $\vec{\tau}_d$) gives C_h exactly. In this complementary problem we have voltage and current as

$$V^{(c,s)} = -\int_{C_h} \vec{E}_s^{(d)} \cdot d\vec{\ell} = -Z_o \int_{C_h} \vec{H}_s \cdot d\vec{\ell}$$

$$I^{(c,s)} = 2\int_{C_e} \vec{H}_s^{(d)} \cdot d\vec{\ell} = -\frac{2}{Z_o}\int_{C_e} \vec{E}_s \cdot d\vec{\ell} \qquad (5.3)$$

Note the reversal of roles of C_e and C_h and the reversal of sense of direction for integration on C_e for now finding the current. From (4.1) the complementary current can also be expressed in terms of the complementary source surface current density

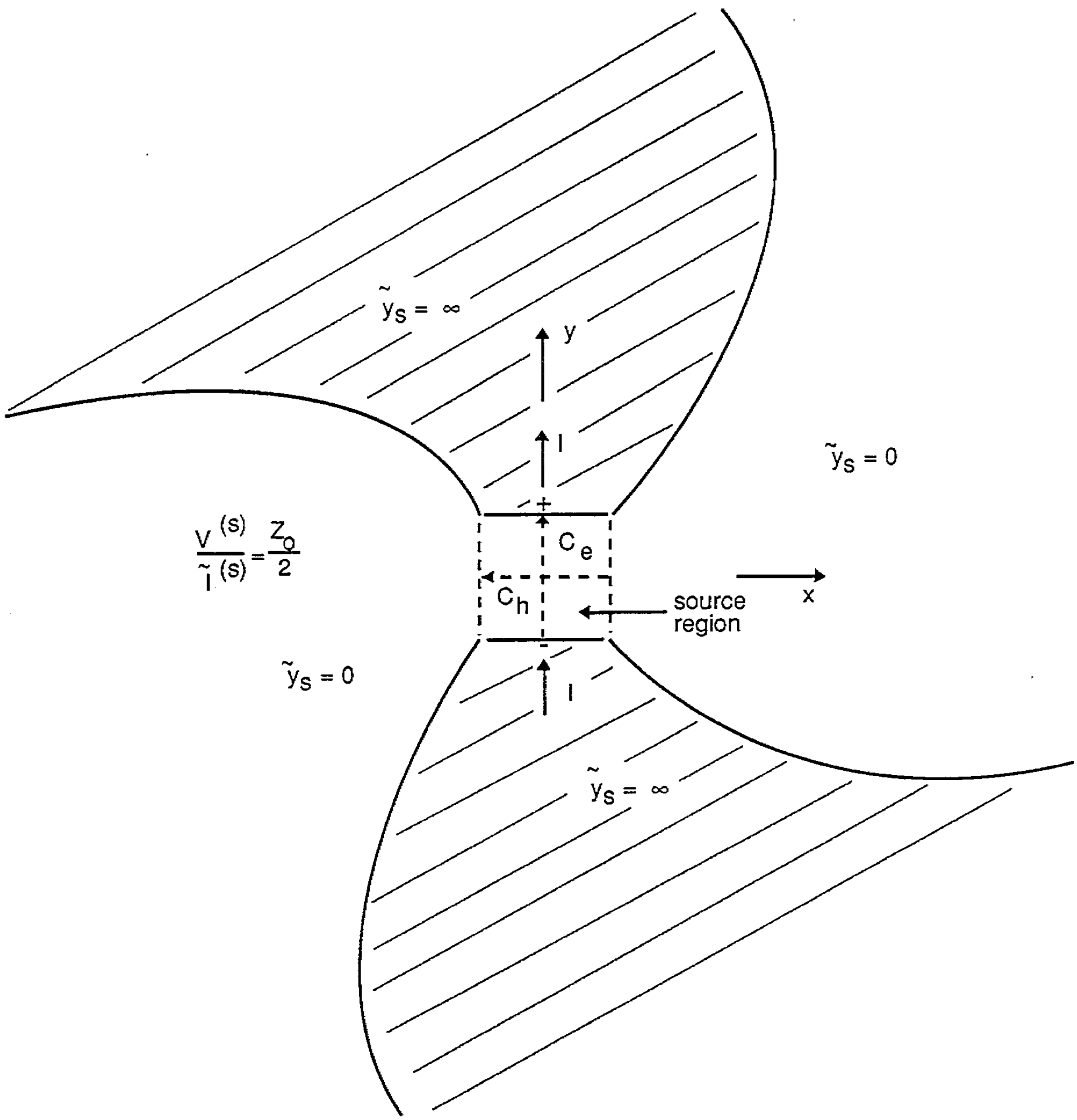


Fig. 5.1. Self-Complementary Antenna

$$I^{(c,s)} = \int_{C_e}\left[\vec{\tau}_d \cdot \vec{J}_s^{(s,c)}\right] \cdot d\vec{\ell} = \int_{C_e} \vec{J}_s^{(s,c)} \cdot \vec{\tau}_d \cdot d\vec{\ell} \tag{5.4}$$

Note that the complementary problem is the same as the original except for the rotation (i.e. self complementarity). Further note that this source region has $\vec{E}_s \times \vec{H}_s$ pointing outward (z positive) with $\vec{H}_s$ then a positive rotation from $\vec{E}_s$. This allows us to equate (with +sign)

$$V^{(c,s)} = V^{(s)} \quad , \quad I^{(c,s)} = I^{(s)} \tag{5.5}$$

Identifying common terms in (5.1) and (5.3) gives

$$V^{(s)} = \frac{Z_o}{2} I^{(c,s)} \quad , \quad V^{(c,s)} = \frac{Z_o}{2} I^{(s)} \tag{5.6}$$

Thus we have

$$\tilde{Z}_{in} \equiv \frac{\tilde{V}^{(s)}}{\tilde{I}^{(s)}} = \frac{Z_o}{2} \tag{5.7}$$

which the usual result for the input impedance of a C_{2c} self-complementary antenna.

B. Self-Complementary Antenna with Exterior Resistive Termination

Extending the usual concept of a self complementary antenna allow more general self-complementary admittances as discussed in section 3. One case of this is illustrated in fig. 5.2. Here we begin with the usual two conductors near z=0 with $\phi_c = \pi/2$. After going out some distance, say roughly Ψ_o, define a boundary with C_4 symmetry., beyond which we have a sheet impedance of $Z_o/2$ (corresponding to $\tilde{y}_s = 1$).

One of the problems with a real self-complementary antenna concerns the requirement to extend to infinite radius ($\Psi = \infty$) in the z=0 plane. At low frequencies the scheme in fig. 5.1 with conductors only has $\tilde{Z}_{in} \to \infty$ as $s \to 0$ if the conductors are truncated. The scheme in fig. 5.2 might have the resistive-sheet terminator truncated at some large radius Ψ_1. In this case $\tilde{Z}_{in}$ is resistive for low frequencies and

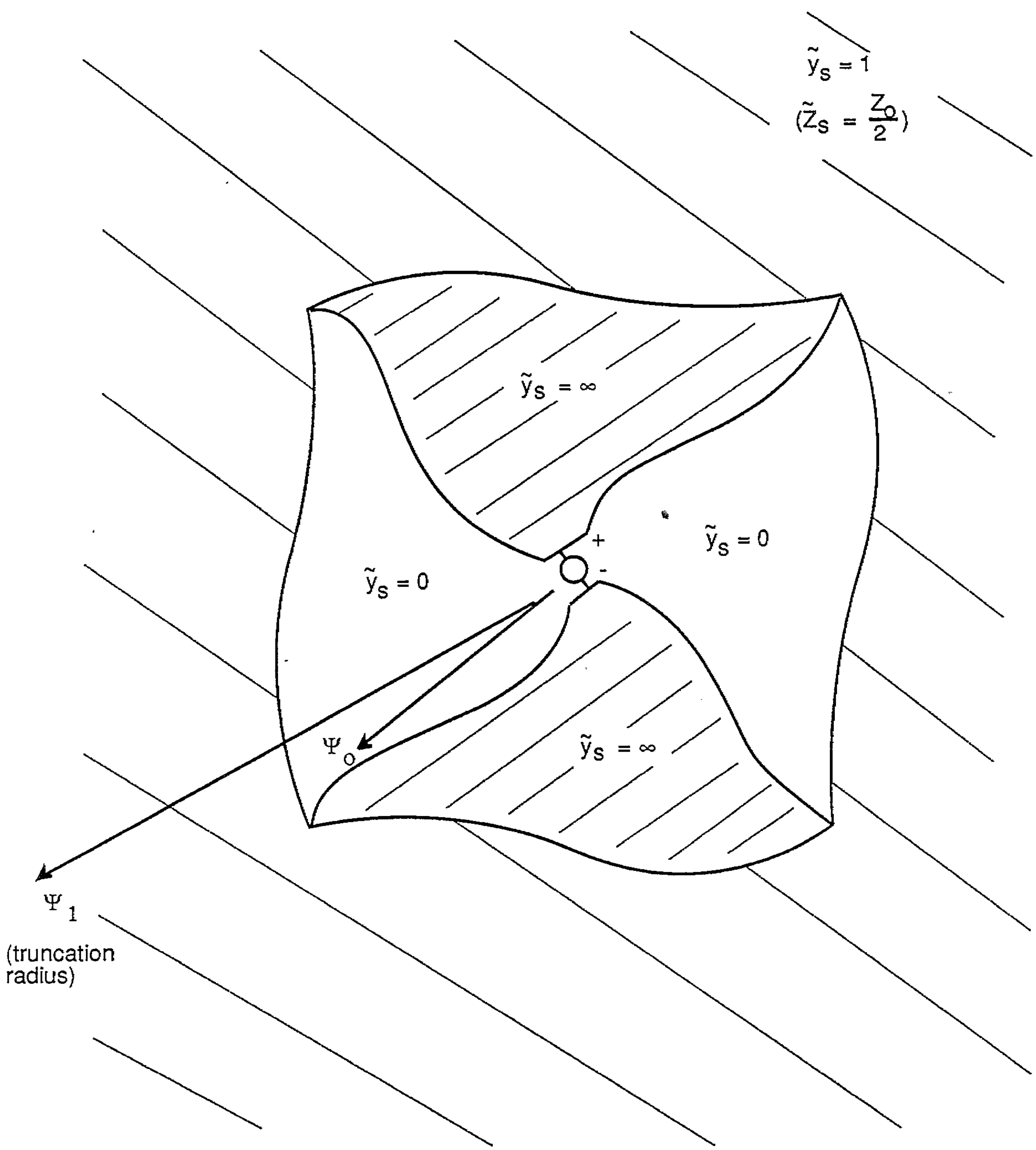


Fig. 5.2. Self-Complementary Antenna with Exterior Resistive Termination

$$\tilde{Z}_{in} \to \frac{Z_o}{2} \qquad \text{as } s \to 0$$

$$\text{and } \frac{\Psi_1}{\Psi_o} \to \infty \tag{5.8}$$

with approximate equality for $\Psi_1 >> \Psi_0$.

Practically this means that in terms of input impedance a "finite-size" self-complementary antenna can be built with the frequency-independent $\tilde{Z}_{in}$ in (5.7) extending down to D.C. Finite Ψ_1 means the value at D.C. is not exactly $Z_0/2$, but by modifying the resistive termination value and/or shape in its outer portions a value of $Z_0/2$ at D.C. may be realized.

Note that many other kinds of self-complementary sheets as discussed in section 4 can be included in this antenna design.

C. Self-Complementary Resistor

A related structure is the self-complementary resistor with C_{2c} symmetry as illustrated in fig. 5.3. As we approach Ψ=0 on the z=0 plane there is some ambiguity due to the self-complementary requirement in rotation at this point. This can be resolved by noting that the scalar self inverse described by $\tilde{y}_s = 1$ (or $\tilde{Z}_s = Z_o / 2$) as discussed in section 3 is an acceptable choice at a point. Considering the dyadic self-rotated inverse we find that $\tilde{y}_{s_1} = \tilde{y}_{s_s}$ so that the <u>only</u> self-complementary choice at z=0 is

$$\tilde{\tilde{y}}_s(0, \phi; s) = \tilde{\tilde{1}}_t$$

$$\tilde{y}_s(0, \phi; s) = 1$$

$$\tilde{Z}_s(0, \phi; s) = \frac{Z_o}{2} \tag{5.9}$$

So let us consider some such region near z=0 characterized by this choice. Considering the case of C_{2c} symmetry let this region have the C_{2c} and C_4 symmetry as in fig. 5.3 with connection to conductors and free space with C_{2c} symmetry outside this region. Let this outside region contain any source regions as well, but the detailed exterior location is unimportant. It is, however, important that there be two each electric and magnetic boundaries alternating around the resistive region.

Another way to look at this case is to take the geometry in fig. 5.2 and perform an inversion of $\Psi \to \Psi^{-1}$. This does not change any angles and such transformations with respect to the cylindrical

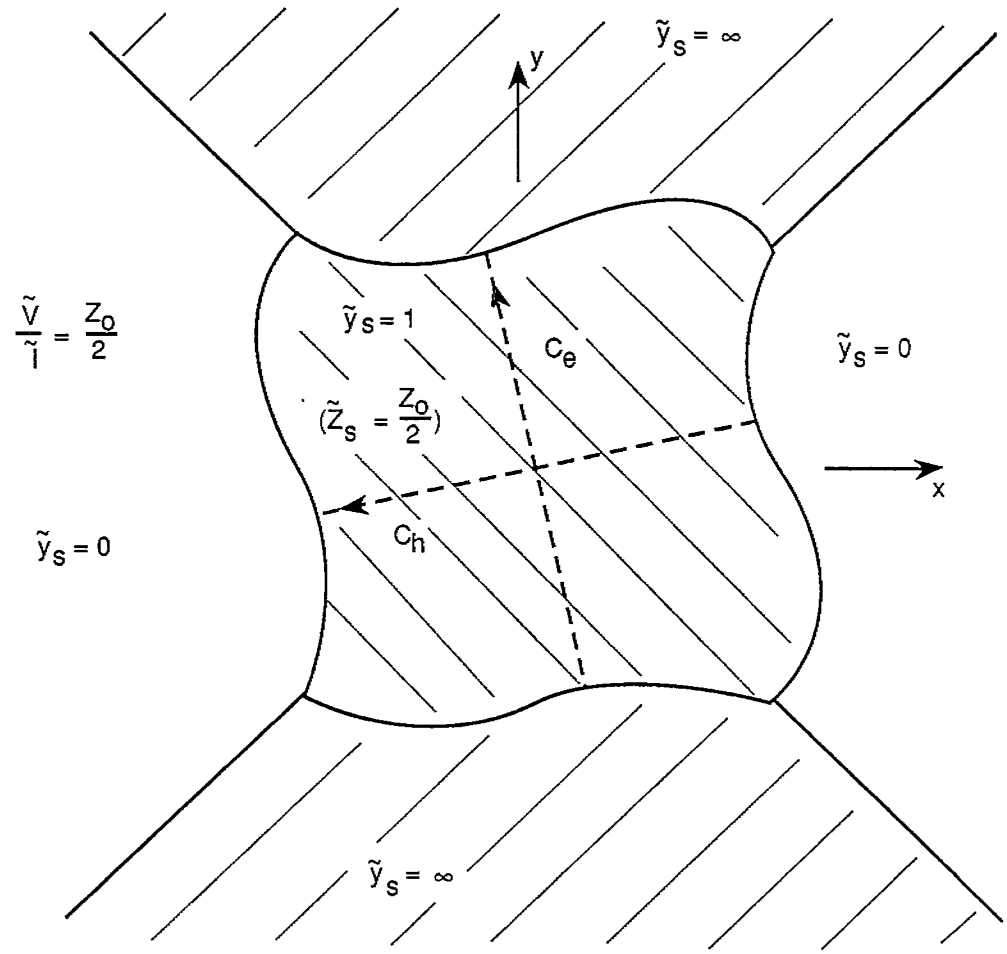


Fig. 5.3. Self-Complementary Resistor

radius Ψ leave C_{Nc} symmetry unchanged. In inversion the exterior resistive termination ($\tilde{y}_s = 1$) maps into the region near Ψ=0.

Considering the C_e and C_h contours in fig. 5.3 note that C_e goes into C_h upon rotation by $\pi/2$, just as in fig. 5.1. The boundaries of the resistive region (two electric and two magnetic) need not be straight but only need to rotate into each other on successive $\pi/2$ rotation.

Now evaluate voltage and current for the resistive region as

$$V = - \int_{C_e} \vec{E}_s \cdot d\vec{\ell} = Z_o \int_{C_e} \vec{H}_s^{(d)} \cdot d\vec{\ell}$$

$$I = 2 \int_{C_h} \vec{H}_s \cdot d\vec{\ell} = \frac{2}{Z_o} \int_{C_h} \vec{E}_s^{(d)} \cdot d\vec{\ell} \tag{5.10}$$

Note that this differs from (5.1) in the reversal of sign for the current since this is a resistive region with

$$\vec{J}_s = \frac{Z_o}{2} \vec{E}_s = 2 \; \vec{\tau}_d \cdot \vec{H}_s \tag{5.11}$$

i.e. with surface current density parallel to $\vec{E}_s$ instead of opposite to it as in a source region.

For the complementary problem there is the choice of sense of rotation in defining voltage and current. For the source problem the rotation sense has been taken as $+\pi/2$ (or increasing ϕ). In (5.1) and (5.3) this gives the source voltages and currents the same sign. For the complementary resistor problem let us take the rotation sense as $-\pi/2$ giving

$$V^{(c)} = \int_{C_h} \vec{E}_s^{(d)} \cdot d\vec{\ell} = Z_o \int_{C_h} \vec{H}_s \cdot d\vec{\ell}$$

$$I^{(c)} = 2 \int_{C_e} \vec{H}_s^{(d)} \cdot d\vec{\ell} = -\frac{2}{Z_o} \int_{C_e} \vec{E}_s \cdot d\vec{\ell} \tag{5.12}$$

Of course, C_e should rotate into C_h in this $-\pi/2$ rotation. Note that we have kept the same orientation for C_e and C_h in figs. 5.1 and 5.3. Again the surface current density and electric field are parallel in this self-complementary resistive region as

$$\vec{J}_s^{(c)} = \frac{Z_o}{2} \vec{E}_s^{(d)} = \vec{\tau}_d \cdot \vec{H}_s^{(d)} \tag{5.13}$$

Now the resistive region has $\tilde{E}_s \times \tilde{H}_s$ pointing inward with $\tilde{H}_s$ then a negative rotation from $\tilde{E}_s$. Consistent with the conventions in (5.10) and (5.12) we can equate

$$V^{(c)} = V \ , \quad I^{(c)} = I \tag{5.14}$$

Identifying common terms in (5.10) and (5.12) gives

$$V = \frac{Z_o}{2} I^{(c)} \quad , \quad V^{(c)} = \frac{Z_o}{2}\, I \tag{5.15}$$

Thus we have the impedance of the resistive region as

$$\tilde{Z} \equiv \frac{\tilde{V}}{\tilde{I}} = \frac{Z_o}{2} \tag{5.16}$$

which we can regard as the resistance of a self-complementary resistor. Note that, strictly speaking, to consider this resistive region to have a uniquely defined impedance in the circuit-element sense, this region should be electrically small for frequencies of interest. Then the electric boundaries can be taken as constant potentials. In this case the rest of the structure outside the self-complementary resistor (including its electric and magnetic boundaries) can in general be neglected.

A special case of such a self-complementary resistor has the shape of a square with conducting contacts along two opposite sides. This evidently has the resistance equal to the sheet resistance (ohms per square for any size square), consistent with (5.16). The more interesting thing is that this result holds for any C_4 shape with C_{2c} boundaries (alternating electric and magnetic).

D. C_4 Geometry for Lumped Impedances

Extending the results of the previous subsection replace the $Z_0/2$ sheet impedance by an arbitrary (uniform) $\tilde{Z}_s$ as indicated in fig. 5.4. Assuming the sheet is sufficiently electrically small (so that one can neglect parasitic capacitances and inductances) and has a C_4 shape with four alternating identical electric and magnetic boundaries, its impedance is found by scaling (quasistatic, multiplying all elementary pieces of the sheet resistance by the same constant) as just

$$\tilde{Z} = \tilde{Z}_s \tag{5.17}$$

This permits a host of shapes (four pointed stars, circles, etc.), all of which can be simply calculated.

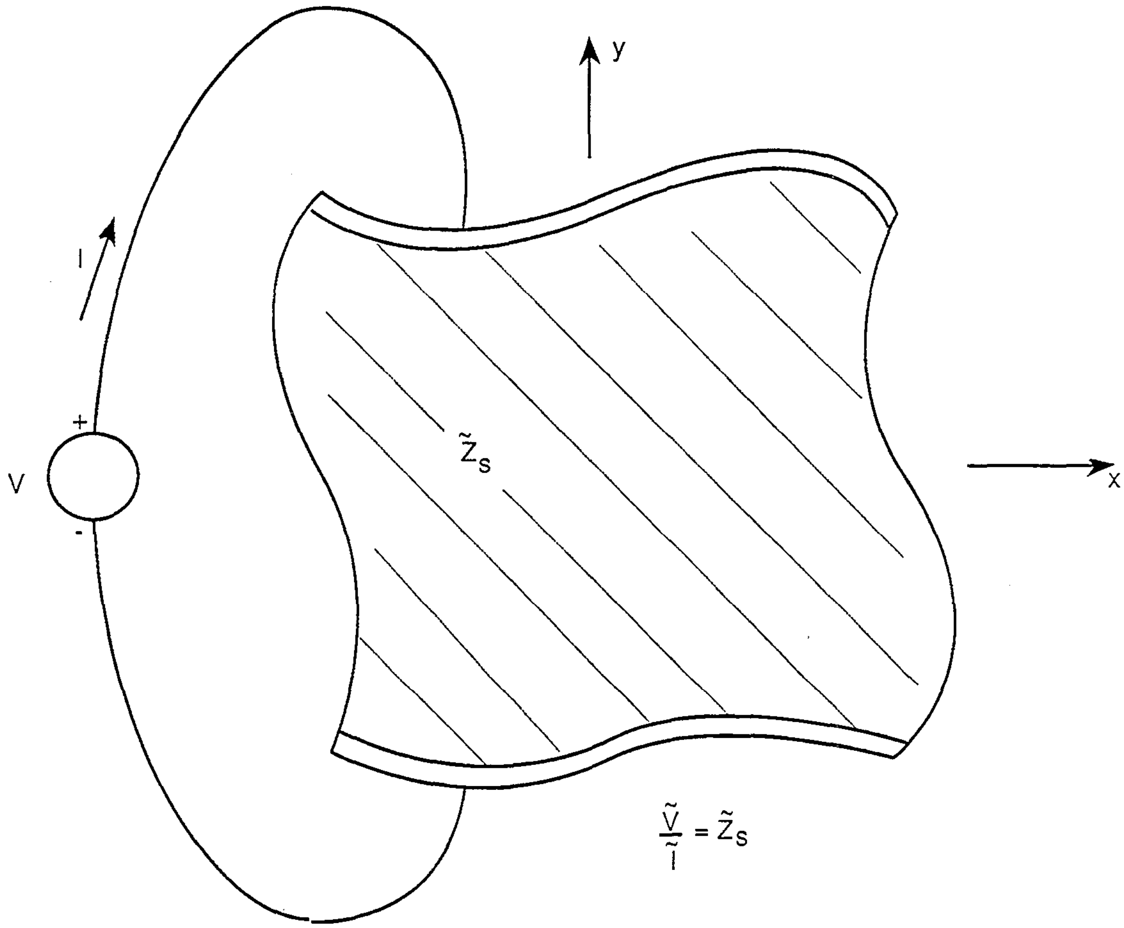


Fig. 5.4. C_4 Impedance Sheet with Alternating Electric and Magnetic Boundaries

VI. Complementary Incident and Scattered Fields

Consider an incident plane wave as

$$\vec{E}^{(inc)}(\vec{r},t) = E_o\, f\left(t - \frac{\vec{1}_1 \cdot r}{c}\right)\vec{1}_2 \quad , \quad \vec{H}^{(inc)} = \frac{E_o}{Z_o}\, f\left(t - \frac{\vec{1}_1 \cdot r}{c}\right)\vec{1}_3$$

$$\tilde{\vec{E}}^{(inc)}(\vec{r},s) = E_o\, \tilde{f}(s)e^{-\gamma\vec{1}_1\cdot\vec{r}}\,\vec{1}_2 \quad , \quad \tilde{\vec{H}}^{(inc)}(\vec{r},s) = \frac{E_o}{Z_o}\, \tilde{f}(s)e^{-\gamma\vec{1}_1\cdot\vec{r}}\,\vec{1}_3$$

$$\vec{1}_1 \times \vec{1}_2 = \vec{1}_3,\ \vec{1}_2 \times \vec{1}_3 = \vec{1}_1,\ \vec{1}_3 \times \vec{1}_1 = \vec{1}_2 \tag{6.1}$$

$\gamma = \frac{s}{c} \equiv$ propagation constant

$\vec{1}_1 \equiv$ direction of incidence

Just as in (4.10) the rotated incident fields are set equal to the complementary fields at $\vec{r}_c$ as

$$\vec{\vec{\tau}}_d \cdot \vec{E}^{(inc)}(\vec{r},t) = \vec{E}^{(d,inc)}(\vec{r}_c,t) = Z_o\, \vec{H}^{(inc)}(\vec{r}_c,t)$$

$$\vec{\vec{\tau}}_d \cdot \vec{H}^{(inc)}(\vec{r},t) = \vec{H}^{(d,inc)}(\vec{r}_c,t) = -\frac{1}{Z_o}\, \vec{E}^{(inc)}(\vec{r}_c,t)$$

$$\vec{r}_c = \vec{\vec{\tau}}_d \cdot \vec{r} \tag{6.2}$$

with $\vec{\vec{\tau}}_d$ taken in the three-dimensional sense of (4.9). Note that the $\pi/2$ (or $-\pi/2$) rotation is the only possible rotation for self complementary fields as discussed in section 4. For a plane wave this is simply related to the right-angle relationship of the fields.

First noting the phase on a constant z plane, and making it the same for $\vec{r}$ and $\vec{r}_c$ in accordance with (6.1) and (6.2) gives

$$\vec{1}_1 \cdot \vec{r} = \vec{1}_1 \cdot \vec{r}_c = \vec{1}_1 \cdot \vec{\vec{\tau}}_d \cdot \vec{r} = z \tag{6.3}$$

consistent with $\vec{\vec{\tau}}_d$ being a rotation for constant z. This establishes

$$\vec{1}_1 = \vec{1}_z \tag{6.4}$$

as the propagation constant. (One could have $-\vec{1}_z$ for $-\pi/2$ rotation.) Furthermore (6.2) gives for the other unit vectors

$$\vec{\tau}_d \cdot \vec{1}_2 = \vec{1}_3 \quad , \quad \vec{\tau}_d \cdot \vec{1}_3 = -\vec{1}_2 \tag{6.5}$$

so that

$$\vec{\tau}_d \cdot = \vec{1}_1 \times \tag{6.6}$$

with only a right-angle relationship allowed.

As discussed in appendix C the incident field gives an equivalent source as

$$\tilde{\vec{J}}_s^{(s)}(\Psi,\ \phi;\ s) = \tilde{\vec{Y}}_s(\Psi,\ \phi;\ s) \cdot \tilde{\vec{E}}_s^{(inc)}(\Psi,\ \phi;\ s) \tag{6.7}$$

The source for the complementary problem is just

$$\tilde{\vec{J}}_s^{(c,s)}(\Psi,\ \phi;\ s) = \tilde{\vec{Y}}_s^{(c)}(\Psi,\ \phi;\ s) \cdot \tilde{\vec{E}}_s^{(d,inc)}(\Psi,\ \phi;\ s) \tag{6.8}$$

As in (4.2) the self-complementary condition equates the complementary sources and fields at $\phi' + \phi_c'$ to the original quantities at ϕ rotated by ϕ_c'. As in (4.6) this angle can be only $\pm\pi/2$.

So we set (two-dimensional sense)

$$\tilde{\vec{Y}}_s^{(c)}\left(\Psi,\ \phi + \frac{\pi}{2};\ s\right) \cdot \tilde{\vec{E}}_s^{(d,inc)}\left(\Psi,\ \phi + \frac{\pi}{2};\ s\right) = \vec{\tau}_d \cdot \tilde{\vec{Y}}_s(\Psi,\ \phi;\ s) \cdot \tilde{\vec{E}}^{(inc)}(\Psi,\ \phi;\ s) \tag{6.9}$$

Replacing $\tilde{\vec{E}}_s^{(d,inc)}$ from (4.2) gives

$$\tilde{\vec{Y}}_s^{(c)}\left(\Psi,\ \phi + \frac{\pi}{2};\ s\right) \cdot \vec{\tau}_d \cdot \tilde{\vec{E}}^{(inc)}(\Psi,\ \phi;\ s) = \vec{\tau}_d \cdot \tilde{\vec{Y}}_s(\Psi,\ \phi;\ s) \cdot \tilde{\vec{E}}^{(inc)}(\Psi,\ \phi;\ s) \tag{6.10}$$

Rotating $\tilde{\vec{E}}^{(inc)}$ by rotating $\vec{1}_2$ (parallel to the x, y plane) then gives

$$\tilde{\vec{Y}}_s^{(c)}\left(\Psi,\ \phi + \frac{\pi}{2};\ s\right) = \vec{\tau}_d \cdot \tilde{\vec{Y}}_s(\Psi,\ \phi;\ s) \cdot \vec{\tau}_d^{\,T} \tag{6.11}$$

This is the same as the self-complementary condition for the sheet admittance in (2.2) except that ϕ_C is limited to $\pi/2$ (or also $-\pi/2$). This is C_{2c} symmetry for the sheet admittance. As in (4.7) and (4.8) C_{Nc} is consistent with this for certain values of N (2, 6, 10, $\cdots$).

Within the above constraints a normally incident plane wave gives a set of self-complementary equivalent sources. Then the results of section 4 apply and the scattered fields are TEM on the z axis as in (4.14) and (4.15). Note that the scattered fields are symmetric with respect to the x, y plane so this result applies to both +z and -z, i.e. forward scattering and backscattering.

A special case has

$$\tilde{y}_s = 1 \quad , \quad \tilde{Z}_s = \frac{Z_o}{2} \tag{6.12}$$

uniformly over S. The normally incident plane wave ($\vec{1}_2$ direction) scatters from S in a very simple form for positive z as

$$\vec{E}^{(sc)}(\vec{r},\, t) = -\frac{1}{2}\, \vec{E}^{(inc)}(\vec{r},t) = -\frac{1}{2}\, E_o f\left(t - \frac{\vec{1}_z \cdot \vec{r}}{c} \right) \vec{1}_2$$

$$\vec{H}^{(sc)}(\vec{r},\, t) = -\frac{1}{2}\, \vec{H}^{(inc)}(\vec{r},t) = -\frac{1}{2}\, \frac{E_o}{Z_o}\, f\left(t + \frac{\vec{1}_z \cdot \vec{r}}{c} \right) \vec{1}_3 \tag{6.13}$$

and for negative z as

$$\vec{E}^{(sc)}(\vec{r},\, t) = -\frac{1}{2}\, E_o\, f\left(t + \frac{\vec{1}_z \cdot \vec{r}}{c} \right) \vec{1}_2$$

$$\vec{H}^{(sc)}(\vec{r},\, t) = +\frac{1}{2}\, \frac{E_o}{Z_o}\, f\left(t + \frac{\vec{1}_z \cdot \vec{r}}{c} \right) \vec{1}_3 \tag{6.14}$$

Note that besides being TEM the polarization (except for sign) coincides with the incident field. Note that the form in (6.12), besides being self complementary is self inverse in the sense of section 3A. Being independent of ϕ this form of sheet admittance has C_∞ symmetry and hence $C_{\infty c}$ symmetry, of which C_{2c} is a subgroup.

VII. Concluding Remarks

As should be clear now the self-complementary character of planar structures has some subtle characteristics. There is the distinction between the general C_{Nc} symmetry of the admittance sheet (or screen on S) and the C_{2c} symmetry of the sources and fields. Noting the special properties of the C_N and C_{Nc} symmetries of various electromagnetic structures, particularly as N-terminal networks, further investigation should be helpful for such cases.

Appendix A. Duality and Combined Fields

Including magnetic as well as electric sources the Maxwell equations in free space are

$$\nabla \times \vec{E} = -\mu_o \frac{\partial}{\partial t} \vec{H} - \vec{J}_m$$

$$\nabla \times \vec{H} = \epsilon_o \frac{\partial}{\partial t} \vec{E} + \vec{J} \quad \text{(A.1)}$$

These are cast in combined form as

$$\vec{E}_q \equiv \vec{E} + q\, j Z_o \vec{H}$$

$$\vec{J}_q \equiv \vec{J} + \frac{qj}{Z_o} \vec{J}_m \quad \text{(A.2)}$$

$$Z_o \equiv \sqrt{\frac{\mu_o}{\epsilon_o}} \equiv \text{wave impedance of free space}$$

$$\cong 377\Omega$$

$$q \equiv \text{separation index}$$

$$= \pm 1$$

These satisfy the combined Maxwell equation

$$\left[\nabla \times \frac{-qj}{c} \frac{\partial}{\partial t}\right] \vec{E}_q = qj\, Z_o\, \vec{J}_q \quad \text{(A.3)}$$

$$c \equiv \frac{1}{\sqrt{\mu_o\, \epsilon_o}} \equiv \text{speed of light}$$

$$\cong 3 \times 10^8 \; m/s$$

Duality refers to the symmetry between the electric and magnetic quantities. This is expressed in combined form by

$$\vec{E}_q^{(d)} = -qj\, \vec{E}_q \equiv \text{dual combined field}$$

$$\vec{J}_q^{(d)} = -qj\, \vec{J}_q \equiv \text{dual combined current density} \quad \text{(A.4)}$$

Note that the dual quantities also solve the combined Maxwell equation (A.3). In terms of separate electric and magnetic parameters we have

$$\vec{E}^{(d)} = Z_o \vec{H}$$

$$\vec{H}^{(d)} = -\frac{1}{Z_o} \vec{E}$$

$$\vec{J}^{(d)} = \frac{1}{Z_o} \vec{J}_m$$

$$\vec{J}_m^{(d)} = -Z_o \vec{J} \qquad (A.5)$$

This dual set also satisfy the Maxwell equations (A.1)

Appendix B. Complementary Sheet Admittances and Surface-Current-Density Sources

Now let us restrict currents to a plane S, the z=0 plane, as in fig. B.1. Let there be an electric surface current density on this plane with boundary conditions

$$\tilde{\vec{J}}_s = \tilde{\vec{\vec{Y}}} \cdot \tilde{\vec{E}} + \tilde{\vec{J}}^{(s)}$$

$$= \vec{1}_z \times \left[\tilde{\vec{H}}_+ - \tilde{\vec{H}}_-\right] \tag{B.1}$$

$\tilde{\vec{J}}_s^{(s)}$ ≡ source electric surface current density

$\tilde{\vec{\vec{Y}}}_s$ ≡ dyadic sheet admittance on plane

$\sim$ ≡ Laplace transform (2 sided) from time t to complex frequency s

Here $\pm$ refers to z=0$\pm$, or just adjacent to S on the indicated side. Note that the tangential magnetic field is discontinuous in passing through the plane (as is the normal electric field).

Consider the case that S (and its complement) correspond to an antenna in transmission (and by reciprocity in reception). Then there is no incident field and the fields are symmetric with respect to S.

With no incident field the wave propagates outward on both sides of S giving symmetric fields [2,4]

$$\vec{r}_m \equiv \vec{\vec{R}} \cdot \vec{r}$$

$$\vec{E}(\vec{r}_m, t) = \vec{\vec{R}} \cdot \vec{E}(\vec{r}, t)$$

$$\vec{H}(\vec{r}_m, t) = -R \cdot \vec{H}(\vec{r}, t) \tag{B.2}$$

$$\vec{\vec{R}} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \vec{\vec{1}}_t - \vec{1}_z\vec{1}_z \equiv \text{ reflection dyad}$$

$$\vec{\vec{1}}_t \equiv \vec{1}_x\vec{1}_x + \vec{1}_y\vec{1}_y = \vec{\vec{1}} - \vec{1}_z\vec{1}_z \equiv \text{ transverse dyad}$$

$$\vec{\vec{1}} \equiv \text{ identity dyad}$$

Note on S it is only the tangential fields of concern in (B.1) for which we can take

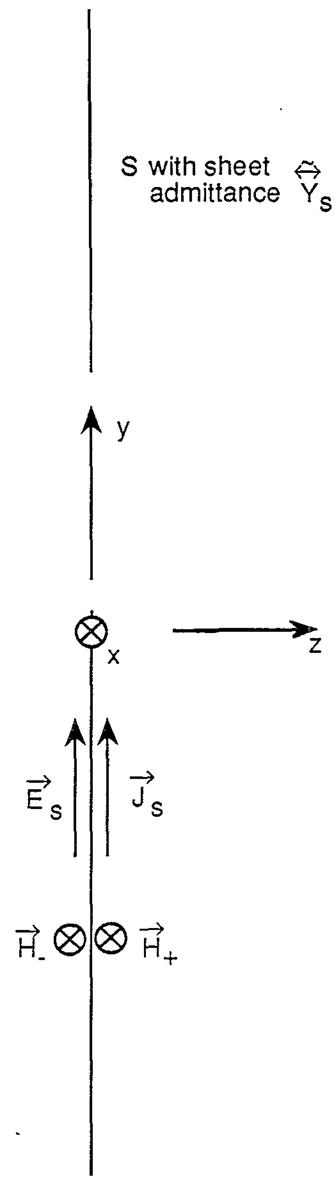


Fig. B.1. Boundary Conditions on Plane

$$\vec{E}_s \equiv \vec{1}_t \cdot \vec{E}$$

$$\vec{H}_s \equiv \vec{1}_t \cdot \vec{H}_+ = -\vec{1}_t \cdot \vec{H}_- \tag{B.3}$$

Then (B.1) becomes

$$\vec{J}_s = \vec{\vec{Y}}_s \cdot \vec{E}_s + \vec{J}_s^{(s)} = 2\,\vec{1}_z \times \vec{H}_s = 2\vec{\tau}_d \cdot \vec{H}_s$$

$$\vec{\tau}_d = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\vec{\tau}_d = \vec{1}_z \times \tag{B.4}$$

Considering now only the tangential fields we can deal with only 2 x 2 matrices without ambiguity as

$$\vec{\vec{Y}}_s \equiv \begin{pmatrix} \tilde{Y}_{s_{x,x}} & \tilde{Y}_{s_{x,y}} \\ \tilde{Y}_{s_{y,x}} & \tilde{Y}_{s_{y,y}} \end{pmatrix}$$

$$\tilde{Y}_{s_{y,x}} \equiv \tilde{Y}_{s_{x,y}} \quad \text{(reciprocity)}$$

$$\vec{1}_t \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\vec{\tau}_d \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \equiv \pi/2 \text{ rotation} \tag{B.5}$$

Then the two forms of $\vec{J}_s$ in (B.4) can be considered as two dimensional, only involving x and y components on or immediately adjacent to S. Note that in this two-dimensional case

$$\vec{1}_t^{-1} = \vec{1}_t$$

$$\vec{\tau}_d^{-1} = \vec{\tau}_d^{T} = -\vec{\tau}_d \tag{B.6}$$

Now consider the complementary problem. For z>0 let us have the dual fields as in (A.5). However, do not introduce a magnetic surface current density on S, but let there be some equivalent electric surface current density satisfying the boundary conditions (B.4). Let this surface current density and the sheet admittance be designated as complementary (not dual) quantities with superscript c giving

$$\tilde{J}_s^{(c)} = \tilde{\tilde{Y}}_s^{(c)} \cdot \tilde{E}_s^{(d)} + \tilde{J}_s^{(c,s)} = 2\ \vec{1}_z \times \tilde{H}_s^{(d)} = 2\tilde{\tau}_d \cdot \tilde{H}_s^{(d)} \qquad \text{(B.7)}$$

Note that this distinction between complementary and dual avoids magnetic currents, but changes signs of tangential fields in passing through S. For convenience we consider z=0+ for these results.

Now replace the quantities in (B.4) by the dual quantities (z = 0+) from (A.5) giving

$$\tilde{J}_s = -Z_o\ \tilde{\tilde{Y}}_s \cdot \tilde{H}_s^{(d)} + \tilde{J}_s^{(s)} = \frac{2}{Z_o}\ \vec{1}_z \times \tilde{E}_s^{(d)} = \frac{2}{Z_o}\ \tilde{\tau}_d \cdot \tilde{E}_s^{(d)} \qquad \text{(B.8)}$$

First assume we are on some region of S with no sources. Then (B.7) and (B.8) give

$$-\frac{1}{2}\ \tilde{\tau}_d \cdot \tilde{\tilde{Y}}_s^{(c)} \cdot \tilde{E}_s^{(d)} = \tilde{H}_s^{(d)}$$

$$\frac{Z_o^2}{2}\ \tilde{\tau}_d \cdot \tilde{\tilde{Y}}_s \cdot \tilde{H}_s^{(d)} = \tilde{E}_s^{(d)} \qquad \text{(B.9)}$$

which can be combined to give

$$-\frac{Z_o^2}{4}\ \tilde{\tau}_d \cdot \tilde{\tilde{Y}}_s^{(c)} \cdot \tilde{\tau}_d \cdot \tilde{\tilde{Y}}_s \cdot \tilde{H}_s^{(d)} = \tilde{H}_s^{(d)} \qquad \text{(B.10)}$$

For arbitrary $\tilde{H}_s$ this implies

$$\frac{Z_o^2}{4}\ \tilde{\tau}_d \cdot \tilde{\tilde{Y}}_s^{(c)} \cdot \tilde{\tau}_d \cdot \tilde{\tilde{Y}}_s = -\vec{1}_t$$

$$\tilde{\tilde{Y}}_s^{(c)} = -\frac{4}{Z_o^2}\ \tilde{\tau}_d \cdot \tilde{\tilde{Y}}_s^{-1} \cdot \tilde{\tau}_d = -\frac{4}{Z_o^2}\ \tilde{\tau}_d \cdot \tilde{\tilde{Y}}_s^{-1} \cdot \tilde{\tau}_d^T \qquad \text{(B.11)}$$

This is the basic complementary admittance relationship derived in [3]. Normalizing the admittance as

$$\tilde{\tilde{y}}_s \equiv \frac{2}{Z_o}\ \tilde{\tilde{Y}}_s \ , \quad \tilde{\tilde{y}}_s^{(c)} \equiv \frac{2}{Z_o}\ \tilde{\tilde{Y}}_s^{(c)} \qquad \text{(B.12)}$$

we have

$$\tilde{\tilde{y}}_s^{(c)} = -\tilde{\tau}_d \cdot \tilde{\tilde{y}}_s^{-1} \cdot \tilde{\tau}_d = \tilde{\tau}_d \cdot \tilde{\tilde{y}}_s^{-1} \cdot \tilde{\tau}_d^T \tag{B.13}$$

Second, assume we are on some region of S with only sources, i.e., let both $\tilde{\tilde{Y}}_s$ and $\tilde{\tilde{Y}}_s^{(c)}$ be zero (no conduction surface current density). This is then a source region. Then (B.7) and (B.8) become

$$\vec{J}_s^{(c,s)} = 2\ \tilde{\tau}_d \cdot \vec{H}_s^{(d)} = -\frac{2}{Z_o}\ \tilde{\tau}_d \cdot \vec{E}_s$$

$$\vec{J}_s^{(s)} = 2\ \tilde{\tau}_d \cdot \vec{H}_s = -\frac{2}{Z_o}\ \tilde{\tau}_d \cdot \vec{E}_s^{(d)} \tag{B.14}$$

Appendix C. Inclusion of Incident Fields

Consider that there are some incident fields to be included in the formalism. Then (B.1) is modified in the form

$$\vec{E} = \vec{E}^{(inc)} + \vec{E}^{(sc)} = \vec{E}_+ \ (\text{on} \ z = 0+)$$

$$\vec{H} = \vec{H}^{(inc)} + \vec{H}^{(sc)} = \vec{H}_+ \ (\text{on} \ z = 0+)$$

$$\tilde{\vec{J}}_s = \tilde{\tilde{Y}}_s \cdot \tilde{\vec{E}}_+ = \tilde{\tilde{Y}}_s \cdot \tilde{\vec{E}}^{(inc)} + \tilde{\tilde{Y}}_s \cdot \tilde{\vec{E}}^{(sc)}$$

$$= \vec{1}_z \times \left[\tilde{\vec{H}}_+ - \tilde{\vec{H}}_-\right] = \vec{1}_z \times \left[\tilde{\vec{H}}_+^{(sc)} - \tilde{\vec{H}}_-^{(sc)}\right]$$

$$= 2 \ \vec{1}_z \times \tilde{\vec{H}}_+^{(sc)} = 2 \ \tilde{\tau}_d \cdot \tilde{\vec{H}}_+^{(sc)} \qquad \text{(C.1)}$$

As we can see from this by comparison to (B.1), the previous results (without incident fields) are extendable to this case by the identification

$$\tilde{\vec{J}}_s^{(s)} = \tilde{\tilde{Y}}_s \cdot \tilde{\vec{E}}^{(inc)} \qquad \text{(C.2)}$$

where the scattered fields correspond to the previous fields. These scattered fields are symmetric with respect to S in the sense of (B.2).

An alternate formulation is in terms of the total field for z=0+ in which case we have

$$\tilde{\vec{J}}_s = \tilde{\tilde{Y}}_s \cdot \tilde{\vec{E}} = 2 \ \tilde{\tau}_d \cdot \tilde{\vec{H}}_+ - \tilde{\vec{J}}_s^{(sc)}$$

$$\tilde{\vec{J}}_s^{(sc)} = -2 \ \tilde{\tau}_d \cdot \tilde{\vec{H}}^{(inc)} \equiv \text{short-circuit surface current density} \qquad \text{(C.3)}$$

So one has a source in terms of the incident electric field in (C.2) or the incident magnetic field in (C.3).

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